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ON THE MODULUS OF SMOOTHNESS AND THE G_α-CONDITIONS IN B-SPACES

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1. Differentiation of real functions on a B-space

Let E be a Banach space with dual E', and let f be a map from E into $\mathbb{R}$. Let $x_o \in E$ and $D \in E'$, then we say that

D is a subdifferential of f at $x_o \in E$, if we have

$$\langle D,y\rangle \leq f(x_o+y) - f(x_o) \qquad \forall \|y\| \leq \delta .$$

for some $\delta > 0$.

D is a Gateaux differential of f at $x_o \in E$, if we have

$$\langle D,y\rangle = \lim_{t\to\infty} t^{-1}(f(x_o+ty) - f(x_o)) \qquad \forall y \in E.$$

D is a Fréchet differential of f at $x_o \in E$, if we have

$$\lim_{y\to 0} \|y\|^{-1} | f(x_o+ty) - f(x_o) - \langle D,y\rangle | = 0.$$

We note that f admits at most one Gateaux (Fréchet) differential at a given point, whereas f may admit many sub-differentials. If f admits D as its Fréchet differential then D is a Gateau differential for f. If D is a Gateaux differential for f, and f is convex, then D is a sub-differential for f.

Now we define the modulus of continuity, ω, and the symmetric modulus of continuity, ρ, by

$$\omega(f,x,t) = \sup_{y\in B}\{|f(x+ty) - f(x)|\} \qquad \forall\, x\in E \quad \forall\, t>0$$

$$\omega(f,A,t) = \sup_{x\in A} \omega(f,x,t) \qquad \forall\, A\subseteq E \quad \forall\, t>0$$

$$\rho(f,x,t) = \sup_{y\in B}\{\tfrac{1}{2}|f(x+ty) + f(x-ty) - 2f(x)|\} \qquad \forall\, x\in E \quad \forall\, t>0$$

$$\rho(f,A,t) = \sup_{x\in A} \rho(f,x,t) \qquad \forall\, A\subseteq E \quad \forall\, t>0$$

Where B is the unit Ball in E, and S is the sphere:

$$B = \{y\in E \mid \|y\| \le 1\}, \qquad S = \{y\in E \mid \|y\| = 1\}.$$

<u>Proposition 1.1</u>. <u>Let</u> $x_o \in E$ <u>and let</u> U <u>be a neighborhood of</u> x_o, <u>so that</u> f <u>admits a subdifferential</u>, $D(x)$, <u>at</u> x <u>for all</u> $x \in U$, <u>and suppose that</u>

(1.1.1) $\exists\, \varepsilon > 0$ <u>so that</u> $t \rightsquigarrow f(x_o+t\theta)$ <u>is continuous in</u> $[-\varepsilon,\varepsilon]$ <u>for all</u> $\theta \in S$,

(1.1.2) $D(x)$ <u>is continuous at</u> $x = x_o$.

<u>Then</u> $D(x_o)$ <u>is a Fréchet differential of</u> f at x_o.

<u>Proof</u>. We may assume that ε in (1.1.1) is taken so small that $B(x_o,\varepsilon) \subseteq U$ ($B(x_o,\varepsilon)$ denote the ball with center x_o and radius ε), and we may also assume that $D(x)$ is bounded in $B(x_o,\varepsilon)$, that is

$$\|D(x)\| \le M \qquad \forall\, x \in B(x_o,\varepsilon).$$

Now let $\Theta \in S$, and put $g(t) = f(x_o+t\Theta)$, and $h(t) = \langle D(x_o+t\Theta),\Theta\rangle$. Then for each $s \in [-\varepsilon,\varepsilon]$ we can find $\delta(s) > 0$, so that

$$\langle D(x_o+s\Theta),y\rangle \leq f(x_o+s\Theta+y) - f(x_o+\Theta) \quad \forall \|y\| \leq \delta(s)$$

Putting $y = u\Theta$ give

$$u\,h(s) \leq g(s+u) - g(s) \qquad \forall |u| \leq \delta(s) \quad \forall |s| \leq \varepsilon.$$

Now $|h(s)| \leq \|D(x_o+s\Theta)\| \leq M$ for $|s| \leq \varepsilon$, and so

$$\underline{D}^+(g,s) = \liminf_{u\to 0+} \frac{g(s+u)-g(s)}{u} \geq -M$$

$$\overline{D}^-(g,s) = \limsup_{u\to 0^-} \frac{g(s+u)-g(s)}{u} \leq M .$$

So by continuity of g and Lemma VII.6.3 in [3], we find that

$$|g(t) - g(s)| \leq M|t-s| \qquad \forall t,s \in [-\varepsilon,\varepsilon].$$

In particular we have that g is absolutely continuous in $[-\varepsilon,\varepsilon]$ and

$$g'(s) = \overline{D}^-(g,s) \leq h(s) \leq \underline{D}^+(g,s) = g'(s)$$

a.e. in $[-\varepsilon,\varepsilon]$. Hence we have for $\Theta \in S$ and $|t| \leq \varepsilon$:

$$f(x_o+t\Theta) - f(x_o) = \int_0^t \langle D(x_o+s\Theta),\Theta\rangle ds.$$

Now let $\|y\| \leq \varepsilon$, and put $t = \|y\|$, $\Theta = t^{-1}y$, then we have

$$\|y\|^{-1} |f(x_o+y) - f(x_o) - \langle D(x_o),y\rangle|$$

$$\leq \frac{1}{t}\int_0^t \|D(x_o+s\Theta) - D(x_o)\|\, ds \leq \omega(D,x_o,t)$$

and by continuity of D at x_o, we have that $D(x_o)$ is a Fréchet differential of f at x_o.

Proposition 1.2. Let f be a convex continuous function from E into $\mathbb{R}$ and U an open nonempty subset of E, so that

$$\frac{1}{t}\,\rho(f,U,t) \xrightarrow[t\to o]{} 0 \tag{1.2.1}$$

Then f admits a Fréchet differential, $D(x)$, everywhere in U.

Proof. Let $x \in U$ and let $\theta \in S$, then the function $g(t) = f(x+t\theta)$ is convex. Hence g has a left and a right derivative, say g^- and g^+, everywhere, and we have

$$g^-(t) \leq g^+(t) \qquad \forall t$$

$$g(t+s) - g(t) - sg^+(t) \geq 0 \qquad \forall t \quad \forall s \geq 0$$

$$g(t-s) - g(t) + sg^-(t) \geq 0 \qquad \forall t \quad \forall s \geq 0$$

Moreover we have

$$0 \leq s(g^+(0) - g^-(0)) \leq g(s) + g(-s) - 2g(0)$$

$$\leq 2\rho(f,U,s)$$

and so by (1.2.1) we have that $g^+(0) = g^-(0)$, and so

$$D_o(x,y) = \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}$$

exists for $x \in U$ and all y, and we have

$$0 \leq f_t(x,y) - D_o(x,y) \leq \frac{2}{t}\,\rho(f,U,t\,\|y\|)$$

for all $t > 0$ and all $x \in U$, $y \in E$, where $f_t(x,y) = \frac{f(x+ty)-f(x)}{t}$.
Hence we have

$$f_t(x,y) \xrightarrow[t\to 0]{} D_o(x,y) \qquad \text{uniformly in } U \times B$$

and so D_o is continuous in $(x,y) \in U \times B$. It is evident that we have

$$D_o(x,\lambda y) = \lambda D_o(x,y) \qquad \forall x \in U \quad \forall y \in E \quad \lambda \in \mathbb{R}$$

Now let $x \in U$ and let $y_1, y_2 \in E$. Let $\varepsilon > 0$ be chosen so that $B(x,\varepsilon) \subseteq U$. If $|t| \leq \varepsilon$ then we have

$$\begin{aligned} |f_t(x,y_1+y_2) - D_o(x,y_1) - D_o(x,y_2)| &\leq |f_t(x+ty_1,y_2) - D_o(x+ty_1,y_2)| + \\ &+ |D_o(x+ty_1,y_2) - D_o(x,y_2)| + |f_t(x,y_1) - D_o(x,y_1)| \\ &\leq \frac{4}{t}\rho(f,U,t(\|y_1\|+\|y_2\|)) + |D_o(x+ty_1,y_2) - D_o(x,y_2)| \end{aligned}$$

So by continuity of D_o and (1.2.1) we find that

$$D_o(x,y_1+y_2) = \lim_{t\to 0} f_t(x,y_1+y_2) = D_o(x,y_1) + D_o(x,y_2)$$

Hence $D(x) = D_o(x,\cdot) \in E'$, and we have that

$$|f(x+y) - f(x) - \langle D(x),y\rangle| \leq 2\rho(f,U,\|y\|) \qquad \forall x \in U \quad \forall y \in E$$

from which it follows that $D(x)$ is the Fréchet differential for f at x.

2. The modulus of smoothness

Let $N_p(x) = \|x\|^p$ for $1 \leq p < \infty$, then we define $D_p(x)$ to be the Fréchet differential of N_p whenever it exists. Note that $D_p(0)$ exists and is equal to 0 for $p > 1$, but does not exist for $p = 1$. Suppose that $x \neq 0$ and $D_p(x)$ exists then straightforward computations show:

(2.1) $D_p(\lambda x)$ exists for $\lambda \neq 0$ and $D_p(\lambda x) = \lambda^{p-1}\operatorname{sign}(\lambda) D_p(x)$

(2.2) $D_q(x)$ exist for all $q > 0$ and

$$D_q(x) = \frac{q}{p}\|x\|^{q-p} D_p(x) = \frac{q}{p}\|x\|^{q-1} D_p\left(\frac{x}{\|x\|}\right)$$

(2.3) $\langle D_p(x), x\rangle = p\|x\|^p$

(2.4) $\|D_p(x)\| = p\|x\|^{p-1}$.

Let $0 < p < \infty$ then we define the p-<u>modulus of smoothness</u>, ρ_p, by

$$\rho_p(t) = \rho(N_p, S, t) = \sup_{x,y \in S} \frac{1}{2}(\|x+ty\|^p + \|x-ty\|^p - 2)$$

and we define, $\omega_p(t)$, by

$$\omega_p(t) = \sup\{\|D_p(x) - D_p(y)\| \mid x,y \in S, \ \|x-y\| \leq t\}$$

for $t > 0$ (if $D_p(x)$ does not exist for some $x \in S$, we put $\omega_p(t) \equiv \infty$). From (2.2) and (2.4) we find

(2.5) $\omega_p(t) = p\,\omega_1(t) \qquad \forall t > 0 \quad \forall p \geq 1$

(2.6) $\omega_p(t) \leq 2p \qquad \forall t > 0 \quad \forall p \geq 1$, if $\omega_1 < \infty$

Let $x,y \in E\setminus\{0\}$, and put $x_o = \|x\|^{-1}$ and $y_o = \|y\|^{-1}y$, then we have

$$x_o - y_o \leq \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}$$

and so we find

(2.7) $$\|D_p(x) - D_p(y)\| \leq p\left| \|x\|^{p-1} - \|y\|^{p-1} \right| + \|y\|^{p-1}\omega_p\left(\frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}\right)$$

for all $x,y \neq 0$ and all $p \geq 1$.

E is called <u>uniformly smooth</u> if $\rho_1(t) = o(t)$ as $t \to 0$, and E is called <u>uniformly</u> p-<u>smooth</u> $(1 < p \leq 2)$, if $\rho_1(t) = 0(t^p)$ as $t \to 0$. E is called a G_α-<u>space</u> $(0<\alpha \leq 1)$, if there exists a map, $G: E \to E'$, so that for some $A > 0$ we have

(2.8) $$\|G(x)\| = \|x\|^{\alpha} \qquad \forall x \in E$$

(2.9) $$\langle G(x), x\rangle = \|x\|^{1+\alpha} \qquad \forall x \in E$$

(2.10) $$\|G(x) - G(y)\| \leq A\|x-y\|^{\alpha} \qquad \forall x,y \in E$$

<u>Lemma 2.1</u>. <u>There exist constants</u> $K_p < \infty$ <u>for</u> $p \geq 1$ so <u>that we have</u>

(2.1.1) $$\rho_1(t) \leq \rho_p(t) \leq K_p\rho_1(t) \qquad \forall\, 0 \leq t \leq 1, \quad p \geq 1$$

<u>Proof</u>. Let $x,y \in S$ and $t \geq 0$, then

$$\|x+ty\| + \|x-ty\| \geq \|2x\| = 2$$

and since $a^q + b^q \leq a^p + b^p$ whenever $1 \leq q \leq p < \infty$ and $a + b \geq 2$, we find

(2.11) $$\rho_q(t) \leq \rho_p(t) \qquad \forall t \geq 0 \ \ \forall 1 \leq q \leq p < \infty$$

Now it is easily seen that there exists constants $c_p < \infty$ so that

$$1 + s^p \leq 2(\frac{1+s}{2})^p + c_p(1-s)^2 \qquad \forall\, 0 \leq s \leq 1$$

Hence we find

$$a^p + b^p \leq 2(\frac{a+b}{2})^p + c_p \max(a^{p-2}, b^{p-2})|a-b|^2 \quad \forall\, a,b \geq 0$$

Let x and y belong to S and $0 \leq t \leq 1$. Now we put $a = \|x+ty\|$ and $b = \|x-ty\|$, then $0 \leq a \leq 2$ and $0 \leq b \leq 2$, and so

$$\begin{aligned}
\tfrac{1}{2}(a^p+b^p-2) &\leq (\tfrac{1}{2}(a+b))^p + 2^{p-3}c_p|a-b|^2 - 1 \\
&\leq (\rho_1(t)+1)^p - 1 + 2^{p-1}c_p t^2 \\
&\leq p(\rho_1(t)+1)^{p-1}\rho_1(t) + 2^{p-1}c_p t^2 \\
&\leq 2^{p-1}p\,\rho_1(t) + 2^{p-1}c_p t^2
\end{aligned}$$

Now from [1] we know that

$$(1+t^2)^{\frac{1}{2}} - 1 \leq \rho_1(t)$$

and since $(\frac{1}{2}t)^2 \leq (1+t^2)-1$ for $0 \leq t \leq 1$, we find

$$\tfrac{1}{2}(a^p+b^p-2) \leq (2^{p-1}p + 2^{p+1}c_p)\rho_1(t)$$

for all $0 \leq t \leq 1$. Hence (2.1.1) holds with $K_p = p2^{p-1} + 2^{p+1}c_p$.

<u>Lemma 2.2</u>. <u>Suppose that</u> $D_1(x)$ <u>exists for all</u> $x \in S$, <u>then there exists constants</u> $A, B \in \mathbb{R}_+$, <u>so that</u>

(2.2.1) $\rho_1(t) \leq A\, t\, \omega_1(t) \qquad \forall\, 0 \leq t \leq 1$

(2.2.2) $t\, \omega_1(t) \leq B\, \rho_1(t) \qquad \forall\, 0 \leq t \leq 1$

<u>Proof</u>. Let $x, y \in S$ and $t > 0$ be given, then

$$\|x+ty\| + \|x-ty\| - 2 = \int_0^t \langle D_1(x+sy) - D_1(x), y\rangle\, ds + \int_0^t \langle D_1(x) - D_1(x-sy), y\rangle\, ds \leq 2t\, \omega_1(2t)$$

so we find

(2.12) $\rho_1(t) \leq t\, \omega_1(2t) \qquad \forall\, t \geq 0 .$

Now let

$$f_t(x,y) = t^{-1}(\|x+ty\| - \|x\|) \quad \text{for} \quad x, y \in E, \quad t>0.$$

Then $\lim_{t \to 0} f_t(x,y) = \langle D_1(x), y\rangle$ for $x \neq 0$ and $y \in E$, and since $t \sim \|x+ty\|$ is convex we have

$$f_t(x,y) - \langle D_1(x), y\rangle \geq 0 \qquad \forall\, t>0$$

$$f_t(x,-y) + \langle D_1(x), y\rangle \geq 0 \qquad \forall\, t>0$$

Hence if $t > 0$ and $x, y \in S$ we have

$$0 \leq f_t(x,y) - \langle D_1(x), y\rangle \leq f_t(x,y) + f_t(x,-y)$$

$$= t^{-1}\{\|x+ty\| + \|x-ty\| - 2\} \leq t^{-1}\, \rho_1(t)$$

Now let x, y and z belong to S, and let $t > 0$, then we we have

$$|\langle D(x)-D(y),z\rangle| \leq |\langle D(x),z\rangle - f_t(x,z)| + |f_t(x,z)-f_t(y,z)|$$

$$+ |f_t(y,z) - \langle D(y),z\rangle|$$

$$\leq 2t^{-1}\rho(t) + |f_t(x,z) - f_t(y,z)|$$

Using the inequality

$$\|u\| + \|v\| \leq \|u+v\| + \|u+v\|\,\rho_1\left(\frac{\|u-v\|}{\|u+v\|}\right)$$

for $u = x+tz$ and $v = y$ gives

$$\|x+tz\| + \|y\| \leq \|x+y+tz\| + \|x+y+tz\|\,\rho_1\left(\frac{\|x-y+tz\|}{x+y+tz}\right)$$

$$\leq \|y+tz\| + \|x\| + (t+2)\,\rho_1(t+\|x-y\|)$$

for $0 < t \leq \frac{1}{2}$ and $\|x-y\| \leq \frac{1}{2}$, since we have

$$\|x+y+tz\| \geq \|2x\| - \|x-y\| - \|tz\| \geq 1$$

for $t \leq \frac{1}{2}$ and $\|x-y\| \leq \frac{1}{2}$. Hence we find

$$f_t(x,z) - f_t(y,z) = t^{-1}(\|x+tz\| + \|y\| - \|y+tz\| - \|x\|)$$

$$\leq 3t^{-1}\,\rho_1(t+\|x-y\|)$$

and similarly we find for $t \leq \frac{1}{2}$ and $\|x-y\| \leq \frac{1}{2}$:

$$f_t(y,z) - f_t(x,z) \leq 3t^{-1}\,\rho_1(t+\|x-y\|).$$

Putting $t = \|x-y\|$ gives

$$|\langle D(x) - D(y),z\rangle| \leq 2\|x-y\|^{-1}\,\rho_1(\|x-y\|) + 3\|x-y\|^{-1}\rho_1(2\|x-y\|)$$

for $\|x-y\| \leq \frac{1}{2}$ and all $z \in S$. Hence we have

(2.13) $$\omega_1(t) \leq 5t^{-1}\rho_1(2t) \qquad \forall\ 0 \leq t \leq \tfrac{1}{2}$$

From the lemma on p.251 in [1] it follows that there exists a constant $C > 0$ so that

$$\rho_1(2t) \leq C\,\rho_1(t) \qquad \forall\ 0 \leq t \leq 1$$

and so (2.2.1) and (2.2.2) follows from (2.12) and (2.13).

<u>Theorem 2.3</u>. <u>The following conditions are equivalent</u>

(2.3.1) E <u>is uniformly smooth</u>.

(2.3.2) $\rho_p(t) = 0(t)$ <u>as</u> $t \to 0$ <u>for some</u> $p \geq 1$.

(2.3.3) $\rho_p(t) = 0(t)$ <u>as</u> $t \to 0$ <u>for all</u> $p \geq 1$.

(2.3.4) $\lim\limits_{t\to 0} \omega_1(t) = 0$.

<u>Proof</u>. (2.3.1), (2.3.2), and (2.3.3) are equivalent by Lemma 2.1, and (2.3.4) implies (2.3.1) by Lemma 2.2. So we have only left to show that (2.3.1) implies (2.3.4). Let

$$U = \{x \in E \mid \tfrac{1}{2} < \|x\| < 2\}$$

then we have for $x \in U$, $y \in B$ and $t > 0$

$$\tfrac{1}{2}(\|x+ty\| + \|x-ty\| - 2\|x\| \leq \|x\|\rho_1\left(\frac{t\|y\|}{\|x\|}\right) \leq 2\rho_1(2t)$$

Hence we have

$$\rho(N_1, U, t) \leq 2\rho_1(2t)$$

and so by Proposition 1.2, we find that (2.3.1) implies that $D_1(x)$ exists everywhere on S, and by Lemma 2.2, we find that $\omega_1(t) \xrightarrow[t\to 0]{} 0$.

Theorem 2.4. *Let* $0 < \alpha \leq 1$, *then the following conditions are equivalent*

(2.4.1) E *is uniformly* $(1+\alpha)$-*smooth*.

(2.4.2) $\rho_p(t) = 0(t^{1+\alpha})$ *as* $t \to 0$ *for some* $p \geq 1$.

(2.4.3) $\rho_p(t) = 0(t^{1+\alpha})$ *as* $t \to 0$ *for all* $p \geq 1$.

(2.4.4) $\omega_1(t) = 0(t^{\alpha})$ *as* $t \to 0$ *for all* $p \geq 1$.

(2.4.5) $\exists c>0$ *so that* $\|D_{1+\alpha}(x) - D_{1+\alpha}(y)\| \leq c\|x-y\|^{\alpha}\ \forall x,y$

(2.4.6) $\exists c>0$ *so that* $\|x+y\|^{1+\alpha} + \|x-y\|^{1+\alpha} \leq 2\|x\|^{1+\alpha} + c\|y\|^{1+\alpha}\ \forall x,y$

(2.4.7) E *is a* G_α-*space*.

Proof. From Lemma 2.1, Lemma 2.2 and Theorem 2.3 it follows that (2.4.1)-(2.4.4) are equivalent.

(2.4.4.) $\Leftrightarrow$ (2.4.5): From (2.5) and (2.7) it follows that

$$\|D_{1+\alpha}(x) - D_{1+\alpha}(y)\| \leq (1+\alpha)\left\{\|x-y\|^{\alpha} + \|y\|^{\alpha}\omega_1\left(\frac{2\|x-y\|}{\|y\|}\right)\right\}$$

$$\leq C\|x-y\|^{\alpha}$$

if $\omega_1(t) = 0(t^{\alpha})$ and $t \to 0$. On the other hand it is evident that (2.4.5) implies (2.4.4).

(2.4.3.) $\Leftrightarrow$ (2.4.6): From the definition of $\rho_{1+\alpha}$ we have

$$\|x+y\|^{1+\alpha} + \|x-y\|^{1+\alpha} \leq 2\|x\|^{1+\alpha} + 2\|x\|^{1+\alpha}\rho_{1+\alpha}\left(\frac{\|y\|}{\|x\|}\right)$$

$$\leq 2\|x\|^{1+\alpha} + C\|y\|^{1+\alpha}$$

if $\rho_{1+\alpha}(t) = 0(t^{1+\alpha})$ as $t \to 0$ (note that we always have that $\rho_p(t) = 0(t^p)$ as $t \to \infty$ for all $p \geq 1$). On the other hand it is evident that (2.4.6) implies (2.4.2).

(2.4.5) $\Leftrightarrow$ (2.4.7): Suppose that (2.4.5) holds and put

$$G(x) = (1+\alpha)^{-1} D_{1+\alpha}(x)$$

then G satisfies (2.8) and (2.9) by (2.3) and (2.4). And (2.10) follows from (2.4.5).

Now suppose that E is a G_α-space, and let $G: E \to E'$ satisfy (2.8) - (2.10). Let $x, y \in E$, then we have

$$\begin{aligned} \langle G(x), y\rangle &= \langle G(x), x+y\rangle - \langle G(x), x\rangle \\ &\leq \|G(x)\| \, \|x+y\| - \|x\|^{1+\alpha} \\ &= \|x\|^{1+\alpha} (\|x\|^{-1} \|x+y\| - 1) \\ &\leq (1+\alpha)^{-1} (\|x+y\|^{1+\alpha} - \|x\|^{1+\alpha}) \end{aligned}$$

where we have used the inequality

$$t-1 \leq (1+\alpha)^{-1} (t^{1+\alpha} - 1)$$

which is valid for $t \geq 0$ and $\alpha \geq 0$. Hence $D(x) = (1+\alpha)G(x)$ is a subdifferential of $\|x\|^{1+\alpha}$ at x for all $x \in E$, and from (2.10) it follows that D is continuous on E. So by Proposition 1.1 it follows that $D = D_{1+\alpha}$ and from (2.10) it follows that (2.4.5) holds, and so (2.4.5) is equivalent to (2.4.7).

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