AARHUS UNIVERSITET

# TENSOR PRODUCTS OF C*-ALGEBRAS

## PART II.

## INFINITE TENSOR PRODUCTS

by

A. Guichardet

Juni 1969

Lecture Notes Series

No. 13

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# INTRODUCTION

The basic idea of these Lectures Notes is to consider the tensor product of an arbitrary family of vector spaces $(E_i)_{i\in I}$ as the inductive limit of the finite tensor products $\underset{i\in J}{\otimes} E_i$, J finite subset of I ; to this end we suppose that we are given for each i, a non zero vector $t_i$ in $E_i$ and we define, for $J \subset K$, a mapping $L_{J,K} : \underset{i\in J}{\otimes} E_i \longrightarrow \underset{i\in K}{\otimes} E_i$ by writing

$$\underset{i\in K}{\otimes} E_i = \Big(\underset{i\in J}{\otimes} E_i\Big) \otimes \Big(\underset{i\in K-J}{\otimes} E_i\Big)$$

$$L_{J,K}(x) = x \otimes \Big(\underset{i\in K-J}{\otimes} t_i\Big) \quad ;$$

we thus obtain an inductive limit which we denote by $\otimes^t E_i$. If each $E_i$ is a Banach space $t_i$ must have norm one ; if $E_i$ is a $*$- algebra $t_i$ must be hermitian and idempotent.

If now we have $C^*$- algebras $A_i$ with non zero projections $e_i$ we can define two tensor products $\overset{\vee}{\otimes}{}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$, which are identical if the $A_i$ are postliminar. Our main results concern the irreducible representations and the characters of $\overset{\alpha}{\otimes}{}^e A_i$ where $\alpha = \vee$ or $*$ ; for instance if each $e_i$ is central every finite character of $\overset{\alpha}{\otimes}{}^e A_i$ is a tensor product of characters and we get a precise description of the topological space $C_1(\overset{\alpha}{\otimes}{}^e A_i)$ (see n.14.3). As for the irreducible representations of $\overset{\alpha}{\otimes}{}^e A_i$ we examine two thoroughly different particular cases : if $e_i$ is "large" in the sense that $A_i$ admits sufficiently many irreducible representations $\pi$ with rank $\pi(e_i) \geqslant 2$, $\overset{*}{\otimes}{}^e A_i$ is antiliminar and, with some further assumptions, admits an irreducible representation which is not a tensor product (see § 11). On the contrary if $e_i$ is "small", i.e. if for each $\pi$ in $\hat{A}_i$ we have rank $\pi(e_i) \leq 1$ and if moreover each $A_i$ is postliminar, then $\overset{*}{\otimes}{}^{e} A_i$ is postliminar, each irreducible representation of it is a tensor product and we get a precise description of the topological space $\widehat{\overset{*}{\otimes}{}^{e} A_i}$ (see n.13.2).

In § 8 we introduce the notion of infinite tensor product of Hilbert algebras, which gives us a very simple method to determine the type of certain infinite tensor products of type I factors (see § 9).

In § 1 we introduce a notion which plays a basic role throughout these lectures : the restricted product of a family of sets $X_i$ with respect to a family of subsets $Y_i$ ; this is the subset $\prod^{(Y_i)} X_i$ of $\prod X_i$ consisting of all families $(x_i)$ such that $x_i \in Y_i$ except for a finite number of indices $i$ ; if each $X_i$ is a topological space and $Y_i$ is open in $X_i$ , the restricted product becomes a topological space in a natural way.

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Given a set I we denote by $\mathcal{F}(I)$ the set of all finite subsets of I ; we say that a property P of an element i of I holds *for almost every i* or *almost everywhere* if P holds for every i lying outside some finite subset. If i is an element of I we denote by $\delta_i$ the real function on I which takes the value 1 at i and 0 at each other point.

If X is a topological space we denote by $\mathcal{K}(X)$ the space of all continuous complex functions on X with compact support.

If A is a $*$ - algebra every hermitian idempotent element in A will be called a projection. We recall the following result concerning infinite products of complex numbers : given a family $(x_i)_{i\in I}$ of non zero complex numbers, the product $\prod_{i\in I} x_i$ is convergent and non zero if and only if we have $\sum_{i\in I} |x_i - 1| < \infty$ .

We finally recall the associativity property for finite tensor products : given a finite family $(E_i)_{i\in I}$ of Banach spaces and a partition $I = \bigcup_{\lambda\in L} I_\lambda$ , there exists an isomorphism F of $\hat{\bigotimes}_{i\in I} E_i$ onto $\hat{\bigotimes}_{\lambda\in L}(\hat{\bigotimes}_{i\in I_\lambda} E_i)$ such that $F(\bigotimes_{i\in I} x_i) = \bigotimes_{\lambda\in L}(\bigotimes_{i\in I_\lambda} x_i)$ for each family $(x_i)$ in $\prod_{i\in I} E_i$ . Similar results hold for the $\vee$ and $*$ tensor products of $C^*$ - algebras.

## § 1. Restricted products of sets and topological spaces.

**Definition 1.** Given a family $(X_i)_{i \in I}$ of sets and for each $i$ a subset $Y_i$ of $X_i$, we call restricted product of the family $(X_i)$ with respect to the family $(Y_i)$ the set of all families $(x_i) \in \prod_{i \in I} X_i$ such that $x_i \in Y_i$ for almost every $i$ ; and we denote it by $\prod_{i \in I}^{(Y_i)} X_i$ . If $Y_i$ is reduced to some point $a_i$ we write $\prod^{(a_i)} X_i$ or $\prod^{a} X_i$ instead of $\prod^{(\{a_i\})} X_i$ .

For each finite subset J of I we denote by $X_{(J)}$ the set of all families $(x_i) \in \prod X_i$ such that $x_i \in Y_i$ for $i \notin J$ , i.e.

$$X_{(J)} = \prod_{i \in J} X_i \times \prod_{i \in I-J} Y_i \ ;$$

then the $X_{(J)}$ are subsets of $\prod^{(Y_i)} X_i$ which is their union, and $X_{(J)} \subset X_{(K)}$ if $J \subset K$ .

Suppose now that each $X_i$ is a topological space and that $Y_i$ is open in $X_i$ ; we shall define a topology on $\prod^{(Y_i)} X_i$ in the following way : we endow each $X_{(J)}$ with the product topology ; then for $J \subset K$ , $X_{(J)}$ is an open topological subspace of $X_{(K)}$ ; we say that a subset U of $\prod^{(Y_i)} X_i$ is open iff for each J , $U \cap X_{(J)}$ is open in $X_{(J)}$ ; we get a topology which is the inductive limit of those of the $X_{(J)}$ , and is stronger than the product topology ; each $X_{(J)}$ appears as an open topological subspace of $\prod^{(Y_i)} X_i$ .

Particular cases.

(i) If $Y_i = X_i$ , the restricted product is identical to the ordinary product.

(ii) If for each i, $X_i$ is locally compact and $Y_i$ compact, the restricted product is locally compact since each $X_{(J)}$ is locally compact.

(iii) If $X_i$ is discrete and $Y_i$ is reduced to some point $a_i$, the restricted product is discrete since each $X_{(J)}$ is discrete.

(iv) If $X_i$ is a locally compact group and $Y_i$ a compact open subgroup, the restricted product is, in a natural way, a locally compact group ; this construction is used in order to define the so called " adele groups " : in the simplest case I is the set of all prime numbers, the $X_i$ are the p-adic fields $Q_p$ and the $Y_i$ - there rings of integers $Z_p$ (see for instance [37], ch. III, § 1).

(v) If $X_i$ is a discrete group and $Y_i$ is reduced to the neutral element, $\prod^{(Y_i)} X_i$ is nothing but the usual restricted product $\prod' X_i$ .

(vi) If $X_i$ is a compact group and $Y_i = X_i$ the restricted product is identical with the ordinary product $\prod X_i$ .

## Restricted products of Borel spaces.

We now suppose that each $X_i$ is a Borel space and $Y_i$ a Borel subset of $X_i$ ; put on each $X_{(J)}$ the product Borel structure ; then for $J \subset K$ , $X_{(J)}$ is a Borel subspace of $X_{(K)}$ ; we define a Borel structure on $X = \prod^{(Y_i)} X_i$ by saying that a subset U of X is Borel iff for each J, $U \cap X_{(J)}$ is Borel in $X_{(J)}$ ; then each $X_{(J)}$ appears as a Borel subspace of X.

## Restricted products of measures.

Suppose that each $X_i$ is locally compact, $Y_i$ compact and open, and that we have a positive Radon measure $\mu_i$ on $X_i$ with $\mu_i(Y_i) = 1$ ; set $\nu_i = \mu_i \mid Y_i$ ; for each $J \in \mathcal{F}(I)$ we can form the product measure

$$\mu_{(J)} = (\bigotimes_{i \in J} \mu_i) \otimes (\bigotimes_{i \in I-J} \nu_i) ;$$

if $J \subset K$ we have $\mu_{(K)} \mid X_{(J)} = \mu_{(J)}$ , hence there exists a unique positive Radon measure $\mu$ on $\prod^{(Y_i)} X_i$ such that $\mu \mid X_{(J)} = \mu_{(J)}$ ; it will be called restricted product of the measures $\mu_i$ .

If in particular $X_i$ is a locally compact group, $Y_i$ a compact open subgroup and $\mu_i$ a left Haar measure on $X_i$, $\mu$ is a left Haar measure on $\prod^{(Y_i)} X_i$ .

Finally if $X_i$ is a Borel space, $Y_i$ a Borel subset and $\mu_i$ a positive Borel measure on $X_i$ with $\mu_i(Y_i) = 1$, the same construction applies and yields a positive Borel measure .

## § 2. Infinite tensor products of vector spaces.

Let us consider an arbitrary family $(E_i)_{i\in I}$ of vector spaces and for each $i$ a non zero element $t_i$ in $E_i$ ; denote by $t$ the family $(t_i)$ ; for each finite subset $J$ of $I$ we set $E_{(J)} = \bigotimes_{i\in J} E_i$ ; for $J \subset K$ we define a linear mapping

$$L_{J,K} : E_{(J)} \longrightarrow E_{(K)}$$

by writing

$$E_{(K)} = E_{(J)} \otimes E_{(K-J)}$$

$$L_{J,K}(x) = x \otimes (\bigotimes_{i\in K-J} t_i) \qquad \forall\ x \in E_{(J)} ;$$

the mappings $L_{J,K}$ are injective and form an inductive system, which means that for $J \subset K \subset M$ we have $L_{J,M} = L_{K,M} \circ L_{J,K}$ .

Definition 2. We shall denote by $\bigotimes_{i\in I}^{(t_i)} E_i$ or $\bigotimes_{i\in I}^{t} E_i$ the inductive limit of the above inductive system, and by $L_J$ the canonical mapping $E_{(J)} \longrightarrow \otimes^t E_i$ ; the $L_J$ will sometimes allow us to consider the $E_{(J)}$ as subspaces of $\otimes^t E_i$ , which is then their union ; if $J$ is reduced to a point $i$ we shall write $L_i$ instead of $L_{\{i\}}$. For each family $(x_i) \in \prod^t E_i$ we denote by $\otimes x_i$ the element $L_J(\bigotimes_{i\in J} x_i)$ where $J$ is an arbitrary finite subset verifying $x_i = t_i\ \forall\ i \notin J$ ; every element of $\otimes^t E_i$ is a linear combination of elements of the form $\otimes x_i$ ; the mapping $(x_i) \longmapsto \otimes x_i$ is multilinear.

## Properties of the infinite tensor product.

### (i) Universal property.

**Proposition 1.** For every multilinear mapping u of $\prod^t E_i$ into a vector space F there exists a unique linear mapping $v : \otimes^t E_i \longrightarrow F$ such that $v(\otimes x_i) = u((x_i))$ for each $(x_i) \in \prod^t E_i$ ; in this way we get a bijective correspondance between the multilinear mappings $\prod^t E_i \longrightarrow F$ and the linear mappings $\otimes^t E_i \longrightarrow F$ .

Proof. Choose a finite subset J of I ; for each family $(x_i)_{i\in J}$ define a family $(x_i')_{i\in I}$ where

$$x_i' = \begin{cases} x_i & \text{if} \quad i \in J \\ t_i & \text{if} \quad i \notin J \end{cases} ;$$

the multilinear mapping

$$\begin{aligned} \prod_{i\in J} E_i &\longrightarrow F \\ (x_i)_{i\in J} &\longmapsto u((x_i')) \end{aligned}$$

gives rise to a linear mapping

$$\begin{aligned} v_J \; : \; E_{(J)} &\longrightarrow F \\ \bigotimes_{i\in J} x_i &\longmapsto u((x_i')) \quad ; \end{aligned}$$

the $v_J$ form an inductive system and v is their inductive limit.

### (ii) Associativity.

For each partition $I = \bigcup_{\lambda\in L} I_\lambda$ there exists an isomorphism $\bigotimes_{i\in I}^{t} E_i \longrightarrow \bigotimes_{\lambda\in L}^{(v_\lambda)} \Big( \bigotimes_{i\in I_\lambda}^{u_\lambda} E_i \Big)$ taking each element of the form $x_i$ into $\bigotimes_{\lambda \in L} ( \bigotimes_{i \in I_\lambda} x_i )$ ; here we have set $u_\lambda = (t_i)_{i \in I_\lambda}$ and $v_\lambda = \bigotimes_{i \in I_\lambda} t_i$ .

(iii) Functorial property.

Let us also consider vector spaces $F_i$ and non zero elements $u_i$ in $F_i$ ; let $v_i$ be a linear mapping $E_i \longrightarrow F_i$ with $v_i(t_i) = u_i$ ; there exists a unique linear mapping $\otimes v_i$ : $\otimes^t E_i \longrightarrow \otimes^u F_i$ such that

$$(\otimes v_i)(\otimes x_i) = \otimes v_i(x_i) \qquad \forall (x_i) \in \prod^t E_i ;$$

if the $v_i$ are injective, $\otimes v_i$ is injective too.

(iv) Bases of $\otimes^t E_i$ .

Suppose that for each i we have a basis of $E_i$ of the form $(e_{i,x_i})$ where the index $x_i$ runs over some set $X_i$, and that $X_i$ contains an element $y_i$ with $e_{i,y_i} = t_i$ ; for each $x = (x_i)$ in $\prod^{(y_i)} X_i$ we set $e_{(x)} = \bigotimes_{i \in I} e_{i,x_i}$ ; then it is easy to verify that the vectors $e_{(x)}$ constitute a basis of $\otimes^t E_i$ .

Bibliography [9].

N.B. There is no pages 7,8,9.

## § 3. Infinite tensor products of algebras.

Let us consider a family of algebras $(A_i)_{i \in I}$ and for each i , a non zero idempotent $e_i$ in $A_i$ ; the finite tensor products $A_{(J)}$ are algebras and the mappings $L_{J,K}$ are morphisms of algebras ; by endowing $\otimes^e A_i$ with the inductive limit structure we get an algebra whose multiplication is characterized by

$$\otimes a_i \cdot \otimes b_i = \otimes a_i b_i \qquad \forall (a_i) , (b_i) \in \Pi^e A_i .$$

Let us now suppose that for each i, $A_i$ is a $\star$ - algebra and $e_i$ a projection (i.e. hermitian idempotent) ; then the $A_{(J)}$ are $\star$ - algebras and the $L_{J,K}$ are morphisms of $\star$ - algebras ; we get a structure of $\star$ - algebra on $\otimes^e A_i$ characterized by

$$(\otimes a_i)^* = \otimes a_i^* .$$

The reader will easily state the properties similar to (ii) and (iii) of § 2.

A particular case. Suppose that $e_i$ is a unit element for $A_i$ ; we then write $\otimes A_i$ instead of $\otimes^e A_i$ ; $\otimes e_i$ is the unit element of $\otimes A_i$ ; the $L_J$ are mutually commuting morphisms of unitary $\star$ - algebras ; moreover $\otimes A_i$ has the following universal property :

Given a unitary $\star$ - algebra B there exists a bijective correspondance between the morphisms (of unitary $\star$ - algebras) u :

$\otimes A_i \longrightarrow B$ and the families of mutually commuting morphisms $u_i : A_i \longrightarrow B$ ; this correspondance is given by

$$u(\otimes a_i) = \prod u_i(a_i) \qquad \forall (a_i) \in \prod^e A_i .$$

Infinite tensor products of representations.

Proposition 2. Let, for each i, $A_i$ an algebra, $e_i$ a non zero idempotent in $A_i$, $E_i$ a vector space, $t_i$ a non zero element of $E_i$, $\pi_i$ a representation of $A_i$ in $E_i$ such that $\pi_i(e_i).t_i = t_i$. Then there exists a unique representation $\pi$ of $\otimes^e A_i$ in the space $\otimes^t E_i$ such that

$$\pi(\otimes a_i).\otimes x_i = \otimes \pi_i(a_i).x_i \qquad \forall (a_i) \in \prod^e A_i, (x_i) \in \prod^t E_i .$$

Proof. Take a family $(a_i) \in \prod^e A_i$ and a finite subset J with $a_i = e_i \;\; \forall \; i \in I-J$ ; write

$$\bigotimes_{i\in I}^{t} E_i = (\bigotimes_{i\in J} E_i) \otimes (\bigotimes_{i\in I-J}^{t} E_i) \;;$$

we have an operator $\bigotimes_{i\in J} \pi_i(a_i)$ in the first factor and, by property (iii) § 2, an operator $\bigotimes_{i\in I-J} \pi_i(a_i)$ in the second factor ; whence an operator in $\bigotimes_{i\in I}^{t} E_i$ which we denote by $u((a_i))$ :

$$u((a_i)).\otimes x_i = \otimes \pi_i(a_i).x_i \qquad \forall (x_i) \in \prod^t E_i \;;$$

the mapping $u : \prod^e A_i \longrightarrow \mathcal{L}(\otimes^t E_i)$ is multilinear, hence defines a linear mapping

$$\pi : \otimes^e A_i \longrightarrow \mathcal{L}(\otimes^t E_i)$$

$$\pi(\otimes a_i).\otimes x_i = \otimes \pi_i(a_i).x_i \;;$$

finally it is easily seen that $\pi$ is a representation.

## § 4. Infinite tensor products of Banach spaces.

Let us consider a family of Banach spaces $E_i$ and for each i, a unit vector $t_i$ in $E_i$ ; let us endow each finite tensor product $E_{(J)}$ with the $\wedge$ crossnorm ; if $J \subset K$ the isomorphism $E_{(K)} \sim E_{(J)} \otimes E_{(K-J)}$ is isometric, thus the $L_{J,K}$ are isometric ; we put on $\otimes^t E_i$ the inductive limit norm, so that each mapping $L_J$ becomes isometric ; we have

$$\| \otimes x_i \| = \prod \| x_i \| \qquad \forall (x_i) \in \prod^t E_i .$$

Definition 3. We shall denote by $\hat{\otimes}^t E_i$ the completion of $\otimes^t E_i$ with respect to the norm defined above ; this is also the inductive limit of the Banach spaces $\hat{\bigotimes}_{i \in J} E_i$ .

Definition of $\otimes x_i$ for certain families $(x_i)$.

Proposition 3. Let $(x_i)$ be a family of vectors $x_i \in E_i$ such that $\sum \| x_i - t_i \| < \infty$ ; the product $\prod \| x_i \|$ exists, and it is null iff one of the $x_i$ is null ; the family of the vectors $L_J( \bigotimes_{i \in J} x_i)$ has a limit in $\hat{\otimes}^t E_i$ , whose norm is equal to $\prod \| x_i \|$ .

Proof. The proposition being trivial if one $x_i$ is null, we can suppose $x_i \neq 0 \ \forall i$ ; then

$$\sum | \| x_i \| - 1 | = \sum | \| x_i \| - \| t_i \| |$$

$$\leq \sum \| x_i - t_i \| < \infty$$

whence $\prod \| x_i \|$ exists and is not null ; the finite products $\prod_{i \in J} \| x_i \|$ are bounded by some constant $k \geqslant 1$ . We must now prove that the vectors $X_J = L_J(\bigotimes_{i \in J} x_i)$ form a Cauchy family, i.e. that for every $\varepsilon > 0$ there exists $K \in \mathcal{F}(I)$ with the following property :

$$J_1 , J_2 \supset K \Longrightarrow \| X_{J_2} - X_{J_1} \| \leqslant \varepsilon \quad ; \tag{1}$$

take K such that

$$J \cap K = \emptyset \Longrightarrow \sum_{i \in J} \| x_i - t_i \| \leqslant \varepsilon / k \quad ;$$

in order to prove (1) we can suppose $J_1 \subset J_2$ ; set $J = J_2 - J_1$ ; we have

$$X_{J_2} - X_{J_1} = L_{J_2}( \bigotimes_{i \in J_2} x_i - ( \bigotimes_{i \in J_1} x_i ) \otimes ( \bigotimes_{i \in J} t_i ))$$

$$\| X_{J_2} - X_{J_1} \| = \prod_{i \in J_1} \| x_i \| . \| \bigotimes_{i \in J} x_i - \bigotimes_{i \in J} t_i \| \quad ;$$

denoting by $i_1, \dots i_n$ the elements of J we can write

$$\bigotimes_{i \in J} x_i - \bigotimes_{i \in J} t_i = x_{i_1} \otimes \dots \otimes x_{i_n} - t_{i_1} \otimes \dots \otimes t_{i_n}$$

$$= (x_{i_1} - t_{i_1}) \otimes x_{i_2} \otimes \dots \otimes x_{i_n} + \dots + t_{i_1} \otimes \dots t_{i_{n-1}} \otimes (x_{i_n} - t_{i_n})$$

$$\| \bigotimes_{i \in J} x_i - \bigotimes_{i \in J} t_i \| \leqslant k \| x_{i_1} - t_{i_1} \| + \dots + k \| x_{i_n} - t_{i_n} \|$$

$$\leqslant \varepsilon / k$$

whence

$$\| X_{J_2} - X_{J_1} \| \leqslant k . \varepsilon / k = \varepsilon .$$

Definition 4. Given a family $(x_i)$ we set

$$x_i = \lim L_J\left( \bigotimes_{i \in J} x_i \right)$$

whenever the righthand side makes sense ; this is the case if $\sum \| x_i - t_i \| < \infty$ ; in any case we have $\| \otimes x_i \| = \prod \| x_i \|$.

Bibliography [9].

## § 5. Infinite tensor products of Banach $*$ - algebras.

Suppose we are given a family of Banach $*$ - algebras $A_i$ and for each i, a projection $e_i$ of norm 1 in $A_i$ ; then $\hat{\otimes}^e A_i$ is a Banach $*$ - algebra, the inductive limit of the finite tensor products $\hat{\otimes}_{i\in J} A_i$ . If in particular $e_i$ is a unit element for $A_i$, $\hat{\otimes}^e A_i$ , which will be denoted $\hat{\otimes} A_i$ , has the following universal property : given a unitary Banach $*$ - algebra B and mutually commuting morphisms (of unitary $*$ - algebras) continuous and with norm 1, $u_i : A_i \longrightarrow B$ , there exists a unique continuous morphism $u : \hat{\otimes} A_i \longrightarrow B$ such that

$$u(\otimes a_i) = \prod u_i(a_i) \qquad \forall (a_i) \in \prod^e A_i .$$

Example 1. We consider the situation of § 1, (iv), set $A_i = L^1(X_i)$ and denote by $e_i$ the characteristic function of $Y_i$ ; this is a projection of norm 1 if the Haar measure $\mu$ on $X_i$ is chosen so that $\mu(Y_i) = 1$.

Theorem 1. There exists an isometric isomorphism w of $\hat{\otimes}^e A_i$ onto $L^1(\prod^{(Y_i)} X_i)$ with the following property : for every family $(a_i)$ in $\prod^e A_i$, $w(\otimes a_i)$ is the function f on $\prod^{(Y_i)} X_i$ defined by

$$f(x) = \begin{cases} \prod a_i(x_i) & \text{if} \quad x = (x_i) \in X_{(J)} \\ 0 & \text{in the opposite case} \end{cases}$$

(we have set $J = \{ i \mid a_i \neq e_i \}$).

Proof. ~~The reader will notice the similarity of this proof [illegible]~~. For every $J \in \mathcal{F}(I)$ we have an isometric isomorphism

$$u_J : \hat{\bigotimes_{i \in J}} A_i \longrightarrow L^1(\prod_{i \in J} X_i)$$

which transforms each element $\bigotimes_{i \in J} a_i$ into the function $(x_i)_{i \in J} \longmapsto \prod_{i \in J} a_i(x_i)$ ; then an isometric morphism

$$v_J : \hat{\bigotimes_{i \in J}} A_i \longrightarrow L^1(\prod_{i \in J} X_i) \hat{\otimes} L^1(\prod_{i \in I-J} Y_i) \sim L^1(X_{(J)})$$

$$a \longmapsto u_J(a) \otimes 1 \quad ;$$

finally extending $v_J(a)$ to a function on $X = \prod^{(Y_i)} X_i$ which is zero outside of $X_{(J)}$, we get an isometric morphism

$$w_J : \hat{\bigotimes_{i \in J}} A_i \longmapsto L^1(X) \ .$$

As easily cheked the $w_J$ form an inductive system and we get an isometric morphism

$$w : \hat{\otimes}^e A_i \longrightarrow L^1(X)$$

which transforms $\otimes a_i$ as indicated in the statement. It remains to be shown that $\operatorname{Im} w$ is dense in $L^1(X)$, or that each function f in $\mathcal{K}(X)$ is a limit of elements in $\operatorname{Im} w$ ; the support of f is included in some $X_{(J)}$ ; by the Stone-Weierstrass theorem we can suppose that f depends only on a finite number of coordinates, i.e. that there exists some $J' \in \mathcal{F}(I)$ such that $f = g \otimes 1$ with $g \in \mathcal{K}(\prod_{i \in J'} X_i)$ ;

we can also suppose that $J' > J$, but in this case $f \in \operatorname{Im} w_{J'}$.

Corollary 1. The $L^1$ algebra of the restricted product of discrete groups $G_i$ is canonically isomorphic to $\hat{\otimes} L^1(G_i)$.

Corollary 2. The $L^1$ algebra of a product of compact groups $G_i$ is canonically isomorphic to $\hat{\otimes}^e L^1(G_i)$ where $e_i$ is the function 1 on $G_i$.

Example 2. We first define the symmetric algebra of a Banach space. Consider a Banach space E and set, for each integer $n > 0$

$$E^{\hat{\otimes} n} = E \hat{\otimes} \ldots\ldots \hat{\otimes} E \quad \text{n-times} ;$$

every permutation s of the set $\{1, \ldots n\}$ gives rise to an automorphism $U_{s,n}$ of $E^{\hat{\otimes} n}$ such that

$$U_{s,n}(x_1 \otimes \ldots \otimes x_n) = x_{s(1)} \otimes \ldots \otimes x_{s(n)} ;$$

the operator $P_n = (n!)^{-1} \sum_s U_{s,n}$ is a projection of norm 1 ; we set $S^nE = \operatorname{Im} P_n =$ the set of all elements in $E^{\hat{\otimes} n}$ which are invariant by all $U_{s,n}$ ; finally we denote by $SE$ the Banach direct sum of all $S^nE$ for $n = 0,1,2,\ldots$, i.e. the set of all sequences $x = (x_n)$ where $x_n \in S^nE$ and $\|x\| = \sum \|x_n\| < \infty$ ; by definition $S^oE = \mathbb{C}$.

It is proved in the courses of Algebra that there exists on the algebraic direct sum $A = \bigoplus_{n=o}^{\infty} E^{\otimes n}$ a structure of commutative algebra such that

$$(x\,y)_n = \sum_{p=0}^{n} P_n(x_p \otimes y_{n-p}) \tag{2}$$

for every $x = (x_n)$ and $y = (y_n)$ in A ; we then have

$$\|(x\,y)_n\| \leqslant \sum_{p=0}^{n} \|x_p\| \cdot \|y_{n-p}\|$$

$$\|x\,y\| = \sum_{n} \|(x\,y)_n\| \leqslant \sum_{p,q} \|x_p\| \cdot \|y_q\| = \|x\| \cdot \|y\|$$

hence the multiplication can be extended to SE which becomes a commutative Banach algebra ; (2) is still valid for x and y $\in$ SE ; SE admits a unit element $\varepsilon = (1,0,0,\dots)$.

For each $a \in E$ it will be convenient to denote by exp a the following element of SE :

$$\exp a = (1,\ a,\ a^{\otimes 2}/2!, \dots,\ a^{\otimes n}/n!, \dots) ;$$

we have

$$\|\exp a\| = e^{\|a\|}$$

and

$$\exp(a+b) = \exp a \cdot \exp b.$$

Each Banach space $S^nE$ is generated by the particular tensors $x^{\otimes n}$ ; then the algebra SE is generated by $\varepsilon$ and E identified with the set of all elements $(0,\ x,\ 0,0,\dots)$.

Proposition 4. The Banach algebra SE possesses the following universal property : given a commutative Banach algebra B with unit, by associating with each morphism of unitary algebras

$v : SE \longrightarrow B$ with $\| v \| \leqslant 1$ , its restriction to E one gets a bijective correspondance between such morphisms v and all the continuous linear mappings $E \longrightarrow B$ of norm $\leqslant 1$ .

Proof. We have to show that every continuous linear mapping $u : E \longrightarrow B$ of norm $\leqslant 1$ can be extended to a morphism v ; we have for each n a multilinear mapping of norm $\leqslant 1$

$$\begin{aligned} E^n &\longrightarrow B \\ (a_1, \dots a_n) &\longmapsto u(a_1) \dots u(a_n) \end{aligned}$$

whence a linear mapping of norm $\leqslant 1$

$$\begin{aligned} v_n : E^{\hat{\otimes} n} &\longrightarrow B \\ a_1 \otimes \dots \otimes a_n &\longmapsto u(a_1) \dots u(a_n) \ ; \end{aligned}$$

it suffices to set, for each $x = (x_n) \in SE$ :

$$v(x) = \sum_{n=0}^{\infty} v_n(x_n) \ .$$

QED

Note that $v(\exp a) = e^{u(a)}$ .

Corollary 3. Let $(E_i)_{i \in I}$ be a family of Banach spaces, E its Banach direct sum, i.e. the set of all families $x = (x_i) \in \Pi E_i$ with $\| x \| = \Sigma \| x_i \| < \infty$ . Then SE is canonically isomorphic to $\hat{\otimes} SE_i$ ; this isomorphism carries each $\exp x$ into $\otimes \exp x_i$ .

The proof is purely categorical : it suffices to remark that SE and $\hat{\otimes}\, SE_i$ are solutions of the same universal problem ; note that $\otimes \exp x_i$ exists because $\sum \| \exp x_i - \varepsilon_i \| < \infty$ since $\| \exp x_i - \varepsilon_i \| = e^{\| x_i \|} - 1 \sim \| x_i \|$.

Remark 0. For $a \in E$ , exp a is nothing but $\sum_{n=0}^{\infty} a^n / n!$, the image of a in the exponential map which can be defined in any Banach algebra.

## § 6. Infinite tensor products of Hilbert spaces.

### n.6.1. Definition and first properties.

Let us consider a family $(H_i)_{i\in I}$ of Hilbert spaces and for each i, a unit vector $t_i$ in $H_i$ ; endow each $H_{(J)}$ with its usual prehilbert structure ; the mappings $L_{J,K}$ are isometric and we can put on $\otimes^t H_i$ the inductive limit prehilbert structure ; each $H_{(J)}$ appears as a subprehilbert space of $\otimes^t H_i$ and we have

$$(\otimes x_i \mid \otimes y_i) = \prod (x_i \mid y_i) \qquad \forall\ (x_i),(y_i) \in \prod^t H_i .$$

**Definition** 5. We shall denote by $\overset{h}{\otimes}{}^t H_i$ the Hilbert completion of the prehilbert space $\otimes^t H_i$ ; it is also the inductive limit of the finite tensor products $\overset{h}{\underset{i\in J}{\otimes}} H_i$ .

It is easy to construct orthonormal bases of $\overset{h}{\otimes}{}^t H_i$ : choose for each i an orthonormal basis $(e_{i,j})_{j\in J_i}$ of $H_i$ with $e_{i,o} = t_i$ ; for each element $f = (f(i))$ in $\prod^o J_i$ set $e_f = \otimes e_{i,f(i)}$ ; then it is easy to verify that the $e_f$ constitute an orthonormal basis of $\overset{h}{\otimes}{}^t H_i$ .

**Associativity**. For each partition $I = \bigcup_{\lambda\in L} I_\lambda$ there exists an isomorphism of $\overset{h}{\underset{i\in I}{\otimes}}{}^t H_i$ onto $\overset{h}{\underset{\lambda\in L}{\otimes}}{}^{(v_\lambda)} ( \overset{h}{\underset{i\in I_\lambda}{\otimes}}{}^{u_\lambda} H_i)$ with the same properties as in § 2 (ii).

Bibliography [9] .

## n.6.2. Definition of $\otimes x_i$ for certain families $(x_i)$.

As in § 4 we set $\otimes x_i = \lim L_J(\bigotimes_{i\in J} x_i)$ whenever this limit exists ; one can prove exactly as in prop. 3 that it does exist if $\sum \| x_i - t_i \| < \infty$ ; but it still exists under more general conditions :

Proposition 5. Let $(x_i)$ be a family of vectors satisfying

$$\sum \big| \, \| x_i \| - 1 \, \big| < \infty \tag{3}$$

$$\sum \big| (x_i \mid t_i) - 1 \big| < \infty \quad ; \tag{4}$$

then $\prod \| x_i \|$ exists, and it is null iff one of the $x_i$ is null. The family of the vectors $L_J(\bigotimes_{i\in J} x_i)$ has a limit in $\overset{h}{\otimes}{}^{t} H_i$ whose norm is $\prod \| x_i \|$ ; moreover we have

$$\lim_J \Big\| \bigotimes_{i\in I-J} x_i - \bigotimes_{i\in I-J} t_i \Big\| = 0 .$$

Proof. As in prop. 3 we can suppose $x_i \neq 0 \ \forall \ i$ ; then $\prod \| x_i \|$ exists and is non null by virtue of (3) ; set $c = \prod \| x_i \|^2$ ; the finite products $\prod_{i\in J} \| x_i \|$ are bounded by some $k > 0$ ; on the other hand we have $(x_i \mid t_i) \neq 0$ almost everywhere and we can suppose $(x_i \mid t_i) \neq 0 \ \forall \ i$ ; by (4), $\prod (x_i \mid t_i)$ has a value $d \neq 0$ .

Take a number $\varepsilon > 0$ ; there exists $J \in \mathcal{F}(I)$ such that $K \supset J$ implies

$$\Big| \prod_{i\in K} \| x_i \|^2 - c \Big| \leqslant \varepsilon c / (4k + \varepsilon)$$

and

$$\left|\prod_{i\in K} (x_i \mid t_i) - d\right| \leqslant \varepsilon|d|\,(8k+\varepsilon) \quad ;$$

setting $L = K - J$ we have

$$\Big\|\bigotimes_{i\in L} x_i - \bigotimes_{i\in L} t_i\Big\|^2 = \Big\|\bigotimes_{i\in L} x_i\Big\|^2 + 1 - 2\,\mathrm{Re}\Big(\bigotimes_{i\in L} x_i \,\Big|\, \bigotimes_{i\in L} t_i\Big)$$

$$\leqslant \Big|\prod_{i\in L} \|x_i\|^2 - 1\Big| + 2\Big|\prod_{i\in L} (x_i \mid t_i) - 1\Big|$$

$$\Big|\prod_{i\in L} \|x_i\|^2 - 1\Big| = \Big|\prod_{i\in K} \|x_i\|^2 - c + c - \prod_{i\in J} \|x_i\|^2\Big| / \prod_{i\in J} \|x_i\|^2$$

$$\leqslant (2\varepsilon c/(4k+\varepsilon))/(c - \varepsilon c/(4k+\varepsilon)) = \varepsilon/2k$$

$$\Big|\prod_{i\in L} (x_i \mid t_i) - 1\Big| = \Big|\prod_{i\in K} (x_i|t_i) - d + d - \prod_{i\in J}(x_i|t_i)\Big| / \Big|\prod_{i\in J}(x_i|t_i)$$

$$\leqslant (2\varepsilon|d|/(8k+\varepsilon))/(d - \varepsilon|d|\,(8k+\varepsilon)) = \varepsilon/4k$$

$$\Big\|\bigotimes_{i\in L} x_i - \bigotimes_{i\in L} t_i\Big\|^2 \leqslant \varepsilon/k \quad ; \qquad (5)$$

then

$$\Big\| L_K\Big(\bigotimes_{i\in K} x_i\Big) - L_J\Big(\bigotimes_{i\in J} x_i\Big)\Big\| = \Big\|\bigotimes_{i\in K} x_i - L_{J,K}\Big(\bigotimes_{i\in J} x_i\Big)\Big\|$$

$$= \prod_{i\in J} \|x_i\| \cdot \Big\|\bigotimes_{i\in L} x_i - \bigotimes_{i\in L} t_i\Big\|$$

$$\leqslant k\,(\varepsilon/k)^{\frac{1}{2}} = (\varepsilon\, k)^{\frac{1}{2}} \;;$$

this proves that $(L_J(\bigotimes_{i\in J} x_i))$ is a Cauchy family. Finally our last assertion is a consequence of (5).

n.6.3. Relations between the various tensor products $\overset{h}{\otimes}{}^{t} H_i$ .

We shall prove that if two families $(t_i)$, $(u_i)$ are sufficiently close, $\overset{h}{\otimes}{}^{t} H_i$ and $\overset{h}{\otimes}{}^{u} H_i$ are canonically isomorphic.

Lemma 1. The following relations between families $(t_i),(u_i)$ of unit vectors

$$\sum_i |1 - (t_i | u_i)| < \infty \tag{6}$$

$$\sum_i (1 - |(t_i | u_i)|) < \infty \tag{7}$$

are equivalence relations ; if we write them respectively $t \approx u$ and $t \sim u$ , we have $t \sim u$ if and only if there exists a family of complex numbers $\alpha_i$ with $|\alpha_i| = 1$ and $(u_i) \approx (\alpha_i t_i)$ .

Proof. These relations are trivially reflexive and symmetric ; let us show that the first one is ~~symmetri~~ transitive : suppose $(u_i) \approx (v_i)$ ; then

$$1 - (t_i | v_i) = 1 - (t_i | u_i) + 1 - (u_i | v_i) + (t_i - u_i | u_i - v_i)$$

$$\sum \| t_i - u_i \|^2 = \sum (2 - 2 \operatorname{Re}(t_i | u_i))$$

$$\leqslant \sum 2 | 1 - (t_i | u_i) | < \infty$$

and similarly

$$\sum \| u_i - v_i \|^2 < \infty$$

hence

$$\sum |(t_i - u_i | u_i - v_i)| \leqslant \sum \| t_i - u_i \| \cdot \| u_i - v_i \| < \infty$$

$$\sum | 1 - (t_i | v_i) | < \infty .$$

We now prove the last assertion ; if $(u_i) \approx (\alpha_i t_i)$ we have

$$\sum (1 - |(t_i|u_i)|) = \sum (1 - |(\alpha_i t_i|u_i)|)$$
$$\leqslant \sum |1 - (\alpha_i t_i|u_i)| < \infty ;$$

conversely suppose $t \sim u$ and set

$$\alpha_i = \begin{cases} |(t_i|u_i)|/(t_i|u_i) & \text{if } (t_i|u_i) \neq 0 \\ 1 & \text{in the opposite case ;} \end{cases}$$

then

$$\sum |1 - (\alpha_i t_i|u_i)| = \sum (1 - |(t_i|u_i)|) < \infty .$$

The transitivity of $\sim$ is now immediate.

*Theorem* 2. Let us suppose $t \sim u$ and more precisely $(u_i) \approx (\alpha_i t_i)$ ; there exists a unique isomorphism $F : \overset{h}{\otimes}{}^t H_i \longrightarrow \overset{h}{\otimes}{}^u H_i$ with the following property : if $\otimes x_i$ exists in the first space, $\otimes \alpha_i x_i$ exists in the second space and is equal to $F(\otimes x_i)$.

*Proof*. The unicity is clear since the $\otimes x_i$ with $(x_i) \in \prod{}^t H_i$ generate the first space. For each $J \in \mathcal{F}(I)$ we define a multilinear mapping

$$\prod_{i\in J} H_i \longrightarrow \overset{h}{\underset{i\in I}{\otimes}}{}^u H_i = (\overset{h}{\underset{i\in J}{\otimes}} H_i) \overset{h}{\otimes} (\overset{h}{\underset{i\in I-J}{\otimes}}{}^u H_i)$$

$$(x_i)_{i\in J} \longmapsto (\underset{i\in J}{\otimes} \alpha_i x_i) \otimes (\underset{i\in I-J}{\otimes} \alpha_i t_i)$$

which makes sense by prop. 5 ; it gives rise to a linear mapping

$$F_J : \overset{h}{\bigotimes_{i\in J}} H_i \longrightarrow \overset{h}{\bigotimes_{i\in I}}{}^{u} H_i$$

$$\bigotimes_{i\in J} x_i \longmapsto \Big(\bigotimes_{i\in J} \alpha_i x_i\Big) \otimes \Big(\bigotimes_{i\in I-J} \alpha_i t_i\Big)$$

which is easily seen to be isometric ; since the $F_J$ form an inductive system we get an isometric linear mapping

$$F : \overset{h}{\bigotimes}{}^{t} H_i \longrightarrow \overset{h}{\bigotimes}{}^{u} H_i$$

$$\otimes x_i \longmapsto \otimes \alpha_i x_i \qquad \forall (x_i) \in \Pi^t H_i .$$

Let us prove that F is onto ; if $(y_i) \in \Pi^u H_i$ we have, for J sufficiently large

$$\begin{aligned} \otimes y_i &= \Big(\bigotimes_{i\in J} y_i\Big) \otimes \Big(\bigotimes_{i\in I-J} u_i\Big) \\ &= \lim \Big(\bigotimes_{i\in J} y_i\Big) \otimes \Big(\bigotimes_{i\in I-J} \alpha_i t_i\Big) \end{aligned}$$

by the last assertion of prop. 5 ; since

$$\Big(\bigotimes_{i\in J} y_i\Big) \otimes \Big(\bigotimes_{i\in I-J} \alpha_i t_i\Big) \in \operatorname{Im} F ,$$

we see that Im F is dense, hence equal to the whole space. Let us now suppose that $\otimes x_i$ exists in $\overset{h}{\otimes}{}^{t} H_i$ ; then

$$\begin{aligned} F(\otimes x_i) &= F\Big(\lim L_J\Big(\bigotimes_{i\in J} x_i\Big)\Big) \\ &= \lim F_J\Big(\bigotimes_{i\in J} x_i\Big) \\ &= \lim \Big(\bigotimes_{i\in J} \alpha_i x_i\Big) \otimes \Big(\bigotimes_{i\in I-J} \alpha_i t_i\Big) ; \end{aligned}$$

and this is equal to $\lim \Big(\bigotimes_{i\in J} \alpha_i x_i\Big) \otimes \Big(\bigotimes_{i\in I-J} u_i\Big)$ since

$$\Big\| \Big(\bigotimes_{i\in J} \alpha_i x_i\Big) \otimes \Big(\bigotimes_{i\in I-J} \alpha_i t_i\Big) - \Big(\bigotimes_{i\in J} \alpha_i x_i\Big) \otimes \Big(\bigotimes_{i\in I-J} u_i\Big) \Big\| =$$

$$\prod_{i\in J} \| x_i \| \cdot \| \bigotimes_{i\in I-J} \alpha_i t_i - \bigotimes_{i\in I-J} u_i \|$$

which tends to 0 by the last assertion of prop. 5 .

Remark 1. The infinite tensor products of Hilbert spaces have been introduced by von Neumann in [44] ; the space $\overset{h}{\otimes}{}^t H_i$ is denoted by him $\prod\limits_{\alpha\in I} \overset{\mathcal{C}}{\otimes} H_\alpha$ where $\mathcal{C}$ is the class of t with respect to the relation $\approx$ ; if we take one element t in each class and the sum of the corresponding tensor products, we get $\prod\limits_{\alpha\in I} \otimes H_\alpha$ ; finally if we take only elements t which lie in some class with respect to $\sim$ , we get $\prod\limits_{\alpha\in I} \overset{\mathcal{C}_w}{\otimes} H_\alpha$.

## n.6.4. Infinite tensor products of operators.

Proposition 6. Suppose we have for each i a continuous linear operator $T_i$ in $H_i$ such that $\prod \| T_i \|$ exists and

$$\sum | \, \| T_i \, t_i \| - 1 \, | < \infty \qquad (8)$$

$$\sum | (T_i \, t_i \, | \, t_i) - 1 \, | < \infty \quad ; \qquad (9)$$

there exists a unique continuous linear operator T in $\overset{h}{\otimes}{}^t H_i$ with the following property : if $\otimes x_i$ exists, $\otimes T_i x_i$ exists and is equal to $T(\otimes x_i)$.

Proof. First take an element x of the algebraic tensor product $\otimes^t H_i$ and write

$$\otimes^t H_i = (\bigotimes_{i\in J} H_i) \otimes (\bigotimes_{i\in I-J}{}^t H_i)$$

$$x = x_J \otimes ( \bigotimes_{i\in I-J} t_i ) \quad ;$$

by prop. 5 we can consider the vector

$$T x = ( \bigotimes_{i\in J} T_i . x_J ) \otimes ( \bigotimes_{i\in I-J} T_i t_i )$$

and we get a linear operator $T$ in $\otimes^t H_i$ ; T is continuous since

$$\| T x \| \leq \prod_{i\in J} \| T_i \| . \| x_J \| . \prod_{i\in I-J} \| T_i t_i \|$$

$$\leq \| x \| . \prod \| T_i \| \quad ;$$

hence it can be extended to a continuous linear operator T in $\overset{h}{\otimes}{}^t H_i$ . Suppose now that $\otimes x_i$ exists ; then

$$T(\otimes x_i) = T(\lim_J L_J( \bigotimes_{i\in J} x_i))$$

$$= \lim_J T(( \bigotimes_{i\in J} x_i) \otimes ( \bigotimes_{i\in I-J} t_i))$$

$$= \lim_J ( \bigotimes_{i\in J} T_i x_i) \otimes ( \bigotimes_{i\in I-J} T_i t_i)$$

and this is equal to $\lim_J ( \bigotimes_{i\in J} T_i x_i) \otimes ( \bigotimes_{i\in I-J} t_i)$ by the same reasoning as in the end of th. 2.

Definition 6. Given a family $(T_i)$ of continuous linear operators in $H_i$, we set

$$\overset{h}{\otimes}{}^t T_i = \text{str.lim.} ( \bigotimes_{i\in J} T_i ) \otimes I$$

whenever this limit exists ; it does exist under the hypothesis of prop. 6.

Proposition 7. In the situation of th. 2, if $\overset{h}{\otimes}{}^{t}\, T_i$ exists then $\overset{h}{\otimes}{}^{u}\, T_i$ exists too and is equal to $F . \overset{h}{\otimes}{}^{t}\, T_i . F^{-1}$ .

In fact it is easy to see that for each finite J, F carries $(\bigotimes_{i\in J} T_i)\otimes I$ into the analogous operator in $\overset{h}{\otimes}{}^{u}\, H_i$ .

### n.6.5. Distributivity of tensor products with respect to Hilbert sums and integrals.

In the following theorem we suppose I countable.

Theorem 3. Let us consider for each i, a standard Borel space $X_i$, a Borel subset $Y_i$, a positive Borel measure $\mu_i$ on $X_i$ with $\mu_i(Y_i) = 1$ , a $\mu_i$ - measurable field of Hilbert spaces $x_i \longmapsto H_{i,x_i}$ , and a square integrable vector field $t_{i,x_i} \in H_{i,x_i}$ where $t_{i,x_i}$ is of norm 1 if $x_i \in Y_i$ and 0 in the opposite case. For each i let us set

$$H_i = \int_{X_i}^{\oplus} H_{i,x_i} . d\,\mu_i(x_i)$$

$$t_i = \int_{X_i}^{\oplus} t_{i,x_i} . d\,\mu_i(x_i) \qquad \text{(unit vector in } H_i\text{)} ;$$

let us set $X = \prod^{(Y_i)} X_i$ and define $\mu$ on X as in § 1 ; finally for each $x = (x_i) \in X$ we set

$$H_{(x)} = \Big( \overset{h}{\bigotimes_{i\in J}} H_{i,x_i} \Big) \overset{h}{\otimes} \Big( \overset{h\,(t_{i,x_i})}{\bigotimes_{i\in I-J}} H_{i,x_i} \Big)$$

where $J = \{ i \mid x_i \notin Y_i \}$. Then one can put on the field $x \longmapsto H_{(x)}$ a structure of $\mu$ - measurable field such that

$\overset{h}{\underset{i\in J}{\otimes}}{}^{t}\, H_i$ is canonically isomorphic to $\int_X^{\oplus} H_{(x)}\cdot d\mu(x)$.

Sketch of the proof. For $J \in \mathcal{F}(I)$ we have, by § 1.3 of Part I, an isomorphism

$$u_J \;:\; \overset{h}{\underset{i\in J}{\otimes}} H_i \longrightarrow K = \int^{\oplus}_{\prod_{i\in J} X_i} \overset{h}{\underset{i\in J}{\otimes}} H_{i,x_i}\cdot d(\underset{i\in J}{\otimes}\mu_i)(x)$$

$$a_i \longmapsto \int^{\oplus} \underset{i\in J}{\otimes} a_{i,x_i}\cdot d(\underset{i\in J}{\otimes}\mu_i)(x)$$

where $a_i = \int^{\oplus}_{X_i} a_{i,x_i}\cdot d\mu_i(x_i)$ ; then we have an isometric mapping

$$u'_J \;:\; K \longrightarrow L = K \overset{\ell}{\otimes} \int^{\oplus}_{\prod_{i\in I-J} Y_i} \overset{h}{\underset{i\in I-J}{\otimes}}{}^{(t_{i,x_i})} H_{i,x_i}\cdot d(\underset{i\in I-J}{\otimes}\mu_i)(x)$$

$$b \longmapsto b \otimes \int^{\oplus} \underset{i\in I-J}{\otimes} t_{i,x_i}\cdot d(\underset{i\in I-J}{\otimes}\mu_i)(x) \quad ;$$

then an isomorphism

$$u''_J \;:\; L \longrightarrow \int^{\oplus}_{X_{(J)}} H_{(x)}\cdot d\mu_{(J)}(x) \quad ;$$

and finally an isometric mapping

$$u'''_J \;:\; \int^{\oplus}_{X_{(J)}} H_{(x)}\cdot d\mu_{(J)}(x) \longrightarrow \int^{\oplus}_{X} H_{(x)}\cdot d\mu(x)$$

consisting in extending each vector field by 0 outside of $X_{(J)}$ ; the mappings $u'''_J \circ u''_J \circ u'_J \circ u_J$ form an inductive system, hence define an isometric mapping

$$u \;:\; \overset{h}{\otimes}{}^{t}\, H_i \longrightarrow \int^{\oplus}_{X} H_{(x)}\cdot d\mu(x) \quad ;$$

one proves that it is surjective by a reasoning similar to that of theorem 1.

Corollary 4. If $\mu_i$ has total mass 1 and $Y_i = X_i$ we get an isomorphism

$$\overset{h}{\underset{i\in I}{\bigotimes}}{}^{t} \int_{X_i}^{\oplus} H_{i,x_i}\cdot d\mu_i(x_i) \sim \int_{\Pi X_i}^{\oplus} \overset{h}{\underset{i\in I}{\bigotimes}}{}^{(t_{i,x_i})} H_{i,x_i}\cdot d(\otimes\mu_i)(x).$$

Corollary 5. If $H_{i,x_i} = \mathbb{C}$ and $t_{i,x_i} = 1$ or $0$ depending on whether $x_i$ belongs to $Y_i$ or not, we get an isomorphism

$$\overset{h}{\bigotimes}{}^{t} L^2(X_i, \mu_i) \sim L^2(X, \mu)$$

where $t_i$ is the characteristic function of $Y_i$.

Corollary 6. Assuming the hypotheses of both corollaries 4 and 5 we have an isomorphism

$$\overset{h}{\bigotimes}{}^{t} L^2(X_i, \mu_i) \sim L^2(\Pi X_i, \otimes\mu_i)$$

where $t_i$ is the function 1 on $X_i$.

Suppose now that each $\mu_i$ has the mass 1 at each point and that $Y_i$ is reduced to some point $a_i$ ; the reader will be able to state a result similar to th. 3 ; strictly speaking this is not a corollary of th. 3 since in our particular case we have not to assume I countable and $X_i$ standard ; we shall only state the following corollary, analogous to cor. 5 :

Corollary 7. Suppose we have for each i, a set $X_i$ and a point $a_i$ of $X_i$ ; then $\ell^2(\prod^{(a_i)} X_i)$ is canonically isomorphic to $\overset{h\,(\delta_{a_i})}{\otimes} \ell^2(X_i)$.

## n.6.6. The Hilbert symmetric space of a Hilbert space.

We shall introduce a notion similar to that of " symmetric algebra of a Banach space " (see § 5, ex. 2), but conveniently adapted to the category of Hilbert spaces. Let H be some Hilbert space ; for each integer $n > 0$ we can consider the Hilbert space

$$H^{\overset{h}{\otimes} n} = H \overset{h}{\otimes} \cdots\cdots \overset{h}{\otimes} H \qquad n - \text{times}$$

and then the closed subspace $S^n H$ consisting of those elements which are invariant by all permutations ; we denote by $SH$ the Hilbert sum of all $S^n H$ , $n = 0,1,2,\ldots$ ; an element of $SH$ is a sequence $X = (X_n)$ with $X_n \in S^n H$ and we have

$$\|X\|^2 = \Sigma \|X_n\|^2 < \infty$$

$$(X \mid Y) = \Sigma (X_n \mid Y_n) \quad .$$

For every x in H we denote by exp x the following element of $SH$ :

$$\exp x = (1,\ x,\ x^{\otimes 2}/(2!)^{\frac{1}{2}}, \ldots\ x^{\otimes n}/(n!)^{\frac{1}{2}}, \ldots)$$

so that we have

$$(\exp x \mid \exp y) = e^{(x \mid y)}$$

$$\| \exp x \|^2 = e^{\| x \|^2}$$

whence it follows that the mapping exp is continuous.

Lemma 2. The elements $\exp x$ are total in SH.

It is sufficient to prove that each element of the form $a^{\otimes n}$ belongs to the closed linear subspace K generated by the $\exp x$ ; let us set for each real number t :

$$f(t) = \exp t\, a \ ;$$

an easy computation shows that

$$f^{(n)}(0) = (n!)^{\frac{1}{2}}\, a^{\otimes n} \ ;$$

now the relation

$$f^{(n)}(0) = \lim_{t=0} n!\,.\,t^{-n} \left(f(t) - f(0) - \ldots - t^{n-1}((n-1)!)^{-1} f^{(n-1)}(0)\right)$$

proves by induction that $f^{(n)}(0) \in K$ .

Proposition 8. Let H be the Hilbert sum of a family of Hilbert spaces $H_i$ ; there exists a unique isomorphism F of ~~$SH_i$~~ SH onto $\overset{h(\varepsilon_i)}{\otimes} SH_i$ with the following property : for each $x = (x_i) \in H$ , $\otimes \exp x_i$ exists in $\overset{h(\varepsilon_i)}{\otimes} SH_i$ and is equal to $F(\exp x)$.

Proof. The unicity is clear. Now if $(x_i) \in H$ we have

$$\| \exp x_i \| - 1 = e^{\frac{1}{2} \| x_i \|^2} - 1 \sim \tfrac{1}{2} \| x_i \|^2$$

$$(\exp x_i \mid \varepsilon_i) - 1 = 0$$

so that $\exp x_i$ exists by prop. 5 ; we have

$$\begin{aligned}(\otimes \exp x_i \mid \otimes \exp y_i) &= \prod (\exp x_i \mid \exp y_i) \\ &= \prod e^{(x_i \mid y_i)} \\ &= e^{\Sigma (x_i \mid y_i)} = e^{(x \mid y)} \\ &= (\exp x \mid \exp y) \ ;\end{aligned}$$

thus there exists an isomorphism F of SH onto the closed linear subspace of $\overset{h\varepsilon}{\otimes} SH_i$ generated by the elements $\otimes \exp x_i$ with $F(\exp x) = \otimes \exp x_i$ ; but the $\otimes \exp x_i$ are total in $\overset{h\varepsilon}{\otimes} SH_i$ .

Remark 2. The space SH is used in Quantum Field Theory for the so called Representations of Commutation Relations ; see for instance [32],[34],[39].

## § 7. Infinite tensor products of von Neumann algebras.

### n.7.1. The concrete tensor product.

Let us consider a family of Hilbert spaces $H_i$ and for each i, a unit vector $t_i$ in $H_i$ and a von Neumann algebra $\mathcal{A}_i$ in $H_i$.

Definition 7. We shall denote by $\overset{c}{\otimes}{}^t \mathcal{A}_i$ the von Neumann algebra in the space $H = \overset{h}{\otimes}{}^t H_i$ which is generated by all operators of the form $\otimes T_i$ where $T_i \in \mathcal{A}_i$ and $T_i = I$ almost everywhere.

Clearly $\overset{c}{\otimes}{}^t \mathcal{A}_i$ also contains every operator $\otimes T_i$ with $T_i \in \mathcal{A}_i$ in the sense of definition 6 ; and in particular every operator $\otimes T_i$ where $T_i \in \mathcal{A}_i$ , $\Pi \| T_i \|$ exists, $\Sigma \, | \, \| T_i t_i \| - 1 \, | < \infty$ and $\Sigma \, | (T_i t_i \mid t_i) - 1 \, | < \infty$ . In the situation of th. 2, $\overset{c}{\otimes}{}^t \mathcal{A}_i$ and $\overset{c}{\otimes}{}^u \mathcal{A}_i$ are spatially isomorphic ; more precisely we have

$$F . \overset{c}{\otimes}{}^t \mathcal{A}_i . F^{-1} = \overset{c}{\otimes}{}^u \mathcal{A}_i \; ;$$

we shall see later (see § 9) that the type of $\overset{c}{\otimes}{}^t \mathcal{A}_i$ depends strongly on the choice of t.

Proposition 9. We have $\overset{c}{\otimes}{}^t \mathcal{A}'_i = ( \overset{c}{\otimes}{}^t \mathcal{A}_i )'$ , ~~the equality holds if all $\mathcal{A}_i$ are semi-finite~~.

First assertion : if $T_i \in \mathcal{A}_i$ , $T'_i \in \mathcal{A}'_i$ and $T_i = T'_i = I$ almost everywhere we have

$$\otimes T_i \cdot \otimes T'_i \;=\; \otimes\, T_i\, T'_i \;=\; \otimes\, T'_i\, T_i \;=\; \otimes\, T'_i \cdot \otimes\, T_i \;.$$

Second assertion : take some operator S in $(\overset{c}{\otimes}\, {}^t\mathcal{A}_i)'$, some weak neighbourhood V of S :

$$V = \{ S' \mid |((S'-S).x_n | y_n)| < 1 \;,\; n = 1,\dots N \}$$

and some $\varepsilon > 0$. There exists $J \in \mathcal{F}(I)$ with the following property :

$$\| P.x_n - x_n \| \leq \varepsilon \;,\; \| P.y_n - y_n \| \leq \varepsilon \;,\; n = 1,\dots N$$

where P is the projection onto the subspace $K = (\overset{\ell}{\bigotimes_{i\in J}} H_i) \otimes (\bigotimes_{i\in I-J} t_i)$; we can write $P = I \otimes Q$ where Q is the projection onto the vector $\bigotimes_{i\in I-J} t_i$ .

We claim that $S_P \in (\overset{c}{\bigotimes_{i\in J}} \mathcal{A}_i)' \otimes I$ ; in fact for $T_i \in \mathcal{A}_i$ and $x_i \in H_i$ with $i \notin J \Longrightarrow T_i = I,\; x_i = t_i$ , we have

$$\begin{aligned} S_P \cdot \otimes T_i \cdot \otimes x_i &= P.S. \otimes T_i \cdot \otimes x_i \\ &= P. \otimes T_i .S. \otimes x_i \\ &= I \otimes Q. \otimes T_i .S. \otimes x_i \\ &= \otimes T_i . I \otimes Q.S. \otimes x_i \\ &= \otimes T_i .P.S. \otimes x_i \\ &= \otimes T_i .S_P. \otimes x_i \;. \end{aligned}$$

Since our assertion is true for finite tensor products (see [48] theorem 12.3 ~~[1], prop. 14, p. 102~~), this implies $S_P \in (\overset{c}{\bigotimes_{i\in J}} \mathcal{A}'_i) \otimes I$ ; now $S_P$, operator in K, has the form $R \otimes I$ where $R \in \overset{c}{\bigotimes_{i\in J}} \mathcal{A}'_i$ ;

we can consider the operator $R \otimes I$ in $\overset{h}{\otimes}{}^{t} H_i$ where I is the identity operator in $\overset{h}{\underset{i \in I-J}{\otimes}}{}^{t} H_i$ ; we have

$$R \otimes I \in \Big( \overset{c}{\underset{i \in J}{\otimes}} \mathcal{A}_i' \Big) \otimes I \subset \overset{c}{\otimes}{}^{t} \mathcal{A}_i' \quad ;$$

on the other hand if $\varepsilon$ is sufficiently small we have $R \otimes I \in V$ since

$$\begin{aligned} |((R \otimes I - S).x_n | y_n)| \leqslant{} & |(R \otimes I.x_n | y_n) - (R \otimes I.Px_n | Py_n)| \\ & + |(R \otimes I.Px_n | Py_n) - (S.Px_n | Py_n)| \\ & + |(S.Px_n | Py_n) - (S.x_n | y_n)| \quad ; \end{aligned}$$

the second member of the righthand side is null while the other two are less than $\varepsilon \|S\| . (\|x_n\| + \|y_n\|)$.

Theorem 4. The von Neumann algebra $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ is a factor if and only if each $\mathcal{A}_i$ is a factor ; it is equal to $\mathcal{L}(H)$ if and only if $\mathcal{A}_i = \mathcal{L}(H_i) \quad \forall i$ .

Second assertion : if for some j, $\mathcal{A}_j'$ contains a non scalar operator $T_j$, $( \overset{c}{\otimes}{}^{t} \mathcal{A}_i )'$ contains the non scalar operator $\otimes T_i$ where $T_i = T_j$ if $i = j$ , $T_i = I$ if $i \neq j$ .

Conversely suppose $\mathcal{A}_i = \mathcal{L}(H_i)$ ; by the preceding proposition we have

$$\Big( \overset{c}{\otimes}{}^{t} \mathcal{A}_i \Big)' = \overset{c}{\otimes}{}^{t} \mathcal{A}_i' = \text{scalars.}$$

First assertion : if for some j the center of $\mathcal{A}_j$ contains a non scalar operator $T_j$, the center of $\overset{c}{\otimes}{}^t\mathcal{A}_i$ contains the above operator $\otimes T_i$. Conversely suppose $\mathcal{A}_i$ is a factor ; by the preceding proposition the von Neumann $\mathcal{A}$ generated by $\overset{c}{\otimes}{}^t\mathcal{A}_i$ and $(\overset{c}{\otimes}{}^t\mathcal{A}_i)'$ contains all operators $\otimes T_iT_i'$ where $T_i \in \mathcal{A}_i$, $T_i' \in \mathcal{A}_i'$, $T_i = T_i' = I$ almost everywhere, hence all operators $\otimes S_i$ where $S_i \in \mathcal{L}(H_i)$, $S_i = I$ almost everywhere ; by the first part of the proof we have $\mathcal{A} = \mathcal{L}(H)$, and $\overset{c}{\otimes}{}^t\mathcal{A}_i$ is a factor.

## n.7.2. The abstract tensor product.

Given a family $(\mathcal{A}_i)_{i\in I}$ of von Neumann algebras and, for each i, a projection $e_i$ in $\mathcal{A}_i$, we shall construct a von Neumann algebra $\overset{a}{\otimes}{}^e\mathcal{A}_i$ which admits as quotients the various concrete tensor products $\overset{c}{\otimes}{}^t\mathcal{A}_i$ .

Let us first define the inductive limit of an inductive system of von Neumann algebras ; let I be a filtering ordered set, $(\mathcal{A}_i)_{i\in I}$ a family of von Neumann algebras, and for $i \leq j$, $M_{i,j}$ a normal morphism $\mathcal{A}_i \longrightarrow \mathcal{A}_j$ (by convention all morphisms of von Neumann algebras preserve the unit elements) such that $M_{j,k} \circ M_{i,j} = M_{i,k}$ for $i \leq j \leq k$ ; denote by $\mathcal{A}$ the algebraic inductive limit of this inductive system, by $M_i$ the canonical morphisms $\mathcal{A}_i \longrightarrow \mathcal{A}$ and by $\pi$ the direct sum of all cyclic representations $\rho$ of $\mathcal{A}$ such that $\rho \circ M_i$ is normal for each i ; the von Neumann $\mathfrak{A}$ generated by $\pi(\mathcal{A})$ will be called the inductive limit of our inductive system ; note that it exists if and only if there exist representations $\rho$ of the above type ; it has the following universal property : given a von Neumann algebra $\mathcal{B}$, by associating to each normal morphism $v : \mathfrak{A} \longrightarrow \mathcal{B}$ the family $(v \circ u_i)$, we get a bijective correspondance between the normal morphisms v and the families of normal morphisms $v_i : \mathfrak{A}_i \longrightarrow \mathcal{B}$ such that $v_j \circ M_{i,j} = v_i$ for $i \leqslant j$ (inductive systems of normal morphisms).

We are now in a position to define $\overset{a}{\otimes}{}^{e}{}_{i}$ ; by realizing each $\mathfrak{A}_i$ in some Hilbert space we can define the finite tensor products $\overset{c}{\bigotimes}_{i \in J} \mathfrak{A}_i$ , which are independant of the chosen realizations and form an inductive system : for $J \subset K$ we write

$$\overset{c}{\bigotimes}_{i\in K} \mathfrak{A}_i = \Big(\overset{c}{\bigotimes}_{i\in J} \mathfrak{A}_i\Big) \overset{c}{\otimes} \Big(\overset{c}{\bigotimes}_{i\in K-J} \mathfrak{A}_i\Big)$$

and define $M_{J,K}$ by

$$M_{J,K}(a) = a \otimes \Big(\bigotimes_{i\in K-J} e_i\Big) .$$

Definition 8. We denote by $\overset{a}{\bigotimes}_{i\in I}{}^{e}\,\mathfrak{A}_i$ the inductive limit of the above inductive system. If $e = I$ we write $\overset{a}{\bigotimes}_{i\in I} \mathfrak{A}_i$ .

Proposition 10. Let, for each i, $H_i$ a Hilbert space, $t_i$ a unit vector in $H_i$, $\pi_i$ a normal representation of $\mathfrak{A}_i$ in $H_i$

with $\pi_i(e_i).t_i = t_i$ ; there exists a normal representation $\pi$ of $\overset{a}{\otimes}{}^{e}\mathcal{A}_i$ in $\overset{h}{\otimes}{}^{t} H_i$ such that $\pi(\otimes a_i) = \otimes \pi_i(a_i)$ for each family $(a_i) \in \Pi^{e}\mathcal{A}_i$ . Moreover $\mathrm{Im}\,\pi = \overset{c}{\otimes}{}^{t}\,\mathrm{Im}\,\pi_i$ .

Proof. Set $H = \overset{h}{\otimes}{}^{t} H_i$ , $\mathcal{B}_i = \mathrm{Im}\,\pi_i$ = von Neumann algebra in $H_i$ ; take some J in $\mathcal{F}(I)$ ; by [1], p. 60 we have a normal morphism

$$u_J : \overset{c}{\underset{i\in J}{\otimes}} \mathcal{A}_i \longrightarrow \overset{c}{\underset{i\in J}{\otimes}} \mathcal{B}_i$$

$$\otimes a_i \longmapsto \otimes \pi_i(a_i) \quad ;$$

on the other hand we can define a normal morphism

$$u'_J : \overset{c}{\underset{i\in J}{\otimes}} \mathcal{B}_i \longrightarrow \mathcal{L}(H)$$

by writing

$$H = \Big(\overset{h}{\underset{i\in J}{\otimes}} H_i\Big) \overset{h}{\otimes} \Big(\overset{h}{\underset{i\in I-J}{\otimes}}{}^{t} H_i\Big)$$

$$u'_J(b) = b \otimes \Big(\underset{i\in I-J}{\otimes} \pi_i(e_i)\Big) \quad ;$$

we get normal morphisms

$$v_J = u'_J \circ u_J : \overset{c}{\underset{i\in J}{\otimes}} \mathcal{A}_i \longrightarrow \mathcal{L}(H)$$

which form an inductive system and define a normal morphism $\pi : \overset{a}{\otimes}{}^{e}\mathcal{A}_i \longrightarrow \mathcal{L}(H)$ such that

$$\pi(\otimes a_i) = \otimes \pi_i(a_i) \qquad \forall\,(a_i) \in \Pi^{e}\mathcal{A}_i \,.$$

Last assertion : clearly we have $\mathrm{Im}\,\pi \subset \overset{c}{\otimes}{}^{t}\mathcal{B}_i$ ; to prove the converse inclusion it suffices to prove that $\otimes b_i \in \operatorname{Im} \pi$ for each family $(b_i)$ with $b_i \in \mathcal{B}_i$ and $b_i = I$ almost everywhere, i.e. for $i \notin J$ ; for each $i \in J$ there exists $a_i$ in $\mathcal{A}_i$ with $\pi_i(a_i) = b_i$ ; take $K \supset J$ and define an element $a_{(K)}$ in $\overset{c}{\underset{i\in K}{\otimes}} \mathcal{A}_i$ by

$$a_{(K)} = \Big( \underset{i\in J}{\otimes} a_i \Big) \otimes \Big( \underset{i\in K-J}{\otimes} I \Big) \ ;$$

since $v_K(a_{(K)})$ belongs to $\operatorname{Im} \pi$ , it suffices to show that $v_K(a_{(K)})$ converges strongly to $\otimes b_i$ , i.e. that

$$v_K(a_{(K)}).x \longrightarrow \otimes b_i.x$$

for each x in H ; by equicontinuity and linearity we can suppose $x = \otimes x_i$ , $x_i = t_i$ for $i \notin K$ ; then

$$\begin{aligned} v_K(a_{(K)}).x &= \Big( \underset{i\in J}{\otimes} \pi_i(a_i).x_i \Big) \otimes \Big( \underset{i\in K-J}{\otimes} x_i \Big) \otimes \Big( \underset{i\in I-K}{\otimes} \pi_i(e_i).t_i \Big) \\ &= \Big( \underset{i\in J}{\otimes} b_i.x_i \Big) \otimes \Big( \underset{i\in K-J}{\otimes} x_i \Big) \otimes \Big( \underset{i\in I-K}{\otimes} t_i \Big) \\ &= \otimes b_i.x_i = \otimes b_i.x \ . \end{aligned}$$

## § 8. Infinite tensor products of Hilbert algebras.

Let us consider for each i, a Hilbert algebra $\mathcal{A}_i$ with Hilbert completion $H_i$, and a projection of norm one $e_i \in \mathcal{A}_i$ ; then the left multiplication operator $U_{e_i}$ is a non zero projection. The algebraic tensor product $\mathcal{A} = \otimes^e \mathcal{A}_i$ is a $*$ - algebra and at the same time a prehilbert space whose Hilbert completion is $H = \overset{h}{\otimes}{}^e H_i$ ; we claim that $\mathcal{A}$ is a Hilbert algebra : the axioms (i),(ii),(iv) of [1], p. 66 are trivially verified ; as for (iii), take an element $a = \otimes a_i$ in $\mathcal{A}$ ; by prop. 6 we can form the continuous operator $\overset{h}{\otimes}{}^e U_{a_i}$ in H and we have, for $(b_i) \in \mathcal{A}$ :

$$(\overset{h}{\otimes}{}^e U_{a_i})(\otimes b_i) = \otimes a_i b_i = \otimes a_i \cdot \otimes b_i \quad ;$$

this proves that $\overset{h}{\otimes}{}^e U_{a_i} = U_{\otimes a_i}$ and that the mapping $b \longmapsto ab$ of $\mathcal{A}$ into itself is continuous for each a of the form $\otimes a_i$ ; the same property holds by linearity for each $a \in \mathcal{A}$ .

We shall denote by $\mathcal{U}_i$ , $\mathcal{V}_i$ , $\mathcal{U}$, $\mathcal{V}$ the von Neumann algebras canonically associated with $\mathcal{A}_i$ and $\mathcal{A}$.

Proposition 11. We have $\mathcal{U} = \overset{c}{\otimes}{}^e \mathcal{U}_i$ , $\mathcal{V} = \overset{c}{\otimes}{}^e \mathcal{V}_i$ .

We have $\mathcal{U} \subset \overset{c}{\otimes}{}^e \mathcal{U}_i$ since $\mathcal{U}$ is generated by the operators $U_{\otimes a_i} = \overset{h}{\otimes}{}^e U_{a_i}$ which belong to $\overset{c}{\otimes}{}^e \mathcal{U}_i$ ; in the same manner $\mathcal{V} \subset \overset{c}{\otimes}{}^{e} \mathcal{V}_i$ ; then we have

$$\mathcal{U} \subset \overset{c}{\otimes}{}^{e} \mathcal{U}_i = \overset{c}{\otimes}{}^{e} \mathcal{V}'_i \subset (\overset{c}{\otimes}{}^{e} \mathcal{V}_i)' \subset \mathcal{V}' = \mathcal{U} .$$

__Example__ 3. We take for $\mathcal{A}_i$ the algebra of all Hilbert-Schmidt operators in some Hilbert space $K_i$ whose dimension $r_i$ is finite or infinite but $> 1$ ; define the scalar product in $\mathcal{A}_i$ by

$$(a \mid b) = s_i^{-1} \operatorname{Tr} ab^*$$

where $s_i$ is some integer verifying $0 < s_i \leqslant r_i$ ; finally we take for $e_i$ a projection of rank $s_i$.

__Proposition__ 12. If I is infinite, the type of the factor $\mathcal{U}$ is

(i) $I_\infty$ if $s_i = 1$

(ii) $II_1$ if $s_i = r_i$ (which implies $r_i < \infty$ )

(iii) $II_\infty$ if $1 < s_i < r_i$ .

__Proof__. We first remark that $r_i$ infinite implies $\mathcal{U}$ infinite. Now choose an orthonormal basis $( \xi_\alpha )$ of $K_i$ such that Im $e_i$ is the subspace generated by $\xi_{\alpha_1}, \ldots \xi_{\alpha_{s_i}}$ ; there exists an isomorphism $F : \mathcal{A}_i \longrightarrow K_i \overset{h}{\otimes} K_i$ with the following properties : for each a in $\mathcal{A}_i$ with matrix $(a_{\alpha\beta})$ we have

$$F(a) = s_i^{-\frac{1}{2}} \sum_{\alpha,\beta} a_{\alpha\beta} \,.\, \xi_\alpha \otimes \xi_\beta \ ;$$

$$F(e_i) = s_i^{-\frac{1}{2}} \sum_{n=1}^{s_i} \xi_{\alpha_n} \otimes \xi_{\alpha_n} \ ;$$

$$F.U_a.F^{-1} = {}^t a \otimes I \ .$$

Suppose now $s_i = 1$ ; we can write

$$H_i = \mathcal{A}_i = H_{i,1} \overset{h}{\otimes} H_{i,2} \qquad \text{with} \quad H_{i,j} = K_i$$

$$e_i = e_{i,1} \otimes e_{i,2}$$

then using the associativity of the tensor products :

$$\overset{h}{\otimes}{}^{e} H_i = \Big(\overset{h}{\otimes}{}^{(e_{i,1})} H_{i,1}\Big) \overset{h}{\otimes} \Big(\overset{h}{\otimes}{}^{(e_{i,2})} H_{i,2}\Big)$$

$$\overset{h}{\otimes}{}^{e} U_{a_i} = \Big(\overset{h}{\otimes}{}^{(e_{i,1})} t_{a_i}\Big) \otimes I$$

$$\mathcal{U} = \Big(\overset{h}{\otimes}{}^{(e_{i,1})} H_{i,1}\Big) \otimes I$$

which proves (i).

Now suppose $s_i > 1$ ; denote by $c_i$ the projection onto $\xi_{\alpha_1}$ ; we have

$$(c_i \mid c_i) = s_i^{-1} < 1 \quad ;$$

then for each $J \in \mathcal{F}(I)$

$$\Big\| \Big(\bigotimes_{i\in J} c_i\Big) \otimes \Big(\bigotimes_{i\in I-J} e_i\Big) \Big\|^2 = \prod_{i \in J} s_i^{-1} \quad ;$$

the left multiplication operator corresponding to the element $\big(\bigotimes_{i\in J} c_i\big) \otimes \big(\bigotimes_{i\in I-J} e_i\big)$ is a trace class projection in $\mathcal{U}$, whose trace is arbitrarily small ; hence $\mathcal{U}$ is continuous.

In case (ii), $e_i$ is a unit element for $\mathcal{A}_i$ , $\otimes^{e} \mathcal{A}_i$ has a unit element, $\mathcal{U}$ is finite and consequently of type $II_1$.

Finally consider the situation (iii) ; if $r_i$ is infinite, $\mathcal{U}$ is infinite, continuous and also semi-finite, hence of type

$II_\infty$ ; if $r_i$ is finite, $\mathcal{A}_i$ has a unit element $1_i$ with

$$(1_i \mid 1_i) = r_i / s_i \ ;$$

by a reasoning quite analogous to the above we can construct projections in $\mathcal{U}$ whose trace is finite and arbitrarily large, so that $\mathcal{U}$ is of type $II_\infty$ .

## § 9. The type of certain infinite tensor products of von Neumann algebras.

We suppose I countable ; for each $i \in I$ we set $H_i = H_{i,1} \overset{h}{\otimes} H_{i,2}$ where $H_{i,1}$ and $H_{i,2}$ are Hilbert spaces having the same dimension $r_i$, $1 < r_i \leq \aleph_0$ ; every element $t_i$ of $H_i$ can be written

$$t_i = \sum_{n=0}^{r_i - 1} \alpha_{i,n} \cdot t_{i,1,n} \otimes t_{i,2,n}$$

where $(t_{i,j,n})$ is an appropriate basis of $H_{i,j}$ and $\alpha_{i,n}$ a positive number with

$$\alpha_{i,0} \geq \alpha_{i,1} \geq \dots\dots \quad \text{and} \quad \sum_n \alpha_{i,n}^2 = 1 .$$

The aim of this paragraph is to determine the type of the factor $\overset{c}{\otimes}{}^t \mathcal{A}_i$ where $\mathcal{A}_i = \mathcal{L}(H_{i,1}) \otimes I$ .

### n.9.1. First method.

We shall use prop. 12 ; we first suppose that for each i, the strictly positive $\alpha_{i,n}$ are equal, let

$$\alpha_{i,0} = \dots\dots = \alpha_{i,s_i-1} = s_i^{-\frac{1}{2}}$$

where $s_i$ is some integer $\leq r_i$ , and

$$\alpha_{i,n} = 0 \quad \text{for} \quad n \geq s_i .$$

Denote by $\mathcal{A}_i$ the Hilbert algebra of the Hilbert-Schmidt operators in $H_{i,1}$ , endowed with the scalar product

$$(a \mid b) = s_i^{-1} \cdot \mathrm{Tr}\, ab^* \quad ;$$

and by $e_i$ the projection corresponding to the subspace of $H_{i,1}$ generated by $t_{i,1,0}, \ldots, t_{i,1,s_i-1}$ ; the isomorphism $\mathcal{A}_i \longrightarrow H_{i,1} \overset{h}{\otimes} H_{i,2}$ described in § 8 gives rise to an isomorphism of the Hilbert completion of $\otimes^e \mathcal{A}_i$ onto $\overset{h}{\otimes}{}^t H_i$ which carries $\mathcal{U}(\otimes^e \mathcal{A}_i)$ into $\overset{c}{\otimes}{}^t \mathfrak{a}_i$ ; thus the type of $\overset{c}{\otimes}{}^t \mathfrak{a}_i$ is given by prop.12.

Suppose now the $\alpha_{i,n}$ arbitrary ; we can replace the family $(t_i)$ by an equivalent family without changing the type of $\overset{c}{\otimes}{}^t \mathfrak{a}_i$ (see § 7 after defin. 7) ; if we set

$$t'_i = t_{i,1,0} \otimes t_{i,2,0}$$

we have $(t_i \mid t'_i) = \alpha_{i,0}$ ; thus if $\sum_i (1 - \alpha_{i,0})$ is finite, t is equivalent to t' and $\overset{c}{\otimes}{}^t \mathfrak{a}_i$ is of type $I_\infty$ ; one can get in a similar way the other results contained in the

Theorem 5. The type of $\overset{c}{\otimes}{}^t \mathfrak{a}_i$ is

(i) $I_\infty$ if $\sum_i (1 - \alpha_{i,0}) < \infty$

(ii) $II_1$ if $r_i < \infty$ and $\sum_i \left(1 - r_i^{-\frac{1}{2}} \sum_{n=0}^{r_i-1} \alpha_{i,n}\right) < \infty$

(iii) $II_\infty$ if there exist integers $s_i$ with $1 < s_i < r_i$ and $\sum_i \left(1 - s_i^{-\frac{1}{2}} \sum_{n=0}^{s_i-1} \alpha_{i,n}\right) < \infty$ .

n.9.2. Second method.

In this number we shall prove that $\overset{c}{\otimes}{}^t \mathfrak{a}_i$ can be obtained by the method used by Murray and von Neumann to construct examples of factors (see [1], p. 132), which will allow us to establish the converses of (i) and (ii) in th. 5. It will be more convenient to write

$$t_i = \sum_{n \in N_i} \alpha_{i,n} \cdot t_{i,1,n} \otimes t_{i,2,n}$$

where $N_i$ is equal to $\mathbb{Z}$ if $r_i = \aleph_0$ and to $\mathbb{Z}/r_i\mathbb{Z}$ if $r_i$ is finite ; we can suppose

$$\alpha_{i,0} \geqslant \alpha_{i,1} \geqslant \alpha_{i,-1} \geqslant \alpha_{i,2} \geqslant \alpha_{i,-2} \geqslant \dots \qquad \text{if} \quad r_i = \aleph_0$$

$$\alpha_{i,0} \geqslant \alpha_{i,1} \geqslant \dots \geqslant \alpha_{i,r_i-1} \qquad \text{if} \quad r_i < \aleph_0 .$$

a) Particular case.

We suppose $\alpha_{i,n} > 0$ for each i and n . Let us denote by $\mu_i$ the measure on $N_i$ having the mass $\alpha^2_{i,n}$ at each point n , by $\delta_{i,n}$ the Dirac function at n ; set

$$K_{i,1} = L^2(N_i, \mu_i)$$

$$K_{i,2} = \ell^2(N_i) \quad ;$$

the elements $\alpha^{-1}_{i,n} \cdot \delta_{i,n} \otimes \delta_{i,n-p}$ constitute an orthonormal basis of $K_{i,1} \overset{h}{\otimes} K_{i,2}$ ; define an isomorphism

$$M_i : H_{i,1} \overset{h}{\otimes} H_{i,2} \longrightarrow K_{i,1} \overset{h}{\otimes} K_{i,2}$$

by

$$M_i(t_{i,1,n} \otimes t_{i,2,p}) = \alpha^{-1}_{i,n} \cdot \delta_{i,n} \otimes \delta_{i,n-p} \quad ;$$

then

$$M_i(t_i) = \sum \delta_{i,n} \otimes \delta_{i,0} = 1 \otimes \delta_{i,0} .$$

The von Neumann algebra $\mathcal{L}(H_{i,1})$ is generated by, on the one hand, the diagonal operators with respect to the basis $(t_{i,1,n})$, and on the other hand the shift operator

$$V_i \ : \ t_{i,1,n} \longmapsto t_{i,1,n+1} \ ;$$

if $T_f$ is the diagonal operator of multiplication by some function f, $M_i$ carries $T_f \otimes I$ into $T_f \otimes I$ ; on the other hand $M_i$ carries $V_i \otimes I$ into $W_i$ defined by

$$W_i(\delta_{i,n} \otimes \delta_{i,p}) = \alpha_{i,n} / \alpha_{i,n+1} \cdot \delta_{i,n+1} \otimes \delta_{i,p+1} \ .$$

The restricted product $\Pi' N_i$ acts in the space $\Pi N_i$ by componentwise addition ; consequently it acts in $L^2(\Pi N_i, \otimes \mu_i)$ by unitary operators $U_m$ for $m \in \Pi' N_i$ ; similarly $\Pi' N_i$ acts in $\ell^2(\Pi' N_i)$ by unitary operators $U'_m$ . Now we have an isomorphism

$$\overset{h}{\otimes} M_i \ : \ \overset{h}{\otimes}{}^{t} H_i = \overset{h}{\otimes}{}^{t}(H_{i,1} \overset{h}{\otimes} H_{i,2}) \longrightarrow \overset{h}{\otimes}{}^{(1 \otimes \delta_{i,o})}(K_{i1} \otimes K_{i2}) \ ;$$

then by the associativity property, an isomorphism

$$\overset{h}{\otimes}{}^{(1 \otimes \delta_{i,o})}(K_{i,1} \overset{h}{\otimes} K_{i,2}) \longrightarrow (\overset{h}{\otimes}{}^{1} K_{i,1}) \overset{h}{\otimes} (\overset{h}{\otimes}{}^{(\delta_{i,o})} K_{i,2}) \ ;$$

finally by corollaries 6 and 7 an isomorphism

$$(\overset{h}{\otimes}{}^{1} K_{i,1}) \overset{h}{\otimes} (\overset{h}{\otimes}{}^{(\delta_{i,o})} K_{i,2}) \longrightarrow L^2(\Pi N_i, \otimes \mu_i) \overset{h}{\otimes} \ell^2(\Pi' N_i) \ ;$$

by composing these three isomorphisms we get a new isomorphism

$$M \ : \ \overset{h}{\otimes}{}^{t} H_i \longrightarrow L^2(\Pi N_i, \otimes \mu_i) \overset{h}{\otimes} \ell^2(\Pi' N_i) \ ;$$

M carries each operator of the form $\otimes(T_{f_i} \otimes I)$ into $T_{\otimes f_i} \otimes I$ ;

each operator $\otimes (V_i^{m_i} \otimes I)$, where $m = (m_i) \in \Pi' N_i$, into the operator $U_m \otimes U_m'$ ; hence M carries $\overset{c}{\otimes}\, {}^t\mathcal{A}_i$ into the von Neumann algebra generated by the operators $T_f \otimes I$ with $f \in L^{\infty}(\Pi N_i, \otimes \mu_i)$ , and $U_m \otimes U_m'$ with $m \in \Pi' N_i$ .

Suppose $\overset{c}{\otimes}\, {}^t\mathcal{A}_i$ is of type I ; by [43], $\otimes \mu_i$ is atomic ; since the point in $\Pi N_i$ which has the largest measure is the point with all components null, we must have $\Pi \alpha_{i,o} > 0$ , or equivalently $\Sigma (1 - \alpha_{i,o}) < \infty$ .

Suppose now $\overset{c}{\otimes}\, {}^t\mathcal{A}_i$ is of type $II_1$ ; by [43], $\otimes \mu_i$ is equivalent to some finite Borel measure, invariant under $\Pi' N_i$ ; on the other hand all $r_i$ are finite , $\Pi N_i$ is a compact group for the product topology, and $\Pi' N_i$ is a dense subgroup acting by translations ; each finite Borel measure on $\Pi N_i$ is a Radon measure, and being invariant under $\Pi' N_i$, it will be invariant by all translations, i.e. equivalent to the Haar measure ; the Haar measure is the product of the measures $\nu_i$ which have a mass $r_i^{-1}$ at each point ; by [41] $\otimes \mu_i \sim \otimes \nu_i$ implies $\sum_i (1 - r_i^{-\frac{1}{2}} \sum_{n=o}^{r_i - 1} \alpha_{i,n}) < \infty$ .

b) General case.

The previous results still hold since we can make all $\alpha_{i,n}$ strictly positive by modifying them sufficiently little to not change the equivalence class of $(t_i)$ nor the nature of the families $(1 - \alpha_{i,0})$ and $(1 - r_1^{-\frac{1}{2}} \Sigma \alpha_{i,n})$ : thus we have proved the following

**Theorem** 6. With the notations of the beginning of § 9, the type of the factor $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ is

(i) $I_\infty$ if and only if $\sum_i (1 - \alpha_{i,0}) < \infty$

(ii) $II_1$ if and only if $r_i < \infty$ and $\sum_i (1 - r_i^{-\frac{1}{2}} \sum_{n=0}^{r_i - 1} \alpha_{i,n}) < \infty$ .

**Remark** 3. It is more difficult to distinguish the types $II_\infty$ and III , C.C.Moore proves in [42] the following result, by means of a deeper analysis of the infinite products of measures : suppose $\alpha_{i,0} \geqslant k$ $\forall i$ for some $k > 0$ , then $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ is of type III if and only if

$$\sum_{i,n} \alpha_{i,n}^2 (\inf(\alpha_{i,0}^2 / \alpha_{i,n}^2 - 1 , c))^2 = \infty$$

for some (or equivalently for all) constants $c > 0$ ; he also states without proof on p. 458 a result equivalent to the third assertion of our th 5 . E.Störmer has proved by another method (see [46]) some results contained in th. 5 and also the following (contained in Moore's theorem) : suppose $r_i$ and $\alpha_{i,n}$ independant of i ; then $\overset{c}{\otimes}{}^{t} \mathcal{A}_i$ is of type III iff there exist at least two distinct and non null $\alpha_{i,n}$ .

Remark 4 (On the isomorphisms between the various factors $\overset{c}{\otimes}\, {}^{t}\mathcal{Q}_i$).

In [35] Araki and Woods study systematically the set E of the isomorphism classes of the factors $\overset{c}{\otimes}\, {}^{t}\mathcal{Q}_i$ which can be obtained by varying the $r_i$ and $t_i$ ; E can be divided into five mutually disjoint subsets $E_1, \ldots E_5$ :

$E_1$ contains only one factor, which is of type I .

$E_2$ contains only one factor, which is of type $II_1$ and hyperfinite.

$E_3$ contains exactly the factors $\mathcal{A}_\lambda$, $\lambda \in ]0,\frac{1}{2}[$, constructed in the following way : $\mathcal{A}_\lambda = \overset{c}{\otimes}\, {}^{t}\mathcal{Q}_i$ where $r_i = 2$ , $\lambda_{i,0} = \lambda$ , $\alpha_{i,1} = 1-\lambda$ ; these factors are of type III ; it had been proved earlier that they are mutually non isomorphic ([45]).

$E_4$ contains factors of various types and among others an uncontable family of type III factors.

$E_5$ contains exactly one factor, which is of type III.

## § 10. Infinite tensor products of $C^*$- algebras. Definition and first properties.

Let us consider a family $(A_i)_{i \in I}$ of $C^*$- algebras and for each i, a non zero projection $e_i$ in $A_i$ ; if we endow each finite tensor product $\underset{i \in J}{\otimes} A_i$ with the $\nu$ crossnorm (resp. the $*$ crossnorm), the morphisms $L_{J,K}$ are isometric, so that we can define the $\nu$ and $*$ norms on $\otimes^e A_i$ ; clearly these are respectively the largest and the smallest $C^*$- crossnorms ; the $C^*$ completions will be denoted by $\overset{\vee}{\otimes}{}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$ ; they can also be considered as the inductive limits of the finite tensor products $\underset{i \in J}{\overset{\vee}{\otimes}} A_i$ and $\underset{i \in J}{\overset{*}{\otimes}} A_i$ ; if each $A_i$ has property (T), $\overset{\vee}{\otimes}{}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$ are identical.

The tensor products $\overset{\vee}{\otimes}{}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$ possess properties of associativity similar to that of § 2.

If $e_i$ is a unit element for $A_i$, we write $\overset{\vee}{\otimes} A_i$ and $\overset{*}{\otimes} A_i$ instead of $\overset{\vee}{\otimes}{}^e A_i$ and $\overset{*}{\otimes}{}^e A_i$ ; $\overset{\vee}{\otimes} A_i$ has the following universal property : given a $C^*$- algebra B with unit, there is a bijective correspondance between the unitary morphisms $u : \overset{\vee}{\otimes} A_i \longrightarrow B$ and the families of commuting unitary morphisms $u_i : A_i \longrightarrow B$ ; it is given by $u(\otimes a_i) = \prod u_i(a_i)$ for each $(a_i)$ in $\prod^e A_i$ .

Proposition 13. Consider a family of Banach $*$- algebras $A_i$ with projections $e_i$ of norm 1 ; denote by $e_i'$ the canonical image of $e_i$ in $C^*(A_i)$ and suppose $e_i' \neq 0$. Then $C^*(\hat{\otimes}^{e} A_i)$ is canonically isomorphic to $\check{\otimes}^{e'} C^*(A_i)$.

In fact we have

$$\check{\otimes}^{e'} C^*(A_i) = \varinjlim \bigotimes_{i\in J} {}^*(A_i) = \varinjlim C^*\Big(\hat{\bigotimes_{i\in J}} A_i\Big)$$

and it is easy to see that the functor $^*$ commutes with the inductive limits.

Corollary 8. Let $(X_i)$ be a family of locally compact groups with compact open subgroups $Y_i$ ; then $C^*(\prod^{(Y_i)} X_i)$ is canonically isomorphic to $\check{\otimes} C^*(X_i)$ where $e_i$ is the characteristic function of $Y_i$.

This is a consequence of th. 1 and prop. 13.

Example . Let $(X_i)$ be a family of locally compact topological spaces with compact open subsets $Y_i$ ; then $\mathcal{C}_0(\prod^{(Y_i)} X_i)$ is canonically isomorphic to $\overset{*}{\otimes}^{e} \mathcal{C}_0(X_i)$ where $e_i$ is the characteristic function of $Y_i$. The proof is quite similar to that of th. 1.

## § 11. Infinite tensor products of representations of $C^*$-algebras.

Proposition 14. Let us consider for each i, a $C^*$- algebra $A_i$, a non zero projection $e_i$ in $A_i$, a Hilbert space $H_i$, a unit vector $t_i$ in $H_i$ and a representation $\pi_i$ of $A_i$ in $H_i$ such that

$$\sum_i ( 1 - \| \pi_i(e_i).t_i \| ) < \infty . \qquad (10)$$

Then there exists a unique representation $\pi$ of $\overset{*}{\otimes}{}^{e} A_i$ in $\overset{h}{\otimes}{}^{t} H_i$ such that $\pi(\otimes a_i) = \otimes \pi_i(a_i)$ for each family $(a_i)$ in $\prod^e A_i$ .

Proof The unicity is clear. To prove the existance take an element in $\otimes^e A_i$ of the form

$$a = a_J \otimes ( \bigotimes_{i \in I-J} e_i )$$

where $J \in \mathcal{F}(I)$ and $a_J \in \bigotimes_{i \in J} A_i$ ; write

$$\overset{h}{\otimes}{}^{t} H_i = ( \overset{h}{\bigotimes_{i\ J}} H_i ) \otimes ( \overset{h}{\bigotimes_{i \in I-J}}{}^{t} H_i ) ;$$

(10) implies $\sum_i | 1 - (\pi_i(e_i).t_i \mid t_i) | < \infty$ and by prop. 6 we can consider the following continuous linear operator in $\overset{h}{\otimes}{}^{t} H_i$ :

$$\pi(a) = ( \bigotimes_{i \in J} \pi_i )(a_J) \otimes ( \overset{h}{\bigotimes_{i \in I-J}}{}^{t} \pi_i(e_i)) ;$$

one can easily check that $\pi$ is a representation and moreover by prop. 6 we have

$$\| \pi(a) \| \leqslant \| a_J \|_* = \| a \|_* ;$$

then $\pi$ extends to a representation of $\overset{*}{\otimes}^{e} A_i$ which has the required properties.

Definition 10. The representation $\pi$ defined in prop. 14 will be denoted by $\overset{*}{\otimes}^{e,t}\pi_i$. The von Neumann algebra it generates is identical with $\overset{c}{\otimes}^{t}\pi_i(A_i)''$ (the proof is the same as for prop. 10) ; thus $\overset{*}{\otimes}^{e,t}\pi_i$ is factorial or irreducible if and only if each $\pi_i$ has the same property.

The kernel of $\overset{*}{\otimes}^{e,t}\pi_i$ depends only on the kernels of the $\pi_i$ ; in particular $\overset{*}{\otimes}^{e,t}\pi_i$ is faithful if and only if each $\pi_i$ is faithful.

On the other hand by prop. 7 if $t \sim t'$, $\overset{*}{\otimes}^{e,t}\pi_i$ and $\overset{*}{\otimes}^{e,t'}\pi_i$ are equivalent ; the following proposition is a partial converse of this result.

Proposition 15. We suppose each $\pi_i$ is irreducible ; then $\overset{*}{\otimes}^{e,t}\pi_i$ and $\overset{*}{\otimes}^{e,t'}\pi_i$ are equivalent if and only if we have $t \sim t'$.

Proof. We suppose $t \not\sim t'$ and prove that the two representations are not equivalent.

a) Particular case.

We suppose $A_i = \mathcal{L}(H_i)$, $\pi_i$ = identity. Since $t \not\sim t'$, we have $\sum_i (1 - |(t_i\ t_i')|) = \infty$ ; there exists a countable subset $I_o \subset I$ such that $\sum_{i \in I_o} (1 - |(t_i | t_i')|) = \infty$ ; it is sufficient to prove that the representations $\overset{*}{\bigotimes}_{i \in I_o}{}^{e,t} \pi_i$ and $\bigotimes_{i \in I_o}{}^{e,t'} \pi_i$ are not equivalent, so that we are led back to the case where I is countable, say $I = \{1, 2, \dots\}$. Denote by $P_i$ the projection operator in $H_i$ onto $t_i$ and set

$$T_n = P_1 \otimes P_2 \otimes \dots \otimes P_n \otimes e_{n+1} \otimes e_{n+2} \otimes \dots \quad ;$$

we shall prove that $(\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n)$ converges strongly to 0 while $(\overset{*}{\otimes}{}^{e,t} \pi_i)(T_n)$ does not, which will establish our result.

To prove that

$$(\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n).x \longrightarrow 0 \qquad \forall\ x \in \overset{h}{\otimes}{}^{t'} H_i$$

we can take x in the algebraic tensor product $\otimes^{t'} H_i$ since our operators have norms $\leqslant 1$ ; then by linearity we can take $x = \otimes x_i$ where $x_i = t_i'$ if i is larger than some number j ; then for $n \geqslant j$ :

$$(\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n).x = (\bigotimes_{i=1}^{j} P_i x_i) \otimes (\bigotimes_{i=j+1}^{n} P_i t_i') \otimes (\bigotimes_{i>n} t_i')$$

$$\| (\overset{*}{\otimes}{}^{e,t'} \pi_i)(T_n).x \| \leqslant \| x \| . \prod_{i=j+1}^{n} |(t_i | t_i')| \quad ;$$

since $\sum_{i=j+1}^{\infty} (1 - |(t_i | t_i')|) = \infty$ , $\prod_{i=j+1}^{\infty} |(t_i | t_i')| = 0$ ,

$\prod_{i=j+1}^{n} |(t_i | t_i')| \longrightarrow 0$ and we are done.

b) General case.

Set $f_i = \pi_i(e_i)$ ; $\otimes^f \mathcal{L}(H_i)$ is included and weakly dense in the tensor product $\overset{a}{\otimes}{}^{f}\mathcal{L}(H_i)$ defined in n.7.2 ; we have a morphism $\otimes\pi_i : \otimes^{e} A_i \longrightarrow \otimes^{f}\mathcal{L}(H_i)$ whose image is weakly dense since each $\pi_i$ is irreducible ; by prop. 10 we have representations u and u' of $\overset{a}{\otimes}{}^{f}\mathcal{L}(H_i)$ in $\overset{h}{\otimes}{}^{t} H_i$ and $\overset{h}{\otimes}{}^{t'} H_i$ ; by the first part of the proof the restrictions of u and u' to $\otimes^{e} A_i$ are not equivalent, but these restrictions are nothing but $\otimes^{e,t}\pi_i$ and $\otimes^{e,t'}\pi_i$ .

Theorem 7. If I is infinite and if each $A_i$ admits sufficiently many irreducible representations $\pi$ with rank $\pi(e_i) \geqslant 2$ , then $\overset{*}{\otimes}{}^{e} A_i$ is antiliminar.

Proof. For each i there exist a set $X_i$ and for each point $x_i$ of $X_i$ an irreducible representation $\pi_{i,x_i}$ of $A_i$ in some Hilbert space $H_{i,x_i}$ such that rank $\pi_{i,x_i}(e_i) \geqslant 2$ and $\underset{x_i \in X_i}{\otimes} \pi_{i,x_i}$ is faithful. We choose a point $y_i$ in $X_i$ and vectors $t_{i,x_i}$ , $t'_{i,x_i}$ in $\operatorname{Im}\pi_{i,x_i}(e_i)$ such that

$$(t_{i,x_i} \mid t'_{i,x_i}) = 0$$

$$\| t_{i,x_i} \| = \| t'_{i,x_i} \| = \begin{cases} 1 & \text{if } x_i = y_i \\ 0 & \text{otherwise} \end{cases} ;$$

for each $x = (x_i)$ in $X = \prod^{(y_i)} X_i$ we set

$$\pi(x) = \overset{*}{\otimes}{}^{e\,(t_{i,x_i})}\pi_i \; , \quad \pi'(x) = \overset{*}{\otimes}{}^{e\,(t'_{i,x_i})}\pi_i$$

which makes sense because $t_{i,x_i}$ and $t'_{i,x_i}$ are unit vectors for almost all i ; $\pi_{(x)}$ and $\pi'_{(x)}$ are irreducible representations of $\overset{*}{\otimes}{}^{e} A_i$ , have the same kernel and are unequivalent by the preceding proposition ; it suffices now to prove that $\bigoplus\limits_{x\in X} \pi_{(x)}$ and $\bigoplus\limits_{x\in X} \pi_{(x)}$ are faithful. Consider the first : as indicated in n.6.5 we have an isomorphism

$$\bigoplus_{x\in X} H_{(x)} \longrightarrow \overset{h}{\underset{i\in I}{\otimes}}{}^{t} \Big( \bigoplus_{x_i\in X_i} H_{i,x_i} \Big) \quad ;$$

as easily verified this isomorphism carries $\bigoplus\limits_{x\in X} \pi_{(x)}$ into $\overset{*}{\underset{i\in I}{\otimes}}{}^{e,t} \Big( \bigoplus\limits_{x_i\in X_i} \pi_{i,x_i} \Big)$ ; since the second representation is faithful, so is the first.

Corollary 9. If I is infinite, $e_i$ = unit element, and $A_i$ has no nonzero commutative two sided closed ideal, $\overset{*}{\otimes} A_i$ is antiliminar.

Corollary 10. We suppose that I is infinite countable, $A_i$ is postliminar separable and admits sufficiently many irreducible representations $\pi$ with rank $\pi(e_i) \geqslant 2$ ; then $\overset{*}{\otimes}{}^{e} A_i$ admits an irreducible representation which is not equivalent to a tensor product of representations.

Proof. Take a partition $I = I_1 \cup I_2$ with $I_1$ and $I_2$ infinite and write $\overset{*}{\otimes}{}^{e} A_i = B_1 \overset{*}{\otimes} B_2$ where $B_j = \overset{*}{\underset{i\in I_j}{\otimes}}{}^{e} A_i$ ;

by Part I, prop. 7, $B_1$ and $B_2$ have property (T) and we can also write $\overset{*}{\otimes}{}^{e} A_i = B_1 \overset{\vee}{\otimes} B_2$ ; $B_1$ and $B_2$ being separable and antiliminar, by Part I, th. 6 the algebra $B_1 \overset{\vee}{\otimes} B_2$ admits an irreducible representation which is not a tensor product of representations of $B_1$ and $B_2$ , and consequently it is not a tensor product of representations of the $A_i$ .

Bibliography [9] .

## § 12. Infinite tensor products of positive functionals on $C^*$ - algebras.

We consider $C^*$ - algebras $A_i$ with non zero projections $e_i$.

Proposition 16. Let $f_i$ be a positive functional on $A_i$ such that $f_i(e_i) = 1$ and $\prod \| f_i \| < \infty$ (note that $\| f_i \| \geqslant 1$). Then there exists a unique positive functional f on $\overset{*}{\otimes}{}^e A_i$ verifying

$$f(\otimes a_i) = \prod f_i(a_i) \qquad \forall (a_i) \in \prod{}^e A_i \quad ;$$

its norm is equal to $\prod \| f_i \|$ ; finally the representation associated with f is equivalent to $\overset{*}{\otimes}{}^{e,t} \pi_i$ where $\pi_i$ is the representation associated with $f_i$ and $t_i$ the corresponding cyclic vector multiplied by $\| f_i \|^{-\frac{1}{2}}$.

Proof. The unicity being trivial we shall prove the existence of f ; we have a multilinear functional on $\prod{}^e A_i : (a_i) \longmapsto \prod f_i(a_i)$ , whence a linear functional f on $\otimes^e A_i$ such that $f(\otimes a_i) = \prod f_i(a_i)$ ; f is positive since its restriction to any finite tensor product is positive ; moreover if a is an element of $\underset{i \in J}{\otimes} A_i$ we have

$$| f(a) | \leqslant \prod_{i \in J} \| f_i \| \, . \, \| a \|_* \leqslant \prod_{i \in I} \| f_i \| \, . \, \| a \|_* ; \qquad (11)$$

hence f extends to a positive functional on $\overset{*}{\otimes}{}^e A_i$ .
We now denote by $H_i$ the space of $\pi_i$, by $x_i$ the corresponding

cyclic vector and set $t_i = x_i . \| f_i \|^{-\frac{1}{2}}$ ; we have

$$f_i(a_i) = (\pi_i(a_i).x_i \mid x_i) \qquad \forall \; a_i \in A_i$$

$$\| t_i \|^2 = \| x_i \|^2 . \| f_i \|^{-1} = 1$$

$$\| \pi_i(e_i).t_i \|^2 = (\pi_i(e_i).t_i \mid t_i)$$

$$= \| f_i \|^{-1}.(\pi_i(e_i).x_i \mid x_i) = \| f_i \|^{-1} ;$$

since $\prod \| f_i \|^{-\frac{1}{2}} > 0$ we have

$$\sum (1 - \| f_i \|^{-\frac{1}{2}}) < \infty$$

$$\sum (1 - \| \pi_i(e_i).t_i \|) < \infty \quad ;$$

by prop. 14 we can form the representation $\pi = \overset{*}{\otimes}{}^{e,t} \pi_i$ ;

since

$$\| x_i \| = (x_i \mid t_i) = \| f_i \|^{\frac{1}{2}}$$

we have

$$\sum (\| x_i \| - 1) < \infty$$

$$\sum ((x_i \mid t_i) - 1) < \infty$$

and we can consider the vector $x = \otimes x_i$ ; it is cyclic for $\pi$ and we have $f(a) = (\pi(a).x \mid x)$ for each a of the form $\otimes a_i$ , hence for each a in $\overset{*}{\otimes}{}^{e} A_i$ .

We finally prove that $\| f \| = \prod \| f_i \|$ ; by (11) we have $\| f \| \leq \prod \| f_i \|$ ; to prove the converse inequality take an $\varepsilon > 0$ , a $J \in \mathcal{F}(I)$ such that

$$\prod_{i \in J} \| f_i \| \geq \prod \| f_i \| . (1+\varepsilon)^{-1} ,$$

and for each $i \in J$ an element $a_i$ in $A_i$ such that

$$\| a_i \| = 1 \quad \text{and} \quad | f_i(a_i) | \geqslant \| f_i \| . (1+\varepsilon)^{-1/n}$$

where $n = \text{card } J$ ; set

$$a = ( \underset{i \in J}{\otimes} a_i ) \otimes ( \underset{i \in I-J}{\otimes} e_i ) \quad ;$$

then $\| a \| = 1$ and

$$\begin{aligned} \| f \| \geqslant | f(a) | &= \prod_{i \in J} | f_i(a_i) | \\ &\geqslant (1+\varepsilon)^{-1} . \prod_{i \in J} \| f_i \| \\ &\geqslant (1+\varepsilon)^{-2} . \prod \| f_i \| . \end{aligned}$$

Definition 11. The positive functional f defined above will be denoted by $\overset{*}{\otimes}{}^{e} f_i$ ; it is factorial or pure if and only if each $f_i$ has the same prpperty ; it is a state if and only if each $f_i$ is a state. In particular if $e_i$ is the identity of $A_i$ one can form the tensor product of an arbitrary family of states.

Proposition 17. Let us consider a $C^*$- algebra $A_o$ , a nonzero projection $e_o$ in $A_o$ and two distinct pure states $f_o$ and $g_o$ on $A_o$ with $f_o(e_o) = g_o(e_o) = 1$ ; let us set $A = \overset{*}{\otimes}{}^{e} A_i$ where $A_i = A_o$ , $e_i = e_o$ , $f = \overset{*}{\otimes}{}^{e} f_i$ where $f_i = f_o$ , $g = \overset{*}{\otimes}{}^{e} g_i$ where $g_i = g_o$ . Then the representations associated with f and g are unequivalent.

Proof. Suppose they are equivalent ; denote by $H_i, \pi_i, x_i$ and $K_i, \rho_i, y_i$ the objects associated with $f_i$ and $g_i$ respectively ; by hypothesis there exists an isomorphism

$$F : \overset{h}{\otimes}{}^{x} H_i \longrightarrow \overset{h}{\otimes}{}^{y} K_i$$

with

$$F.\otimes\pi_i(a_i).F^{-1} = \otimes\rho_i(a_i) \qquad \forall\,(a_i) \in \Pi^e A_i \ ;$$

if $a_i = e_i$ except for one index j, $\otimes\pi_i(a_i)$ and $\otimes\rho_i(a_i)$ are multiples of $\pi_j(a_j)$ and $\rho_j(a_j)$, so that $\pi_j$ and $\rho_j$ have a common multiple ; since they are irreducible they must be equivalent and we can realize them in some common Hilbert space $H_o$ with two vectors $x_o$ and $y_o$ which are nonproportional since $f_o \neq g_o$ ; by prop. 15, $\overset{*}{\otimes}{}^{e,x}\pi_i$ and $\overset{*}{\otimes}{}^{e,y}\rho_i$ are unequivalent, which is a contradiction.

Remark 5. E.Störmer proves in [46] that the above result still holds when $f_o$ and $g_o$ are not pure, and that one can replace in the conclusion the word " unequivalent " by " non quasi-equivalent ". In the same paper he also studies the states of $\overset{*}{\otimes} A_i$ which are " symmetric ", i.e. invariant by all the automorphisms of $\overset{*}{\otimes} A_i$ determined by permutations of I ; he proves in particular that the extremal symmetric states are exactly the states $\overset{*}{\otimes} f_i$ where $f_i = f_o$ , $f_o$ a state of $A_o$ ; and he determines the type of such a state $\overset{*}{\otimes} f_i$ when $f_o$ is factorial.

In [40] A.Hulanicki and R.R.Phelps prove the following result : consider some group $G_0$ of automorphisms of $A_0$ ; then $G_0^I$ acts by automorphisms in $\overset{*}{\otimes} A_i$ ; let G be the group of automorphisms of $\overset{*}{\otimes} A_i$ generated by $G_0^I$ and the permutation automorphisms ; then the extremal G-invariant states are exactly the states $\overset{*}{\otimes} f_i$ where $f_i = f_0$ , $f_0$ a $G_0$-invariant state of $A_0$.

## § 13. Study of the case where $e_i$ has rank $\leqslant 1$ .

### n.13.1. Definitions and examples.

In the preceding paragraphs we have seen several properties of $\overset{*}{\otimes}^{e} A_i$ in the case where $e_i$ is "large" in a certain sense (for instance prop. 17, th. 7, cor. 9 and 10); in this paragraph we shall be concerned with the thoroughly different case where $e_i$ is "small".

Definition 12. Given a $C^*$- algebra A, a projection e in A is said to have rank $\leqslant 1$ if for every irreducible representation $\pi$ of A the projection $\pi(e)$ has rank $\leqslant 1$.

By [2], 4.2.6 each projection of rank $\leqslant 1$ is contained in the largest liminar ideal of A ; consequently it must be 0 if A is antiliminar. On the other hand if a projection e lies in some closed two sided ideal I of A, it has rank $\leqslant 1$ in I iff it has rank $\leqslant 1$ in A (in fact for each irreducible representation $\pi$ of A, $\pi | I$ is either null or irreducible); finally if f is a projection of rank $\leqslant 1$ in A, its canonical image in A/I has also rank $\leqslant 1$ .

Example 5.

(i) If A is commutative every projection has rank $\leqslant 1$.

(ii) If A is elementary (i.e. of the form $\mathcal{L}\mathcal{C}(H)$ with H a Hilbert space), every projection which has rank 1 in the usual sense has rank $\leqslant 1$ in our sense. If we set $A_i = \mathcal{LC}(H_i)$ and take $e_i$ of rank 1, $\overset{*}{\otimes}{}^{e} A_i$ is nothing but $\mathcal{LC}(\overset{h}{\otimes}{}^{t} H_i)$ where $t_i \in \mathrm{Im}\, e_i$ ; in fact if $(a_i) \in \Pi^e A_i$ , $\otimes a_i$ belongs to $\mathcal{LC}(\overset{h}{\otimes}{}^{t} H_i)$ since it is of the form $(\underset{i \in J}{\otimes} a_i) \otimes (\underset{i \in I-J}{\otimes} e_i)$ where both factors are compact ; it follows that $\overset{*}{\otimes}{}^{e} A_i$ is included in $\mathcal{LC}(\overset{h}{\otimes}{}^{t} H_i)$ , but being irreducible it must be equal to it.

(iii) Let A be the $C^*$- algebra defined by a continuous field of $C^*$- algebras $(A(t), \Theta)$ (see[2], 10.4.1); $e = (e(t))$ an element of A such that each $e(t)$ is a projection of rank $\leqslant 1$ in $A(t)$. Then e is a projection of rank $\leqslant 1$ ; in fact one obtains all irreducible representations $\pi$ of A in the following manner : taking an index t and an irreducible representation $\rho$ of $A(t)$, and setting

$$\pi(a) = \rho(a(t)) \qquad \text{for each } a \in A .$$

In particular if the $A(t)$ are elementary one can take $e = (e(t))$ where rank $e(t) = 0$ or 1.

(iv) Let G be a locally compact group containing a compact open subgroup K with the following property : for each irreducible representation $\pi$ of G in a space H, the space of all vectors in H invariant by $\pi(K)$ has dimension $\leqslant 1$ . Then the characteristic function e of K has rank $\leqslant 1$ in $C^*(G)$ ; in fact it is known that $\pi(e) = \int_K \pi(k).dk$ is the projection onto the space of all vectors invariant under $\pi(K)$. Here dk is the normalized Haar measure of K.

(v) Let p and q be two projections in a Hilbert space H ; then the sub-$C^*$ - algebra A of $\mathcal{L}(H)$ generated by p and q is postliminar and p and q have rank $\leqslant 1$ in A (G.K.Pedersen, Oral communication).

## n.13.2. Irreducible representations of $\overset{*}{\otimes}{}^{e} A_i$ .

In this number we consider a tensor product $A = \overset{*}{\otimes}{}^{e} A_i$ where each $e_i$ has rank $\leqslant 1$ in $A_i$ ; we denote by $Y_i$ the set of all $\pi$ in $\hat{A}_i$ such that $\pi(e_i) \neq 0$ ; it is open in $A_i$. We define a mapping

$$F : \prod^{(Y_i)} \hat{A}_i \longrightarrow \hat{A}$$

in the following manner : take an element $\Pi = (\pi_i)$ in $\prod^{(Y_i)} \hat{A}_i$ and a finite subset $J \subset I$ such that $i \notin J$ implies $\pi_i \in Y_i$ ; then for $i \notin J$ , $\pi_i(e_i)$ has rank 1, we can take a unit vector $t_i$ in $\operatorname{Im} \pi_i(e_i)$ and form the representation $\otimes_{i \in I-J}^{e,t} \pi_i$ , which is independant of the choice of $t_i$ in $\operatorname{Im} \pi_i(e_i)$ ; now by writing $A = (\overset{*}{\underset{i \in J}{\otimes}} A_i) \overset{*}{\otimes} (\overset{*}{\underset{i \in I-J}{\otimes}}{}^{e} A_i)$

we can form the representation $(\overset{*}{\bigotimes}_{i\in J} \pi_i) \overset{*}{\otimes} (\overset{*}{\bigotimes}_{i\in I-J}^{e,t} \pi_i)$ of A ; this is an element of $\hat{A}$ which is independant of the choice of J (easy verification) and which we shall denote by $F(\Pi)$.

Lemma 3. Denote by A a $C^*$- algebra, by e a nonzero projection in A, by S the set of all pure states f on A verifying $f(e) = 1$ , by Y the set of all $\pi$ in $\hat{A}$ verifying $\pi(e) \neq 0$ , by M the canonical mapping $f \longmapsto \pi_f$ of $P(A)$ onto $\hat{A}$ . Then $M|S$ maps S onto Y and is open.

Proof. We have $M(S) \subset Y$ since

$$1 = f(e) = (\pi_f(e).x_f | x_f) \Longrightarrow \pi_f(e) \neq 0 \quad ;$$

we have $M(S) = Y$ : in fact if $\pi$ belongs to Y we can take a unit vector x in $\operatorname{Im} \pi(e)$ and setting $f = \omega_x \circ \pi$ we have $f(e) = 1$ and $\pi = \pi_f$ . To prove that $M|S$ is open, denote by T the set of all f in $P(A)$ verifying $f(e) \neq 0$ ; to each f in T we associate the state $L(f)$ defined by

$$L(f)(a) = f(eae) / f(e) \quad ;$$

then L is a continuous mapping of T into S ; if f is in S we have $L(f) = f$ since writing $f = \omega_x \circ \pi$ we have

$$f(e) = 1 = (\pi(e).x|x)$$

$$\pi(e).x = x$$

$$L(f)(a) = (\pi(eae).x|x) = (\pi(a).x|x) = f(a) \; ;$$

this proves that L maps T onto S. Let U be an open set in S ; $L^{-1}(U)$ is open in T, hence in $P(A)$ since T is open ; $M(L^{-1}(U))$ is equal to $M(U)$ since we have $M(f) = M(L(f))$ for each f in T ; since M is open, $M(U)$ is open and $M|S$ is open.

Proposition 18. The mapping $F : \prod^{(Y_i)} \hat{A}_i \longrightarrow \hat{A}$ is injective and bicontinuous.

Proof of the injectivity. Suppose $F(\Pi) = F(\Pi')$ and take j in I ; there is $J \in \mathcal{F}(I)$ such that $j \in J$ and

$$i \notin J \implies \pi_i \text{ and } \pi'_i \in Y_i \quad ;$$

we can write

$$F(\Pi) = \Big( \mathop{\overset{*}{\otimes}}_{i\in J} \pi_i \Big) \overset{*}{\otimes} \Big( \mathop{\overset{*}{\otimes}}_{i\in I-J}^{e,t} \pi_i \Big)$$

and similarly for $F(\Pi')$ ; since the assertion is true for the finite tensor products we have $\mathop{\overset{*}{\otimes}}_{i\in J} \pi_i = \mathop{\overset{*}{\otimes}}_{i\in J} \pi'_i$ and then $\pi_j = \pi'_j$.

Proof of the continuity. It is sufficient to prove that for each J, $F|X_{(J)}$ is continuous ; but $F|X_{(J)}$ is the composition of the following mappings :

$$(\pi_i)_{i\in I} \xmapsto{a} ((\pi_i)_{i\in J}, (\pi_i)_{i\in I-J}) \xmapsto{b} \Big( \mathop{\overset{*}{\otimes}}_{i\in J} \pi_i, \mathop{\overset{*}{\otimes}}_{i\in I-J}^{e,t} \pi_i \Big)$$

$$\xmapsto{c} \Big( \mathop{\overset{*}{\otimes}}_{i\in J} \pi_i \Big) \overset{*}{\otimes} \Big( \mathop{\overset{*}{\otimes}}_{i\in I-J}^{e,t} \pi_i \Big) \quad ;$$

a is trivially continuous, c is continuous by Part I, prop. 5 ; as for b, $(\pi_i)_{i\in J} \longmapsto \overset{*}{\otimes}_{i\in J} \pi_i$ is continuous by the same result, and it remains to be shown that $(\pi_i)_{i\in I-J} \longmapsto \overset{*}{\otimes}^{e,t}_{i\in I-J} \pi_i$ is continuous ; in other words we are led back to prove the continuity of the mapping F in the case where $\pi_i(e_i) \neq 0 \quad \forall i$. Denote by $S_i$ the set of all pure states f of $A_i$ verifying $f(e_i) = 1$ ; by lemma 3 the mapping $M_i : S_i \longrightarrow Y_i$ is open ; then the mapping $M = (M_i) : \Pi S_i \longrightarrow \Pi Y_i$ is open ; consider the following diagramm

$$\begin{tikzcd}
\Pi S_i \arrow[r, "G"] \arrow[d, "M"'] & P(A) \arrow[d] \\
\Pi Y_i \arrow[r, "F"'] & \hat{A}
\end{tikzcd}$$

since it is commutative we have only to show that the mapping $G : (f_i) \longmapsto \overset{*}{\otimes}^e f_i$ is continuous, i.e. that for each a in A the mapping $(f_i) \longmapsto (\overset{*}{\otimes}^e f_i)(a)$ is continuous ; since all our positive functionals have norm 1, by equicontinuity we can suppose $a \in \otimes^e A_i$ , then by linearity $a = \otimes a_i$ , $(a_i) \in \Pi^e A_i$ ; then the assertion is trivial.

*Proof of the bicontinuity*. Take $J \in \mathcal{F}(I)$ ; an element $\Pi$ of $\Pi^{(Y_i)} A_i$ belongs to $X_{(J)}$ iff $F(\Pi)$ is not identically zero on $\overset{*}{\otimes}_{i\in J} A_i$ ; thus we see that $F(X_{(J)})$ is open in Im F ;

it suffices to prove that $F^{-1}$ is continuous on $F(X_{(J)})$, or that each mapping $F(\Pi) \longmapsto \pi_j$ is continuous on $F(X_{(J)})$ ; we can suppose $j \in J$ and write

$$F(\Pi) = \Big(\overset{*}{\underset{i\in J}{\otimes}} \pi_i\Big) \overset{*}{\otimes} \Big(\overset{*}{\underset{i\in I-J}{\otimes}}{}^{e,t} \pi_i\Big) \ ;$$

by Part I, th. 5, $\overset{*}{\underset{i\in J}{\otimes}} \pi_i$ is a continuous function of $F(\Pi)$ and $\pi_j$ a continuous function of $\overset{*}{\underset{i\in J}{\otimes}} \pi_i$ .

Theorem 8. If each $A_i$ is postliminar and $e_i$ has rank $\leqslant 1$, the mapping $F$ is a homeomorphism of $\Pi^{(Y_i)} \hat{A}_i$ onto $\widehat{\overset{*}{\otimes}{}^{e} A_i}$ ; moreover $\overset{*}{\otimes}{}^{e} A_i$ is postliminar.

Proof. We shall prove that each factor representation $\pi$ of $A$ is of type I and also that if $\pi$ is irreducible, it is equivalent to a tensor product of representations. There exists a $J \in \mathcal{F}(I)$ such that $\pi \,|\, \overset{*}{\underset{i\in J}{\otimes}} A_i \neq 0$ , i.e.

$$\pi\big(a_{(J)} \otimes \big(\underset{i\in I-J}{\otimes} e_i\big)\big) \neq 0$$

for some $a_{(J)}$ in $\overset{*}{\underset{i\in J}{\otimes}} A_i$ ; since this algebra is postliminar we can write by Part I, prop. 2, $\pi = \pi_1 \overset{*}{\otimes} \pi_2$ where $\pi_1$ is some factor representation of $\overset{*}{\underset{i\in J}{\otimes}} A_i$ and $\pi_2$ some factor representation of $\overset{*}{\underset{i\in I-J}{\otimes}}{}^{e} A_i$ ; $\pi_1$ is of type I and if moreover $\pi$ is irreducible, $\pi_1$ is irreducible too and is a tensor product of irreducible representations of the $A_i$, $i \in J$ ; on the other hand

$$\pi(a_{(J)} \otimes (\bigotimes_{i \in I-J} e_i)) = \pi_1(a_{(J)}) \otimes \pi_2(\bigotimes_{i \in I-J} e_i)$$

implies $\pi_2(\bigotimes_{i \in I-J} e_i) \neq 0$ ; we are thus led to prove the following assertion :

If $\pi(\otimes e_i) \neq 0$ , $\pi$ is of type I ; if $\pi$ is irreducible it is equivalent to a tensor product of representations.

We denote by H the space of $\pi$ and choose a unit vector u in $\operatorname{Im} \pi(\otimes e_i) \subset H$ ; for each $j \in I$ we can write (since $A_j$ is postliminar) :

$$H = K_j \overset{h}{\otimes} K'_j$$

$$\pi = \rho_j \overset{*}{\otimes} \rho'_j$$

$$\pi(\otimes e_i) = \rho_j(e_j) \otimes \rho'_j(\bigotimes_{i \neq j} e_i)$$

where $\rho_j$ is a factor representation of $A_j$ in $K_j$ ; $\rho_j$ is a multiple of some irreducible representation $\pi_j$ in a space $H_j$ and we can write

$$H = H_j \overset{h}{\otimes} L_j \overset{h}{\otimes} K'_j$$

$$\pi = \pi_j \overset{*}{\otimes} I \overset{*}{\otimes} \rho'_j$$

$$\pi(\otimes e_i) = \pi_j(e_j) \otimes I \otimes \rho'_j(\bigotimes_{i \neq j} e_i) \quad ;$$

setting $H'_j = L_j \overset{h}{\otimes} K'_j$ and $\pi'_j = I \otimes \rho'_j$ we obtain

$$H = H_j \overset{h}{\otimes} H'_j$$

$$\pi = \pi_j \overset{*}{\otimes} \pi'_j$$

$$\pi(\otimes e_i) = \pi_j(e_j) \otimes \pi'_j(\bigotimes_{i \neq j} e_i) \quad ;$$

since $\pi_j(e_j)$ has rank 1, u has the form $t_j \otimes t'_j$ where $t_j$ is some unit vector in $\operatorname{Im} \pi_j(e_j)$.

Consider now a finite subset J of I ; by the same procedure as before we can write

$$H = H_{(J)} \overset{h}{\otimes} H'_{(J)}$$

$$\pi = \pi_{(J)} \overset{*}{\otimes} \pi'_{(J)}$$

$$\pi(\otimes e_i) = \pi_{(J)}(\underset{i \in J}{\otimes} e_i) \otimes \pi'_{(J)}(\underset{i \in I-J}{\otimes} e_i)$$

where $\pi_{(J)}$ is an irreducible representation of $\underset{i \in J}{\overset{*}{\otimes}} A_i$ in $H_{(J)}$ ; $\pi_{(J)}$ is a tensor product of irreducible representations $\sigma_j$ of the $A_j$, $j \in J$ ; for each $j \in J$ the restriction $\sigma'_j$ of $\pi_{(J)}$ to $A_j$ is a multiple of $\sigma_j$ ; write

$$A = A_j \overset{*}{\otimes} (\underset{i \in J-j}{\overset{*}{\otimes}} A_i) \overset{*}{\otimes} (\underset{i \in I-J}{\overset{*}{\otimes}}{}^{e} A_i)$$

and choose approximate identities $(u_s)$ and $(v_t)$ of the second and the third factors in the righthand side ; for each $a_j$ in $A_j$ we have

$$\pi(a_j \otimes u_s \otimes v_t) = \pi_j(a_j) \otimes \pi'_j(u_s \otimes v_t)$$

which converges strongly to $\pi_j(a_j) \otimes I$ in $H_j \overset{h}{\otimes} H'_j$ ; we have also

$$\pi(a_j \otimes u_s \otimes v_t) = \pi_{(J)}(a_j \otimes u_s) \otimes \pi'_{(J)}(v_t)$$

which converges strongly to $\sigma'_j(a_j) \otimes I$ in $H_{(J)} \overset{h}{\otimes} H'_{(J)}$ ;

this proves that $\pi_j$ and $\sigma'_j$ have a common multiple ; consequently $\pi_j$ is equivalent to $\sigma_j$ ; we thus can write

$$H = \Big( \overset{h}{\bigotimes_{i\in J}} H_i \Big) \overset{h}{\otimes} H'_{(J)} \tag{12}$$

$$\pi = \Big( \overset{*}{\bigotimes_{i\in J}} \pi_i \Big) \overset{*}{\otimes} \pi'_{(J)} \tag{13}$$

$$\pi(\otimes e_i) = \Big( \bigotimes_{i\in J} \pi_i(e_i) \Big) \otimes \pi'_{(J)}\Big( \bigotimes_{i\in I-J} e_i \Big) \quad ;$$

$$u = \Big( \bigotimes_{i\in J} t_i \Big) \otimes t'_{(J)}$$

where $t'_{(J)}$ is some unit vector in $\mathrm{Im}\,\pi'_{(J)}\big( \bigotimes_{i\in I-J} e_i \big)$.

If $K \supset J$ we have

$$\begin{aligned} u &= \Big( \bigotimes_{i\in K} t_i \Big) \otimes t'_{(K)} \\ &= \Big( \bigotimes_{i\in J} t_i \Big) \otimes \Big( \bigotimes_{i\in K-J} t_i \Big) \otimes t'_{(K)} \end{aligned}$$

whence

$$t'_{(J)} = \Big( \bigotimes_{i\in K-J} t_i \Big) \otimes t'_{(K)} \quad . \tag{14}$$

Define an isometric linear mapping $U_J : \overset{h}{\bigotimes_{i\in J}} H_i \longrightarrow H$ by writing (12) and

$$U_J(x) = x \otimes t'_{(J)} \quad ;$$

the various $U_J$ form an inductive system : in fact denoting by $L_{J,K}$ the canonical mapping $\overset{h}{\bigotimes_{i\in J}} H_i \longrightarrow \overset{h}{\bigotimes_{i\in K}} H_i$ we have

$$\begin{aligned} U_K(L_{J,K}(x)) &= x \otimes \Big( \bigotimes_{i\in K-J} t_i \Big) \otimes t'_{(K)} \\ &= x \otimes t'_{(J)} \qquad \text{(by (14))} \\ &= U_J(x) \quad ; \end{aligned}$$

this inductive system gives rise to an isometric linear mapping $U : \overset{h}{\otimes}{}^{t} H_i \longrightarrow H$ . We now prove that U intertwines the representations $\overset{*}{\otimes}{}^{e,t}\pi_i$ and $\pi$ , i.e. that

$$U((\overset{*}{\otimes}{}^{e,t}\pi_i)(a).x) = \pi(a).U.x$$

for each a in A and x in $\overset{h}{\otimes}{}^{t} H_i$ ; we can take $a = \otimes a_i$ with $(a_i) \in \Pi^e A_i$ and $x = \otimes x_i$ with $(x_i) \in \Pi^t H_i$ ; we have $a_i = e_i$ and $x_i = t_i$ if i belongs to the complement of some finite J , then

$$\begin{aligned} U((\overset{*}{\otimes}{}^{e,t}\pi_i)(a).x) &= U(\otimes \pi_i(a_i).x_i) \\ &= (\underset{i\in J}{\otimes} \pi_i(a_i).x_i) \otimes t'_{(J)} \\ &= ((\underset{i\in J}{\otimes} \pi_i)(\underset{i\in J}{\otimes} a_i).\underset{i\in J}{\otimes} x_i) \otimes \pi'_{(J)}(\underset{i\in I-J}{\otimes} e_i).t'_{(J)} \\ &= \pi(a).((\underset{i\in J}{\otimes} x_i)\otimes t'_{(J)}) \qquad \text{(by (13))} \\ &= \pi(a).U.x. \end{aligned}$$

Thus U intertwines $\overset{*}{\otimes}{}^{e,t}\pi_i$ and $\pi$ ; since the first representation is irreducible and the second is factorial, it is of type I ; if moreover $\pi$ is irreducible, it is equivalent to $\overset{*}{\otimes}{}^{e,t}\pi_i$ .

Corollary 11. Let for each i, $G_i$ be a postliminar locally compact group containing a compact open subgroup $K_i$ with the property indicated in example 5 (iv). Then the locally compact group $\Pi^{(K_i)} G_i$ is postliminar and its spectrum is homeomorphic to $\prod^{(Y_i)} \hat{G}_i$ where $Y_i$ is the set of all $\pi$ in $\hat{G}_i$ such that the space of all vector invariant by $\pi(K_i)$ has dimension 1.

Interesting applications of this result to adele groups can be found in [37], ch.III, § 3, n.3.

Corollary 12. If $G_i$ is a commutative locally compact group and $K_i$ a compact open subgroup, the dual group of $\prod^{(K_i)} G_i$ is isomorphic and homeomorphic to $\prod^{(L_i)} \hat{G}_i$ where $L_i$ is the subgroup orthogonal to $K_i$ .

In fact a character x of $G_i$ verifies $\int_{K_i} x(k).dk = 0$ if and only if it is trivial on $K_i$.

Another corollary has been stated in example 4.

### n.13.3. The Plancherel measure class of $\prod^{(K_i)} G_i$ .

In this number we suppose I countable and consider for each i a separable postliminar locally compact group $G_i$ with compact open subgroup $K_i$ such that for each $\pi$ in $\hat{G}_i$ the space of all vectors invariant under $\pi(K_i)$ has dimension $\leqslant 1$ ; we denote by $Y_i$ the set of all $\pi$ such that the above space has dimension 1 ; we set $G = \prod^{(K_i)} G_i$ , $X = \prod^{(Y_i)} \hat{G}_i$ . We recall that given a separable postliminar group G, the Plancherel measure class of G is the measure class on $\hat{G}$ corresponding to the central desintegration of the left regular representation of G .

Proposition 19. One can choose for each i , a measure $\lambda_i$ in the Plancherel measure class of $G_i$ in such a way that $\lambda_i(Y_i) = 1$ and that the homeomorphism $F : X \longrightarrow \hat{G}$ of corollary 11 carries the restricted product of the $\lambda_i$ (see definition in § 1) into a measure belonging to the Plancherel measure class of G .

We shall need the following lemma :

Lemma 4. Let G be a separable locally compact group, K a compact open subgroup, e the characteristic function of K considered as an element of $L^1(G)$, t the same function considered as an element of $L^2(G) = H$ , $\pi$ the left regular representation of G in H ; let us write the central desintegration of $\pi$ .

$$H = \int_X^{\oplus} H_x . d\mu(x)$$

$$\pi = \int^{\oplus} \pi_x . d\mu(x)$$

where $\mu$ is some Borel measure on some standard Borel space X ; t admits a decomposition $\int^{\oplus} t_x . d\mu(x)$ ; then the sets

$$X_1 = \{ x \mid \pi_x(e) \neq 0 \}$$

$$X_2 = \{ x \mid t_x \neq 0 \}$$

are identical up to negligible sets.

*Proof of the lemma.* Since $\pi(e).t = t$ we have $\pi_x(e).t_x = t_x$ almost everywhere and $X_2$ is almost contained in $X_1$. To prove the converse inclusion denote by $\mathfrak{Z}$ the algebra of all diagonalizable operators, by $\mathfrak{Z}'$ the algebra of all decomposable operators, by $\mathfrak{A}$ the von Neumann algebra generated by $\pi(G)$, by $\rho$ the right regular representation of G in H :

$$(\rho(g).f)(g') = \Delta(g)^{\frac{1}{2}} . f(g'g) \quad ;$$

set $L = \operatorname{Im} \pi(e)$ ; we can write

$$L = \int^{\oplus} L_x . d\mu(x) \qquad \text{with} \quad L_x = \operatorname{Im} \pi_x(e) \ ;$$

L is the set of all f in H which are constant on the right cosets Kg ; since G is separable and K open these cosets form a countable set, say $Kg_0, Kg_1, \dots$ with $g_0$ = neutral element ; L has an orthonormal basis $w_o, w_1, \ldots$ where $w_n$ is the characteristic function of $Kg_n$, $w_o = t$ ; $w_n$ admits a decomposition $\int^{\oplus} w_n(x).d\mu(x)$ ; we have

$$\Delta(g_n)^{\frac{1}{2}} . \rho(g_n)^{-1} \in \mathcal{A}' \subset \mathcal{Z}'$$

hence $\Delta(g_n)^{\frac{1}{2}} . \rho(g_n)^{-1}$ admits a decomposition $\int^{\oplus} T_n(x).d\mu(x)$ ; on the other hand we have

$$\Delta(g_n)^{\frac{1}{2}} . \rho(g_n)^{-1} . w_o = w_n$$

hence for almost every x we have

$$T_n(x).t_x = w_n(x) \qquad \forall\ n ;$$

since for almost every x the $w_n(x)$ generate $L_x$, we see that for almost every x

$$t_x = 0 \Longrightarrow L_x = 0 \Longrightarrow \pi_x(e) = 0 .$$

Proof of the proposition.

Denote by $\pi_i$ and $\pi$ the left regular representations of $G_i$ in $H_i = L^2(G_i)$ and of G in $H = L^2(G)$, by $t_i$ the characteristic function of $K_i$ considered as an element of $H_i$, by $\mu_i$ a left Haar measure on $G_i$ with $\mu_i(K_i) = 1$ ; by § 1 the restricted product $\mu$ of the $\mu_i$ is a left Haar measure on G ; by corollary 5 we have an isomorphism

$$U : \overset{h}{\otimes}{}^{t} H_i \longrightarrow H$$

with the following property : if $f_i \in H_i$ and $f_i = t_i$ almost everywhere, Uf is the function defined by $Uf(g) = \prod f_i(g_i)$ ; it is easy to check that U carries $\otimes \pi_i(g_i)$ into $\pi(g)$ for each $g = (g_i)$ in G .

Take an arbitrary measure $\lambda_i$ in the Plancherel class og $G_i$ ; we can write the central desintegration of $\pi_i$ :

$$H_i = \int^{\oplus}_{\hat{G}_i} H_{i,\rho_i} . d\lambda_i(\rho_i)$$

$$\pi_i = \int^{\oplus} \pi_{i,\rho_i} . d\lambda_i(\rho_i)$$

where $\pi_{i,\rho_i}$ is some multiple of $\rho_i$ ; by lemma 4, $t_i$ admits a decomposition $\int^{\oplus} t_{i,\rho_i} . d\lambda_i(\rho_i)$ such that $t_{i,\rho_i} \neq 0$ iff $\rho_i \in Y_i$ ; then we can replace $\lambda_i$ by an equivalent measure which we still denote by $\lambda_i$, and suppose $\| t_{i,\rho_i} \| = 1$ for each $\rho_i \in Y_i$ ; since $\| t_i \| = 1$ we have $\lambda_i(Y_i) = 1$ and we can form the restricted product $\lambda$ of the $\lambda_i$ .

By theorem 3 we have an isomorphism

$$V : \overset{h}{\otimes}{}^{t} H_i \longrightarrow \int^{\oplus}_X \overset{h}{\otimes}{}^{(t_{i,\rho_i})} H_{i,\rho_i} . d\lambda(\rho) ;$$

as easily seen V carries the representation $\otimes^t \pi_i$ into $\int^{\oplus}_X \otimes^{(t_{i,\rho_i})} \pi_{i,\rho_i} . d\lambda(\rho)$ , the proof will be complete if we show that for each $\rho \in X$ , $\otimes^{(t_{i,\rho_i})} \pi_{i,\rho_i}$ is a multiple of $F(\rho)$. Since $\pi_{i,\rho_i}$ is a multiple of $\rho_i$ we can write

$$H_{i,\rho_i} = K_{i,\rho_i} \overset{h}{\otimes} K'_{i,\rho_i}$$

$$\pi_{i,\rho_i} = \rho_i \otimes I$$

then

$$\pi_{i,\rho_i}(e_i) = \rho_i(e_i) \otimes I \quad ;$$

since $t_{i,\rho_i} \in \mathrm{Im}\, \pi_{i,\rho_i}(e_i)$ and rank $\pi_i(e_i) \leqslant 1$ we can write

$$t_{i,\rho_i} = s_{i,\rho_i} \otimes s'_{i,\rho_i}$$

with $s_{i,\rho_i} \in \mathrm{Im}\, \rho_i(e_i)$ , then by virtue of the associativity

$$\overset{h}{\otimes}^{(t_{i,\rho_i})} H_{i,\rho_i} = \left(\overset{h}{\otimes}^{(s_{i,\rho_i})} K_{i,\rho_i}\right) \overset{h}{\otimes} \left(\overset{h}{\otimes}^{(s'_{i,\rho_i})} K'_{i,\rho_i}\right)$$

$$\otimes^{(t_{i,\rho_i})} \pi_{i,\rho_i} = \left(\otimes^{(s_{i,\rho_i})} \rho_i\right) \otimes I$$

$$= F(\rho) \otimes I \; .$$

## § 14. Infinite tensor products of traces on $C^*$- algebras.

### n.14.1. Definition.

Let us consider for each i a $C^*$- algebra $A_i$ , a non zero projection $e_i$ in $A_i$ and a semi-finite lower semi-continuous (s.f.l.s.c.) trace $f_i$ on $A_i$ such that $f_i(e_i) = 1$ ; denote by $m_i$, $n_i$, $N_i$ the associated ideals (cf.[2], 6.1.2 and 6.2.1), by $\mathcal{A}_i$ the Hilbert algebra $n_i / N_i$ , by $\dot{a}$ the canonical image in $\mathcal{A}_i$ of any element a in $n_i$ , by $H_i$ the Hilbert completion of $\mathcal{A}_i$ , by $\mathcal{U}_i$ the left von Neumann algebra associated with $\mathcal{A}_i$ , by $t_i$ the natural trace on $\mathcal{U}_i$ , by $\pi_i$ the representation of $A_i$ in $H_i$ defined by $f_i$ ; $e_i$ is a projection of norm 1 in $\mathcal{A}_i$ and we can form the Hilbert algebra $\mathcal{A} = \otimes^{(e_i)} \mathcal{A}_i$ ; its Hilbert completion is $H = \overset{h}{\otimes}{}^{(\dot{e}_i)} H_i$ and the left von Neumann algebra of $\mathcal{A}$ is $\mathcal{U} = \overset{c}{\otimes}{}^{(\dot{e}_i)} \mathcal{U}_i$ ; denote by t its natural trace and by $\mathcal{N}$ the ideal of Hilbert-Schmidt operators for t ; since $\pi_i(e_i).\dot{e}_i = \dot{e}_i$ we can form the representation $\pi = \otimes^{e(\dot{e}_i)} \pi_i$ which generates the von Neumann algebra $\mathcal{U}$ ; for each family $(a_i)$ in $\Pi^e A_i$ with $a_i \in n_i$ we have

$$\pi(\otimes a_i) = U_{\otimes \dot{a}_i} \in \pi(A) \cap \mathcal{N} ;$$

since the operators $\pi(\otimes a_i)$ generate $\mathcal{U}$ , we see that the pair $(\pi, t)$ is a traced representation ; hence it defines a s.f.l.s.c. trace f on $A = \overset{*}{\otimes}{}^{e} A_i$ : $f = t \circ \pi$ ; if $(a_i) \in \Pi^{e} A_i$ and $a_i \in m_i^{+}$ we can write $a_i = b_i^2$ where $b_i \in n_i^{+}$ and we have

$$\begin{aligned} f(\otimes a_i) &= f((\otimes b_i)^2) = t(\pi((\otimes b_i)^2)) \\ &= t((U_{\otimes \dot{b}_i})^2) = (\otimes \dot{b}_i \mid \otimes \dot{b}_i) \\ &= \Pi (\dot{b}_i \mid \dot{b}_i) = \Pi\, t_i((U_{\dot{b}_i})^2) \\ &= \Pi\, t_i(\pi_i(b_i)^2) \\ &= \Pi\, t_i(\pi_i(a_i)) = \Pi\, f_i(a_i) . \end{aligned}$$

Let us now suppose that the $f_i$ are finite and $\Pi \| f_i \| < \infty$ ; the definition ideal m of f contains each element $\otimes a_i$ with $(a_i) \in \Pi^{e} A_i^{+}$ , hence contains $\otimes^{e} A_i$ ; on the other hand, by prop. 16, f is continuous on $\otimes^{e} A_i$ ; since it is l.s.c., it must be finite and hence equal to the positive functional $\overset{*}{\otimes}{}^{e} f_i$ . Thus we have proved the following

Proposition 20. Given for each i a s.f.l.s.c. trace $f_i$ on $A_i$ such that $f_i(e_i) = 1$ we can construct canonically a s.f.l.s.c. trace f on $\overset{*}{\otimes}{}^{e} A_i$ with the following properties :

(i) $f(\otimes a_i) = \Pi f_i(a_i)$ if $a_i \in A_i^{+}$ , $a_i = e_i$ almost everuwhere and $f_i(a_i) < \infty$

(ii) the representation associated with f is quasi-equivalent to $\overset{*}{\otimes}{}^{e,\dot{e}}\,\pi_i$ where $\pi_i$ is the representation associated with $f_i$ and $\dot{e}_i$ the canonical image of $e_i$ in the space of $\pi_i$ .

If each $f_i$ is finite and $\prod \| f_i \| < \infty$ , f is nothing but the central positive functional $\overset{*}{\otimes}{}^e\, f_i$ .

Suppose now that $f_i(e_i) = 1$ only for almost all i ; taking J in $\mathcal{F}(I)$ such that $i \notin J \Longrightarrow f_i(e_i) = 1$ we can write

$$A = \left( \overset{*}{\underset{i \in J}{\otimes}} A_i \right) \overset{*}{\otimes} \left( \overset{*}{\underset{i \in I-J}{\otimes}}{}^{e} A_i \right)$$

and consider the tensor product of the traces $\overset{*}{\underset{i\in J}{\otimes}} f_i$ (defined in Part I, prop. 21) and f (defined in prop. 20).

Definition 13. The above s.f.l.s.c. trace on $\overset{*}{\otimes}{}^e A_i$ will be denoted by $\overset{*}{\otimes}{}^e f_i$ ; it is a character if and only if each $f_i$ is a character. If each $f_i$ is finite and $\prod \| f_i \| < \infty$ , $\overset{*}{\otimes}{}^e f_i$ is nothing but the central positive functional $\overset{*}{\otimes}{}^e f_i$ .
By composing with the canonical morphism $\overset{\vee}{\otimes}{}^e A_i \longrightarrow \overset{*}{\otimes}{}^e A_i$ we also get a trace $\overset{\vee}{\otimes}{}^e f_i$ on $\overset{\vee}{\otimes}{}^e A_i$ which has the same properties.
In the remainder of this paragraph we shall prove that certain s.f.l.s.c. traces on $\overset{*}{\otimes}{}^e A_i$ or $\overset{\vee}{\otimes}{}^e A_i$ are tensor products of traces.

### n.14.2. Type I characters of $\overset{*}{\otimes}{}^{e} A_i$ .

**Theorem** 9. We suppose each $A_i$ postliminar ; let f be a character on $\overset{*}{\otimes}{}^{e} A_i$ which is of type I and satisfies the following condition : there exists a family $(a_i)$ in $\prod^e A_i^+$ such that $0 < f(\otimes a_i) < \infty$ . Then f is a tensor product.

Proof. Let $J = \{ i \mid a_i \neq e_i \}$ ; we can write

$$A = \overset{*}{\otimes}{}^{e} A_i = (\underset{i\in J}{\overset{*}{\otimes}} A_i) \overset{*}{\otimes} (\underset{i\in I-J}{\overset{*}{\otimes}}{}^{e} A_i)$$

$$a_i = (\underset{i\in J}{\otimes} a_i) \otimes (\underset{i\in I-J}{\otimes} e_i) \quad ;$$

by Part I, prop. 22, f is the tensor product of two characters $f_1$ , $f_2$ of $\underset{i\in J}{\overset{*}{\otimes}} A_i$ and $\underset{i\in I-J}{\overset{*}{\otimes}}{}^{e} A_i$ respectively, and $f_1$ is a tensor product of characters of the $A_i$, $i\in J$ ; on the other hand we have $0 < f_2(\underset{i\in I-J}{\otimes} e_i) < \infty$ , so that we are led back to prove the theorem in the case where $0 < f(\otimes e_i) < \infty$ .

Let $\pi$ be an irreducible representation of A in a space H such that $f = Tr \circ \pi$ where Tr is the usual trace in H ; for each j we can write

$$H = H_j \overset{h}{\otimes} H'_j$$

$$\pi = \pi_j \overset{*}{\otimes} \pi'_j$$

where $\pi_j$ and $\pi'_j$ are irreducible representations of $A_j$ and

$\overset{*}{\bigotimes}{}^{e}_{i \neq j} A_i$ ; then

$$\pi(\otimes e_i) = \pi_j(e_j) \otimes \pi'_j(\bigotimes_{i \neq j} e_j) \tag{15}$$

$$\mathrm{Tr}\,\pi(\otimes e_i) = \mathrm{Tr}\,\pi_j(e_j) \cdot \mathrm{Tr}\,\pi'_j(\bigotimes_{i \neq j} e_j) \quad ;$$

hence $\mathrm{Tr}\,\pi_j(e_j)$ is a strictly positive integer.

Consider now a finite subset $J$ of $I$ ; by the same procedure as in th. 8 we can write

$$H = (\overset{h}{\bigotimes_{i \in J}} H_i) \overset{h}{\otimes} H'_{(J)}$$

$$\pi = (\overset{*}{\bigotimes_{i \in J}} \pi_i) \overset{*}{\otimes} \pi'_{(J)} \quad ;$$

then

$$\mathrm{Tr}\,\pi(\otimes e_i) = \prod_{i \in J} \mathrm{Tr}\,\pi_i(e_i) \cdot \mathrm{Tr}\,\pi'_{(J)}(\bigotimes_{i \in I-J} e_i) \; ;$$

the second factor in the righthand side is a strictly positive integer, so that

$$\prod_{i \in J} \mathrm{Tr}\,\pi_i(e_i) \leqslant \mathrm{Tr}\,\pi(\otimes e_i) \quad ;$$

since $J$ is arbitrary $\mathrm{Tr}\,\pi_i(e_i)$ must be equal to 1 for almost every $i$ ; by taking off again a finite set of indices we can suppose $\mathrm{Tr}\,\pi_i(e_i) = 1 \;\; \forall\, i$ . Then $\pi_i(e_i)$ is a one dimensional projection ; we choose a unit vector $u$ in $\mathrm{Im}\,\pi(\otimes e_i)$ ; by (15) $u$ is of the form $u = t_j \otimes t'_j$ where $t_j$ is a unit vector in $\mathrm{Im}\,\pi_j(e_j)$ ; now the same reasoning as in th. 8 applies to prove that $\pi$ is equivalent to $\overset{*}{\otimes}{}^{e,t} \pi_i$ ; each pair $(\pi_i, \mathrm{Tr})$ is a traced representation since $\mathrm{Tr}\,\pi_i(e_i)$

$= 1$ ; set $f_i = \mathrm{Tr} \circ \pi_i$ ; we shall prove that $f = \overset{*}{\otimes}{}^{e} f_i$ ; $f_i$ defines a Hilbert algebra $\mathcal{A}_i$, and a representation $\rho_i$ of $A_i$ in $K_i = \overline{\mathcal{A}}_i$ ; let $\dot{e}_i$ be the canonical image of $e_i$ in $\mathcal{A}_i$ ; we can identify $\mathcal{A}_i$ with a dense subalgebra of the algebra of all Hilbert-Schmidt operators in $H_i$, $K_i$ with the space $H_i \overset{h}{\otimes} H_i$ , $\dot{e}_i$ (the projection onto $t_i$) with $t_i \otimes t_i$ , $\rho_i$ with $\pi_i \otimes I$ ; then $\overset{h}{\otimes}{}^{e} K_i$ is canonically isomorphic to $(\overset{h}{\otimes}{}^{t} H_i) \overset{h}{\otimes} (\overset{h}{\otimes}{}^{t} H_i)$ ; the representation associated with $\overset{*}{\otimes}{}^{e} f_i$ is quasi-equivalent to $\overset{*}{\otimes}{}^{e,\dot{e}} \rho_i$ , hence to $(\overset{*}{\otimes}{}^{e,t} \pi_i) \otimes I$ , then to $\overset{*}{\otimes}{}^{e,t} \pi_i$ and to $\pi$ ; but the representation associated with $f$ is also quasi-equivalent to $\pi$ , and this shows that $f = \overset{*}{\otimes}{}^{e} f_i$.

*Corollary* 13. Consider a family of postliminar locally compact groups $G_i$ with compact open subgroups $K_i$, and an irreducible representation $\pi$ of $\prod^{(K_i)} G_i$ ; suppose that there exists an integrable function $f$ on $\prod^{(K_i)} G_i$ of the form $f = \otimes f_i$ such that $0 < \mathrm{Tr}\, \pi(f^* f) < \infty$ and that for almost every $i$, $f_i$ is the characteristic function of $K_i$ . Then $\pi$ is equivalent to a tensor product of irreducible representations $\pi_i$ and for almost every $i$ the space of all vectors invariant by $\pi_i(K_i)$ has dimension 1.

Interesting applications of this result to adele groups can be found in [37], ch.III, § 3, n.5.

### n.14.3. Characters of $\overset{\alpha}{\otimes}{}^e A_i$ when $e_i$ is central.

In this number we suppose that for each i, $e_i$ belongs to the center of $A_i$ ; then if $\pi$ is a factor representation of $A_i$, $\pi(e_i)$ must be equal to 0 or I ; if f is a character of $A_i$ and $0 < f(e_i) < \infty$ , f is finite ; if moreover f is normed we have $f(e_i) = 1$.

Example 6. If G is a locally compact group and K an invariant compact open subgroup, its characteristic function is central in $C^*(G)$.

Proposition 21. Let f be a character of $\overset{\alpha}{\otimes}{}^e A_i$ ($\alpha = \nu$ or $*$ ) with the following property : there exists a family $(a_i)$ in $\prod^e A_i^+$ such that $0 < f(\otimes a_i) < \infty$ . Then f is a tensor product in the sense of definition 13.

Proof. For the same reason as in th. 9 we can suppose that $0 < f(\otimes e_i) < \infty$ ; since $\otimes e_i$ belongs to the center of $A = \overset{\alpha}{\otimes}{}^e A_i$ , f is finite and we can suppose it is normed ; it defines a representation $\pi$ in a space H and a finite normed trace t on the factor $\mathcal{Q} = \pi(A)''$ ; we have $\pi(\otimes e_i) = I$. The canonical morphisms $L_j : A_j \longrightarrow A$ are commuting ; set $\pi_j = \pi \circ L_j$ , representation of $A_j$ ; the von Neumann algebras $\pi_j(A_j)'' = \mathcal{Q}_j$ are included in $\mathcal{Q}$ and commuting .

Consider a family $(a_i)$ in $\prod^e A_i$ with $a_i = e_i$ for $i \notin J$ ; we have

$$\prod_{i \in J} L_i(a_i) = \Big( \bigotimes_{i \in J} a_i e_i \Big) \otimes \Big( \bigotimes_{i \in I-J} e_i \Big)$$

$$= \otimes a_i \,.\, \otimes e_i$$

hence

$$\bigotimes_{i \in J} \pi_i(a_i) = \pi(\otimes a_i) . \pi(\otimes e_i) = \pi(\otimes a_i) \ ;$$

then the $\mathcal{a}_i$ generate $\mathcal{a}$ and consequently are factors ; the von Neumann algebra $\mathcal{a}_{(J)}$ generated by the $\mathcal{a}_i$ with $i \in J$ is also a factor ; set $t_i = t \mid \mathcal{a}_i$ , $t_{(J)} = t \mid \mathcal{a}_{(J)}$ , $f_i = t_i \circ \pi_i$ , character of $A_i$ ; we want to prove that $f = \overset{*}{\otimes}{}^e f_i$ ; it suffices to prove that $f(\otimes a_i) = \prod f_i(a_i)$ with $a_i = e_i$ for $i \notin J$ ; then

$$f(\otimes a_i) = t(\pi(\otimes a_i)) = t\Big( \prod_{i \in J} \pi_i(a_i)\Big)$$

$$= t_{(J)}\Big( \prod_{i \in J} \pi_i(a_i)\Big) \ ;$$

by Part I, lemma 13 we get

$$f(\otimes a_i) = \prod_{i \in J} t_i(\pi_i(a_i)) = \prod_{i \in J} f_i(a_i) \ .$$

QED

We shall now investigate the finite characters of A ; we denote by $U_i$ the set of all $f \in C_1(A_i)$ such that $f(e_i) = 1$ ; $U_i$ is open since for each f in $C_1(A_i)$ we have $f(e_i) = 0$ or $1$ . For each family $F = (f_i)$ in $\prod^{(U_i)} C_1(A_i)$, we denote by $T(F)$ the character $\overset{\alpha}{\otimes}{}^{e} f_i$ .

Theorem 10. The mapping T is a homeomorphism of $\prod^{(U_i)} C_1(A_i)$ onto $C_1(\overset{\alpha}{\otimes}{}^{e} A_i)$ .

T is injective : suppose $T(F) = T(F')$, take $j$ in $I$ and $J$ in $\mathcal{F}(I)$ such that $j \in J$ and $f_i(e_i) = f'_i(e_i) = 1$ for $i \notin J$ ; for each a in $\overset{\alpha}{\bigotimes}_{i\in J} A_i$ we have

$$\begin{aligned} (\overset{\alpha}{\bigotimes_{i\in J}} f_i)(a) &= (T(F))(a \otimes (\bigotimes_{i\in I-J} e_i)) \\ &= (T(F'))(a \otimes (\bigotimes_{i\in I-J} e_i)) \\ &= (\bigotimes_{i\in J} f'_i)(a) \end{aligned}$$

and this implies $f_j = f'_j$ .

T is surjective by prop. 21.

T is continuous : we must prove that for each $J \in \mathcal{F}(I)$, T is continuous on $X_{(J)} = \prod_{i\in J} C_1(A_i) \times \prod_{i\in I-J} U_i$ ; then T is the composition of the following mappings :

$$F \xmapsto{a} ((f_i)_{i\in J}, (f_i)_{i\in I-J}) \xmapsto{b} (\overset{\alpha}{\bigotimes_{i\in J}} f_i, \overset{\alpha}{\bigotimes_{i\in I-J}}{}^{e} f_i)$$

$$\xmapsto{c} (\overset{\alpha}{\bigotimes_{i\in J}} f_i) \overset{\alpha}{\otimes} (\overset{\alpha}{\bigotimes_{i\in I-J}}{}^{e} f_i) \;;$$

a is clearly continuous ; b is the direct product of two mappings $b_1$ , $b_2$ ; $b_1$ and c are continuous by Part I, prop. 5 ;

the proof of the continuity of $b_2$ is the same as for the continuity of G in prop. 18.

*T is bicontinuous* : $T(X_{(J)})$ is open in $C_1(A)$ since a character $f = \overset{\alpha}{\otimes}{}^{e} f_i$ belongs to $T(X_{(J)})$ iff $f_i(e_i) = 0$ $\forall\ i \in I-J$ , which is equivalent to f non zero on the subalgebra $\left( \overset{\alpha}{\underset{i\in J}{\otimes}} A_i \right) \overset{\alpha}{\otimes} \left( \underset{i\in I-J}{\otimes} e_i \right)$ . Thus it is sufficient to prove that $F^{-1}$ is continuous on $T(X_{(J)})$, i.e. that for each j, the mapping $\overset{\alpha}{\otimes}{}^{e} f_i \longmapsto f_j$ is continuous on this subset ; we can suppose $j \in J$ ; then our mapping is the composition of the following ones :

$$\overset{\alpha}{\otimes}{}^{e} f_i = \left( \overset{\alpha}{\underset{i\in J}{\otimes}} f_i \right) \overset{\alpha}{\otimes} \left( \overset{\alpha\, e}{\underset{i\in I-J}{\otimes}} f_i \right) \longmapsto \overset{\alpha}{\underset{i\in J}{\otimes}} f_i \longmapsto f_j$$

and both are continuous by Part I, th. 10.

*Corollary* 14. If $e_i$ is the identity of $A_i$, $C_1(\overset{\alpha}{\otimes} A_i)$ is canonically isomorphic to $\prod C_1(A_i)$.

*Corollary* 15. We suppose $A_i$ separable and I countable ; then $\widehat{(\overset{\alpha}{\otimes}{}^{e} A_i)}_f$ is Borel isomorphic to $\prod^{(Y_i)} \widehat{(A_i)}_f$ where $Y_i$ is the set of all $\pi$ in $\widehat{(A_i)}_f$ with $\pi(e_i) \neq 0$.

In fact it is easy to see that the mapping $\widehat{(\overset{\alpha}{\otimes}{}^{e} A_i)}_f \longrightarrow \prod^{(Y_i)} \widehat{(A_i)}_f$ is Borel ; on the other hand both spaces are standard.

QED

Given a locally compact group G we denote by E(G) the set of all extremal continuous positive definite functions $\varphi$ on G with $\varphi(e_o) = 1$ , $e_o$ = neutral element ; there is a bijection $E(G) \longleftrightarrow C_1(C^*(G))$, to each $\varphi$ in E(G) corresponding the character $a \longmapsto f_\varphi(a) = \int a(g).\varphi(g).dg$. If K is a compact open subgroup of G and e the characteristic function of K, we have $f_\varphi(e) = 1$ iff $\int \varphi(k).dk = 1$ where dk is the normalized Haar measure of K ; and this is equivalent to $\varphi(k) = 1 \ \forall\ k \in K$ since $|\varphi(g)| \leqslant 1$ for each g in G.

Corollary 16. Consider for each i a locally compact group $G_i$ and an invariant compact open subgroup $K_i$ ; then $E(\prod^{(K_i)} G_i)$ is in a canonical bijection with $\prod^{(Y_i)} E(G_i)$ where $Y_i$ is the set of all $\varphi$ in $E(G_i)$ verifying $\varphi(k) = 1 \ \forall\ k \in K_i$ .

Corollary 17. If $G_i$ is compact and $K_i = G_i$, $E(\prod G_i)$ is in a canonical bijection with $\prod^{(\varepsilon_i)} E(G_i)$ where $\varepsilon_i$ is the function 1 .

Corollary 18. If $G_i$ is discrete and $K_i$ is reduced to the neutral element, $E(\prod' G_i)$ is in a canonical bijection with $\prod E(G_i)$ .

Bibliography [9],[47] .

## Supplementary bibliography for Part II.

[32] H.Araki. A lattice of von Neumann algebras associated with the Quantum Theory of a free Bose field. J.Math.Phys., t.4, 1963, p.1343-1362.

[33] H.Araki. A classification of factors.II. To appear in Publ. Res.Inst.Math.Sci.Kyoto Univ.,

[34] H.Araki - E.J.Woods. Complete Boolean algebras of type I factors. Ibid., t.2, 1966, p.157-242.

[35] H.Araki - E.J.Woods. A classification of factors. Ibid. to appear.

[36] D.J.C.Bures. Certain factors constructed as infinite tensor products. Compos.Math., t.15, 1963, p.169-191.

[37] I.M.Gelfand - M.I.Graïev - I.I.Piatetski-Shapiro. Generalized functions. T. 6. Academic Press.

[38] A.Guichardet. Sur la catégorie des algèbres de von Neumann. Bull.Sci.Math., t.40, 1966, p.41-64.

[39] A.Guichardet. Algèbres d'observables associées aux relations de commutation. A.Colin.

[40] A.Hulanicki - R.R.Phelps. Some applications of tensor products of partially ordered linear spaces. J.Funct.Anal., t.2, 1968, p.177-201.

[41] S.Kakutani. On equivalence of infinite product measures. Ann.Math., t.49, 1948, p.214-226.

[42] C.C.Moore. Invariant measures on product spaces. Fifth Berkeley Symp.Math.Stat.Proba., t.2, Part 2, p.447-459.

[43] F.J.Murray - J.von Neumann. On rings of operators. Ann.Math., t.37, 1936, p.116-226.

[44] J.von Neumann. On infinite direct products. Compos.Math., t.6, 1938, p.1-77.

[45] R.T.Powers. Representations of uniformly hyperfinite algebras and their associated von Neumann rings. Ann.Math., t.86, 1967, p.138-171.

[46] E.Störmer. Symmetric states of infinite tensor products of $C^*$- algebras. To appear in J.Funct.Anal.

[47] E.Thoma. Uber positiv-definit Klassenfunktionen abzählbarer Gruppen. Math.Zeitsch., t.84, 1964, p.389-402.

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