AARHUS UNIVERSITET

TENSOR PRODUCTS OF C*-ALGEBRAS

PART I.

FINITE TENSOR PRODUCTS

by

A. Guichardet

April 1969

Lecture Notes Series

No. 12.

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Introduction

Given two C^*- algebras A_1 and A_2 , we can form their algebraic tensor product $A_1 \otimes A_2$ and look for reasonable norms on it such that the completion is a C^*- algebra ; more precisely we shall study the C^*- crossnorms and define in a natural manner two such norms : the largest one and the smallest one ; their properties are rather similar to those of the norms $\wedge$ and $\hat{\wedge}$ which arise in the case of Banach spaces . Then we shall be concerned with the properties of the completions of $A_1 \otimes A_2$ with respect to these norms, in particular their types (liminar, postliminar, antiliminar, with continuous trace), representations, states and traces. The results are still valid for the tensor product of an arbitrary finite number of C^*- algebras, but we consider the case of two C^*- algebras for the sake of simplicity of notations.

Throughout these lectures we shall use the notations of Dixmier's book on C^*- algebras ; moreover $S(A)$ will denote the set of all states of a C^*- algebra A , endowed with the weak topology ; it is compact if A has a unit element, but not necessarily locally compact in the general case ; $C_1(A)$ will denote the set of all normed characters (or central states) of A, also endowed with the weak topology ;it is not locally compact in general, even when A has a unit element (see [7], § 3, prop.6), but is Polish if A is separable (see [2], 7.4.2). For each Banach $*$- algebra A with approximate identity we denote by $C^*(A)$ the enveloping C^*- algebra of A.

A C^*- algebra is said to be <u>simple</u> if it has no proper closed two-sided ideal ; a state f on a C^*- algebra is <u>factorial</u> if the associated representation π_f is factorial. All our vector spaces and algebras are complex ; for any two vector spaces or algebras E_1 , E_2, we denote by $E_1 \otimes E_2$ their <u>algebraic</u> tensor product ; every element x of this space can be written, but not in a unique manner, as $x = \sum_{n=1}^{N} x_{1,n} \otimes x_{2,n}$.

§ 1. Preliminaries.

n.1.1. Tensor products of Banach spaces.

If E_1 and E_2 are Banach spaces, we say that a seminorm p on $E_1 \otimes E_2$ is a cross seminorm (resp. a subcross seminorm) if

$$p(x_1 \otimes x_2) = (\text{resp.} \leq) \ \|x_1\| \cdot \|x_2\| \quad \forall\, x_i \in E_i \, .$$

The $\wedge$ crossnorm.

There exists a largest subcross seminorm on $E_1 \otimes E_2$ (which is in fact a crossnorm) :

$$\|x\|_\wedge = \inf \sum_{n=1}^{N} \|x_{1,n}\| \cdot \|x_{2,n}\|$$

where the inf is taken for all families $(x_{1,n}, x_{2,n})$ satisfying $x = \sum_{n=1}^{N} x_{1,n} \otimes x_{2,n}$. The completion of $E_1 \otimes E_2$ for this crossnorm will be denoted by $E_1 \hat{\otimes} E_2$; it possesses the following universal property : for each continuous bilinear mapping u of $E_1 \times E_2$ into a Banach space F there exists a unique continuous linear mapping $v : E_1 \hat{\otimes} E_2 \longrightarrow F$ such that $v(x_1 \otimes x_2) = u(x_1, x_2)$.

Functorial properties. For $i = 1,2$ let u_i be a continuous linear mapping of E_i into a Banach space F_i ; the linear mapping $u_1 \otimes u_2 : E_1 \otimes E_2 \longrightarrow F_1 \otimes F_2$ can be extended in a continuous linear mapping $u_1 \hat{\otimes} u_2 : E_1 \hat{\otimes} E_2 \longrightarrow F_1 \hat{\otimes} F_2$; evidently

$$(u_1 \hat{\otimes} u_2)(x_1 \otimes x_2) = u_1(x_1) \otimes u_2(x_2) \, .$$

If u_i is surjective and F_i has the quotient norm of E_i, then $u_1 \hat{\otimes} u_2$ is surjective, $F_1 \hat{\otimes} F_2$ has the quotient norm of $E_1 \hat{\otimes} E_2$ and moreover

$$\operatorname{Ker} u_1 \hat{\otimes} u_2 = \overline{\operatorname{Ker} u_1 \otimes E_2 + E_1 \otimes \operatorname{Ker} u_2}\ .$$

Example 1. If X is a measurable space with a measure μ , and if E is a Banach space, $L^1(X, \mu) \hat{\otimes} E$ is canonically isomorphic to $L^1(X, \mu, E)$, the space of all μ-integrable mappings of X into E ; this isomorphism carries each element of the form $f \otimes \xi$ into the mapping $x \longmapsto f(x).\xi$. In particular

$$L^1(X_1, \mu_1) \hat{\otimes} L^1(X_2, \mu_2) \sim L^1(X_1 \times X_2, \mu_1 \otimes \mu_2)\ .$$

The $\hat{\hat{}}$ crossnorm.

If f_i is a linear functional on E_i we can consider the linear functional $f_1 \otimes f_2$ on $E_1 \otimes E_2$ characterized by the property

$$(f_1 \otimes f_2)(x_1 \otimes x_2) = f_1(x_1).f_2(x_2)\ ;$$

the $\hat{\hat{}}$ norm is defined by

$$\| x \|_{\hat{\hat{}}} = \sup |(f_1 \otimes f_2)(x)|$$

where the sup is taken for all f_i which are continuous and of norm ≤ 1 ; this is the smallest crossnorm which is reasonable in a certain sense (see [6]) ; the completion of $E_1 \otimes E_2$ for this norm will be denoted by $E_1 \hat{\hat{\otimes}} E_2$.

Functorial properties. If we have continuous linear mappings $u_i : E_i \longrightarrow F_i$ there is a continuous linear mapping $u_1 \hat{\hat{\otimes}} u_2$: $E_1 \hat{\hat{\otimes}} E_2 \longrightarrow F_1 \hat{\hat{\otimes}} F_2$; if u_i is isometric, $u_1 \hat{\hat{\otimes}} u_2$ is also isometric.

Example 2. If X is a locally compact topological space we denote by $\underline{C}_o(X)$ the space of all complex continuous functions f on X which vanish at infinity ; that means that for each number

$a > 0$ the set of $x \in X$ such that $|f(x)| \geq a$ is compact ; this is a Banach space for the sup-norm ; if X is compact we write $\underline{C}(X)$ instead of $\underline{C}_0(X)$. Then if E is a Banach space, $\underline{C}_0(X) \hat{\hat{\otimes}} E$ is canonically isomorphic to $\underline{C}_0(X,E)$; in particular

$$\underline{C}_0(X_1) \hat{\hat{\otimes}} \underline{C}_0(X_2) \sim \underline{C}_0(X_1 \times X_2) .$$

Definition 1. If $\| \quad \|_\alpha$ is any subcross norm on $E_1 \otimes E_2$ we denote by $E_1 \overset{\alpha}{\otimes} E_2$ the completion of $E_1 \otimes E_2$ for this norm.

Bibliography. [6],[18].

n.1.2. Tensor products of Banach $*$-algebras.

(For the definition of Banach $*$-algebras, see [2], 1.2.1 ; see also [4] and [5].)

Let us consider two Banach algebras A_1 and A_2 ; $A_1 \otimes A_2$ is an algebra with a multiplication verifying

$$(a_1 \otimes a_2).(b_1 \otimes b_2) = a_1b_1 \otimes a_2b_2 \quad ;$$

the $\hat{\hat{}}$ norm is not always a norm of algebra, i.e. does not always verify $\|a\,b\|_{\hat{\hat{}}} \leq \|a\|_{\hat{\hat{}}} . \|b\|_{\hat{\hat{}}}$ (see [5]) ; but it is easy to see that the $\wedge$ norm does ; if A_1 and A_2 are Banach $*$-algebras $A_1 \otimes A_2$ is a normed $*$-algebra with an involution satisfying

$$(a_1 \otimes a_2)^* = a_1^* \otimes a_2^* \quad ;$$

then $A_1 \hat{\otimes} A_2$ is a Banach $*$-algebra ; it possesses the following universal property : let us say that two morphisms u_1 and u_2 of A_1 and A_2 into a Banach $*$-algebra B commute if

$$u_1(a_1).u_2(a_2) = u_2(a_2).u_1(a_1) \qquad \forall\ a_i \in A_i \ ;$$

then for each pair (u_1, u_2) of commuting continuous morphisms there exists a unique continuous morphism $v : A_1 \hat{\otimes} A_2 \longrightarrow B$ such that

$$v(a_1 \otimes a_2) = u_1(a_1).u_2(a_2) \qquad \forall\, a_i \in A_i .$$

Conversely if A_1 and A_2 admit unit elements, each continuous morphism v can be obtained in this manner.

Example 3. If G_1 and G_2 are locally compact groups, the Banach $*$-algebra $L^1(G_1 \times G_2)$ is canonically isomorphic to the tensor product $L^1(G_1) \hat{\otimes} L^1(G_2)$.

n.1.3. Tensor products of Hilbert spaces and von Neumann algebras.

(See [1])

The Hilbert tensor product of two Hilbert spaces H_1, H_2 will be denoted by $H_1 \overset{h}{\otimes} H_2$; we recall that it is the completion of $H_1 \otimes H_2$ for a scalar product which verifies

$$(x_1 \otimes x_2 \mid y_1 \otimes y_2) = (x_1 \mid y_1).(x_2 \mid y_2) .$$

If $\mathcal{A}_i$ is a von Neumann algebra in H_i the von Neumann algebra in $H_1 \overset{\ell}{\otimes} H_2$ generated by all operators $a_1 \otimes a_2$ with $a_i \in \mathcal{A}_i$ will be denoted by $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2$; it is a factor iff $\mathcal{A}_1$ and $\mathcal{A}_2$ are factors ; it is equal to $\mathcal{L}(H_1 \overset{\ell}{\otimes} H_2)$ iff $\mathcal{A}_i = \mathcal{L}(H_i)$.

Proposition 0. Let $\mathcal{A}$ be a factor in a Hilbert space H and $\mathcal{A}'$ its commutant ; the morphism $u : \mathcal{A} \otimes \mathcal{A}' \longrightarrow \mathcal{L}(H)$ defined by $u(a \otimes a') = a\, a'$ is injective ; in particular if a and a' are not zero, a a' is also not zero.

For the proof see [1], p. 31, exercice 6.

Distributivity with respect to Hilbert integrals.

For $i = 1,2$ let X_i be a measurable space with a measure μ_i , $s_i \longmapsto H_{i,s_i}$ a μ_i-measurable field of Hilbert spaces ; it is easy to construct on the field $(s_1,s_2) \longmapsto H_{1,s_1} \overset{\ell}{\otimes} H_{2,s_2}$ a structure of $\mu_1 \otimes \mu_2$-measurable field of Hilbert spaces and an isomorphism

$$U : \int^{\oplus} H_{1,s_1}\, d\mu_1(s_1) \overset{\ell}{\otimes} \int^{\oplus} H_{2,s_2}\, d\mu_2(s_2) \longrightarrow \iint^{\oplus} H_{1,s_1} \overset{\ell}{\otimes} H_{2,s_2} \cdot d(\mu_1 \otimes \mu_2)(s_1,s_2)$$

with the following property : for each vector $x_i = \int^{\oplus} x_{i,s_i} \cdot d\mu_i(s_i)$ one has

$$U(x_1 \otimes x_2) = \iint^{\oplus} x_{1,s_1} \otimes x_{2,s_2} \cdot d(\mu_1 \otimes \mu_2)(s_1,s_2) .$$

§ 2. Representations of the algebraic tensor product of two C^*-algebras.

We consider two C^*-algebras A_1 and A_2 .

n.2.1. Tensor products of representations.

Let π_i be a representation of A_i in a Hilbert space H_i ; we can form the representation of $A_1 \otimes A_2$ in $H_1 \overset{h}{\otimes} H_2$ defined by

$$(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2) \ ;$$

the von Neumann algebra generated by this representation is clearly $\pi_1(A_1)'' \overset{c}{\otimes} \pi_2(A_2)''$; it follows that $\pi_1 \otimes \pi_2$ is factorial (resp. irreducible) iff π_1 and π_2 have the same property. The equivalence (resp. quasiequivalence) class of $\pi_1 \otimes \pi_2$ depends only on the analogous classes for π_1 and π_2 .

If we have a measurable field of representations π_{i,s_i} , we can write, with the notations of n.1.3

$$\int^{\oplus} \pi_{1,s_1} \, d\mu_1(s_1) \otimes \int^{\oplus} \pi_{2,s_2} \, d\mu_2(s_2) \simeq$$

$$\iint^{\oplus} \pi_{1,s_2} \otimes \pi_{2,s_2} \, . \, d(\mu_1 \otimes \mu_2)(s_1,s_2) \ ;$$

in particular for discrete sums

$$(\bigoplus_s \pi_{1,s}) \otimes (\bigoplus_t \pi_{2,t}) \simeq \bigoplus_{s,t} (\pi_{1,s} \otimes \pi_{2,t}) .$$

n.2.2. Restrictions of a representation of $A_1 \otimes A_2$.

Definition 2. We shall say that a representation π of $A_1 \otimes A_2$ is a subcross representation if

$$\|\pi(a_1 \otimes a_2)\| \leqslant \|a_1\| . \|a_2\| \qquad \forall\ a_i \in A_i .$$

If A_1 and A_2 have unit elements e_1 and e_2, every representatiom of $A_1 \otimes A_2$ is subcross because we have

$$\begin{aligned} \|\pi(a_1 \otimes a_2)\| &= \|\pi(a_1 \otimes e_2 \,.\, e_1 \otimes a_2)\| \\ &\leqslant \|\pi(a_1 \otimes e_2)\| . \|\pi(e_1 \otimes a_2)\| \\ &\leqslant \|a_1\| . \|a_2\| \end{aligned}$$

since $a_1 \longmapsto \pi(a_1 \otimes e_2)$ and $a_2 \longmapsto \pi(e_1 \otimes a_2)$ are representations of A_1 and A_2 respectively.

Proposition 1. To each subcross representation π of $A_1 \otimes A_2$ in a Hilbert space H one can associate canonically representations π_1 and π_2 of A_1 and A_2 in H such that

$$\pi(a_1 \otimes a_2) = \pi_1(a_1).\pi_2(a_2) = \pi_2(a_2).\pi_1(a_1) \qquad (1)$$

for each a_i in A_i ; moreover one has

$$\pi_1(a_1) = \text{strong limit of } \pi(a_1 \otimes v_t)$$

$$\pi_2(a_2) = \text{strong limit of } \pi(u_s \otimes a_2)$$

where (u_s) and (v_t) are arbitrary approximate identities of A_1 and A_2 . Finally if π is faithful or non degenerate or factorial, π_1 and π_2 have the same property.

Proof. We choose an approximate identity (u_s) of A_1 and prove that for each a_2 in A_2 , $\pi(u_s \otimes a_2)$ has a strong limit ; set $H_1 = \pi(A_1 \otimes A_2).H$; we must prove that $\pi(u_s \otimes a_2).x$ has a limit for each $x \in \overline{H}_1$; since the family $\pi(u_s \otimes a_2)$ is bounded (because π is subcross), we can take x in H_1 ; by linearity we can suppose x has the following form

$$x = \pi(b_1 \otimes b_2).y \quad \text{where} \quad b_i \in A_i \ , \ y \in H \ ;$$

then

$$\pi(u_s \otimes a_2).x = \pi(u_s b_1 \otimes a_2 b_2).y \ ;$$

this converges to $\pi(b_1 \otimes a_2 b_2).y$ since

$$\|\pi(u_s b_1 \otimes a_2 b_2).y - \pi(b_1 \otimes a_2 b_2).y\| = \|\pi((u_s b_1 - b_1) \otimes a_2 b_2).y\|$$

$$\leq \|u_s b_1 - b_1\| . \|a_2 b_2\| . \|y\| \ ;$$

we have thus proved that $\pi(u_s \otimes a_2)$ has a strong limit which is independant of the approximate identity (u_s) ; denote it by $\pi_2(a_2)$; as easily verified π_2 is a representation ; define π_1 in an analogous manner ; to prove (1) :

$$\begin{aligned}\pi_1(a_1).\pi_2(a_2) &= \lim \pi(a_1 \otimes v_t). \lim \pi(u_s \otimes a_2) \\ &= \lim \pi(a_1 u_s \otimes v_t a_2) \\ &= \pi(a_1 \otimes a_2)\end{aligned}$$

since

$$\|\pi(a_1 u_s \otimes v_t a_2) - \pi(a_1 \otimes a_2)\|$$

$$\leq \|\pi((a_1 u_s - a_1) \otimes v_t a_2)\| + \|\pi(a_1 \otimes (v_t a_2 - a_2))\|$$

$$\leq \|a_1 u_s - a_1\| . \|a_2\| + \|a_1\| . \|v_t a_2 - a_2\| .$$

The last assertion is trivial.

<u>Remark</u> 1. From the above proof we also deduce the following : if A_i is a Banach $*$-algebra with approximate identity, to each

subcross representation π of $A_1 \otimes A_2$ one can associate representations π_1 and π_2 of A_1 and A_2 verifying (1).

Definition 3. The representations π_1 and π_2 associated with π will be called the restrictions of π to A_1 and A_2 ; if A_1 and A_2 have unit elements, they are nothing but the usual restrictions of π .

Proposition 2. If π_1 (or π_2) is a type I factor representation, π is equivalent to a tensor product of representations. In fact we can write

$$H = H_1 \overset{h}{\otimes} H_2$$

$$\pi_1(a_1) = \rho_1(a_1) \otimes I$$

$$\pi_2(a_2) = I \otimes \rho_2(a_2)$$

where ρ_i is some representation of A_i in H_i ; then

$$\pi(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2) .$$

Proposition 3. If π_i is a representation of A_i with π_1 non degenerate, the restriction of $\pi_1 \otimes \pi_2$ to A_2 is a multiple of π_2. In fact this restriction is given by

$$\begin{aligned} \rho_2(a_2) &= \lim (\pi_1 \otimes \pi_2)(u_s \otimes a_2) \\ &= \lim \pi_1(u_s) \cdot \pi_2(a_2) \\ &= I \otimes \pi_2(a_2) . \end{aligned}$$

Lemma 1. If A_1 has no unit element, each C^* subcross norm p on $A_1 \otimes A_2$ can be extended to a C^* subcross norm $\tilde{p}$ on $\tilde{A}_1 \otimes A_2$. The same holds for A_2.

Proof. Choose a non degenerate isometric representation π of $A_1 \otimes A_2$ in a space H ; its restrictions π_1 and π_2 are faithful and non degenerate ; π_1 extends to a representation $\tilde{\pi}_1$ of $\tilde{A}_1$, which is faithful because $\pi_1(A_1)$ does not contain the scalars ; the bilinear mapping

$$\tilde{A}_1 \times A_2 \longrightarrow \mathcal{L}(H)$$

$$((a_1,h_1),a_2) \longmapsto \tilde{\pi}_1(a_1,h_1).\pi_2(a_2)$$

gives rise to a linear mapping

$$\rho : \quad \tilde{A}_1 \otimes A_2 \longrightarrow \mathcal{L}(H)$$

$$\sum_{n=1}^{N} (a_{1,n},h_{1,n}) \otimes a_{2,n} \longmapsto \sum_{n=1}^{N} \pi_1(a_{1,n},h_{1,n}).\pi_2(a_{2,n}) \;;$$

ρ is easily verified to be a representation. We now prove that ρ is <u>faithful</u> : take an element b in Ker ρ ; for each element a in $A_1 \otimes A_2$ we have

$$b\,a \in \operatorname{Ker} \rho \cap (A_1 \otimes A_2) \;=\; \operatorname{Ker} \pi = \{0\} \;;$$

then for each x in $H \overset{\ell}{\otimes} H$:

$$0 = (\tilde{\pi}_1 \otimes \pi_2)(ba).x = (\tilde{\pi}_1 \otimes \pi_2)(b).(\pi_1 \otimes \pi_2)(a).x \;;$$

since π_1 and π_2 are non degenerate, the same holds for $\pi_1 \otimes \pi_2$, the elements $(\pi_1 \otimes \pi_2)(a).x$ are dense in $H \overset{\ell}{\otimes} H$, and we see that $(\tilde{\pi}_1 \otimes \pi_2)(b) = 0$; since $\tilde{\pi}_1 \otimes \pi_2$ is faithful, $b = 0$ and ρ is faithful.

Now setting $\tilde{p}(a) = \| \rho(a) \|$ for each a in $\tilde{A}_1 \otimes A_2$ we get a C^* norm which clearly extends p and is subcross since

$$\begin{aligned}\tilde{p}((a_1,h_1) \otimes a_2) &= \| \rho((a_1,h_1) \otimes a_2) \| \\ &\leqslant \| \tilde{\pi}_1(a_1,h_1) \| . \| \pi_2(a_2) \| \\ &\leqslant \| (a_1,h_1) \| . \| a_2 \| .\end{aligned}$$

<u>Bibliography</u> [8], [20].

§ 3. The $\vee$ crossnorm.

n.3.1. Definition of the ν crossnorm.

We consider two C^*-algebras A_1 and A_2 ; denote by $\| \quad \|_\nu$ the LUB of all C^*-subcross seminorms on $A_1 \otimes A_2$; this is clearly a C^*-subcross seminorm majorized by $\| \quad \|_\wedge$; this is in fact a crossnorm because if π_i is a faithful representation of A_i , $\pi_1 \otimes \pi_2$ is a faithful representation of $A_1 \otimes A_2$ and

$$\|(\pi_1 \otimes \pi_2)(a_1 \otimes a_2)\| = \|a_1\| . \|a_2\| .$$

If A_1 and A_2 have units each C^*-seminorm on $A_1 \otimes A_2$ is majorized by $\| \quad \|_\nu$ since each representation is subcross. The elementary properties of the ν norm are summarized in the following :

<u>Theorem</u> 1. The LUB of all C^*-subcross seminorms on $A_1 \otimes A_2$ is a C^*-crossnorm $\| \quad \|_\nu$; the representations of the completion $A_1 \overset{\vee}{\otimes} A_2$ are in bijective correspondance with the subcross representations of $A_1 \otimes A_2$, and in particular with all representations of $A_1 \otimes A_2$ if A_1 and A_2 have units . The C^*-algebra $A_1 \overset{\vee}{\otimes} A_2$ has the following universal property : if we have two commuting morphisms u_i of A_i into some C^*-algebra B, there exists a unique morphism $u : A_1 \overset{\vee}{\otimes} A_2 \longrightarrow B$ such that $u(a_1 \otimes a_2) = u_1(a_1).u_2(a_2)$.

Note that the representations of $A_1 \overset{\vee}{\otimes} A_2$ are also the same as those of $A_1 \hat{\otimes} A_2$, so that $A_1 \overset{\vee}{\otimes} A_2$ is the enveloping C^*-algebra of $A_1 \hat{\otimes} A_2$. More generally we have the following :

<u>Proposition</u> 4. If A_1 and A_2 are Banach $*$-algebras with approximate identities, $C^*(A_1 \hat{\otimes} A_2)$ is canonically isomorphic to $C^*(A_1) \overset{\vee}{\otimes} C^*(A_2)$.

Proof. Let u_i and u the canonical morphisms $A_i \longrightarrow C^*(A_i)$ and $A_1 \hat{\otimes} A_2 \longrightarrow C^*(A_1 \hat{\otimes} A_2)$; take some faithful representation π of $C^*(A_1) \check{\otimes} C^*(A_2)$ in a Hilbert space H ; let π_i be the restriction of π to $C^*(A_i)$; $\pi_1 \circ u_1$ and $\pi_2 \circ u_2$ are commuting morphisms of A_1 and A_2 into $\mathcal{L}(H)$; hence there exists a representation ρ of $A_1 \hat{\otimes} A_2$ in H such that

$$\rho(a_1 \otimes a_2) = \pi_1(u_1(a_1)).\pi_2(u_2(a_2)) \quad ;$$

now there exists a representation $\tilde{\rho}$ of $C^*(A_1 \hat{\otimes} A_2)$ in H with

$$\tilde{\rho}(u(a)) = \rho(a) \qquad \forall\ a \in A_1 \hat{\otimes} A_2 \quad ;$$

in particular

$$\tilde{\rho}(u(a_1 \otimes a_2)) = \rho(a_1 \otimes a_2) = \pi_1(u_1(a_1)).\pi_2(u_2(a_2))$$
$$= \pi(u_1(a_1) \otimes u_2(a_2)) \quad ;$$

this proves that $\operatorname{Im} \tilde{\rho} = \operatorname{Im} \pi$; we must now show that $\tilde{\rho}$ is faithful or, equivalently, that for each representation σ of $C^*(A_1 \hat{\otimes} A_2)$ in a space K there exists a representation $\bar{\sigma}$ of $\operatorname{Im} \tilde{\rho}$ in K such that $\bar{\sigma} \circ \tilde{\rho} = \sigma$; set $\tau = \sigma \circ u$; by remark 1, τ admits restrictions τ_1 and τ_2 ; τ_i extends to a representation $\tilde{\tau}_i$ of $C^*(A_i)$ in K ; since $\tilde{\tau}_1$ and $\tilde{\tau}_2$ commute, they define a representation $\tilde{\tau}$ of $C^*(A_1) \check{\otimes} C^*(A_2)$ in K with

$$\tilde{\tau}(u_1(a_1) \otimes u_2(a_2)) = \tau_1(a_1).\tau_2(a_2)$$
$$= \tau(a_1 \otimes a_2)$$
$$= \sigma(u(a_1 \otimes a_2)) \quad ;$$

set $\bar{\sigma} = \tilde{\tau} \circ \pi^{-1}$; we have

$$\bar{\sigma}(\tilde{\rho}(u(a_1 \otimes a_2))) = \tilde{\tau}(u_1(a_1) \otimes u_2(a_2)) = \sigma(u(a_1 \otimes a_2))$$

whence $\bar{\sigma} \circ \tilde{\rho} = \sigma$.

<u>Corollary</u> 1. If G_1 and G_2 are locally compact groups, $C^*(G_1 \times G_2)$ is canonically isomorphic to $C^*(G_1) \check{\otimes} C^*(G_2)$.

<u>Bibliography</u> [10].

n.3.2. <u>Tensor products of states and representations</u>.

If π_i is a representation of a C^*-algebra A_i, the algebraic tensor product $\pi_1 \otimes \pi_2$ can be extended to a representation $\pi_1 \check{\otimes} \pi_2$ of $A_1 \check{\otimes} A_2$; this representation has the same properties as $\pi_1 \otimes \pi_2$ (see n.2.1.). On the other hand each representation π of $A_1 \check{\otimes} A_2$ admits restrictions π_1 and π_2 which have properties analogous to those of § 2.2. Consider now two states f_1, f_2 on A_1, A_2 ; $f_1 \otimes f_2$ is continuous for the ν norm because setting $f_i = \omega_{x_i} \circ \pi_i$ we have

$$f_1 \otimes f_2 = \omega_{x_1 \otimes x_2} \circ (\pi_1 \otimes \pi_2) \quad ;$$

its extension to $A_1 \check{\otimes} A_2$ will be denoted by $f_1 \check{\otimes} f_2$; it is easy to see that

$$\pi_{f_1 \check{\otimes} f_2} \simeq \pi_{f_1} \check{\otimes} \pi_{f_2} \quad ;$$

consequently $f_1 \check{\otimes} f_2$ is pure (resp. factorial) iff f_i is.

<u>Proposition</u> 5. The mappings $(f_1, f_2) \longmapsto f_1 \check{\otimes} f_2$ of $S(A_1) \times S(A_2)$ into $S(A_1 \check{\otimes} A_2)$ and $(\pi_1, \pi_2) \longmapsto \pi_1 \check{\otimes} \pi_2$ of $\hat{A}_1 \times \hat{A}_2$ into $\widehat{A_1 \check{\otimes} A_2}$ are continuous.

<u>Proof</u>. For the first mapping we must show that the mapping $(f_1, f_2) \longmapsto (f_1 \check{\otimes} f_2)(a)$ is continuous for each a in $A_1 \check{\otimes} A_2$; by equicontinuity we can take a in $A_1 \otimes A_2$ and the assertion becomes trivial.

The second mapping is obtained by passing to the quotients in the following commutative diagramm :

$$\begin{tikzcd} P(A_1) \times P(A_2) \arrow[r] \arrow[d, "T"'] & P(A_1 \check{\otimes} A_2) \arrow[d] \\ \hat{A}_1 \times \hat{A}_2 \arrow[r] & \widehat{A_1 \check{\otimes} A_2} \end{tikzcd}$$

and T is open as the direct product of two open mappings (see [2], 3.4.11).

§ 4. Definition and first properties of the $*$ crossnorm.

n.4.1. Definition of the $*$ crossnorm.

Lemma 2. Let A_i be a concrete C^*-algebra in a Hilbert space H_i ; realize $A_1 \otimes A_2$ in $H_1 \overset{h}{\otimes} H_2$ with the operator norm $\| \ \|$; then for each state f_i on A_i and each a in $A_1 \otimes A_2$ we have $|(f_1 \otimes f_2)(a)| \leqslant \|a\|$; for each representation π_i of A_i we have $\|(\pi_1 \otimes \pi_2)(a)\| \leqslant \|a\|$.

Proof. We have the first inequality for each pair of vector states since $\omega_{x_1} \otimes \omega_{x_2} = \omega_{x_1 \otimes x_2}$; then for each pair (f_1, f_2) where the states f_1 and f_2 have the form

$$f_1 = \omega_{x_{1,1}} + \dots + \omega_{x_{1,n_1}} \quad \text{with} \quad \Sigma \|x_{1,i}\|^2 = 1$$

$$f_2 = \omega_{x_{2,1}} + \dots + \omega_{x_{2,n_2}} \quad \text{with} \quad \Sigma \|x_{2,j}\|^2 = 1$$

because

$$\begin{aligned} |(f_1 \otimes f_2)(a)| &\leqslant \Sigma |(\omega_{x_{1,i}} \otimes \omega_{x_{2,j}})(a)| \\ &= \Sigma |\omega_{x_{1,i} \otimes x_{2,j}}(a)| \\ &\leqslant \Sigma \|x_{1,i}\|^2 . \|x_{2,j}\|^2 . \|a\| = \|a\| ; \end{aligned}$$

anf finally for each pair of states (f_1, f_2) by continuity since the f_i of the previous form are dense in $S(A_i)$ (see [2], 3.4.4).

Second assertion : π_i being a sum of cyclic representations we can assume π_i is cyclic, in a space K_i, with a cyclic vector x_i which defines a state f_i of A_i ; denote by f the extension of $f_1 \otimes f_2$ to A , the uniform closure of $A_1 \otimes A_2$ in $\mathcal{L}(H_1 \overset{h}{\otimes} H_2)$ and consider the representation π_f of A in a space H_f ; as in [2], 2.4.2 one can construct an isomorphism of H_f onto $K_1 \overset{l}{\otimes} K_2$ which carries $\pi_f(a).x_f$ into the vector $(\pi_1 \otimes \pi_2)(a).(x_1 \otimes x_2)$ for each a in $A_1 \otimes A_2$, and $\pi_f(a)$ into $(\pi_1 \otimes \pi_2)(a)$; then

$$\|(\pi_1 \otimes \pi_2)(a)\| = \|\pi_f(a)\| \leq \|a\| .$$

<u>Lemma</u> 3. Let A_1 and A_2 be abstract C^*-algebras, π_i and ρ_i representations of A_i with $\operatorname{Ker} \pi_i \subset \operatorname{Ker} \rho_i$; then

$$\|(\pi_1 \otimes \pi_2)(a)\| \geq \|(\rho_1 \otimes \rho_2)(a)\| \quad \forall\, a \in A_1 \otimes A_2 .$$

<u>Proof</u>. Denote by H_i and K_i the spaces of π_i and ρ_i and set

$$B_i = \pi_i(A_i) \in \mathcal{L}(H_i) \quad ;$$

there exists a representation σ_i of B_i in K_i such that $\rho_i = \sigma_i \circ \pi_i$; we have the following mappings

$$A_1 \otimes A_2 \xrightarrow{\pi_1 \otimes \pi_2} B_1 \otimes B_2 \xrightarrow{\sigma_1 \otimes \sigma_2} \mathcal{L}(K_1 \overset{l}{\otimes} K_2)$$

and their composition is $\rho_1 \otimes \rho_2$; by the preceding lemma, for each a in $A_1 \otimes A_2$:

$$\|(\rho_1 \otimes \rho_2)(a)\| = \|(\sigma_1 \otimes \sigma_2)((\pi_1 \otimes \pi_2)(a))\|$$
$$\leq \|(\pi_1 \otimes \pi_2)(a)\| .$$

QED

We are now in a position to define the $*$ crossnorm :

<u>Theorem</u> 2. Let A_1 and A_2 be two C^*-algebras and a an element of their algebraic tensor product ; for all faithful representations π_i of A_i the number $\|(\pi_1 \otimes \pi_2)(a)\|$ has the same value, which we shall denote by $\|a\|_*$; $\|\ \ \|_*$ is a C^*-crossnorm ; for each representation ρ_i of A_i , $\rho_1 \otimes \rho_2$ is continuous

for that norm ; finally $\|a\|_* = \sup\limits_{f_i \in \hat{A}_i} \|(f_1 \otimes f_2)(a)\|$.

<u>Definition</u> 4. The completion of $A_1 \otimes A_2$ for the $*$ norm is denoted by $A_1 \overset{*}{\otimes} A_2$; for each representation π_i or state f_i of A_i , $\pi_1 \overset{*}{\otimes} \pi_2$ and $f_1 \overset{*}{\otimes} f_2$ are the extensions of $\pi_1 \otimes \pi_2$ and $f_1 \otimes f_2$ to $A_1 \overset{*}{\otimes} A_2$.

The identity mapping of $A_1 \otimes A_2$ extends to a morphism $A_1 \overset{\vee}{\otimes} A_2 \longrightarrow A_1 \overset{*}{\otimes} A_2$, so that the second algebra appears as a quotient of the first.

<u>Example</u> 4. $\mathcal{LC}(H_1) \overset{*}{\otimes} \mathcal{LC}(H_2)$ is nothing but $\mathcal{LC}(H_1 \overset{\ell}{\otimes} H_2)$.

<u>Theorem</u> 3. If A_1 or A_2 is postliminar the $*$ and $\vee$ norms are identical.

In fact for each a in $A_1 \otimes A_2$ we have

$$\|a\|_\vee = \sup \|\pi(a)\| \quad \text{where } \pi \in \widehat{A_1 \overset{\vee}{\otimes} A_2} ,$$

but by proposition 2 such a π is equivalent to a tensor product, and we get $\|a\|_\vee \leqslant \|a\|_*$.

QED

For each locally compact group G we denote by $C^*_r(G)$ the image of $C^*(G)$ in the left regular representation of G in the space $L^2(G)$.

<u>Proposition</u> 6. If G_1 and G_2 are locally compact groups, $C^*_r(G_1 \times G_2)$ is canonically isomorphic to $C^*_r(G_1) \overset{*}{\otimes} C^*_r(G_2)$.

Set $G = G_1 \times G_2$, π_i and π = left regular representation of G_i and G , U = canonical isomorphism of $L^2(G_1) \overset{\ell}{\otimes} L^2(G_2)$ onto $L^2(G)$; for each f_i in $L^1(G_i)$, U carries the operator $(\pi_1 \otimes \pi_2)(f_1 \otimes f_2)$ into $\pi(f)$ where $f(x_1,x_2) = f_1(x_1).f_2(x_2)$; these elements $\pi(f)$ are contained and total in $C^*_r(G)$, thus the uniform closure of $(\pi_1 \otimes \pi_2)(L^1(G_1) \otimes L^1(G_2))$ is carried by U into $C^*_r(G)$; but this uniform closure is equal to $C^*_r(G_1) \overset{*}{\otimes} C^*_r(G_2)$.

Corollary 2. If G_1 and G_2 are amenable, the $*$ and $\vee$ norms on $C^*(G_1) \otimes C^*(G_2)$ are equal.

By [15] a group G is amenable iff $C^*_r(G) \sim C^*(G)$; on the other hand $G_1 \times G_2$ is amenable if G_1 and G_2 are.

Remark 2. Consider a locally compact group G and a C^*-algebra A ; $C^*(G) \overset{\vee}{\otimes} A$ is nothing but the crossed [redacted]product $C^*(G,A)$ defined by Zeller-Meier and Leptin among others, where the action of G in A is trivial ; analogously $C^*_r(G) \overset{*}{\otimes} A$ is equal to $C^*_r(G,A)$; if moreover G is amenable we have $C^*(G,A) \sim C^*_r(G,A)$, so that the $*$ and $\vee$ norms on $C^*(G) \otimes A$ are identical.

Bibliography [24],[28].

n.4.2. The fundamental property of the $*$ norm.

Theorem 4. The $*$ norm is the smallest C^*- subcross norm on $A_1 \otimes A_2$.

Proof. By lemma 1 we can suppose that A_1 and A_2 have units e_1 and e_2. Let p be a C^*- subcross norm on $A_1 \otimes A_2$; we have to prove that for each π_i in $\hat{A}_i$:

$$\|(\pi_1 \otimes \pi_2)(a)\| \leqslant p(a) \qquad \forall\ a \in A_1 \otimes A_2 \ ; \tag{2}$$

let E be the set of all pairs (π_1, π_2) in $\hat{A}_1 \times \hat{A}_2$ for which (2) holds ; E is closed because the mapping $(\pi_1, \pi_2) \longmapsto \pi_1 \overset{\vee}{\otimes} \pi_2$ of $\hat{A}_1 \times \hat{A}_2$ into $\widehat{A_1 \overset{\vee}{\otimes} A_2}$ is continuous (cf. proposition 5) and the mapping $\pi \longmapsto \|\pi(a)\|$ of $\widehat{A_1 \overset{\vee}{\otimes} A_2}$

into $\mathbb{R}_+$ is lower semicontinuous ([2], 3.3.2). We will now show that E is dense in $\hat{A}_1 \times \hat{A}_2$; suppose it is not dense ; its complement contains a non empty elementary open set of the form $U_1 \times U_2$, where U_i is the set of all π_i which are not identically zero on some ideal I_i of A_i ; choose a non zero positive a_i in I_i ; then

$$(\pi_1, \pi_2) \in E \Longrightarrow \pi_1(a_1) \text{ or } \pi_2(a_2) = 0$$

$$\Longrightarrow (\pi_1 \otimes \pi_2)(a_1 \otimes a_2) = 0 .$$

Denote by A the completion of $A_1 \otimes A_2$ under the norm p, by B the commutative sub C^*-algebra of A_1 generated by a_1 and e_1 , by $\overline{B \otimes A_2}$ the closure of $B \otimes A_2$ in A , by ρ an irreducible representation of $\overline{B \otimes A_2}$ in a space K such that $\rho(a_1 \otimes a_2) \neq 0$; since $\rho \mid B$ is factorial and of type I we can write

$$K = K_1 \overset{\ell}{\otimes} K_2$$

$$\rho(b_1 \otimes b_2) = \rho_1(b_1) \otimes \rho_2(b_2) \qquad \forall\, b_1 \in B , b_2 \in A_2$$

where ρ_1 and ρ_2 are irreducible representations of B and A_2 in K_1 and K_2. On the other hand by [2], 2.1o.2, there exist a Hilbert space $H \supset K$ and an irreducible representation π of A in H such that K is invariant under $\pi(\overline{B \otimes A_2})$ and $(\pi \mid \overline{B \otimes A_2})_K = \rho$; $\pi \mid A_2$ is factorial, and of type I since its restriction to K is $\rho \mid A_2$, which is a multiple of ρ_2 ; thus π is a tensor product of two irreducible representations π_1, π_2 of A_1, A_2 ; we have $(\pi_1, \pi_2) \in E$ and $(\pi_1 \otimes \pi_2)(a_1 \otimes a_2)$ is not 0 since its restriction to K is $\rho(a_1 \otimes a_2) \neq 0$; so we have got a contradiction.

Corollary 3. If A_1 and A_2 are simple, $A_1 \overset{*}{\otimes} A_2$ is simple too.

It is sufficient to show that every irreducible representation π of $A_1 \overset{*}{\otimes} A_2$ is faithful ; denote by π_1 and π_2 the restrictions of π ; $\pi | A_1 \otimes A_2$ is composed of the two following mappings :

$$A_1 \otimes A_2 \xrightarrow{u} \pi_1(A_1)'' \otimes \pi_2(A_2)'' \xrightarrow{v} \mathcal{L}(H)$$

where $u(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ and $v(T_1 \otimes T_2) = T_1 T_2$; u is faithful because π_1 and π_2 are, and v is faithful by proposition 0 ; thus $\pi | A_1 \otimes A_2$ is faithful and $a \longmapsto \|\pi(a)\|$ is a C^*- subcross norm on $A_1 \otimes A_2$; then $\|\pi(a)\| \geqslant \|a\|_*$ for each a in $A_1 \otimes A_2$ and consequently for each a in $A_1 \overset{*}{\otimes} A_2$.

Bibliography [20].

n.4.3. The property (T).

Definition 5. A C^*-algebra A is said to have property (T) if for every C^*-algebra B the $*$ and $\vee$ norms on $A \otimes B$ coincide. Then by theorem 4 all C^*- subcross norms are identical. By theorem 3 every postliminar C^*-algebra has property (T), and by remark 2 so does the C^*-algebra of every amenable locally compact group.

Proposition 7. If A is the closure of the union of a family of sub C^*-algebras A_i which have property (T), then A has also property (T).

The union $\cup A_i \otimes B$ is dense in $A \otimes B$ for the topology of the $\vee$ norm ; the $*$ and $\vee$ norms are continuous functions for this topology ; on the other hand the restrictions of these norms to each $A_i \otimes B$ are C^*- subcross norms, and consequently must coincide.

Remark 3. It is unknown whether the property (T) passes to the quotients.

Example 5 (of a C^*-algebra not having property (T)).

Denote by G the free group with two generators u and v, by π the left regular representation of G in $H = L^2(G)$, by A the C^*-algebra generated by $\pi(G)$, by U the automorphism $f \longmapsto \check{f}$ of H, which carries π into the right regular representation, and by ρ the representation of $A \otimes A$ in H defined by $\rho(a_1 \otimes a_2) = a_1.Ua_2U$. The $*$ norm on $A \otimes A$ is the operator norm in the Hilbert space $H \overset{\ell}{\otimes} H \sim L^2(G \times G)$; we shall prove that ρ is not continuous for this norm.

Suppose ρ is continuous ; choose an ε with $0 < \varepsilon < 1/12$; let δ be the Dirac function at the unit element on G, considered as an element of H, $\pi^2 = \pi \otimes \pi$ the left regular representation of $G \times G$; it is easy to chek that

$$(\omega_\delta \circ \rho)(\pi^2(g,g)) = 1 \qquad \forall g \in G \; ;$$

on the other hand the von Neumann algebra $\mathcal{B}$ generated by $A \otimes A$ is standard and it follows that every normal state of $\mathcal{B}$ is vectorial ; every state of $A \overset{*}{\otimes} A$ can be extended to a state of $\mathcal{B}$, and thus is a weak limit of vector states ; in particular there is a normed vector x in $H \overset{\ell}{\otimes} H$ verifying

$$|\omega_x(\pi^2(u,u)^{-1}) - (\omega_\delta \circ \rho)(\pi^2(u,u)^{-1})| \leqslant \varepsilon^2/2$$

and the same for v ; this is equivalent to

$$|(\pi^2(u,u)^{-1}.x \mid x) - 1| < \varepsilon^2/2 \quad ;$$

it follows that

$$\| \pi^2(u,u)^{-1}.x - x \| \leqslant \varepsilon$$

and the same for v. For every subset E of $G \times G$ let P_E be the projection in $L^2(G \times G)$ associated with E ; we have

$$P_{(g_1,g_2)E} = \pi^2(g_1,g_2).P_E.\pi^2(g_1,g_2)^{-1} \quad \forall\ g_1,g_2 \in G ;$$

hence

$$(P_{(u,u)E}\ x \mid x) = (P_E.\pi^2(u,u)^{-1}.x \mid \pi^2(u,u)^{-1}.x)$$

$$|(P_E\ x \mid x) - P_{(u,u)E}\ x \mid x)| \leqslant |(P_E\ x \mid x) - (P_E\ x \mid \pi^2(u,u)^{-1}x)|$$

$$+ |(P_E\ x \mid \pi^2(u,u)^{-1}x) - (P_E\ \pi^2(u,u)^{-1}\ x \mid \pi^2(u,u)^{-1}x)|$$

$$\leqslant 2\varepsilon$$

$$(P_{(u,u)E}\ x \mid x) \geqslant (P_E\ x \mid x) - 2\varepsilon \qquad \forall\ E \quad . \quad (3)$$

Take $E = B \times G$ where B is the set of all words of the form $v^m u^n v^p \ldots$ with $m \neq 0$; the sets $(u,u)^n E$ are mutually disjoint, so that

$$\sum_{n=0}^{2} (P_{(u,u)^n E}\ x \mid x) \leqslant 1 \quad ; \qquad (4)$$

by virtue of (3)

$$(P_{(u,u)^n E}\ x \mid x) \geqslant (P_E\ x \mid x) - 2n\varepsilon \quad ;$$

it follows from (4) that

$$3\ (P_E\ x \mid x) - 6\varepsilon \leqslant 1$$

or

$$(P_{B \times G}\ x \mid x) \leqslant 1/3 + 2\varepsilon \quad ;$$

in the same manner we get by taking $E = A \times B$, $A = G-B$:

$$(P_{A \times G}\ x \mid x) \leqslant 1/3 + 2\varepsilon \quad ;$$

but $P_{B \times G} + P_{A \times G} = I$, so that

$$1 = (x \mid x) \leqslant 2/3 + 4\varepsilon < 1$$

which is absurd.

<u>Remark</u> 4. The analogous of corollary 3 for $A_1 \check{\otimes} A_2$ is not true. In fact denote by $\mathcal{A}$ the von Neumann algebra (factor of type II_1) generated by A ; it is simple by [1], p. 275, cor. 3 ; let σ be the representation of $\mathcal{A} \check{\otimes} \mathcal{A}$ in H defined by

$$\sigma(a_1 \otimes a_2) = a_1 \, a_2 \quad ;$$

consider the following diagramm

$$\begin{tikzcd}
A \check{\otimes} A \arrow[r, "\tau"] \arrow[d] & \mathcal{A} \check{\otimes} \mathcal{A} \arrow[r, "\sigma"] \arrow[d, "\chi"] & \mathcal{L}(H) \\
A \overset{*}{\otimes} A \arrow[r] & \mathcal{A} \overset{*}{\otimes} \mathcal{A} &
\end{tikzcd}$$

if χ is injective, $\sigma \circ \tau = \rho$ will pass to the quotient in a mapping $A \overset{*}{\otimes} A \longrightarrow \mathcal{L}(H)$; but we have just proved that this is not the case ; then χ is not injective and $\mathcal{A} \check{\otimes} \mathcal{A}$ is not simple.

<u>Bibliography</u> [20] .

§ 5. Tensor products of states and of continuous linear functionals.

n.5.1. Tensor products of continuous linear functionals.

Proposition 8. If f_i is a continuous linear functional on A_i, one has

$$|(f_1 \otimes f_2)(a)| \leqslant \|f_1\| \cdot \|f_2\| \cdot \|a\|_* \qquad \forall\, a \in A_1 \otimes A_2 .$$

Proof. We can suppose $\|f_i\| = 1$; if we embed A_i in its enveloping von Neumann algebra A_i'' , $A_1 \overset{*}{\otimes} A_2$ is embedded in the algebra $A_1'' \overset{*}{\otimes} A_2''$; let $f_i = u_i \varphi_i$ be the polar decomposition of f_i where u_i is a partially isometric element in A_i'' and φ_i a state ; for each $a = \Sigma\, a_{1,n} \otimes a_{2,n} \in A_1 \otimes A_2$ we have

$$\begin{aligned} (f_1 \otimes f_2)(a) &= \Sigma\, f_1(a_{1,n}) \cdot f_2(a_{2,n}) \\ &= \Sigma\, \varphi_1(u_1\, a_{1,n}) \cdot \varphi_2(u_2\, a_{2,n}) \\ &= (\varphi_1 \otimes \varphi_2)((u_1 \otimes u_2)a) \end{aligned}$$

whence by lemma 2

$$|(f_1 \otimes f_2)(a)| \leqslant \|(u_1 \otimes u_2)a\|_* \leqslant \|a\|_* .$$

Corollary 4. The $*$ norm is not smaller than the $\hat{\wedge}$ norm.

Proposition 9. If A_1 (or A_2) is commutative, the norms $\vee$, $*$ and $\hat{\wedge}$ are identical.

By theorem 3 the first two are identical ; writing $A_1 = \underline{C}_o(X)$ we have $A_1 \hat{\otimes} A_2 \sim \underline{C}_o(X, A_2)$; hence the norm $\hat{\wedge}$ is a C^*-crossnorm and by theorem 4 is greater than the $*$ norm.

<u>Definition</u> 6. If $\| \ \|_\alpha$ is any C^*- crossnorm on $A_1 \otimes A_2$ and if f_i is a continuous linear functional on A_i , by proposition 8, $f_1 \otimes f_2$ extends to a continuous linear functional on $A_1 \overset{\alpha}{\otimes} A_2$ (see definition 1), which we denote by $f_1 \overset{\alpha}{\otimes} f_2$; we have

$$\| f_1 \overset{\alpha}{\otimes} f_2 \| = \| f_1 \| . \| f_2 \| .$$

In the same manner if π_i is a representation of A_i we get a representation $\pi_1 \overset{\alpha}{\otimes} \pi_2$ of $A_1 \overset{\alpha}{\otimes} A_2$; if f_i is a state, $f_1 \overset{\alpha}{\otimes} f_2$ is also a state ; by the proof of lemma 2 , $\pi_{f_1 \overset{\alpha}{\otimes} f_2}$ is equivalent to $\pi_{f_1} \overset{\alpha}{\otimes} \pi_{f_2}$; hence $f_1 \overset{\alpha}{\otimes} f_2$ is factorial (resp. pure) iff f_i has the same property.

<u>Proposition</u> 10. For each non zero element a in $A_1 \overset{*}{\otimes} A_2$ there exist pure states f_1 and f_2 such that $(f_1 \otimes f_2)(a) \neq 0$.

<u>Proof</u>. Realize A_i in some Hilbert space H_i and $A_1 \overset{*}{\otimes} A_2$ in $H_1 \overset{l}{\otimes} H_2$; there exist vectors x and y in $H_1 \overset{l}{\otimes} H_2$ such that $(a\, x \mid y) = 0$; then there exist vectors x_i and y_i in H_i such that

$$(a.x_1 \otimes x_2 \mid y_1 \otimes y_2) \neq 0$$

i.e.

$$(\omega_{x_1, y_1} \otimes \omega_{x_2, y_2})(a) \neq 0 ;$$

$\omega_{x_i ; y_i}$ is a linear combination of states, hence there exist states f_1 and f_2 such that $(f_1 \otimes f_2)(a) \neq 0$; finally f_i is a weak limit of linear combinations of pures states.

<u>Proposition</u> 11. For every a in $A_1 \otimes A_2$ we have

$$\| a \|_*^2 = \sup\, (f_1 \otimes f_2)(b^* a^* a\, b)$$

where f_i is a state of A_i and b an element of $A_1 \otimes A_2$ with $(f_1 \otimes f_2)(b^* b) \leqslant 1$.

Proof. Clearly the left handside is greater than the right one. To prove the converse inequality realize A_i in a Hilbert space H_i and $A = A_1 \otimes A_2$ in $H = H_1 \overset{\ell}{\otimes} H_2$; by decomposing H_i in a direct sum of cyclic subspaces we can suppose A_i admits a cyclic unit vector x_i ; then $x = x_1 \otimes x_2$ is cyclic for A ; let $f_i = \omega_{x_i}$; for each $\varepsilon > 0$ there exists an y in H with $\|y\| \leqslant 1$ and $\|a\,y\| \geqslant \|a\| - \varepsilon$; there exist a $b \in A$ with $\|y - b\,x\| \leqslant \varepsilon$ and $\|b\,x\| \leqslant 1$; then

$$\|a\| - \varepsilon \leqslant \|a\,y\|$$
$$\leqslant \|a\,(y - b\,x)\| + \|a\,b\,x\|$$

$$\|a\,b\,x\| \geqslant \|a\| - \varepsilon - \|a\|\varepsilon$$

$$((f_1 \otimes f_2)(b^* a^* a\, b))^{\frac{1}{2}} = (b^* a^* a\, b\, x \,|\, x)^{\frac{1}{2}}$$
$$= \|a\,b\,x\|$$
$$\geqslant \|a\| - \varepsilon - \|a\|\varepsilon ;$$

and finally

$$(f_1 \otimes f_2)(b^* b) = \|b\,x\|^2 \leqslant 1 .$$

Bibliography [24],[28].

n.5.2. Restrictions of states.

Consider some C^*- crossnorm $\|\ \|_\alpha$ on $A_1 \otimes A_2$, some state f on $A_1 \overset{\alpha}{\otimes} A_2$, $f = \omega_x \circ \pi$, and the restriction π_i of π to A_i ; set $f_i = \omega_x \circ \pi_i$; π_{f_i} is equivalent to the subrepresentation of π_i in the subspace $\overline{\pi_i(A_i).x}$; the central support of this subspace in $\pi_i(A_i)'$ is I because it contains all vectors $\pi_j(a_j).\pi_i(a_i).x$ where $j \neq i$; hence π_{f_i} is quasiequivalent to π_i . We call f_i the restriction of f to A_i ;

we have

$$f_1(a_1) = \lim f(a_1 \otimes v_t)$$

$$f_2(a_2) = \lim f(u_s \otimes a_2)$$

where (u_s) and (v_t) are approximate identities (arbitrary) of A_1 and A_2 ; if f is factorial, f_1 and f_2 are also factorial ; finally the restrictions of a tensor product $f_1 \overset{\alpha}{\otimes} f_2$ are f_1 and f_2 .

<u>Proposition</u> 12. Let f be a pure state of $A_1 \overset{\alpha}{\otimes} A_2$, f_1 and f_2 its restrictions ; if f_1 (or f_2) is pure we have $f = f_1 \overset{\alpha}{\otimes} f_2$.

<u>Proof</u>. The projection E onto the subspace $\overline{\pi_1(A_1).x}$ is minimal in $\pi_1(A_1)'$, hence for each T in $\pi_1(A_1)'$, T_E is a scalar h(T) ; if $T = \pi_2(a_2)$ we have

$$\begin{aligned} h(\pi_2(a_2)) &= (\pi_2(a_2)_E . x \mid x) \\ &= (\pi_2(a_2). x \mid x) = f_2(a_2) \end{aligned}$$

then

$$\begin{aligned} f(a_1 \otimes a_2) &= (\pi_2(a_2) . \pi_1(a_1).x \mid x) \\ &= (\pi_2(a_2)_E . \pi_1(a_1)_E . x \mid x) \\ &= f_2(a_2).(\pi_1(a_1).x \mid x) \\ &= f_2(a_2).f_1(a_1) . \end{aligned}$$

<u>Proposition</u> 13. In order that every pure state of $A_1 \overset{\alpha}{\otimes} A_2$ be a tensor product, it is necessary and sufficient that A_1 or A_2 is commutative.

Sufficiency : suppose A_1 is commutative and take a pure state f of $A_1 \overset{\alpha}{\otimes} A_2$; the first restriction π_1 of π_f is a multiple of some character χ , so that the first restriction f_1

of f is equal to χ ; by proposition 12 we have $f = f_1 \overset{\alpha}{\otimes} f_2$.

Necessity : suppose A_i is not commutative ; then it admits an irreducible representation π_i in a space H_i of dimension ≥ 2 ; set $\pi = \pi_1 \overset{\alpha}{\otimes} \pi_2$, take a vector x in $H_1 \overset{\alpha}{\otimes} H_2$ which is not decomposable, and set $f = \omega_x \circ \pi$; we shall prove that the pure state f is not a tensor product ; suppose the contrary : $f = f_1 \overset{\alpha}{\otimes} f_2$; then π is equivalent to $\pi_{f_1} \overset{\alpha}{\otimes} \pi_{f_2}$, hence π_i is equivalent to π_{f_i} ; there exists $x_i \in H_i$ such that $f_i = \omega_{x_i} \circ \pi_i$; then

$$f = (\omega_{x_1} \circ \pi_1) \overset{\alpha}{\otimes} (\omega_{x_2} \circ \pi_2) = \omega_{x_1 \otimes x_2} \circ \pi ;$$

this implies that x is proportional to $x_1 \otimes x_2$, which is a contradiction.

§ 6. Functorial properties of $A_1 \check{\otimes} A_2$ and $A_1 \overset{*}{\otimes} A_2$.

Let us consider morphisms $u_i : A_i \longrightarrow B_i$ where A_i and B_i are C^*-algebras ; the function on $A_1 \otimes A_2$: $a \longmapsto \|(u_1 \otimes u_2)(a)\|$ is a C^*- subcross seminorm, consequently less than $\|a\|_v$; thus $u_1 \otimes u_2$ can be extended to a morphism

$$u_1 \check{\otimes} u_2 : \; A_1 \check{\otimes} A_2 \longrightarrow B_1 \check{\otimes} B_2 \; .$$

On the other hand realizing B_1 and B_2 in some Hilbert spaces we get a morphism

$$u_1 \overset{*}{\otimes} u_2 : \; A_1 \overset{*}{\otimes} A_2 \longrightarrow B_1 \overset{*}{\otimes} B_2 \; ;$$

clearly if u_1 and u_2 are onto, $u_1 \check{\otimes} u_2$ and $u_1 \overset{*}{\otimes} u_2$ are also onto ; if u_1 and u_2 are injective, the same holds for $u_1 \overset{*}{\otimes} u_2$.

Remark 5. It is not known whether u_1 and u_2 being injective implies $u_1 \check{\otimes} u_2$ is injective.

Proposition 14. Suppose $\operatorname{Im} u_i$ is a closed twosided ideal of B_i ; then we have

$$\operatorname{Ker} u_1 \check{\otimes} u_2 \;=\; \overline{\operatorname{Ker} u_1 \otimes A_2 \;+\; A_1 \otimes \operatorname{Ker} u_2}$$

where the bar means the closure in $A_1 \check{\otimes} A_2$.

Proof. Set $A = A_1 \check{\otimes} A_2$, $u = u_1 \check{\otimes} u_2$, $I_i = \operatorname{Ker} u_i$, $J_i = \operatorname{Im} u_i$, $I = \overline{I_1 \otimes A_2 + A_1 \otimes I_2}$; I is a closed twosided ideal of A ~~[illegible]~~. ~~[illegible]~~ ; the canonical decomposition of u_i

$$A_i \xrightarrow{\;u_i'\;} J_i \xrightarrow{\;u_i''\;} B_i$$

gives rise to the following decomposition :

$$A \xrightarrow{\;u'\;} J_1 \check{\otimes} J_2 \xrightarrow{\;u''\;} B_1 \check{\otimes} B_2$$

where $u' = u_1' \check{\otimes} u_2'$ and $u'' = u_1'' \check{\otimes} u_2''$.

We first prove that u'' is injective ; it suffices to show that every non degenerate subcross representation π of $J_1 \otimes J_2$ can be extended to a subcross representation ρ of $B_1 \otimes B_2$; denote by π_1 and π_2 the (non degenerate) restrictions of π ; they can be extended to representations ρ_1 and ρ_2 of B_1 and B_2 with the same weak closures (cf.[2], 2.10.4) ; ρ_1 and ρ_2 are commuting and define a representation ρ of $B_1 \otimes B_2$ which has the desired properties.

We have now to prove our proposition in the case where u_1 and u_2 are surjective ; denote by w the canonical morphism of A onto A/I ; clearly Ker $u \supset I$ and it is sufficient to prove that

$$\| u(a) \|_\nu \geqslant \| w(a) \| \qquad \forall\, a \in A_1 \otimes A_2 \quad ;$$

$w(a_1 \otimes a_2)$ depends only on $u_1(a_1)$ and $u_2(a_2)$, let

$$w(a_1 \otimes a_2) = v(u_1(a_1), u_2(a_2))$$

where v is a bilinear mapping $B_1 \times B_2 \longrightarrow A/I$; v defines a linear mapping (which is a morphism) $v' : B_1 \otimes B_2 \longrightarrow A/I$ with

$$v'(u(a)) = w(a) \qquad \forall\, a \in A_1 \otimes A_2 \quad ;$$

take b_i in B_i and a_i in A_i with $u_i(a_i) = b_i$; we have

$$\| v'(b_1 \otimes b_2) \| = \| v'(u(a_1 \otimes a_2)) \| = \| w(a_1 \otimes a_2) \|$$

$$\leqslant \| a_1 \| \cdot \| a_2 \| \,;$$

since the norm of B_i is the quotient norm of the norm of A_i we get

$$\| v'(b_1 \otimes b_2) \| \leqslant \| b_1 \| \cdot \| b_2 \|$$

so that the function on $B_1 \otimes B_2$: $b \longmapsto \| v'(b) \|$ is a C^*-

subcross seminorm ; then for every $a \in A_1 \otimes A_2$ we have

$$\| w(a) \| = \| v'(u(a)) \| \leqslant \| u(a) \|_{\vee} .$$

Corollary 5. If J_i is a closed twosided ideal of A_i , $J_1 \overset{\vee}{\otimes} J_2$ can be identified with a closed twosided ideal of $A_1 \overset{\vee}{\otimes} A_2$.

Corollary 6. Consider morphisms $u_i : A_i \longrightarrow B_i$ and suppose A_1 (or A_2) is postliminar ; then

$$\operatorname{Ker} u_1 \overset{*}{\otimes} u_2 = \operatorname{Ker} u_1 \overset{\vee}{\otimes} u_2 = \operatorname{Ker} u_1 \overset{*}{\otimes} A_2 + A_1 \overset{*}{\otimes} \operatorname{Ker} u_2 .$$

Proof. The right handside is a closed twosided ideal by [2], 1.8.4 ; set $I_i = \operatorname{Ker} u_i$, $C_i = \operatorname{Im} u_i$; I_1 and C_1 are postliminar ; the canonical decomposition of u_i gives rise to the following commutative diagramm :

$$\begin{tikzcd}
 & & B_1 \overset{\vee}{\otimes} B_2 \arrow[dd] \\
\left\{\begin{array}{l} A_1 \overset{\vee}{\otimes} A_2 \\ = A_1 \overset{*}{\otimes} A_2 \end{array}\right\} \arrow[r, "u'"] & \left\{\begin{array}{l} C_1 \overset{\vee}{\otimes} C_2 \\ = C_1 \overset{*}{\otimes} C_2 \end{array}\right\} \arrow[ur, "u''_{\vee}"] \arrow[dr, "u''_{*}"'] & \\
 & & B_1 \overset{*}{\otimes} B_2
\end{tikzcd}$$

$u''_{\vee}$ is injective since u''_{*} is injective, and we have

$$\begin{aligned}
\operatorname{Ker} u_1 \overset{\vee}{\otimes} u_2 &= \operatorname{Ker} u_1 \overset{*}{\otimes} u_2 = \operatorname{Ker} u' \\
&= \overline{I_1 \otimes A_2} + \overline{A_1 \otimes I_2} \\
&= I_1 \overset{*}{\otimes} A_2 + A_1 \overset{*}{\otimes} I_2 .
\end{aligned}$$

Bibliography [10].

§ 7. Study of the representations of $A_1 \overset{\alpha}{\otimes} A_2$.

n.7.1. The mappings Π, Π_c and $\bar{\Pi}$.

As before $A = A_1 \overset{\alpha}{\otimes} A_2$ denotes the completion of $A_1 \otimes A_2$ for some C^*- crossnorm $\| \quad \|_\alpha$; A is a quotient of $A_1 \overset{\vee}{\otimes} A_2$, hence Prim A is a closed subset of Prim $(A_1 \overset{\vee}{\otimes} A_2)$, $\hat{A}$ a closed subset of $\widehat{A_1 \overset{\vee}{\otimes} A_2}$, and $\hat{A}$ a Borel subset of $\widehat{A_1 \overset{\vee}{\otimes} A_2}$.

The mapping $(\pi_1, \pi_2) \longmapsto \pi_1 \overset{\alpha}{\otimes} \pi_2$ (see definition 6) gives rise to two mappings :

$$\Pi_c \quad : \quad \hat{A}_1 \times \hat{A}_2 \longrightarrow \hat{A}$$

$$\Pi \quad : \quad \hat{A}_1 \times \hat{A}_2 \longrightarrow \hat{A} \quad ;$$

analogously associating to every representation π of A its restrictions we get a mapping

$$\bar{\Pi} \quad : \quad \hat{A} \longrightarrow \hat{A}_1 \times \hat{A}_2 \quad ;$$

$\bar{\Pi} \circ \Pi$ is the identity, so that Π is injective and $\bar{\Pi}$ surjective.

Theorem 5. The mapping Π_c is bicontinuous ; its image is dense if $\alpha = *$.

Proof. It is continuous by proposition 5 ; the last assertion follows from the last assertion of theorem 2 ; let us now prove that the mapping $T : \pi_1 \otimes \pi_2 \longmapsto \pi_1$ is continuous ; let U be an open subset of A_1, the set of all π_1 which are not identically zero on some subset I_1 of A_1 ; $T^{-1}(U)$ is the set of all $\pi_1 \otimes \pi_2$ which are not identically zero on $I_1 \otimes A_2$, hence it is open.

Theorem 6. If A_1 (or A_2) is postliminar, Π_c and Π are bijective ; moreover if A_1 and A_2 are separable the following conditions are equivalent :

(i) A_1 or A_2 is postliminar

(ii) the mapping $\Pi_o : \hat{A}_1 \times \hat{A}_2 \longrightarrow \widehat{A_1 \check{\otimes} A_2}$ is bijective

(iii) the mapping $\Pi : \hat{A}_1 \times \hat{A}_2 \longrightarrow \widehat{A_1 \check{\otimes} A_2}$ is bijective.

Proof. The first assertion has been proved in proposition 2; clearly (iii) implies (ii) ; to prove that (ii) implies (i) suppose A_1 and A_2 are separable and non postliminar ; by [3], th.1 there exists a representation π_i of A_i such that $\pi_i(A_i)'$ is a type II_1 hyperfinite factor ; by [9], lemme 2.1, there exist a Hilbert space K, a factor $\mathcal{B}$ in K and two isomorphisms

$$F_1 : \pi_1(A_1)'' \longrightarrow \mathcal{B}$$

$$F_2 : \pi_2(A_2)'' \longrightarrow \mathcal{B}' ;$$

$F_1 \circ \pi_1$ and $F_2 \circ \pi_2$ are commuting representations and define a representation of $A_1 \check{\otimes} A_2$ in K, which is irreducible and not equivalent to any tensor product since its restrictions are not of type I.

Remark 6. It is not known whether one can replace $\vee$ by any α in (ii) and (iii).

Bibliography [8],[14] .

n.7.2. Borel properties of Π_o and Π .

In this and the following numbers we suppose A_1 and A_2 separable ; for each $n = 1,2,\ldots \aleph_o$ we take a Hilbert space H_n of dimension n and identify $H_n \overset{\ell}{\otimes} H_m$ with H_{nm} by means of a fixed isomorphism ; the spaces $Fac_n(A_i)$, $Fac_n(A)$, $Fac\,(A_i)$, $Fac\,(A)$ are endowed with their usual Borel structures ; Θ_i and Θ are the canonical mappings $Fac\,(A_i) \longrightarrow \hat{A}_i$ and

$\mathrm{Fac}\,(A) \longrightarrow \hat{A}$; $\mathcal{R}_i$ and $\mathcal{R}$ are the quasi-equivalence relations in $\mathrm{Fac}\,(A_i)$ and $\mathrm{Fac}\,(A)$; $(\mathrm{Fac}\,(A_1) \times \mathrm{Fac}\,(A_2))/(\mathcal{R}_1 \times \mathcal{R}_2)$ has the quotient Borel structure of the product Borel structure and $\hat{A}_1 \times \hat{A}_2$ has the product Borel structure ; the canonical bijection

$$(\mathrm{Fac}\,(A_1) \times \mathrm{Fac}\,(A_2))/(\mathcal{R}_1 \times \mathcal{R}_2) \longrightarrow \hat{A}_1 \times \hat{A}_2 \tag{5}$$

is easily seen to be Borel.

<u>Lemma</u> 4. The mapping $(\pi_1, \pi_2) \longmapsto \pi_1 \overset{\alpha}{\otimes} \pi_2$ of $\mathrm{Fac}\,(A_1) \times \mathrm{Fac}\,(A_2)$ into $\mathrm{Fac}\,(A)$ is Borel.

It suffices to show that for each n and m the mapping $\mathrm{Fac}_n(A_1) \times \mathrm{Fac}_m(A_2) \longrightarrow \mathrm{Fac}_{nm}(A)$ is Borel ; or that are Borel the mappings

$$(\pi_1, \pi_2) \longmapsto (\pi_1 \otimes \pi_2)(a).x$$

where $a \in A$, $x \in H_n \overset{\ell}{\otimes} H_m$; or the mappings

$$(\pi_1, \pi_2) \longmapsto (\pi_1 \otimes \pi_2)(a_1 \otimes a_2).(x_1 \otimes x_2)$$
$$= \pi_1(a_1).x_1 \otimes \pi_2(a_2).x_2$$

but these mappings are continuous.

<u>Remark</u> 7. We do not know whether Π is Borel, because we do not know whether the mapping (5) is biborel.

<u>Lemma</u> 5. The restriction mapping $\pi \longmapsto (\pi_1, \pi_2)$ of $\mathrm{Fac}\,(A)$ into $\mathrm{Fac}\,(A_1) \times \mathrm{Fac}\,(A_2)$ is Borel.

It is sufficient to show that for each n the mapping $\mathrm{Fac}_n(A) \longrightarrow \mathrm{Fac}_n(A_1) \times \mathrm{Fac}_n(A_2)$ is Borel ; or that the mappings $\pi \longmapsto \pi_1(a_1).x_1$ are Borel for $a_1 \in A_1$, $x_1 \in H_n$;

but such a mapping is the pointwise limit of the mappings $\pi \longmapsto \pi(a_1 \otimes v_t).x$ and we can choose a countable approximate identity (v_t).

<u>Proposition</u> 15. The mapping $\bar{\Pi}$ is Borel.

In fact the composed mapping

$$\mathrm{Fac}\,(A) \longrightarrow \mathrm{Fac}\,(A_1) \times \mathrm{Fac}\,(A_2) \longrightarrow (\mathrm{Fac}\,(A_1) \times \mathrm{Fac}\,(A_2)) / (\mathcal{R}_1 \times \mathcal{R}_2)$$
$$\longrightarrow \hat{A}_1 \times \hat{A}_2$$

is Borel, and $\bar{\Pi}$ is obtained from it by passing to the quotients.

<u>Proposition</u> 16. The image of Π is a Borel subset of $\hat{A}$.

It suffices to prove that the set $E = \Theta^{-1}(\mathrm{Im}\,\Pi)$ is Borel in $\mathrm{Fac}\,(A)$; for each π in $\mathrm{Fac}\,(A)$ set $f(\pi) = \pi_1 \otimes \pi_2$ where π_1 and π_2 are the restrictions of π ; f is a Borel mapping ; we have

$$E = \{\pi \mid f(\pi) \text{ is quasi-equivalent to } \pi\}$$
$$= \{\pi \mid (\pi, f(\pi)) \in \text{graph of } \mathcal{R}\} ;$$

but this graph is Borel by [2], 7.2.3, and the mapping $\pi \longmapsto (\pi, f(\pi))$ is Borel.

<u>Bibliography</u> [8],[14].

n.7.3. <u>Product of measures on</u> $\hat{A}_1$ <u>and</u> $\hat{A}_2$.

Given two standard Borel measures μ_1 and μ_2 on $\hat{A}_1$ and $\hat{A}_2$ we shall construct a "product" measure on $\hat{A}$; the construction is made somewhat difficult by the fact quoted in remark 7.

<u>Proposition</u> 17. There exists a standard Borel subset E_i of A_i, carrying μ_i and such that $\Pi | E_1 \times E_2$ is a Borel isomorphism of $E_1 \times E_2$ onto a standard Borel subset of $\hat{A}$; if one takes E_i with these properties and sets $\mu = \Pi(\mu_1 \otimes \mu_2)$, the quasi-equivalence class $\int^{\oplus} \pi . d\mu(\pi)$ is the tensor product of the quasi-equivalence classes $\int^{\oplus} \pi_1 . d\mu_1(\pi_1)$ and $\int^{\oplus} \pi_2 . d\mu_2(\pi_2)$; moreover μ is central iff μ_1 and μ_2 are central.

<u>Proof</u>. By [12], th. 6.3 , there exist a standard Borel subset E_i carrying μ_i and a Borel mapping $R_i : E_i \longrightarrow \mathrm{Fac}\ (A_i)$ such that $\Theta_i \circ R_i$ = identity ; consider the (Borel) composed mapping

$$E_1 \times E_2 \xrightarrow{a} \mathrm{Fac}\ (A_1) \times \mathrm{Fac}\ (A_2) \xrightarrow{b} \mathrm{Fac}\ (A) \xrightarrow{c} \hat{A}$$

where $a = R_1 \times R_2$, b = tensor product ,and c = restriction of Θ to $\mathrm{Im}\ (b \circ a)$; $c \circ b \circ a$ is the restriction of Π to $E_1 \times E_2$, hence $b \circ a$ is injective and its image meets each quasi-equivalence class in one point at most ; by [2], B 21 , this image is Borel standard and $b \circ a$ is a Borel isomorphism ; by [2], 7.2.3, c is a Borel isomorphism and its image, which is nothing but $\Pi(E_1 \times E_2)$, is Borel standard ; we have thus proved the first assertion.

The measure μ defined in the statement is carried by $E = \Pi(E_1\ E_2)$; let R be the inverse mapping of c ; the quasi-equivalence classes $\int^{\oplus} \pi . d\mu(\pi)$ and $\int^{\oplus} \pi_i . d\mu_i(\pi_i)$ contain respectively the representations $\rho = \int_E^{\oplus} R(\pi) . d\mu(\pi)$ and $\rho_i = \int_{E_i}^{\oplus} R_i(\pi_i) . d\mu_i(\pi_i)$, but by n.2.1 we have

$$\rho_1 \otimes \rho_2 \simeq \int_{E_1 \times E_2}^{\oplus} R_1(\pi_1) \otimes R_2(\pi_2) . d(\mu_1 \otimes \mu_2)(\pi_1, \pi_2) \ ; \quad (6)$$

transporting by means of Π we get

$$\rho_1 \otimes \rho_2 \simeq \int_E^{\oplus} R(\pi).d\mu(\pi) = \rho \tag{7}$$

which proves the second assertion.

As for the last assertion denote by $\mathcal{A}(\pi)$, $\mathcal{A}(\pi_i)$, $\mathcal{A}$ and $\mathcal{A}_i$ the von Neumann algebras generated by $R(\pi)$, $R_i(\pi_i)$, ρ and ρ_i ; to say that μ is central amounts to say that $\mathcal{A} = \int^{\oplus}\mathcal{A}(\pi).d\mu(\pi)$; but according to (6) and (7), $\mathcal{A}$ can be identified with $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2$ and $\int^{\oplus}\mathcal{A}(\pi).d\mu(\pi)$ with the tensor product $\int^{\oplus}\mathcal{A}(\pi_1).d\mu_1(\pi_1) \overset{c}{\otimes} \int^{\oplus}\mathcal{A}(\pi_2).d\mu_2(\pi_2)$; since we have always $\mathcal{A} \subset \int^{\oplus}\mathcal{A}(\pi).d\mu(\pi)$ and $\mathcal{A}_i \subset \int^{\oplus}\mathcal{A}(\pi_i).d\mu_i(\pi_i)$ we shall have $\mathcal{A} = \int^{\oplus}\mathcal{A}(\pi).d\mu(\pi)$ iff $\mathcal{A}_i = \int^{\oplus}\mathcal{A}(\pi_i).d\mu_i(\pi_i)$, the last assertion being a consequence of the

<u>Lemma</u> 6. If $\mathcal{A}_i \subset \mathcal{B}_i$ are von Neumann algebras in a Hilbert space H_i, $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2 = \mathcal{B}_1 \overset{c}{\otimes} \mathcal{B}_2$ implies $\mathcal{A}_i = \mathcal{B}_i$.

<u>Proof</u> : suppose $\mathcal{A}_1 \neq \mathcal{B}_1$; there exists $T \in \mathcal{B}_1$, $T \notin \mathcal{A}_1$; then $T \otimes I \notin \mathcal{A}_1 \overset{c}{\otimes} \mathcal{L}(H_2)$ as is easily deduced from the matrix representation of the elements of $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{L}(H_2)$.

<u>Bibliography</u> [8].

n.7.4. <u>Some properties of</u> $\pi_1 \overset{\alpha}{\otimes} \pi_2$ <u>and</u> $A_1 \overset{\alpha}{\otimes} A_2$.

<u>Lemma</u> 7. Consider C^*- algebras A_i and B_i in a Hilbert space H_i and $A_1 \overset{*}{\otimes} A_2$, $B_1 \overset{*}{\otimes} B_2$ as C^*- algebras in $H_1 \overset{\ell}{\otimes} H_2$; then $B_1 \overset{*}{\otimes} B_2 \subset A_1 \overset{*}{\otimes} A_2$ implies $B_i \subset A_i$.

Suppose $B_1 \not\subset A_1$; there exists $a \in B_1$, $a \notin A_1$; there exists a continuous linear functional f_1 on $\mathcal{L}(H_1)$ which is zero on A_1 but not on a ; take a continuous linear form f_2 on $\mathcal{L}(H_2)$ which is non zero on B_2 ; by proposition 8 we have a continuous linear functional $f_1 \overset{*}{\otimes} f_2$ on $\mathcal{L}(H_1) \overset{*}{\otimes} \mathcal{L}(H_2)$; it is zero on $A_1 \overset{*}{\otimes} A_2$ but not on $B_1 \overset{*}{\otimes} B_2$ - which is a contradiction.

Definition 7. An irreducible representation π of a C^*- algebra A in a space H is traceable (resp. compact) if $\pi(A)$ contains (resp. is equal to) $\mathcal{L}\mathcal{C}(H)$.

Proposition 18. Let π_i be an irreducible representation of A_i ; $\pi_1 \overset{\alpha}{\otimes} \pi_2$ is traceable (resp. compact) if and only if π_1 and π_2 have the same property.

It is known that $\pi_1(a_1) \otimes \pi_2(a_2)$ is compact iff $\pi_1(a_1)$ and $\pi_2(a_2)$ are ; hence $\pi_1 \overset{\alpha}{\otimes} \pi_2$ is compact iff π_1 and π_2 are, and $\pi_1 \overset{\alpha}{\otimes} \pi_2$ is traceable if π_1 and π_2 are ; the converse is a consequence of lemma 7 and example 4.

Theorem 7. The C^*- algebra $A_1 \overset{\alpha}{\otimes} A_2$ is liminar (resp. postliminar) if and only if A_1 and A_2 have the same property.

Proof. Suppose A_i is liminar ; every $\pi \in \widehat{A_1 \overset{\alpha}{\otimes} A_2}$ is of the form $\pi_1 \overset{\alpha}{\otimes} \pi_2$, hence compact since π_1 and π_2 are compact ; consequently $A_1 \overset{\alpha}{\otimes} A_2$ is liminar. Conversely if $A_1 \overset{\alpha}{\otimes} A_2$ is liminar, for each $\pi_i \in \hat{A}_i$, π_i is compact since $\pi_1 \overset{\alpha}{\otimes} \pi_2$ is and A_i is liminar.

Suppose now A_i is postliminar ; each factor representation π of $A_1 \overset{\alpha}{\otimes} A_2$ is of the form $\pi_1 \overset{\alpha}{\otimes} \pi_2$, hence is of type I ; by [17], $A_1 \overset{\alpha}{\otimes} A_2$ is postliminar. Conversely if $A_1 \overset{\alpha}{\otimes} A_2$ is postliminar, for each $\pi_i \in \hat{A}_i$, $\pi_1 \overset{\alpha}{\otimes} \pi_2$ is of type I ; it follows ([16], ch.3, § 4) that π_i is of type I ; hence A_i is postliminar.

<u>Proposition</u> 19. Each compact irreducible representation of $A_1 \overset{\alpha}{\otimes} A_2$ is the tensor product of two compact irreducible representations.

It suffices to prove that if $\mathcal{A}$ is a continuous factor in some Hilbert space H, and S and T are non zero elements in $\mathcal{A}$ and $\mathcal{A}'$ respectively, then ST is not compact ; we can suppose S and T are positive because ST compact implies $S\,T\,T^*S^* = S\,S^*T\,T^*$ compact ; then there exist spectral projections P_E and P_F of S and T such that

$$S_E \geqslant h > 0 \quad \text{and} \quad T_F \geqslant k > 0 \ ;$$

for $x \in E \cap F$ we have $Tx \in E$, hence

$$\| STx \| \geqslant h\| Tx \| \geqslant hk\| x \| \ ;$$

$E \cap F$ is infinite dimensional since there exist projections $Q_1, Q_2, \dots$ in $\mathcal{A}'$, non zero, mutually orthogonal, whose sum is P_F , and we have $P_E Q_n \neq 0 \quad \forall n$; this proves that ST is not compact.

<u>Bibliography</u> [8],[22],[28],[29].

<u>Remark</u> 8. It is not known whether every traceable irreducible representation of $A_1 \overset{\alpha}{\otimes} A_2$ is a tensor product.

§ 8. Further results on the type of $A_1 \overset{\alpha}{\otimes} A_2$.

n.8.1. Antiliminar algebras.

We shall make use of the following construction : let f be a continuous linear functional on A_1 ; denote by R_f the mapping $f \otimes I$ of $A_1 \otimes A_2$ into $\mathbb{C} \otimes A_2 \sim A_2$; for each $a = \sum_{n=1}^{N} a_{1,n} \otimes a_{2,n}$ in $A_1 \otimes A_2$ we have

$$R_f(a) = \sum_{n=1}^{N} < f, a_{1,n} > . a_{2,n} \quad ;$$

and for each continuous linear functional g on A_2 :

$$< g, R_f(a) > = \sum < f, a_{1,n} > . < g, a_{2,n} >$$

$$= < f \otimes g , a > ;$$

hence by proposition 8 :

$$\| R_f(a) \| = \sup_{\|g\| \leq 1} |< g, R_f(a) >| \leqslant \| f \| . \| a \|_* \quad ;$$

thus R_f extends to a continuous linear mapping $R_f : A_1 \overset{*}{\otimes} A_2 \longrightarrow A_2$ and we have

$$< g, R_f(a) > = < f \overset{*}{\otimes} g , a > \tag{8}$$

for each a in $A_1 \overset{*}{\otimes} A_2$ and g in A_2' ; note that $\| R_f \| \leqslant \| f \|$, that R_f is surjective and that it is positive if f is positive.

In the same manner one can define mappings $L_g : A_1 \overset{*}{\otimes} A_2 \longrightarrow A_1$.

We set $A = A_1 \overset{*}{\otimes} A_2$.

Lemma 8. For every non zero a in A there exists a pure state f on A_1 such that $R_f(a) \neq 0$.

This is a consequence of proposition 10.

~~Lemma~~

Lemma 9. If I is a twosided ideal of A , $\overline{R_f(I)}$ is a twosided ideal of A_2.

Proof. We have to show that

$$x \in I \ , \ a_2 \in A_2 \implies a_2 \ R_f(x) \in \overline{R_f(I)} \quad ;$$

take an $\varepsilon > 0$; there exist an $y = \sum y_{1,n} \otimes y_{2,n}$ in $A_1 \otimes A_2$ with $\| x - y \| \leqslant \varepsilon$, and an a_1 in A_1 with $\| a_1 \| \leqslant 1$ and

$$\| a_1 \ y_{1,n} - y_{1,n} \| \leqslant \varepsilon / \sum_m \| y_{2,m} \| \quad \forall n \quad ;$$

we then have

$$\| a_2 \ R_f(x) - a_2 \ R_f(y) \| \leqslant \| a_2 \| \cdot \| f \| \cdot \varepsilon$$

$$\begin{aligned} \| a_2 \ R_f(y) - R_f((a_1 \ a_2)y) \| &= \| R_f(\sum y_{1,n} \otimes a_2 y_{2,n} - \sum a_1 y_{1,n} \otimes a_2 y_{2,n}) \| \\ &= \| R_f(\sum (y_{1,n} - a_1 y_{1,n}) \otimes a_2 y_{2,n}) \| \\ &\leqslant \| a_2 \| \cdot \| f \| \cdot \varepsilon \end{aligned}$$

$$\begin{aligned} \| R_f((a_1 \otimes a_2)y) - R_f((a_1 \otimes a_2)x) \| &= \| R_f((a_1 \otimes a_2)(y-x)) \| \\ &\leqslant \| a_2 \| \cdot \| f \| \cdot \varepsilon \end{aligned}$$

whence

$$\| a_2 \ R_f(x) - R_f((a_1 \otimes a_2)x) \| \leqslant 3 \| a_2 \| \cdot \| f \| . \varepsilon \ ;$$

since $R_f((a_1 \otimes a_2)x)$ belongs to $R_f(I)$ and since ε is arbitrary, the assertion follows.

Lemma 10. Denote by H_f , π_f , x_f respectively the Hilbert space, representation and cyclic vector associated with the state f of A_1 , by ρ a representation of A_2 in a space K, and by y an element of K ; for each $a \in A$ we have

$$< \omega_y \ , \rho(R_f(a)) > \ = \ < \omega_{x_f} \overset{*}{\otimes} \omega_y \ , \ (\pi_f \overset{*}{\otimes} \rho)(a) > . \qquad (9)$$

It is sufficient to prove (9) when a has the form $a_1 \otimes a_2$; in that case we have

$$\begin{aligned} < \omega_y , \rho(R_f(a)) > &= < \omega_y , \rho(< f, a_1 > . a_2) > \\ &= < f, a_1 > . < \omega_y , \rho(a_2) > \\ &= < \omega_{x_f} , \pi_f(a_1) > . < \omega_y , \rho(a_2) > \\ &= < \omega_{x_f} \otimes \omega_y , (\pi_f \otimes \rho)(a_1 \otimes a_2) > . \end{aligned}$$

Lemma 11. If $\overline{f \text{ is}}$ a pure state and I a liminar closed twosided ideal of A , $R_f(I)$ is liminar.

Proof. We have to show that for each irreducible representation ρ of A_2 in a space K, one has

$$\rho(R_f(I)) \subset \mathcal{L}\mathcal{C}(K) \quad ;$$

since I is liminar we have

$$(\pi_f \overset{*}{\otimes} \rho)(I) \subset \mathcal{L}\mathcal{C}(H_f \overset{\ell}{\otimes} K) \quad ;$$

by (9)

$$< g , \rho(R_f(a)) > \;=\; < \omega_{x_f} \overset{*}{\otimes} g , (\pi_f \overset{*}{\otimes} \rho)(a) > \tag{10}$$

for every vector state g on $\mathcal{L}(K)$; this is still true for every state of the form $g = \omega_{y_1} + \ldots . \omega_{y_n}$, then, by continuity, for every state of $\mathcal{L}(K)$ (see [2], 3.4.4) ; and finally for every continuous linear functional g on $\mathcal{L}(K)$. If g is null on $\mathcal{L}\mathcal{C}(K)$, $\omega_{x_f} \overset{*}{\otimes} g$ is null on $\mathcal{L}\mathcal{C}(H_f \overset{\ell}{\otimes} K)$, hence on the set $(\pi_f \overset{*}{\otimes} \rho)(I)$; by (10), g is null on $\rho(R_f(I))$; and this proves our assertion.

Theorem 8. The C^*- algebra $A_1 \overset{*}{\otimes} A_2$ is antiliminar if and only if A_1 or A_2 is antiliminar.

The condition is necessary because if A_i contains a non zero liminar closed twosided ideal I_i , $I_1 \overset{*}{\otimes} I_2$ is a non zero liminar closed twosided ideal of A. Conversely suppose A_2 is antiliminar ; let I be a liminar closed twosided ideal of A ; by lemma 11, $R_f(I)$ is null for each pure state f of A_1 , and by lemma 8 this implies that I is null.

Remark 9. If for some C^*- crossnorm $\| \quad \|_\alpha$, $A_1 \overset{\alpha}{\otimes} A_2$ is anti-liminar, A_1 or A_2 is antiliminar ; in fact if A_i contains a non zero liminar ideal I_i , $\overline{I_1 \otimes I_2}$ (the closure in $A_1 \overset{\alpha}{\otimes} A_2$) is a non zero closed twosided ideal in $A_1 \overset{\alpha}{\otimes} A_2$, and is liminar since it is ~~a quotient of~~ isomorphic to $I_1 \overset{*}{\otimes} I_2$. It is not known whether A_1 or A_2 antiliminar implies $A_1 \overset{\alpha}{\otimes} A_2$ antiliminar.

Bibliography [22].

n.8.2. Algebras with continuous trace.

For any C^*- algebra A we denote by p_A the set of all positive elements a of A for which the function on $\hat{A}$: $\pi \longmapsto Tr\ \pi(a)$ is finite and continuous;and by m_A the linear subspace spanned by p_A ; we recall that m_A is a twosided ideal and that A is said to be with continuous trace if m_A is dense in A (see [2], 4.5.2).

We set $A = A_1 \overset{*}{\otimes} A_2$.

Lemma 12. If f is a pure state of A_1 we have $R_f(p_A) \subset p_{A_2}$.

Proof. Choose an orthonormal basis (x_i) of H_f with $x_0 = x_f$, and set $f_i = \omega_{x_i} \circ \pi_f$, $f_0 = f$; let ρ be an irreducible representation of A_2 in a space K , (y_j) an orthonormal basis of K ; for every a in p_A , $Tr((\pi_f \overset{*}{\otimes} \rho)(a))$ is finite ; but

$$\mathrm{Tr}((\pi_f \overset{*}{\otimes} \rho)(a)) = \sum_{i,j} ((\pi_f \overset{*}{\otimes} \rho)(a).x_i \otimes y_j \mid x_i \otimes y_j)$$
$$= \sum_{i,j} < \omega_{x_i} \otimes \omega_{y_j} , (\pi_f \overset{*}{\otimes} \rho)(a) >$$

by (9)

$$= \sum_{i,j} < \omega_{y_j} , \rho(R_{f_i}(a)) >$$
$$= \sum_{i} \mathrm{Tr}\, \rho(R_{f_i}(a))$$
$$= \mathrm{Tr}\, \rho(R_f(a)) + \sum_{i \neq 0} \mathrm{Tr}\, \rho(R_{f_i}(a)) \quad ;$$

since $R_f(a)$ and $R_{f_i}(a)$ are positive, we see that $\mathrm{Tr}\, \rho(R_f(a))$ and $\mathrm{Tr}\, \rho(R_{f_i}(a))$ are finite ; the function $\rho \longmapsto \mathrm{Tr}((\pi_f \overset{*}{\otimes} \rho)(a))$ is finite and continuous since $a \in p_A$; for each i the function $\rho \longmapsto \mathrm{Tr}\, \rho(R_{f_i}(a))$ is lower semicontinuous ([2], 3.5.9), hence the function $\sum_{i \neq 0} \mathrm{Tr}\, \rho(R_{f_i}(a))$ is l.s.c.; $\mathrm{Tr}\, \rho(R_f(a))$ is upper semicontinuous, and also l.s.c., hence continuous ; this proves that $R_f(a) \in p_{A_2}$.

<u>Theorem</u> 9. Let $\| \quad \|_\alpha$ be any C^*- crossnorm on $A_1 \otimes A_2$; $A_1 \overset{\alpha}{\otimes} A_2$ is a C^*- algebra with continuous trace if and only if A_1 and A_2 have the same property.

Sufficiency : since m_{A_i} is dense in A_i , $m_{A_1} \otimes m_{A_2}$ is dense in $A_1 \overset{\alpha}{\otimes} A_2$ and generated by elements $a_1 \otimes a_2$ with $a_i \in p_{A_i}$; each $\pi \in \widehat{A_1 \overset{\alpha}{\otimes} A_2}$ is of the form $\pi_1 \otimes \pi_2$; for $a_i \in p_{A_i}$ we have

$$\mathrm{Tr}\, \pi(a_1 \otimes a_2) = \mathrm{Tr}\, \pi_1(a_1).\mathrm{Tr}\, \pi_2(a_2) \quad ;$$

since $\widehat{A_1 \overset{\alpha}{\otimes} A_2}$ is homeomorphic to $\hat{A}_1 \times \hat{A}_2$, we see that the function $\pi \longmapsto \mathrm{Tr}\, \pi(a_1 \otimes a_2)$ is finite and continuous.

Necessity : $A_1 \overset{\alpha}{\otimes} A_2$ is liminar, hence identical to $A_1 \overset{*}{\otimes} A_2$; take a pure state f of A_1 ; by lemma 12, $R_f(m_A) \subset m_{A_2}$; since

m_A is dense in A and R_f is surjective, m_{A_2} is dense in A_2 .

Remark 10. A similar result holds for the C^*- algebras with generalized continuous trace (cf. [22], th.2).

n.8.3. The largest postliminar ideal of $A_1 \overset{*}{\otimes} A_2$.

Proposition 20. Suppose A_1 (or A_2) has property (T) ; let K_i be the largest postliminar ideal of A_i ; then the largest postliminar ideal of $A_1 \overset{*}{\otimes} A_2$ is $K_1 \overset{*}{\otimes} K_2$.

Proof. First, $K_1 \overset{*}{\otimes} K_2$ is a postliminar ideal in $A_1 \overset{*}{\otimes} A_2$; now consider the following ideals

$$K_1 \overset{*}{\otimes} K_2 = K_1 \overset{\prime}{\otimes} K_2 \subset A_1 \overset{*}{\otimes} K_2 = A_1 \overset{\prime}{\otimes} K_2 \subset A_1 \overset{*}{\otimes} A_2 = A_1 \overset{\nu}{\otimes} A_2 ;$$

$A_1 \overset{*}{\otimes} A_2 / A_1 \overset{*}{\otimes} K_2$ and $A_1 \overset{*}{\otimes} K_2 / K_1 \overset{*}{\otimes} K_2$ are respectively isomorphic to $A_1 \overset{*}{\otimes} A_2 / K_2$ and $A_1 / K_1 \overset{*}{\otimes} K_2$ (cf. proposition 14), hence antiliminar by theorem 8 ; $A_1 \overset{*}{\otimes} A_2 / A_1 \overset{*}{\otimes} K_2$ is isomorphic to $(A_1 \overset{*}{\otimes} A_2 / K_1 \overset{*}{\otimes} K_2) / (A_1 \overset{*}{\otimes} K_2 / K_1 \overset{*}{\otimes} K_2)$; this shows that $A_1 \overset{*}{\otimes} A_2 / K_1 \overset{*}{\otimes} K_2$ is antiliminar ; hence $K_1 \overset{*}{\otimes} K_2$ is the largest postliminar ideal in $A_1 \overset{*}{\otimes} A_2$.

Remark 11. It is not known whether the above proposition still holds without assumption.

Bibliography [22].

§ 9. Temsor products of traces.

Given two von Neumann algebras $\mathcal{A}_1$, $\mathcal{A}_2$ and two faithful normal semifinite traces t_1, t_2 on $\mathcal{A}_1$, $\mathcal{A}_2$, onecan construct canonically a faithful normal semifinite trace $t_1 \overset{c}{\otimes} t_2$ on $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2$ such that

$$(t_1 \overset{c}{\otimes} t_2)(a_1 \otimes a_2) = t_1(a_1).t_2(a_2) \qquad \forall\ a_i \in \mathcal{A}_i^+$$

(here and in the sequel we agree that $0.\infty = \infty .0 = 0$) ; in order to do this, denote by m_i the definition ideal of t_i and form the tensor product of the Hilbert algebras $m_1^{\frac{1}{2}}$, $m_2^{\frac{1}{2}}$; the von Neumann algebra $\mathcal{U}(m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}})$ is isomorphic to $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2$ and it suffices to transport by means of this isomorphism the natural trace of $\mathcal{U}(m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}})$.

We consider some C^*- crossnorm $\| \quad \|_\alpha$ on $A_1 \otimes A_2$.

Proposition 21. Let f_i be a semifinite lower semicontinuous (s.f.l.s.c.) trace on A_i ; one can construct canonically a s.f.l.s.c. trace f on $A_1 \overset{\alpha}{\otimes} A_2$ such that the representation associated with f is quasi-equivalent to the tensor product of those associated with f_1 and f_2 , and that $f(a_1 \otimes a_2) = f_1(a_1).f_2(a_2)$ for every a_i in A_i^+ .

Proof. Denote by m_i the definition ideal of f_i , by π_i the representation associated with f_i , by $\mathcal{A}_i$ the von Neumann algebra generated by $\pi_i(A_i)$, by t_i the faithful normal semi-finite (f.n.s.f.) trace on $\mathcal{A}_i$ such that $f_i = t_i \circ \pi_i$; the pair $(\pi_1 \overset{\alpha}{\otimes} \pi_2 , t_1 \overset{c}{\otimes} t_2)$ is a traced representation (see definition 6.6.1 in [2]) : for if $a_i \in m_i^+$ we have

$$(t_1 \overset{c}{\otimes} t_2)((\pi_1 \overset{\alpha}{\otimes} \pi_2)(a_1 \otimes a_2)) = t_1(\pi_1(a_1)).t_2(\pi_2(a_2)) < +\infty$$

and this proves that the trace class operators in $\mathrm{Im}(\pi_1 \overset{\alpha}{\otimes} \pi_2)$ generate the von Neumann algebra $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2$. It is then sufficient to set $f = (t_1 \overset{c}{\otimes} t_2) \circ (\pi_1 \overset{\alpha}{\otimes} \pi_2)$.

<u>Definition</u> 8. The trace f constructed above is called the <u>tensor product of</u> f_1 <u>and</u> f_2 and denoted by $f_1 \overset{\alpha}{\otimes} f_2$; it is a character iff f_1 and f_2 are characters. On the other hand it is immediately seen that $f_1 \overset{\alpha}{\otimes} f_2$ is finite if f_1 and f_2 are finite ; and conversely that if $f_1 \overset{\alpha}{\otimes} f_2$ is finite and f_1 and f_2 are not identically zero, they are finite ; in this case $f_1 \overset{\alpha}{\otimes} f_2$ is nothing but the tensor product of the central positive functionals f_1 and f_2.

<u>Lemma</u> 13. Let $\mathcal{A}$ be a factor, t a f.n.s.f. trace on $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ two factors included in $\mathcal{A}$, commuting and generating $\mathcal{A}$; suppose that $0 < f(a_1 \; a_2) < +\infty$ for at least one pair (a_1, a_2) in $\mathcal{A}_1^+ \times \mathcal{A}_2^+$. Then there exist a f.n.s.f. trace t_i on $\mathcal{A}_i$ and an isomorphism of $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2$ onto $\mathcal{A}$ carrying $t_1 \overset{c}{\otimes} t_2$ into t and $a_1 \otimes a_2$ into $a_1 a_2$ for each a_i in $\mathcal{A}_i$.

<u>Proof</u>. Denote by E_1 the (non empty) set of all $a_1 \in \mathcal{A}_1^+$ such that $0 < t(a_1 a_2) < +\infty$ for at least one a_2 in $\mathcal{A}_2^+$; define E_2 in a similar manner ; let a_2 be an element of E_2 ; the function on $\mathcal{A}_1^+ : a_1 \longmapsto t(a_1 a_2)$ is a trace which is normal (the verification is immediate), semifinite (since it assumes a value which is neither zero nor infinite) and faithful since $t(a_1 a_2) = 0$ implies $a_1 a_2 = 0$ which in turn implies $a_1 = 0$ (cf. prop. 0) ; choose a fixed f.n.s.f. trace t_1 on $\mathcal{A}_1$ with definition ideal m_1 ; for each a_1 in $\mathcal{A}_1^+$ we have

$$t(a_1 \; a_2) = k_2(a_2) . t_1(a_1) \tag{11}$$

where $k_2(a_2)$ is a strictly positive number ; in the same manner for $a_1 \in E_1$ and $a_2 \in \mathcal{A}_2^+$

$$t(a_1a_2) = k_1(a_1).t'_2(a_2)$$

where t'_2 is a f.n.s.f. trace on $\mathcal{A}_2$ and $k_1(a_1) > 0$; let m_2 be the definition ideal of t'_2 .

If $a_1 \in m_1^+ - 0$ and $a_2 \in E_2$, we have $0 < t_1(a_1) < +\infty$ whence by (11) , $0 < t(a_1a_2) < +\infty$; this proves that $m_1^+ - 0 \subset E_1$; in the same manner $m_2^+ - 0 \subset E_2$. Take a_i in $m_i^+ - 0$; since a_i lies in E_i we have

$$t(a_1a_2) = k_2(a_2).t_1(a_1) = k_1(a_1).t'_2(a_2) \ ;$$

thus $k_1(a_1) / t_1(a_1) = k_2(a_2) / t'_2(a_2)$ is a number k independant of a_1 and a_2 ; if we set $t_2 = k\, t'_2$ we get

$$t(a_1a_2) = t_1(a_1).t_2(a_2)$$

for $a_i \in m_i^+ - 0$, then, by linearity, for $a_i \in m_i$; and also, by semifiniteness, for $a_i \in \mathcal{A}_i^+$.

Denote by H, H_1, H_2 the Hilbert completions of the Hilbert algebras $m^{\frac{1}{2}}$, $m_1^{\frac{1}{2}}$, $m_2^{\frac{1}{2}}$; the bilinear mapping $(a_1,a_2) \longmapsto a_1a_2$ gives rise to a linear mapping $F : m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}} \longrightarrow m^{\frac{1}{2}}$; it is easily verified that F is an isometric $*$- morphism ; then it can be extended to an isomorphism F of $H_1 \overset{\ell}{\otimes} H_2$ onto some closed subspace H' of H ; $F(m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}})$ is invariant by the left multiplication operators U_b where b is in $\mathcal{A}_1$ or $\mathcal{A}_2$; the same holds for H', so that H' is invariant by all U_b where $b \in \mathcal{A}$; similarly H' is invariant by all V_b where $b \in \mathcal{A}$; hence $H' = H$, $F(m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}})$ is a dense sub Hilbert algebra of $m^{\frac{1}{2}}$, and

$$\mathcal{U}(F(m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}})) = \mathcal{U}(m^{\frac{1}{2}}) ;$$

finally the desired isomorphism is obtained by composing the following ones :

$$\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2 \longrightarrow \mathcal{U}(m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}}) \longrightarrow \mathcal{U}(F(m_1^{\frac{1}{2}} \otimes m_2^{\frac{1}{2}})) \longrightarrow$$

$$\longrightarrow \mathcal{U}(m^{\frac{1}{2}}) \longrightarrow \mathcal{A} .$$

<u>Proposition</u> 22. Every character f of $A_1 \overset{\alpha}{\otimes} A_2$ such that $0 <$ $f(a_1 \otimes a_2) < +\infty$ for at least one pair $(a_1, a_2) \in A_1^+ \times A_2^+$, is the tensor product of two characters.

<u>Proof</u>. The character f defines a representation π and a trace t on $\mathcal{A} = \pi(A_1 \overset{\alpha}{\otimes} A_2)''$; let π_i be the restrictions of π, $\mathcal{A}_i$ the factor generated by $\pi_i(A_i)$, t_i the trace constructed in lemma 13 ; the pair (π_i, t_i) is a traced representation : in fact if $0 < f(a_1 \otimes a_2) < +\infty$ we have

$$0 < t(\pi(a_1 \otimes a_2)) = t(\pi_1(a_1) \cdot \pi_2(a_2))$$
$$= t_1(\pi_1(a_1)) \cdot t_2(\pi_2(a_2)) < +\infty$$

whence $0 < t_i(\pi_i(a_i)) < +\infty$ and our assertion follows by [2], 6.7.2. Denote by f_i the character $t_i \circ \pi_i$, by ρ_i the representation associated with f_i, by $\mathcal{B}_i$ the factor $\rho_i(A_i)''$ and by s_i the corresponding trace on $\mathcal{B}_i$; there exists an isomorphism $\mathcal{B}_i \longrightarrow \mathcal{A}_i$ carrying ρ_i in π_i and s_i in t_i, whence an isomorphism $\mathcal{B}_1 \overset{c}{\otimes} \mathcal{B}_2 \longrightarrow \mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2$ carrying $\rho_1 \overset{\alpha}{\otimes} \rho_2$ in $\pi_1 \overset{\alpha}{\otimes} \pi_2$ and $s_1 \overset{c}{\otimes} s_2$ in $t_1 \overset{c}{\otimes} t_2$; by composing with the isomorphism $\mathcal{A}_1 \overset{c}{\otimes} \mathcal{A}_2 \longrightarrow \mathcal{A}$ of lemma 13, we see that $f_1 \overset{\alpha}{\otimes} f_2 = f$.

<u>Theorem</u> 10. The mapping $(f_1, f_2) \longmapsto f_1 \overset{\alpha}{\otimes} f_2$ is a homeomorphism of $C_1(A_1) \times C_1(A_2)$ onto $C_1(A_1 \overset{\alpha}{\otimes} A_2)$.

Proof. It is bijective by proposition 22 and continuous by proposition 5 ; we must now prove that the mapping $f_1 \otimes f_2 \longmapsto f_1(a_1)$ is continuous for every $a_1 \in A_1^+$; choose an increasing approximate identity (v_t) of A_2 ; $f_1(a_1)$ is the limit of the filtering family $(f_1 \otimes f_2)(a_1 \otimes v_t)$, hence $f_1 \otimes f_2 \longmapsto f_1(a_1)$ is l.s.c. ; the same holds for $f_1 \otimes f_2 \longmapsto f_2(a_2)$; in order to prove the continuity we can suppose that $f_1 \otimes f_2$ is in the neighbourhood of some element $f_1^\circ \otimes f_2^\circ$; there exists a_2 in A_2 such that $f_2^\circ(a_2) > 0$; since $f_1 \otimes f_2 \longmapsto f_2(a_2)$ is l.s.c., $f_1 \otimes f_2 \# f_1^\circ \otimes f_2^\circ$ implies $f_2(a_2) > 0$; then

$$f_1(a_1) = (f_1 \otimes f_2)(a_1 \otimes a_2) / f_2(a_2)$$

which proves that $f_1 \otimes f_2 \longmapsto f_1(a_1)$ is u.s.c., and finally continuous.

Corollary 7. Suppose A_1 and A_2 are separable and set $A = A_1 \overset{\alpha}{\otimes} A_2$; then Π induces a Borel isomorphism $\widehat{(A_1)_f} \times \widehat{(A_2)_f} \longrightarrow \widehat{A}_f$.

The restriction of $\overline{\Pi}$ to $\widehat{A}_f$ is Borel by proposition 15 and injective by theorem 10 ; $\widehat{A}_f$ and $\widehat{(A_1)_f} \times \widehat{(A_2)_f}$ are standard by [2], 7.4.3 ; then $\overline{\Pi} \mid \widehat{A}_f$ is a Borel isomorphism by [2], B 21.

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Table of content