The Dual of Noncommutative Orlicz-Lorentz Space*

HAN Ya-zhou

(*College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, China*)

Abstract: It is shown that the dual space of noncommutative Orlicz-Lorentz space $\Lambda_{\varphi,\omega}(\mathcal{M})$ is $M_{\varphi_*,\omega}(\mathcal{M})$, where $\mathcal{M}$ is a semifinite von Neumann algebra and has no minimal projection, φ is an N-function satisfying the Δ_2-condition and ω is a regular weight function. These results are noncommutative analogues of well known characterisations in the setting of classical Orlicz-Lorentz space.

Key words: von Neumann algebra, noncommutative Orlicz-Lorentz space, dual space

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非交换 Orlicz-Lorentz 空间的对偶空间

韩亚洲

(新疆大学 数学与系统科学学院，新疆 乌鲁木齐 830046)

摘　要： 在这篇文章中我们证明了当 φ是满足Δ_2条件的N-函数且 ω 是正则的权函数时, 非交换 Orlicz-Lorentz 空间 $\Lambda_{\varphi,\omega}(\mathcal{M})$ 的对偶空间是 $M_{\varphi_*,\omega}(\mathcal{M})$，这里 $\mathcal{M}$ 是不含最小投影算子的半有限 von Neumann 代数.

关键词： von Neumann 代数; 非交换 Orlicz-Lorentz空间; 对偶空间

0 Introduction

Let (Ω,Σ,μ) be a complete σ-finite measure space and $L^0(\mu)$ be the space of all μ-measurable functions defined on Ω. Let $\varphi:[0,\infty)\to[0,\infty)$ be an Orlicz function (i.e., a convex function which assumes value zero only at zero) and $\omega:(0,\infty)\to(0,\infty)$ be a weight function (i.e., nonincreasing and locally intergrable with respect to the measure m and such that $\int_0^\infty \omega\,\mathrm{d}m=\infty$), then the Orlicz-Lorentz function space $\Lambda_{\varphi,\omega}$ on (Ω,μ) is the set of all $f\in L^0(\mu)$ such that $\int_\Omega \varphi(\lambda f^*)\omega\,\mathrm{d}m<\infty$ for some $\lambda>0$, where for any $f\in L^0(\mu)$, f^* denotes the nonincreasing rearrangement of f defined by $f^*(t)=\inf\{\lambda>0:\mu_f(\lambda)\le t\}$ for any $t>0$ (by convention $\inf\emptyset=\infty$). We know that $\Lambda_{\varphi,\omega}$ is a symmetric function space with the fatou property, equipped with the norm $\|f\|=\inf\{\lambda>0:\varrho_\varphi(\frac{f^*}{\lambda})\le 1\}$, where $\varrho_\varphi(f)=\int_0^\infty\varphi(f^*)\omega\,\mathrm{d}m$. If $\varphi(t)=t$, then $\Lambda_{\varphi,\omega}$ is the Lorentz space Λ_ω.[cf. [1, 2]].

If (X,Σ,ν) is a nonatomic measure space, then we have the following results: let either $\varphi(t)=t$ or φ be an N-function satisfying the Δ_2-condition and let ω be an regular weight function, then $\Lambda_{\varphi,\omega}(\mathbb{R}^+)^*=M_{\varphi_*,\omega}(\mathbb{R}^+)$.

The main result of this paper is the noncommutative analogue to the dual space of the classical Orlicz-Lorentz function space.

The paper is organized as follows. Section 1 consists of some preliminaries and notations, including the noncommutative Lorentz spaces and their elementary properties. Section 2 presents some results about $\Lambda_{\varphi,\omega}(\mathcal{M})$. In Section 3 we prove the main result of this paper.

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Biography: Han Ya-zhou(1982-), male, master, E-mail: hanyazhou@xju.edu.cn.

1 Preliminaries

In this section, we collect some basic facts and notion that will be used for what follows. Throughout this paper, we denote by $\mathcal{M}$ a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace τ, $\mathcal{M}_+$ the set of all nonnegative operators in $\mathcal{M}$, and $\mathcal{M}_{\text{proj}}$ the lattice of (orthogonal) projections in $\mathcal{M}$. For standard facts concerning von Neumann algebras, we refer to [3, 4]. The closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. Let x be affiliated with $\mathcal{M}$, then x is said to be τ-measurable if for every $\varepsilon > 0$ there exists a $P \in \mathcal{M}_{proj}$ such that $P(H) \subseteq D(x)$ and $\tau(P^\perp) < \varepsilon$ (where for any projection P we let $P^\perp = I - P$). The set of all τ-measurable operators will be denoted by $\widetilde{\mathcal{M}}$. The set $\widetilde{\mathcal{M}}$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product. For every $x \in \widetilde{\mathcal{M}}$, there is a unique polar decomposition $x = u|x|$ where $|x| \in \widetilde{\mathcal{M}}_+$ and u is a partial isometry operator. Let $r(x) = u^*u$ and $l(x) = uu^*$. We call $r(x)$ and $l(x)$ the right and left supports of x, respectively. For a positive self-adjoint operator x affiliated with $\mathcal{M}$, we set

$$\tau(x) = \sup_n \tau\Big(\int_0^n \lambda \, dE_\lambda\Big) = \int_0^\infty \lambda \, d\tau(E_\lambda),$$

where $0 \le x = \int_0^\infty \lambda \, dE_\lambda$ is the spectral decomposition of x. For $0 < p < \infty$, $L^p(\mathcal{M})$ is defined as the set of all τ-measurable operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty.$$

In addition, we put $L^\infty(\mathcal{M}) = \mathcal{M}$ and denote by $\|\cdot\|_\infty (= \|\cdot\|)$ the usual operator norm. It is well known that $L^p(\mathcal{M})$ is a Banach space under $\|\cdot\|_p$ $(1 \le p \le \infty)$ satisfying all the expected properties such as duality.

Let x be a τ-measurable operator and $t > 0$. The " tth singular number (or generalized s-number) of x" $\mu_t(x)$ is defined by

$$\mu_t(x) = \inf\{\|xe\| : e \in \mathcal{M}_{proj}, \tau(I - e) \le t\}$$

See [5] for more information about generalized $s-$number. For $x, y \in \widetilde{\mathcal{M}}$, we shall say that x is submajorized by y, written $x << y$, if and only if

$$\int_0^t \mu_s(x)\, ds \le \int_0^t \mu_s(y)\, ds, \text{for all } t > 0.$$

A normed linear subspace $E \subseteq \widetilde{\mathcal{M}}$ is called rearrangement invariant if and only if $x \in \widetilde{\mathcal{M}}, y \in E$ and $\mu_.(x) \le \mu_.(y)$ implies that $\|x\|_E \le \|y\|_E$ and $x \in E$; symmetric if and only if $x, y \in E$ and $x << y$ implies $\|x\|_E \le \|y\|_E$; fully symmetric if and only if $x \in \widetilde{\mathcal{M}}, y \in E$ and $x << y$ implies $\|x\|_E \le \|y\|_E$ and $x \in E$; properly symmetric if E is symmetric, rearrangement invariant and intermediate for Banach couple $(L^1(\mathcal{M}), \mathcal{M})$. Let E be a noncommutative symmetric space. The norm on E is said to have the Beppo-Levi property if and only if $0 \le x_\alpha \uparrow_\alpha \subseteq E$, $\sup_\alpha \|x_\alpha\|_\alpha < \infty$ implies $\sup_\alpha x_\alpha$ exists in E. The norm on E is said to be order continuous if $\|x_\alpha\|_E \downarrow_\alpha 0$ whenever $x_\alpha \downarrow_\alpha 0$. If the norm on E is order continuous, then every continuous linear functional on E is normal, and in this case, the Banach dual E^* may be identified with the associate space E'. See [6, 7] for more information about this.

A Banach space $(E, \|\cdot\|_E)$ is called locally uniformly convex if the conditions $x_n, x \in E, \|x_n\|_E \to \|x\|_E, \|x_n + x\|_E \to 2\|x\|_E$ imply $\|x_n - x\|_E \to 0$. $(E, \|\cdot\|_E)$ is said to be uniformly convex if the conditions $x_n, y_n \in E, \|x_n\|_E \le 1, \|y_n\|_E \le 1, \|x_n + y_n\|_E \to 2$ imply $\|x_n - x\|_E \to 0$. It is clear that in those two definitions it is sufficient to require only $\|x_n\|_E = \|x\|_E = 1$. $(E, \|\cdot\|_E)$ is strictly convex if for every $x, y \in X$ with $\|x\|_E = \|y\|_E = 1$, $x \ne y$ implies $\|\frac{x+y}{2}\|_E < 1$ holds.

In the following we will write $f \asymp g$ for nonnegative functions f and g whenever $C_1 f \le g \le C_2 f$ for some $C_j > 0, j = 1\ 2$. Given $\varphi : [0, \infty) \to [0, \infty)$ an Orlicz function (i.e., it is a convex function, takes value zero only at zero) and $\omega : (0, \infty) \to (0, \infty)$ a weight function (i.e., it is a non-increasing function and locally integrable and $\int_0^\infty \omega \, dt = \infty$). Let φ be an Orlicz function and we denote the Young conjugate of φ by φ_*, i.e.,

$$\varphi_*(t) = \sup\{st - \varphi(s) : s \ge 0\}, \text{for all } t \ge 0.$$

We still further say that φ is an N-function whenever $\lim_{t \to 0} \frac{\varphi(t)}{t} = 0$ and $\lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty$.

Definition 1 Let $\mathcal{M}$ be a semifinite von Neumann algebra. Given $\varphi:[0,\infty)\to[0,\infty)$ an Orlicz function and $\omega:(0,\infty)\to(0,\infty)$ a weight function, the noncommutative Orlicz-Lorentz space $\Lambda_{\varphi,\omega}(\mathcal{M})$ is defined by

$$\Lambda_{\varphi,\omega}(\mathcal{M})=\{x\in\widetilde{\mathcal{M}}:\|x\|<\infty\}$$

where the functional $\|\cdot\|$ on $\widetilde{\mathcal{M}}$ is defined by

$$\|x\|=\inf\{\lambda>0:\varrho_\varphi(\frac{x}{\lambda})=\int_0^\infty\varphi(\frac{\mu_t(x)}{\lambda})\omega(t)\,\mathrm{d}t\le 1\}.$$

It is clear that if $\varphi(t)=t$, then $\Lambda_{\varphi,\omega}(\mathcal{M})$ is the noncommutative Lorentz space $\Lambda_\omega(\mathcal{M})$.

Proposition 1 For $\varrho_\varphi(x)=\int_0^\infty\mu_t(\varphi(|x|))\omega(t)\,\mathrm{d}t=\int_0^\infty\varphi(\mu_t(x))\omega(t)\,\mathrm{d}t$, we have

(i): $\varrho_\varphi(x)=0$ if and only if $x=0$,

(ii):$\varrho_\varphi(x)=\varrho_\varphi(|x|)$,

(iii): $\varrho_\varphi(\alpha x+\beta y)\le\varrho_\varphi(\alpha x)+\varrho_\varphi(\beta y)$ for $\alpha+\beta=1,\alpha,\beta\ge 0$.

Proof (i) and (ii) are all evident. (iii): Letting $\alpha+\beta=1$ and $\alpha,\beta\ge0$, by Theorem 4.4 of [5] and Proposition 3.6 of [Chapter 2, [8]] and the properties of convex function, we have

$$\begin{aligned}\varrho_\varphi(\alpha x+\beta y)&=\int_0^\infty\varphi(\mu_s(\alpha x+\beta y))\omega(s)\,\mathrm{d}s\\&\le\int_0^\infty\varphi(\mu_s(\alpha x)+\mu_s(\beta y))\omega(s)\,\mathrm{d}s\\&\le\alpha\int_0^\infty\varphi(\mu_s(x))\omega(s)\,\mathrm{d}s+\beta\int_0^\infty\varphi(\mu_s(y))\omega(s)\,\mathrm{d}s\\&=\alpha\varrho_\varphi(x)+\beta\varrho_\varphi(y).\end{aligned}$$

Proposition 2 $\Lambda_{\varphi,\omega}(\mathcal{M})$ is a symmetric operator space with the Luxemburg norm: $\|x\|=\inf\{\lambda>0:\int_0^\infty\varphi(\frac{\mu_t(x)}{\lambda})\omega(t)\,\mathrm{d}t\le 1\}$.

Proof It is clear that $\|x\|\ge0$ and $\|x\|=0$ if and only if $x=0$. For, every $\alpha\in\mathbb{C}$, we obtain

$$\begin{aligned}\|\alpha x\|&=\inf\{\lambda>0:\int_0^\infty\varphi(\frac{\mu_t(\alpha x)}{\lambda})\omega(t)\,\mathrm{d}t\le1\}\\&=|\alpha|\inf\{\lambda'>0:\int_0^\infty\varphi(\frac{\mu_t(x)}{\lambda'})\omega(t)\,\mathrm{d}t\le1\}=|\alpha|\|x\|.\end{aligned}$$

Since $\varphi(\frac{\mu_t(x)}{\|x\|+\frac1n})\uparrow_n\varphi(\frac{\mu_t(x)}{\|x\|})$, then we have $\int_0^\infty\varphi(\frac{\mu_t(x)}{\|x\|+\frac1n})\omega(t)dt\le1,n=1,2,\ldots\ldots$. Therefore, it follows that

$$\int_0^\infty\varphi(\frac{\mu_t(x)}{\|x\|})\omega(t)\,\mathrm{d}t\le1.$$

Let $x,y\in\Lambda_{\varphi,\omega}(\mathcal{M})$, we know that

$$\begin{aligned}\int_0^\infty\varphi(\frac{\mu_t(x+y)}{\|x\|+\|y\|})\omega(t)\,\mathrm{d}t&=\varrho_\varphi(\frac{x+y}{\|x\|+\|y\|})\\&\le\frac{\|x\|}{\|x\|+\|y\|}\varrho_\varphi(\frac{x}{\|x\|})+\frac{\|y\|}{\|x\|+\|y\|}\varrho_\varphi(\frac{y}{\|y\|})\\&=\frac{\|x\|}{\|x\|+\|y\|}\int_0^\infty\varphi(\frac{\mu_t(x)}{\|x\|})\omega(t)\,\mathrm{d}t\\&+\frac{\|y\|}{\|x\|+\|y\|}\int_0^\infty\varphi(\frac{\mu_t(y)}{\|y\|})\omega(t)\,\mathrm{d}t\le1.\end{aligned}$$

Then we have $\|x+y\|\le\|x\|+\|y\|$.

If $\int_0^t\mu_s(x)\,\mathrm{d}s\le\int_0^t\mu_s(y)\,\mathrm{d}s,t>0$, then by Proposition 1.2 of [9] we know that

$$\int_0^t\varphi(\mu_s(x))\,\mathrm{d}s\le\int_0^t\varphi(\mu_s(y))\,\mathrm{d}s,t>0,$$

for any Orlicz function φ. Thus, by Proposition 3.6 of [Chapter 2, [8]],

$$\int_0^\infty \varphi(\mu_s(x))\omega(s)\,\mathrm{d}s \le \int_0^\infty \varphi(\mu_s(y))\omega(s)\,\mathrm{d}s,$$

which tells us that $\Lambda_{\varphi,\omega}(\mathcal{M})$ is a symmetric operator space.

From the above Proposition and Theorem 2.1 of [10], we have the following result.

Proposition 3 $\Lambda_{\varphi,\omega}(\mathcal{M})$ is a Banach space with the Luxemburg norm.

Let $\rho : I \to (0,\infty)$ be a concave function, then the Marcinkiewicz space M_ρ is defined by

$$M_\rho = \{f \in L^0 : \|f\|_{M_\rho} = \sup_{t\in I} \frac{\int_0^t f^*(t)\,\mathrm{d}t}{\rho(t)}\}$$

and the Marcinkiewicz space M_S with $S(t) = \int_0^t \omega(s)\,\mathrm{d}s$ is the associate space (=köthe dual space) of $\wedge_\omega$. We define noncommutative Marcinkiewicz space $M_S(\mathcal{M})$ by

$$M_S(\mathcal{M}) = \{x \in \widetilde{\mathcal{M}} : \|x\|_{M_S(\mathcal{M})} = \|\mu_t(x)\|_{M_S} < \infty\}.$$

It is clear that $M_S(\mathcal{M})$ is a noncommutative symmetric Banach function space[cf.[6], P745]. In what follows, given an Orlicz function φ, we define

$$I(f) = \int_0^\infty \varphi_*(\frac{f^*(t)}{\omega(t)})\omega(t)\,\mathrm{d}t,\ f \in L^0(\mathbb{R}^+),$$

and

$$M_{\varphi_*,\omega} = \{f \in L_0(\mathbb{R}^+) : I(\frac{f}{\lambda}) < \infty \text{ for some } \lambda > 0\}.$$

In the space $M_{\varphi_*,\omega}$ we define $\|f\|_{M_{\varphi_*,\omega}} = \inf\{\lambda > 0 : I(\frac{f}{\lambda}) \le 1\}$, then we get $\|\cdot\|_{M_{\varphi_*,\omega}}$ is a quasinorm, if ω is regular. Moreover, if $\varphi(t) = t$, we obtain that

$$M_{\varphi_*,\omega} = \{f \in L_0(\mathbb{R}^+) : \|f\|_{M_{\varphi_*,\omega}} = \sup_{t>0} \frac{f^*(t)}{\omega(t)} < \infty\}.$$

and $M_S = M_{\varphi_*,\omega}$, where ω is regular.

Definition 2 For noncommutative Orlicz-Lorentz space $\Lambda_{\varphi,\omega}(\mathcal{M})$, we define the associate "norm" by

$$\|x\|_{\Lambda'} = \sup\{\tau(|xy|) : \|y\|_E \le 1\}.$$

The associate space of $\Lambda_{\varphi,\omega}(\mathcal{M})$ is

$$\Lambda_{\varphi,\omega}(\mathcal{M})' = \{x \in \widetilde{\mathcal{M}} : \|x\|_{\Lambda'} < \infty\}.$$

See [5] for more information about associate space of properly symmetric Banach space.

Remark 1 Theorem 2.2 of [Chapter 3, [8]] showed that each rearrangement invariant Banach function space $\Lambda_{\varphi,\omega}(\mathbb{R}^+)$ is necessarily intermediate for the pair ($L^1(\mathbb{R}^+)$, $L^\infty(\mathbb{R}^+)$), then it follows immediately that $\Lambda_{\varphi,\omega}(\mathbb{R}^+)$ is a properly symmetric. Banach Space Therefore, by Theorem 5.6 of [6], we have $\Lambda_{\varphi,\omega}(\mathcal{M})' = (\Lambda_{\varphi,\omega})'(\mathcal{M})$. Moreover, by Proposition 5.4 of [6], we have $\Lambda_{\varphi,\omega}(\mathcal{M})'$ is a properly symmetric Banach space.

2 Some results of $\Lambda_{\varphi,\omega}(\mathcal{M})$

Lemma 1 If φ satisfies condition Δ_2, and $\int_0^\infty \omega(t)\,\mathrm{d}t = \infty$, then $\|x_n\| \to 0, n \to \infty$ if and only if $\varrho_\varphi(x_n) \to 0, n \to \infty$.

Proof If $\|x_n\| \to 0, n \to \infty$, then we have $\|\mu_t(x_n)\|_{\Lambda_{\varphi,\omega}(\mathbb{R}^+)} \to 0, n \to \infty$. By Theorem 2.5 (c) of [2], we get $\varrho_\varphi(\mu_t(x_n)) \to 0, n \to \infty$, which implies $\varrho_\varphi(x_n) \to 0, n \to \infty$. On the other hand, if $\varrho_\varphi(x_n) \to 0, n \to \infty$, we know that $\varrho_\varphi(\mu_t(x_n)) \to 0, n \to \infty$. Moreover, by Theorem 2.5(c) of [2], we obtain $\varrho_\varphi(\lambda\mu_t(x_n)) \to 0, n \to \infty$ holds for all $\lambda > 0$, this tells us that $\|x_n\| \to 0, n \to \infty$.

Proposition 4 If φ satisfies condition Δ_2, and $\int_0^\infty \omega(t)\,\mathrm{d}t = \infty$, then

$$K = \left\{ x : \begin{array}{l} x = \sum_{k=1}^n c_k E_k, c_k \in \mathbb{C}, \\ E_k \in \mathcal{M}_{proj}, E_k \perp E_j, if\ k \ne j, \tau(E_k) < \infty,\ j,k = 1,2,\cdots,n \end{array} \right\}$$

is dense in $\Lambda_{\varphi,\omega}(\mathcal{M})$.

Proof If $x\in\Lambda_{\varphi,\omega}(\mathcal{M})$, we have

$$\|x\|=\inf\{\lambda>0:\varrho_{\varphi}(\frac{x}{\lambda})=\int_0^{\infty}\varphi(\frac{\mu_t(x)}{\lambda})\omega(t)\,\mathrm{d}t\le 1\}<\infty,$$

which implies $\mu_t(x)\to 0, t\to\infty$.

If $y\in\Lambda_{\varphi,\omega}(\mathcal{M}), y\ge 0$, let $y=\int_0^{\infty}\lambda\,\mathrm{d}E_\lambda$ be the spectral decomposition of y. Then by Proposition 3.2 [5], we have that $y_n=\int_{\frac{1}{n}}^{n}\lambda\,\mathrm{d}E_\lambda\,(n=1,2,\cdots)$ converges to y in the measure topology. On the other hand $\tau(supp|y_n|)<\infty$, $y_n\le y\,(n=1,2,\cdots)$. Let

$$y_{n,m}=\sum_{j=0}^{m-1}[\frac{1}{n}+\frac{n-\frac{1}{n}}{m}j]E_{\left[\frac{1}{n}+\frac{n-\frac{1}{n}}{m}j,\frac{1}{n}+\frac{n-\frac{1}{n}}{m}(j+1)\right)}(y).$$

Then $\|y_n-y_{n,m}\|_\infty\to 0, m\to\infty$, and $y_{n,m}\le y_n$. So we get

$$\mu_t(y_n-y_{n,m})\le\|y_n-y_{n,m}\|_\infty\to 0, m\to\infty$$

and $\mu_t(y_n-y_{n,m})\le 2\mu_{\frac{t}{2}}(y_n)$. Since $y_n\in\Lambda_{\varphi,\omega}(\mathcal{M})$, using Lebesgue's dominated convergence theorem, we obtain

$$\varrho_{\varphi}(y_n-y_{n,m})=\int_0^{\infty}\varphi(\mu_t(y_n-y_{n,m}))\omega(t)\,\mathrm{d}t\to 0, m\to\infty,$$

which implies $\|y_n-y_{n,m}\|\to 0, n\to\infty$. Similarly, $\|y-y_n\|\to 0, n\to\infty$. Hence, it follows that $y\in\overline{K}$.

For $y\in\Lambda_{\varphi,\omega}(\mathcal{M})$, we have

$$y=Re(y)+iIm(y)=Re^{+}(y)-Re^{-}(y)+i(Im^{+}(y)-Im^{-}(y)),$$

and $Re^{+}(y), Re^{-}(y), Im^{+}(y), Im^{-}(y)$ are positive operators in $\Lambda_{\varphi,\omega}(\mathcal{M})$. So using the result of above, we obtain the desired result.

Proposition 5 If φ satisfies condition Δ_2, and $\int_0^{\infty}\omega(t)\,\mathrm{d}t=\infty$, then

(i) there does not exist an isometric copy of l^1 containing in $\Lambda_{\varphi,\omega}(\mathcal{M})$.

(ii) there does not exist an isometric copy of l^∞ containing in $\Lambda_{\varphi,\omega}(\mathcal{M})$.

(iii)there does not exist an isometric copy of c_0 containing in $\Lambda_{\varphi,\omega}(\mathcal{M})$.

Proof (i) and (ii) are immediate consequences of Theorem 2.4 of [2], Theorem 3.7, Corollary 3.8, Theorem 4.8 and Corollary 4.10 of [11]. (iii) follows immediately from Theorem 4.8 of [7] and Theorem 2.4 of [2].

Proposition 6 If φ satisfies condition Δ_2, and $\int_0^{\infty}\omega(t)\,\mathrm{d}t=\infty$, then $\Lambda_{\varphi,\omega}(\mathcal{M})$ is reflexive and $\Lambda_{\varphi,\omega}(\mathcal{M})$ has the Beppo-Levi property and the norms on $\Lambda_{\varphi,\omega}(\mathcal{M})$ and $\Lambda_{\varphi,\omega}(\mathcal{M})^*$ are order continuous.

Proof It follows immediately from Theorem 4.7 of [7] and Remark 1.

Proposition 7 Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. If φ satisfies condition Δ_2, and $\int_0^{\infty}\omega(t)\,\mathrm{d}t=\infty$, then $\Lambda_{\varphi,\omega}(\mathcal{M})$ is separable.

Proof It is an immediate consequence of Proposition 6.9 and Corollary 6.2 of [11] and Theorem 2.4 of [2].

Proposition 8 If φ satisfies condition Δ_2, then $\rho_{\varphi}(x)=1$ if and only if $\|x\|=1$.

Proof It is an immediate result of Theorem 2.5 of [2].

Proposition 9 Let φ and φ_* satisfy the Δ_2-condition, φ be strictly convex, then

(i) $\Lambda_{\varphi,\omega}(\mathcal{M})$ is uniformly convex.

(ii) $\Lambda_{\varphi,\omega}(\mathcal{M})$ is reflexive and strictly convex.

Proof (i): It follows immediately from Theorem 7 of [12] and Theorem 3.1 of [13]. (ii): By Theorem 4.8 of [7] and Theorem 7 of [12], we have $\Lambda_{\varphi,\omega}(\mathcal{M})$ is reflexive. From Theorem 5.2.5 and Theorem 5.2.6 of [14] and (i), we obtain $\Lambda_{\varphi,\omega}(\mathcal{M})$ is strictly convex.

3 The dual of $\Lambda_{\varphi,\omega}(\mathcal{M})$

Theorem 1 Let ω be a regular weight function and let either $\varphi(t)=t$ or φ be an N-function, then $\Lambda_{\varphi,\omega}(\mathcal{M})'=M_{\varphi_*,\omega}(\mathcal{M})$.

Proof It is an immediate result of Theorem 2 of [1] and Remark 1.

Theorem 2 Let ω be a regular weight function and let φ be an Orlicz function. Then the following holds:

(i) If $0 < \lim_{t\to 0} \frac{\varphi(t)}{t} < \infty$, then $\varphi(t) \asymp t$ and $(\Lambda_{\varphi,\omega}(\mathcal{M}))' = M_S(\mathcal{M})$.

(ii) If $0 < \lim_{t\to 0} \frac{\varphi(t)}{t}$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$, then there exists an N-function ϕ such that $\phi(t) \asymp t^2$ for t small enough and $\phi(t) \asymp \varphi(t)$ for t large enough, and

$$(\Lambda_{\varphi,\omega}(\mathcal{M}))' = M_S(\mathcal{M}) + M_{\phi_*,\omega}(\mathcal{M}).$$

(iii) If $0 = \lim_{t\to 0} \frac{\varphi(t)}{t}$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} < \infty$, then there exists an N-function ϕ such that $\phi(t) \asymp \varphi(t)$ for t small enough and $\phi(t) \asymp t$ for t large enough, and

$$(\Lambda_{\varphi,\omega}(\mathcal{M}))' = M_S(\mathcal{M}) \cap M_{\phi_*,\omega}(\mathcal{M}).$$

Proof It now follows from [106, Theorem 2.2, [8]], that $M_S(\mathbb{R}^+)$ and $M_{\phi_*,\omega}(\mathbb{R}^+)$ are exact interpolation spaces for the couple $(L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$. Then by Proposition 3.1 of [15], we obtain $M_S(\mathcal{M}) + M_{\phi_*,\omega}(\mathcal{M}) = (M_S + M_{\phi_*,\omega})(\mathcal{M})$. on the other hand, it is clear that $(M_S \cap M_{\phi_*,\omega})(\mathcal{M}) = M_S(\mathcal{M}) \cap M_{\phi_*,\omega}(\mathcal{M})$. From the above discussion and Theorem 3 of [1]and Remark 1, we complete the proof.

Theorem 3 Let either $\varphi(t) = t$ or φ be an N-function satisfying the Δ_2-condition and let ω be a regular weight function, then $\Lambda_{\varphi,\omega}(\mathcal{M})^* = M_{\varphi_*,\omega}(\mathcal{M})$.

Proof Under the given assumptions and Theorem 2.4 of [2], we have $\Lambda_{\varphi,\omega}(\mathbb{R}^+)$ is a separable space, which implies the norm on $\Lambda_{\varphi,\omega}(\mathbb{R}^+)$ is order continuous. Therefore, Proposition 3.6 of [6] implies that the norm on the space $\Lambda_{\varphi,\omega}(\mathcal{M})$ is order continuous and so Theorem 5.11 of [6] shows that the dual Space $\Lambda_{\varphi,\omega}(\mathcal{M})^*$ is identified with $\Lambda_{\varphi,\omega}(\mathcal{M})'$. On the other hand, by Theorem 4 of [2] and Remark 1, we have

$$\Lambda_{\varphi,\omega}(\mathcal{M})' = M_{\varphi_*,\omega}(\mathcal{M}).$$

Hence the required result follows.

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