O homeomorfji pewnych przestrzeni. — The Homeomorphy of certain Spaces.

Note

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It is known that spaces L^p, $p \geqslant 1$ i. e. metric spaces, the elements of which are functions $f(x)$, such that $|f(x)|^p$ is integrable in the interval $(0, 1)$, are all homeomorphic with L[1]. Recently the notion of spaces L^p was generalised as follows by W. Orlicz[2]. Consider any function $M(u)$ with the following properties:

1) $M(u)$ continuous in $(-\infty, +\infty)$.
2) $M(0) = 0$, $M(-u) = M(u)$, $M(u) > 0$ if $u > 0$.
3) $\frac{M(u)}{u} \to \infty$ if $|u| \to \infty$.
4) $M(u)$ is convex and thus increasing for $u > 0$.
5) $M(2u) \leq K M(u)$.

If we write $N(v) = Max\,[u\,|v| - M(u)]$ for $u \geqslant 0$, then $N(v)$ is a convex function.

A function $f(x)$ belongs to the space $L(M)$ if it satisfies the inequality

$$\int_0^1 M[f(x)]\, dx < \infty.$$

A space $L(M)$ thus defined is a space of type (B) i. e. a vectorial, normalisable and complete space. The norm of $f(x)$ is defin-

[1]) S. Mazur, St. Math. Vol. I. (1929) p. 83.

[2]) W. Orlicz, Über eine gewisse Klasse von Räumen vom Typus B. Bull. Ac. Pol. d. Sc. et L. A. 1932.

ed by

$$\|f(x)\| = Max \int_0^1 f(x)\, g(x)\, dx$$

for all functions $g(x)$ satisfying

$$\int_0^1 N\,[g(x)]\, dx \leq 1.$$

This norm possesses the following properties:

a) $\int_0^1 M[f_n(x)]\, dx \leq C$ is equivalent to $\|f_n\| \leq C^1$.

b) $\lim \int M(f_p - f_q)\, dx = 0$ is equivalent to $\lim \|f_p - f_q\| = 0$.

The question arises as to whether the spaces $L(M)$ are homoemorphic. The answer is in the affirmative as may be established in the following manner.

Theorem. *If $M(u)$ satisfies the conditions mentioned above, the space $L(M)$ is homeomorphic with L_1-space.*

We prove first a lemma.

Lemma. Given a sequence of functions $\{f_n\}$ and a function $f(x)$ belonging to $L(M)$ and such that

a) any subsequence of $\{f_n\}$ contains another subsequence $\{f_{m_n}\}$ converging almost everywhere to $f(x)$ and

b) $\int_0^1 M(f_n)\, dx \to \int_0^1 M(f)\, dx,$

then

$$\int_0^1 M(f_n - f)\, dx \to 0.$$

To prove this lemma, suppose first that $f_n \to f$ almost everywhere. Then, for any $\eta > 0$ a set E exists whose measure is greater than $1 - \eta$, such that $f_n \to f$ uniformly in E; hence, given $\varepsilon' > 0$, N exists, such that for $n > N$, $|f - f_n| < \varepsilon'$. But

$$\int_0^1 M(f_n - f)\, dx = \int_E + \int_{CE}$$

and for $n > N$

$$\int_E M(f_n - f)\,dx < \varepsilon$$

where $\varepsilon = M(\varepsilon')$. From

$$M(f_n - f) \leq M[2\,max\,(|f|, |f_n|] \leq K[M(f) + M(f_n)]$$

we obtain

$$\int_{CE} M(f_n - f) \leq K\left[\int_{CE} M(f) + \int_{CE} M(f_n)\right].$$

Further

$$\int_E M(f_n) + \int_{CE} M(f_n) \to \int_E M(f) + \int_{CE} M(f)$$

hence for large n

$$\left|\int_{CE} M(f_n) - \int_{CE} M(f)\right| < 2\,\varepsilon$$

on account of $(M[f_n(x)] - M[f(x)]) < \varepsilon$ when x is contained in set E. Finally we have

$$\left|\int_0^1 M(f_n - f)\right| < \varepsilon + K\left[2\int_{CE} M(f) + 2\varepsilon\right]$$

and

$$\left|\int_0^1 M(f_n - f)\right| < \varepsilon + 4\,K\varepsilon$$

if

$$\int_{CE} M(f) < \varepsilon.$$

Returning to the general case, assume that $\int_0^1 M(f_n - f)\,dx$ does not converge to zero. Then, in consequence of the boundedness relation

$$\left|\int_0^1 M(f - f_n)\right| \leq K\left[\int M(f_n) + \int M(f)\right] < KC$$

there is a sequence n_k for which

$$\int M(f - f n_k) \to B \neq 0.$$

But a subsequence of n_k exists, which we may denote by m_k, for which

$$f - f_{m_k} \to 0$$

almost everywhere; in virtue of what has been proved

$$\int M(f_n - f) \to 0.$$

Now this is in contradiction with the hypothesis $B \neq 0$ so that the lemma is true.

We can now prove the theorem. We denote any function belonging to L by $f(x)$ and to $L(M)$ by $\varphi(x)$. Let

$$\varphi(x) \equiv M^{-1}(|f|) \, sgn \, f$$

correspond to $f(x)$ and

$$f(x) \equiv M(\varphi) \, sgn \, \varphi$$

correspond to $\varphi(x)$. The correspondences thus established are uniquely defined and we have to prove that they are continuous.

Indeed, if $\int |f - f_n| \to 0$, then

$$\int M(\varphi - \varphi_n) \to 0.$$

The sequence f_n being asymptotically convergent to $f(x)$, we see that $\varphi_n(x)$ converges asymptotically to $\varphi(x)$ and by reason of

$$M(\varphi_n) = |f_n|$$

we have $\int M(\varphi_n) \to \int M(\varphi)$.

Thus by the lemma we have

$$\int M(\varphi_n - \varphi) \to 0.$$

Now let us assume that

$$\int M(\varphi - \varphi_n) \to 0.$$

Then $f_n \to f$ asymptotically by the same argument as above. We have further

$$\int |f_n| = \int M(\varphi) \to \int M(\varphi) = \int |f|$$

and by the lemma we obtain

$$\int |f_n - f| \to 0.$$