Math. Japonica **32**, No. 2 (1987), 227–237.

AN ESTIMATION OF THE MODULUS OF CONVEXITY IN A CLASS OF ORLICZ SPACES

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(Received 1985, December 26)

Abstract. It is given an estimation of the modulus of convexity in the class of Orlicz spaces L^{Φ} generated by Orlicz functions Φ satisfying condition Δ_2 for all $u \in R$ and such that the function $\Phi(\sqrt{u})$ is convex on R_+. The modulus of convexity of the Orlicz space $L^p \cap L^q$, $2 \leqslant p \leqslant q < \infty$, generated by the Orlicz function $\Phi(u) = \max(|u|^p, |u|^q)$ is estimated. Relationships between uniform convexity of a modular and of a modular norm generated by it are discussed.

Introduction. The notion of uniform convexity of Banach spaces has introduced J. A. Clarkson in [1]. He has proved that the classical real or convex Lebesgue spaces L^p, $1 \leqslant p \leqslant \infty$, are uniformly convex for $1 < p < \infty$. Next, many mathematicians have given some simplifications of the Clarkson proof. A very simple proof has given O. Hanner in [3]. Estimations for the modulus of convexity of L^p, $1 < p < \infty$, may be chosen in [2] and [9]. The best result concerning uniform convexity of L^p, $1 < p \leqslant 2$, has obtained A. Meir [11]. Papers [6] and [12] concern uniform convexity of Orlicz spaces while papers [4] and [5] concern uniform convexity of Musielak-Orlicz spaces. In [4] it is proved that the modulus of convexity of the Luxemburg norm $\| \ \|_{\Phi}$ in a uniformly convex Orlicz space can be estimated whenever an estimation of the modulus of convexity for the modular I_{Φ} is known. In this paper an estimation for the modulus of convexity of the modular I_{Φ} for the class of Orlicz functions Φ such that $\Phi(\sqrt{u})$ is a convex function on R_+ is given. The result from [4] concerning the estimation of the function $q(\epsilon)$, $0 < \epsilon < \infty$, such that $\|f\|_{\Phi} \geqslant \epsilon$ implies $I_{\Phi}(f) \geqslant q(\epsilon)$ is improved. Next, these results are applied to give an estimation of the modulus $\delta_{\| \|_{\Phi}}(\epsilon)$.

AMS (1980) *Subject Classification*. 46E30.

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Moreover, it is proved that the modulus of convexity for the Orlicz space $L^p \cap L^q$, $2 \leqslant p \leqslant q < \infty$, generated by the Orlicz function $\Phi(u) = \max(|u|^p, |u|^q)$, is nonsmaller than $1 - \frac{1}{2}\sqrt[q]{2^q - \epsilon^q}$ $(0 < \epsilon \leqslant 2)$. Finally, relationships between uniform convexity of a modular and of a modular norm generated by it and various notions of uniform convexity of a modular m are discussed.

Now, we shall give some denotations and definitions. Throughout this paper R denotes the real line, $R_+ = [0, \infty)$, (T, Σ, μ) denotes a space of positive measure, Φ denotes an *Orlicz function,* i.e. $\Phi(0) = 0$, Φ is convex, even and not identically equal to zero or infinity for $u > 0$. L^Φ denotes the corresponding *Orlicz space,* i.e. L^Φ contains of all Σ-measurable functions f defined on T for which there is $\lambda > 0$ such that $I_\Phi(\lambda f) = \int_T \Phi(\lambda f(t)) d\mu < \infty$ (see [7], [10], [13] and [14]). With respect to the *Luxemburg norm* $\| \ \|_\Phi$ defined by

$$\|f\|_\Phi = \inf\{\lambda > 0 : I_\Phi(\lambda^{-1} f) \leqslant 1\}$$

L^Φ is a Banach space (see [10]).

We say an Orlicz function Φ satisfies *condition* Δ_2 for all $u \in R$ if there is a constant $K \geqslant 2$ such that $\Phi(2u) \leqslant K\Phi(u)$ for all $u \in R$.

The *moduli of convexity* of a modular m (for definition of the modular see [13-15]) and of a norm $\| \ \|$ are defined for $0 < \epsilon \leqslant 1$ and $0 < \epsilon \leqslant 2$, respectively, by

$$\delta_m(\epsilon) = \inf\{1 - m(\frac{f+g}{2}) : m(f) \leqslant 1,\ m(g) \leqslant 1,\ m(\frac{f-g}{2}) \geqslant \epsilon\}$$

assuming $\inf \phi = 1$ (see [4]),

$$\delta_{\| \|}(\epsilon) = \inf\{1 - \|\frac{f+g}{2}\| : \|f\| \leqslant 1,\ \|g\| \leqslant 1,\ \|f-g\| \geqslant \epsilon\} \text{ (see [2]).}$$

A norm $\| \ \|$ (a modular m) is said to be *uniformly convex* if its modulus of convexity is positive for $0 < \epsilon \leqslant 2$ (for $0 < \epsilon \leqslant 1$).

Results. Before we shall prove the main theorem of this paper, we shall give three auxilliary lemmas.

Lemma 1. *Let Φ be an Orlicz function such that the function $\Phi(\sqrt{u})$ is convex on R_+. Then $\delta_{I_\Phi}(\epsilon) \geqslant \epsilon$ for $0 < \epsilon \leqslant 1$.*

Proof. We have by convexity and super-additivity of the function $\Phi(\sqrt{u})$ (see [8]),

$$\frac{\Phi(|z+w|)+\Phi(|z-w|)}{2} = \frac{\Phi(\sqrt{(z+w)^2})+\Phi(\sqrt{(z-w)^2})}{2}$$

$$\geqslant \Phi\left(\sqrt{\frac{(z+w)^2+(z-w)^2}{2}}\right) = \Phi(\sqrt{z^2+w^2})$$

$$\geqslant \Phi(\sqrt{z^2})+\Phi(\sqrt{w^2}) = \Phi(z)+\Phi(w)$$

for any $w, z \in R$. Hence it follows that for any $f, g \in L^\Phi$,

$$(1) \qquad I_\Phi(f+g)+I_\Phi(f-g) \geqslant 2I_\Phi(f)+2I_\Phi(g).$$

Defining $f_1 = \frac{1}{2}(f+g)$, $g_1 = \frac{1}{2}(f-g)$, we get $f_1+g_1 = f$ and $f_1 - g_1 = g$. So, by (1),

$$I_\Phi(f)+I_\Phi(g) = I_\Phi(f_1+g_1)+I_\Phi(f_1-g_1) \geqslant 2I_\Phi(f_1)+2I_\Phi(g_1), \text{ i.e.}$$

$$(2) \qquad I_\Phi((f+g)/2) \leqslant \frac{1}{2}\{I_\Phi(f)+I_\Phi(g)\} - I_\Phi((f-g)/2).$$

Assuming additionally that $I_\Phi(f) \leqslant 1$ and $I_\Phi(g) \leqslant 1$, we get $I_\Phi((f+g)/2) \leqslant 1-\epsilon$ whenever $I_\Phi((f-g)/2) \geqslant \epsilon$. It means that I_Φ is uniformly convex and $\delta_{I_\Phi}(\epsilon) \geqslant \epsilon$.

For a fixed *Orlicz function* vanishing only at zero and for any $\sigma \in (0, 1)$, we denote by f_σ the function from $R \setminus \{0\}$ into R_+ defined by

$$f_\sigma(u) = \Phi(u/(1-\sigma))/\Phi(u).$$

There holds the following

Lemma 2 (see [4], Lemma 2.3). *Let Φ be an Orlicz function satisfying condition Δ_2 for all $u \in R$ and define the function $p : (0,1) \longrightarrow (0,1)$, by*

$$p(\epsilon) = \sup\ \{\sigma \in (0,1) : \sup_{u>0} f_\sigma(u) \leqslant \frac{1}{1-\epsilon}\}.$$

Then for any $f \in L^{\Phi}$ *and* $\epsilon \in (0,1)$, *we have* $\|f\|_{\Phi} \leqslant 1 - p(\epsilon)$ *whenever* $I_{\Phi}(f) \leqslant 1 - \epsilon$.

Lemma 3. *Let* Φ *be an Orlicz function satisfying condition* Δ_2 *for all* $u \in R$. *If for* $\epsilon > 0$,

$$q(\epsilon) = \inf_{u>0} \{\Phi(u)/\Phi(u/\epsilon)\},$$

then $q(\epsilon) > 0$, $q(\epsilon) \leqslant K$ for $0 < \epsilon \leqslant 2$ *(*K *is the constant from condition* Δ_2*) and* $I_{\Phi}(f) \geqslant q(\epsilon)$ *whenever* $\|f\|_{\Phi} \geqslant \epsilon$.

Proof. It follows from condition Δ_2 that for any $\epsilon > 0$ there is $K_\epsilon > 0$ such that $\Phi(\frac{u}{\epsilon}) \leqslant K_\epsilon \Phi(u)$ for all $u \in R$. Hence $q(\epsilon) \geqslant K_\epsilon^{-1} > 0$. Moreover, we have for $0 < \epsilon \leqslant 2$ and $u \in R$,

$$\Phi(\frac{u}{\epsilon}) \geqslant \Phi(\frac{u}{2}) \geqslant K^{-1}\Phi(u), \text{ i.e. } [\Phi(u)/\Phi(u/\epsilon)] \leqslant K \text{ and } q(\epsilon) \leqslant K.$$

It is clear that $\Phi(\frac{u}{\epsilon}) \leqslant (q(\epsilon))^{-1}\Phi(u)$ for all $u \in R$, $0 < \epsilon \leqslant 2$. Thus $I_{\Phi}(f) < q(\epsilon)$ implies $I_{\Phi}(\frac{f}{\epsilon}) \leqslant (q(\epsilon))^{-1} I_{\Phi}(f) < 1$, i.e. $\|f\|_{\Phi} < \epsilon$.

Combining the above three lemmas, we obtain the following

Theorem 1. *Let* Φ *be an Orlicz function satisfying condition* Δ_2 *for all* $u \in R$ *and such that* $\Phi(\sqrt{u})$ *is a convex function on* R_+. *Then* L^{Φ} *is uniformly convex and* $\delta_{\|\,\|_{\Phi}}(\epsilon) \geqslant p(q(\frac{\epsilon}{2}))$ *for any* $0 < \epsilon \leqslant 2$.

Proof. Assume that $\|f\|_{\Phi} \leqslant 1$, $\|g\|_{\Phi} \leqslant 1$ and $\|f-g\|_{\Phi} \geqslant \epsilon$. Then by condition Δ_2 for all $u \in R$ and Lemmas 2 and 3, we have $I_{\Phi}(f) \leqslant 1$, $I_{\Phi}(g) \leqslant 1$ and $I_{\Phi}(\frac{f-g}{2}) \geqslant q(\frac{\epsilon}{2})$. Applying Lemma 1, we get $I_{\Phi}(\frac{f+g}{2}) \leqslant 1 - q(\frac{\epsilon}{2})$. Next, by Lemma 2, we obtain $\|\frac{f+g}{2}\|_{\Phi} \leqslant 1 - p(q(\frac{\epsilon}{2}))$. It is the desired result.

The following example is an illustration of our method of estimation for the modulus of convexity in considered class of Orlicz spaces.

Example 1. *Let* $\Phi(u) = \max(|u|^p, |u|^q)$, *where* $2 \leqslant p \leqslant q < \infty$. *This function satisfies the assumptions of Theorem 1. We shall show that* $\delta_{\|\,\|_{\Phi}}(\epsilon)$

$\geqslant 1 - \frac{1}{2} \sqrt[q]{2^q - \epsilon^q}$ *for* $0 < \epsilon \leqslant 2$.

Indeed, we have for $0 < \epsilon \leqslant 1$,

$$\begin{aligned} \Phi(u)/\Phi(u/\epsilon) &= \epsilon^p && \text{if } 0 < u \leqslant \epsilon \\ &= \epsilon^q u^{p-q} && \text{if } \epsilon < u \leqslant 1 \\ &= \epsilon^q && \text{if } u > 1, \end{aligned}$$

and for $\epsilon > 1$,

$$\begin{aligned} \Phi(u)/\Phi(u/\epsilon) &= \epsilon^p && \text{if } 0 < u \leqslant 1 \\ &= \epsilon^p u^{q-p} && \text{if } 1 < u \leqslant \epsilon \\ &= \epsilon^q && \text{if } u > \epsilon. \end{aligned}$$

Hence, $q(\epsilon) = \min(\epsilon^p, \epsilon^q)$ for $0 < \epsilon < \infty$.

Let $0 < \epsilon < 1$ and $\sigma \in (0, 1)$. We have

$$\begin{aligned} f_\sigma(u) &= (1-\sigma)^{-p} && \text{if } 0 < u \leqslant 1 - \sigma \\ &= (1-\sigma)^{-q} u^{p-q} && \text{if } 1 - \sigma < u \leqslant 1 \\ &= (1-\sigma)^{-q} && \text{if } u > 1. \end{aligned}$$

Thus, $\sup_{u>0} f_\sigma(u) = (1-\sigma)^{-q}$. Hence it follows that

$$p(\epsilon) = \sup\{\sigma \in (0,1) : (1-\sigma)^{-q} \leqslant (1-\epsilon)^{-1}\} = 1 - \sqrt[q]{1-\epsilon}.$$

Applying Theorem 1, we get for $0 < \epsilon \leqslant 2$,

$$\delta_{\|\|_\Phi}(\epsilon) \geqslant p\left(q\left(\frac{\epsilon}{2}\right)\right) = 1 - \sqrt[q]{1 - (\epsilon/2)^q} = 1 - \frac{1}{2} \sqrt[q]{2^q - \epsilon^q}.$$

Note. For $p = q$ it is a classical result for L^p spaces, $2 \leqslant p < \infty$.

Main notions that will be used in the following may be chosen in [15]. X denotes a *modular space,* i.e. a real vector space equipped with a convex and left-continuous *modular* m. Elements of X will be denoted by x, y (the letters f, g are reserved for denotation of functions). The *modular norm* $|||\ |||$ is defined by

$$|||x||| = \inf\{\lambda > 0 : m(\lambda^{-1}x) \leqslant 1\}.$$

H. Nakano [15] has assumed that a modular m is said to be *uniformly convex* if for any two $\epsilon, \gamma > 0$ we can find $\delta > 0$ such that if $m(x-y) \geqslant \epsilon$ and $\max(m(x), m(y)) \leqslant \gamma$, then

$$m(\frac{1}{2}(x+y)) \leqslant \frac{1}{2}\{m(x)+m(y)\} - \delta.$$

If m satisfies the conditions for uniform convexity in the sense of Nakano with $m(\frac{1}{2}(x-y)) \geqslant \epsilon$ instead of $m(x-y) \geqslant \epsilon$', then we say that m is *uniformly convex in the modified sense of Nakano.*

This new property of m is weaker than uniform convexity in the sense of Nakano.

Corollary 1. (a). *If Φ is an Orlicz function such that $\Phi(\sqrt{u})$ is a convex function on R_+, then I_Φ is uniformly convex in the modified sense of Nakano with $\delta(\epsilon, \gamma) = \epsilon$.*

(b). *If additionally, Φ satisfies condition Δ_2 for all $u \in R$, then I_Φ is uniformly convex in the sense of Nakano with $\delta(\epsilon, \gamma) = \frac{\epsilon}{K}$, where $K = \sup\{\Phi(2u)/\Phi(u) : u > 0\}$.*

(c). *If $\lim_{n\to\infty} x^*(x_n) = x^*(x)$ for any linear modular bounded functional x^* over X* (see [15], p.206) *and* $\overline{\lim_{n\to\infty}}\, m(x_n) \leqslant m(x)$, then $\lim_{n\to\infty} m(\frac{x_n - x}{2}) = 0$.

Proof. Property (a) follows by inequality (2), p.3. For the proof of (b), assume additionally that Φ satisfies condition Δ_2 for all $u \in R$. Then $I_\Phi(x-y) \geqslant \epsilon$ implies $\epsilon \leqslant I_\Phi(x-y) = I_\Phi(2\frac{x-y}{2}) \leqslant K I_\Phi(\frac{x-y}{2})$. Hence we get $I_\Phi(\frac{x-y}{2}) \geqslant \frac{\epsilon}{K}$. Now, it suffices to apply property (a).

The proof of (c) is analogous to the proof of Th. 1, p.227 in [15].

We say a modular m is *uniformly convex in the second modified sense of Nakano* if for any $\epsilon, \gamma > 0$ there exists $\delta(\epsilon, \gamma) \in (0, 1)$ such that $\max(m(x), m(y)) \leqslant \gamma$ and $m(x-y) \geqslant \epsilon$ implies

$$m(\frac{x+y}{2}) \leqslant \frac{1-\delta}{2} \{m(x)+m(y)\}.$$

This property is weaker than uniform convexity in the Nakano sense and there holds the following

Remark 1. *If m is a modular satisfying the condition :*

(Λ_2) *For any $\epsilon, \gamma > 0$ there is $K(\epsilon, \gamma) > 0$ such that $m(x) \leqslant \gamma$ implies $m(2x) \leqslant K m(x) + \epsilon$.*

then all three uniform convexities of Nakano type for m are equivalent.

Proof. It is obvious that uniform convexity in the Nakano sense implies the uniform convexity of m in the modified sense of Nakano. Conversely, if m is uniformly convex in the last sense and $m(x-y) \geqslant \epsilon$, $\max(m(x), m(y)) \leqslant \gamma$, then $m(\frac{x-y}{2}) \leqslant \gamma$ and so

$$\epsilon \leqslant m(x-y) = m(2\frac{x-y}{2}) \leqslant K(\frac{\epsilon}{2}, \gamma)\, m(\frac{x-y}{2}) + \frac{\epsilon}{2},$$

i.e. $m(\frac{x-y}{2}) \geqslant \frac{\epsilon}{2K}$. Hence it follows that $m(\frac{x+y}{2}) \leqslant \frac{1}{2}\{m(x)+m(y)\} - \delta(\frac{\epsilon}{2K}, \gamma)$.

Now, assume that m is uniformly convex in the sense of Nakano and $\max(m(x), m(y)) \leqslant \gamma$, $m(x-y) \geqslant \epsilon$. Then

$$\begin{aligned} m(\frac{x+y}{2}) &\leqslant \frac{1}{2}\{m(x)+m(y)\} - \delta(\epsilon, \gamma) \\ &\leqslant \frac{1}{2}\{m(x)+m(y)\} - \frac{\delta(\epsilon, \gamma)}{2\gamma}\{m(x)+m(y)\} \\ &= \frac{1-\gamma^{-1}\delta(\epsilon, \gamma)}{2}\{m(x)+m(y)\}. \end{aligned}$$

It means that m is uniformly convex in the second modified sense of Nakano. Conversely, assuming that m is uniformly convex in the last sense, we have

$$m(\frac{x+y}{2}) \leqslant \frac{1-\delta(\epsilon, \gamma)}{2}\{m(x)+m(y)\},$$

whenever $m(x-y) \geqslant \epsilon$ and $\max(m(x), m(y)) \leqslant \gamma$. Since

$$\frac{\epsilon}{2K(\frac{\epsilon}{2}, \gamma)} \leqslant m(\frac{x-y}{2}) \leqslant \frac{1}{2}\{m(x)+m(y)\},$$

so

$$m(\frac{x+y}{2}) \leqslant \frac{1}{2}\{m(x)+m(y)\} - \frac{\epsilon\delta(\epsilon, \gamma)}{2K(\frac{\epsilon}{2}, \gamma)},$$

i.e. m is uniformly convex in the Nakano sense.

Remark 2. *Every uniformly convex Orlicz function* $\Phi : X \to R_+$, *i.e. Orlicz function such that for any* $\epsilon \in (0, 1)$ *there is* $\delta(\epsilon) \in (0, 1)$ *such that* $\Phi(x-y) \geqslant \epsilon \max(\Phi(x), \Phi(y))$ *implies* $\Phi(\frac{x+y}{2}) \leqslant \frac{1-\delta}{2}\{\Phi(x)+\Phi(y)\}$ *(for definition and examples see* [4 – 5]*), is a uniformly convex modular on* X *in the second modified sense of Nakano.*

Proof. If $\Phi(x-y) \geqslant \epsilon$ and $\max(\Phi(x), \Phi(y)) \leqslant \gamma$, then

$$\Phi(x-y) \geqslant \frac{\min(\epsilon, 2^{-1})}{\gamma} \max(\Phi(x), \Phi(y)).$$

Thus

$$\Phi(\frac{x+y}{2}) \leqslant \frac{1-\delta(\gamma^{-1}\min(\epsilon, 2^{-1}))}{2}\{\Phi(x)+\Phi(y)\}.$$

Corollary 2. *Every modular* m *as in Remark* 2 *has the following useful property: if* $\lim_{n\to\infty} x^*(x_n) = x^*(x)$ *for any linear and modular bounded functional* x^* *over* X *and* $\overline{\lim_{n\to\infty}}\, m(x_n) \leqslant m(x)$, *where* $m(x) > 0$, *then* $\lim_{n\to\infty} m(x-x_n) = 0$.

The proof is analogous to that of Th. 1, p. 227 in [15].

Notes 1. (a) *All above considered uniform convexities (i.e. uniform convexities of Nakano type) are very strong properties. The definition of uniform convexity* *of a modular* m *given on page* 2 *is a weaker one.*

(b) *If* m *is a uniformly convex modular in the sense of definition given on page* 2, *then for any* x, $x_n \in X$, $n = 1, 2, \ldots$, *such that* $x^*(x_n) \to x^*(x)$ *as* $n \to \infty$ *for any linear and modular bounded functional* x^* *over* X, *and* $\overline{\lim_{n\to\infty}}\, m(x_n) \leqslant m(x) \leqslant 1$, *we have* $m(\frac{1}{2}(x_n - x)) \to 0$ *as* $n \to \infty$.

Proof. (a) follows immediately from definitions. Property (b) may be proved in an analogous way as Th. 1, p.227 in [15].

Remark 3. (a) *No norm on* X *is a uniformly convex modular in the second modified sense of Nakano (so also in the modified sense of Nakano and in the sense of Nakano).*

(b) *Any uniformly convex norm on* X *has the following property: for any* $\epsilon, \gamma > 0$ *there is* $\delta(\epsilon, \gamma) \in (0, 1)$ *such that* $\|x - y\| \geqslant \epsilon$ *and* $\max(\|x\|, \|y\|) \leqslant \gamma$ *imply*

$$\left\| \frac{x+y}{2} \right\| \leqslant \frac{1-\delta}{2} \max(\|x\|, \|y\|).$$

(c) *The inequality* $\delta_X(\epsilon) \leqslant \frac{\epsilon}{2}$ *holds for any normed space* X *and any* $\epsilon \in (0, 2]$.

Proof. (a). Let $x_0 \in X$, $\|x_0\| = 1$ and $X_0 = \{\lambda x_0 : \lambda \in R\}$. Let $\alpha, \beta \geqslant 0$, $x = \alpha x_0$, $y = \beta x_0$. We have

$$\left\| \frac{x+y}{2} \right\| = \left\| \frac{\alpha+\beta}{2} x_0 \right\| = \frac{\alpha+\beta}{2} = \frac{\|x\| + \|y\|}{2}.$$

It means that X is not uniformly convex in the Nakano sense.

(b) If $(X, \|\cdot\|)$ is uniformly convex, then for any $\epsilon \in (0, 2]$ there is $\delta(\epsilon) \in (0, 1)$ such that $\max(\|x\|, \|y\|) \leqslant 1$ and $\|x - y\| \geqslant \epsilon$ imply $\|x + y\| \leqslant 2(1 - \delta)$ (see [9]). Assume that $0 < a = \max(\|x\|, \|y\|) \leqslant \gamma$ and $\|x - y\| \geqslant \epsilon$. We have $\max(\|\frac{x}{a}\|, \|\frac{y}{a}\|) \leqslant 1$ and $\|\frac{x-y}{a}\| \geqslant \frac{\epsilon}{\gamma}$. Hence $\|x + y\| \leqslant 2a(1 - \delta(\frac{\epsilon}{\gamma}))$.

(c) Let $(R, |\cdot|)$ be the real line equipped with the norm $|x| = x$ for $x \geqslant 0$

and $|x| = -x$ for $x < 0$. Since $|x+y| + |x-y| = 2\max(|x|, |y|)$ for any $x, y \in R$, so $|x-y| \geqslant \epsilon$ and $\max(|x|, |y|) \leqslant 1$ imply $|\frac{x+y}{2}| = \max(|x|, |y|) - |\frac{x-y}{2}| \leqslant 1 - \frac{\epsilon}{2}$. Moreover, for $x = 1$, $y = 1 - \epsilon$, we have $x - y = |x - y| = \epsilon$ and $|\frac{x+y}{2}| = 1 - \frac{\epsilon}{2}$. These facts mean that $\delta_R(\epsilon) = \frac{\epsilon}{2}$ for any $\epsilon \in (0, 2]$. Since $(R, |\ |)$ can be isometrically embedded into any Banach space X, so we obtain $\delta_X(\epsilon) \leqslant \frac{\epsilon}{2}$ for any $\epsilon \in (0, 2]$ and any normed space X.

Theorem 2. (a) *For any modular m, uniform convexity of the modular norm $|||\ |||$ implies uniform convexity of the modular m and the inequality $\delta_m(\epsilon) \geqslant \delta_{|||\ |||}(2\epsilon)$ holds for any $\epsilon \in (0, 1]$.*

(b) *If m is uniformly simple, uniformly finite and uniformly convex, then $|||\ |||$ is uniformly convex.*

Proof. It is well known (see [7], [10], [13], and [15]) that for any $x \in X$, we have $m(x) \leqslant 1$ if and only if $|||x||| \leqslant 1$ and $m(x) \leqslant 1$ implies $m(x) \leqslant |||x|||$. Assume that $m(\frac{x-y}{2}) \geqslant \epsilon$ and $\max(m(x), m(y)) \leqslant 1$. Then $|||\frac{x-y}{2}||| \geqslant \epsilon$ and $\max(|||x|||, |||y|||) \leqslant 1$. Thus, by uniform convexity of $|||\ |||$, we get

$$m\left(\frac{x+y}{2}\right) \leqslant \left|\left|\left|\frac{x+y}{2}\right|\right|\right| \leqslant 1 - \delta_{|||\ |||}(2\epsilon).$$

It means that $\delta_m(\epsilon) \geqslant \delta_{|||\ |||}(2\epsilon)$ and m is uniformly convex.

(b) The proof may be proceeding in an analogous manner to that of Th. 3, p. 227 [15].

Remark 4. *For every modular m we have $|||x||| = 1$ whenever $m(x) = 1$.*

Prooof. If $m(r^{-1}x) = \infty$ for any $0 < r < 1$, then $|||x||| = 1$, by the definition of the modular norm $|||\ |||$. Assume that $m(r^{-1}x) < \infty$ for some $r \in (0, 1)$. Then $m(\lambda x)$ is a finite and convex function of λ on the interval $(0, r^{-1})$. Therefore, $m(\lambda x)$ is a strictly increasing function of λ in some neighbourhood of $\lambda_0 = 1$. Hence, we have for any $\lambda > 1$, $m(\lambda x) > m(x) = 1$. Hence it follows that $|||x|||$

$= 1.$

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