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ON COMPARISON OF ORLICZ SPACES AND ORLICZ CLASSES

by

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ABSTRACT. In the paper [3] were investigated spaces L_Φ of Orlicz type. Here there are given necessary and sufficient conditions for inclusions of such Orlicz spaces and Orlicz classes generated by different $\mathcal{N}'$-functions Φ and Ψ. In the first part of this paper it is considered the case of nonatomic submeasure ν_M, in the second one the case of purely atomic ν_M. This note is also a partial generalization of the results obtained by A. Kozek in [4] and I. V. Šragin in [7].

Let R be the set of real numbers, N the set of integers, X a real Banach space and (T, Q, M) a measure space, where Q is a σ-algebra of subsets of T, M is a family of countably additive measures $\mu : Q \to [0, \infty]$. The set function $\nu_M : Q \to [0, \infty]$ defined by

$$\nu_M(A) = \sup_{\mu \in M} \mu(A)$$

is a submeasure ([2], [3]).

Let Q_c be the ring of all subsets from Q on which the submeasure ν_M is order continuous, i.e. if $A \in Q_c$ then $\nu_M(A_n) \to 0$ for all sequences $(A_n) \subset A$ such that $A_1 \supset \supset A_2 \supset \ldots$ and $\bigcap_{i=1}^{\infty} A_i = \phi$.

We say that $A \in Q$ is an atom of any submeasure $\eta : Q \to [0, \infty]$ iff $\eta(A) > 0$ and $\eta(B) = 0$ or $\eta(T \setminus B) = 0$ for all $B \subset A$, $B \in Q$. The submeasure η is called nonatomic if the set T contains no atoms of η. In the following we always suppose that there exists a sequence (T_j) such that $T_j \in Q_c$, $T_1 \subset T_2 \subset \ldots$ and $\nu_M(T \setminus \bigcup_{j=1}^{\infty} T_j) = 0$. In the case of ν_M nonatomic it is $\nu_M(T_j) < \infty$, by Theorem 5.5 in [2]. We say that a property holds almost everywhere (a.e.) in T if it occurs for all $t \in T \setminus A$, where $\nu_M(A) = 0$.

A function $\Phi : X \times T \to [0, \infty]$ is called $\mathcal{N}'$-function if

(a) Φ is $\mathcal{B} \times Q$-measurable, where $\mathcal{B}$ is the Borel σ-algebra in X,

(b) the function $\Phi(\cdot, t) : X \to [0, \infty]$ is lower semicontinuous, convex, even (i.e. $\Phi(-x, t) = \Phi(x, t)$ for all $x \in X$) and $\Phi(0, t) = 0$, for a.e. $t \in T$.

In the sequel, we will assume occasionally the following conditions about the $\mathcal{N}'$-function Φ

(c) (see [3]) $\lim_{||x|| \to \infty} \Phi(x, t) = \infty$, for a.e. $t \in T$,

(d) (see [3]) $\inf_{||x|| = r} \Phi(x, t) > 0$, for each $r > 0$ and a.e. $t \in T$

(e) $\sup_{||x|| = r} \Phi(x, t) < \infty$, for all $r > 0$ and a.e. $t \in T$.

We denote by $S(X)$ the set of all measurable [3] functions $x: T \to X$. The functional

$$I_{\Phi}(x) = \sup_{\mu \in M} \int_T \Phi(x(t), t)\, d\mu,$$

is defined on the whole $S(X)$ and it is a pseudomodular [5]. The modular space L_{Φ}, called the Orlicz space, it is the following set

$$L_{\Phi} = \{x \in S(X) : I_{\Phi}(kx) < \infty \text{ for some } k \in R\}.$$

On the space L_{Φ} one can define the Luxemburg norm [3], i.e. for all $x \in L_{\Phi}$

$$\|x\|_{\Phi} = \inf\{\varepsilon > 0 : I_{\Phi}(x/\varepsilon) \leqslant 1\}.$$

Next, we introduce the following classes of functions

$$\operatorname{dom} I_{\Phi} = \{x \in S(X) : I_{\Phi}(x) < \infty\},$$

$$(\Phi)_b = \{x \in S(X) : I_{\Phi}(bx) < \infty\}$$

for $b \in (0, \infty)$. The set dom I_{Φ} is called the Orlicz class. We also define for $t \in T$

$$\operatorname{dom} \Phi(\cdot, t) = \{x \in X : \Phi(x, t) < \infty\}.$$

1. LEMMA 1.1. Suppose X is separable, Φ, Ψ are $\mathcal{N}'$-functions and b is any positive number. Moreover, let dom $\Psi(\cdot, t) = X$ a.e. in T. Then there exists a family (x_i) of measurable functions such that

(i) $x_i(t) \in \operatorname{dom} \Phi(\cdot, t)$ a.e. *in* $\in T$,

(ii) $cl(x_i(t)) = \operatorname{dom} \Phi(\cdot, t)$ in the norm of X for a.e. $t \in T$,

(iii) for all $i \in N$, the set function $A \mapsto I_{\Psi}(bx_i\chi_A)$ is ν_M-absolutely continuous, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$I_{\Psi}(bx_i\chi_B) < \varepsilon$$

if $\nu_M(B) < \delta$.

PROOF. By Theorem 1 in [6], there is a family (y_m) of measurable functions such that $y_m(t) \in \operatorname{dom} \Phi(\cdot, t)$ and $cl(y_m(t)) = \operatorname{dom} \Phi(\cdot, t)$ for a.e. $t \in T$. Let for $m, k \in N$ be

$$C_{m,k} = \{t \in T : \Psi(by_m(t), t) \leqslant k\}.$$

Putting $(x_i) = (y_m\chi_{C_{m,k}})$ we get easily the desired conditions.

LEMMA 1.2. Let X be separable, Φ, Ψ be $\mathcal{N}'$-functions where Φ is continuous at zero and dom $\Phi(\cdot, t) = X$ for a.e. $t \in T$. Moreover, let (c_n) be an arbitrary sequence of positive numbers and let the functions f_n be given as follows $f_n(t) = \sup_{x \in X}\{\Psi(c_n x, t) - 2^n\Phi(x, t)\}$. Then $f_n(t) = \sup_{i \in N}\{\Psi(c_n x_i(t), t) - 2^n\Phi(x_i(t), t)\}$, where x_i are the functions from the above lemma for any b.

The proof is analogous to that of Lemma 1.7.2 in [4].

LEMMA 1.3. Let us assume the submeasure ν_M be nonatomic. Let (α_i) be a sequence of positive numbers and $a_i : T \to R$ be measurable, nonnegative and finite functions such that

(i) $\sup_{\mu \in M} \int_T a_i(t)\,d\mu \geqslant 2^i \alpha_i$ for all $i \in N$,

(ii) the set functions $A \mapsto \nu_M^i(A) = \sup_{\mu \in M} \int_A a_i(t)\,d\mu$, $A \in Q$, are ν_M-absolutely continuous.

Then there exist an increasing sequence (i_k) of integers and the family (A_k) of pairwise disjoint sets from Q such that

$$\sup_{\mu \in M} \int_{A_k} a_{i_k}(t)\,d\mu = \alpha_{i_k}$$

for all $k \in N$.

PROOF. By (ii), the submeasures ν_M^i are order continuous on every $T_j (j \in N)$. Moreover, ν_M^i are nonatomic as ν_M-absolutely continuous, where ν_M is nonatomic (Theorem 9 in [1]). Hence and by Theorem 10 in [1], the submeasures ν_M^i have the Darboux property, i.e. for every $0 < b < \nu_M^i(T)$ one can choose a set $B \in Q$ such that $\nu_M^i(B) = b$.

Now, we will find the desired sequences (A_k) and (i_k) by induction. In the following we will show only two steps of this induction. Let $i = 1$. Then we choose $B \in Q$ such that

$$\nu_M^1(B_1) = \alpha_1 .$$

If $i \geqslant 2$ then

$$\nu_M^i(B_1) \geqslant (\tfrac{1}{2}) 2^i \alpha_i \quad \text{or} \quad \nu_M^i(T \setminus B_1) \geqslant (\tfrac{1}{2}) 2^i \alpha_i .$$

Therefore, there exists an infinite subsequence (a_k^1) of $(a_i)_{i \geqslant 2}$ and the respective subsequence (α_k^1) of $(2^i \alpha_i)_{i \geqslant 2}$ such that

$$\sup_{\mu \in M} \int_{B_1} a_k^1(t)\,d\mu \geqslant (\tfrac{1}{2}) \alpha_k^1 \tag{1.1}$$

or

$$\sup_{\mu \in M} \int_{T \setminus B_1} a_k^1(t)\,d\mu \geqslant (\tfrac{1}{2}) \alpha_k^1 . \tag{1.2}$$

If (1.2) is satisfied then we put $A_1 = B_1$, whereas if the condition (1.1) is fulfilled then we find $A_1 \subset T \setminus B_1$ such that $\nu_M^1(A_1) = \alpha_1$. Thus we found the set A_1 and $i_1 = 1$.

In the second step we repeat the above construction replacing T, (a_i), $(2^i \alpha_i)$ by $T \setminus A_1$, (a_k^1), $((1/2)\alpha_k^1)$. Thus, by construction of A_1 it follows

$$\sup_{\mu \in M} \int_{T \setminus A_1} a_k^1(t)\,d\mu \geqslant (\tfrac{1}{2}) \alpha_k^1 \tag{1.3}$$

for all $k \in N$. Let $2^{i_0 - 1} \alpha_{i_0}$ be the first element of the sequence $((1/2)\alpha_k^1)$, we have $i_0 \geqslant 2$. Hence there is a set $B_2 \in T \setminus A_1$ such that

$$\sup_{\mu \in M} \int_{B_2} a_{i_0}(t)\,d\mu = \alpha_{i_0} .$$

By (1.3), there exist infinite subsequences (a_k^2) of (a_k^1) and (α_k^2) of (α_k^1) such that

$$\sup_{\mu\in M}\int_{B_2} a_k^2(t)\,d\mu \geqslant (\tfrac{1}{4})\,\alpha_k^2 \tag{1.4}$$

or

$$\sup_{\mu\in M}\int_{(T\setminus A_1)\setminus B_2} a_k^2(t)\,d\mu \geqslant (\tfrac{1}{4})\,\alpha_k^2 . \tag{1.5}$$

The first elements of (a_k^2), (α_k^2) can't be smaller than a_3, α_3. Now, if (1.5) is fulfilled then we put $A_2=B_2$, while if (1.4) is satisfied then one can choose $A_2\subset(T\setminus A_1)\setminus B_2$ such that

$$\sup_{\mu\in M}\int_{A_2} a_{i_0}(t)\,d\mu=\alpha_{i_0}.$$

Putting $i_2=i_0$ we finish the second step of our induction. The thesis is now easily obtained by induction.

THEOREM 1.4. Let X be separable, ν_M nonatomic submeasure, and let Φ and Ψ be $\mathcal{N}'$-functions, where Φ is continuous at zero and dom $\Psi(\cdot, t)=X$ a.e. in T. Under these assumptions, if B is an arbitrary subset of $(0, \infty)$ and dom $I_\Phi\subset\bigcup_{b\in B}(\Psi)_b$, then there are constants b, k and a nonnegative function $f\colon T\to R$ such that $\sup_{\mu\in M}\int_T f(t)\,d\mu<$ $<\infty$ and the following inequality

$$\Psi(bx, t)\leqslant k\Phi(x, t)+f(t) \tag{1.6}$$

is satisfied for all $x\in X$ and a.e. $t\in T$.

PROOF. First we choose a decreasing sequence (b_n) of elements from B in such a manner that for every $b\in B$ there is $n'\in N$ such that $b_n\leqslant b$ for all $n\geqslant n'$. Let (x_i) be the family of functions from Lemma 1.1 under $b=b_1$. Denoting

$$f_n(t)=\sup_{x\in X}\{\Psi(b_n x, t)-2^n\Phi(x, t)\}$$

we have

$$f_n(t)=\sup_{i\in N}\{\Psi(b_n x_i(t), t)-2^n\Phi(x_i(t), t)\},$$

by Lemma 1.2 where we put $c_n=b_n$.

It can be easily proved that the thesis of our theorem is satisfied iff there exists $n\in N$ such that

$$\sup_{\mu\in M}\int_T f_n(t)\,d\mu<\infty.$$

Suppose, for a contrary, $\sup_{\mu\in M}\int_T f_n(t)\,d\mu=\infty$. Then, for all $n\in N$

$$\int_T f_n(t)\,d\mu_n>2^n n$$

for some $\mu_n\in M$. Let now

$$l_{n,k}(t)=\max\{0, \Psi(b_n x_i(t), t)-2^n\Phi(x_i(t), t)\colon\ i=1, \ldots, k\}.$$

Since $l_{n,k}(t)\uparrow f_n(t)$ a.e. in T, then there is an index $N(n)$ such that, denoting $l_n(t)=$ $=l_{n,N(n)}(t)$ we get

$$\int_T l_n(t)\,d\mu_n \geqslant 2^n n \tag{1.7}$$

for each $n\in N$. Let for $k=1,\ldots,N(n)$

$$B_{n,k}=\{t\in T:\ \Psi(b_n x_k(t),t)-2^n\Phi(x_k(t),t)=l_n(t)\}.$$

Putting

$$\bar{x}_n(t)=\begin{cases} x_1(t) & t\in B_{n,1}\\ x_k(t) & t\in B_{n,k}\setminus\bigcup_{i=1}^{k-1}B_{n,i}, \quad k=2,\ldots,N(n)\\ 0 & \text{otherwise}\end{cases}$$

we obtain

$$\Psi(b_n\bar{x}_n(t),t)-2^n\Phi(\bar{x}_n(t),t)=l_n(t)\geqslant 0. \tag{1.8}$$

Hence and by (1.7) we have

$$\sup_{\mu\in M}\int_T \Psi(b_n\bar{x}_n(t),t)\,d\mu\geqslant 2^n n.$$

From the definition of $\bar{x}_n$ and the condition (iii) of Lemma 1.1 it is seen that the set function $A\mapsto I_\Psi(b_n\bar{x}_n\chi_A)$, $(A\in Q)$, is ν_M-absolutely continuous. Therefore taking $a_n(t)=$ $=\Psi(b_n\bar{x}_n(t),t)$ and $\alpha_n=n$ in Lemma 1.3, we find a family of pairwise disjoint sets (A_k) and a sequence (n_k) of integers such that

$$I_\Psi(b_{n_k}\bar{x}_{n_k}\chi_{A_k})=n_k. \tag{1.9}$$

Now, we define for $t\in T$

$$x(t)=\sum_{k=1}^{\infty}\bar{x}_{n_k}(t)\,\chi_{A_k}(t).$$

Since (1.8) is satisfied, so

$$\Phi(\bar{x}_n(t),t)\leqslant\left(\frac{1}{2^n}\right)\Psi(b_n\bar{x}_n(t),t)$$

for all $n\in N$ and a.e. $t\in T$.

Thus

$$I_\Phi(x)\leqslant\sum_{k=1}^{\infty}I_\Phi(\bar{x}_{n_k}\chi_{A_k})\leqslant\sum_{k=1}^{\infty}(\tfrac{1}{2})^{n_k}I_\Psi(b_{n_k}\bar{x}_{n_k}\chi_{A_k})=\sum_{k=1}^{\infty}n_k/2^{n_k}<\infty,$$

by (1.9). So $x\in\operatorname{dom} I_\Phi$.

Let now $b\in B$ be arbitrary. We find $k'\in N$ such that $b_{n_k}\leqslant b$ for all $k\geqslant k'$. Hence and by (1.9) we get

$$I_\Psi(bx)\geqslant\sup_{\mu\in M}\sum_{k=k'}^{\infty}\int_{A_k}\Psi(b_{n_k}\bar{x}_{n_k}(t),t)\,d\mu\geqslant I_\Psi(b_{n_k}\bar{x}_{n_k}\chi_{A_k})=n_k$$

for each $k\geqslant k'$. Therefore $I_\Psi(bx)=\infty$ and so $x\notin\bigcup_{b\in B}(\Psi)_b$. This contradiction finishes the proof of our theorem.

COROLLARY 1.5. Let Φ, Ψ be $\mathcal{N}'$-functions and let the following conditions be satisfied:

(i) The relation $\Psi<\Phi$ is fulfilled, i.e. there exist a constant c and a nonnegative function $h: T\to R$ such that $\sup_{\mu\in M}\int_T h(t)\,d\mu<\infty$ and

$$\Psi(cx, t)\leqslant\Phi(x, t)+h(t)$$

for all $x\in X$ and a.e. $t\in T$.

(ii) There is a number $d>0$ such that

$$\|x\|_\Psi\leqslant d\|x\|_\Phi$$

for all $x\in S(X)$.

(iii) $L_\Phi\subset L_\Psi$.

There holds (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

If we aditionally assume ν_M is a nonatomic submeasure, X is separable, $\Phi(\cdot, t)$ is continuous at zero and dom $\Psi(\cdot, t)=X$ for a.e. $t\in T$, then (iii)$\Rightarrow$(i).

PROOF. (i)$\Rightarrow$(ii). Suppose $k=\sup_{\mu\in M}\int_T h(t)\,d\mu\geqslant 1$. Then for $\varepsilon>0$ and $x\in S(X)$ we get

$$I_\Psi((c/k)(x/\varepsilon))\leqslant(1/k)\,I_\Phi(x/\varepsilon)+1\,.$$

If $I_\Psi(x/\varepsilon)\leqslant 1$ then $I_\Psi((c/k+1))\,(x/\varepsilon))\leqslant 1$. Hence

$$\|(c/k+1))\,x\|_\Psi\leqslant\|x\|_\Phi\,.$$

Now, putting $d=(k+1)/c$ we get the condition (ii). The implication (ii)$\Rightarrow$(iii) is evident. To prove (iii)$\Rightarrow$(i) let us notice dom $I_\Phi\subset L_\Psi$ iff $L_\Phi\subset L_\Psi$ and if $B=(0, \infty)$ then $L_\Psi=\bigcup_{b\in B}(\Psi)_b$. So, if we put $B=(0, \infty)$ in Theorem 1.4 then we get the inequality (1.6) with some constants b, k and a function f. Hence and by convexity of Ψ we get (i) immediately, taking $c=b\min(1/k, 1)$ and $h=f\min(1/k, 1)$.

Putting $B=\{1\}$ in Theorem 1.4, we get immediately the next corollary.

COROLLARY 1.6. Let Φ, Ψ be $\mathcal{N}'$-functions. Let following conditions be satisfied:

(i) There are a constant k and a nonnegative function $h: T\to R$ such that $\sup_{\mu\in M}\int_T h(t)\,d\mu<\infty$ and

for all $x\in X$, a.e. $t\in T$.

$$\Psi(x, t)\leqslant k\Phi(x, t)+h(t)$$

(ii) dom $I_\Phi\subset$dom I_Ψ.

Then (i)$\Rightarrow$(ii) holds. Moreover, if we suppose that the submeasure ν_M is nonatomic, X is separable, $\Phi(\cdot, t)$ is continuous at zero and dom $\Psi(\cdot, t)=X$ a.e. in T, then the conditions (i) and (ii) are equivalent.

LEMMA 1.7. Let Φ, Ψ be $\mathcal{N}'$-functions, X be separable and $\nu_M(T_j)<\infty$. If dom $I_\Phi\subset$dom I_Ψ then there is a family (w_i) of measurable functions satisfying the conditions (i) - (iii) of Lemma 1.1 under $b=1$.

PROOF. Let us take a sequence (y_m) of measurable functions such that $y_m(t)\in$ $\in$dom $\Phi(\cdot, t)$ for each $m\in N$ and $cl(y_m(t))=$dom$\Phi(\cdot, t)$ for all $t\in T\setminus A$, where

$\nu_M(A)=0$. First, let

$$D_{m,j,k}=\{t\in T_j:\ \Phi(y_m(t),t)\leqslant k\}.$$

Since $z_i=y_m\chi_{D_{m,j,k}}\in\operatorname{dom} I_\Phi$, so $z_i\in\operatorname{dom} I_\Psi$. Hence there is a set B such that $\nu_M(B)=0$ and $\Psi(z_i(t),t)<\infty$ for all $t\in T\setminus B$ and every $i\in N$. Now, put

$$C_{m,j,k}=D_{m,j,k}\cap\{t\in T_j:\ \Psi(y_m(t),t)\leqslant k\}.$$

Take the family $(y_m\chi_{C_{m,j,k}}: m,j,k\in N)$ as (w_i). It is enough to show that w_i satisfy the condition (ii). Let then $t_0\in T\setminus(A\cup B)$ and $m\in N$ be arbitrary. Hence $t_0\in T_j$ for some $j\in N$ and one can choose $l\in N$ such that $t_0\in D_{m,j,l}$. Since $y_m\chi_{D_{m,j,l}}$ is some function z_i, so $\Psi(y_m(t_0),t_0)<n$ for some $n\in N$. Next, taking $k=\max(l,n)$ we have $t_0\in C_{m,j,k}$. So, we found a function $w_i=y_m\chi_{C_{m,j,k}}$ such that $x_i(t_0)=y_m(t_0)$. Hence $cl(x_i(t))=\operatorname{dom}\Phi(\cdot,t)$ for a.e. $t\in T$.

REMARK 1.8. The above lemma allows us to prove the implication (ii)⇒(i) of Corollary 1.6, omitting the assumption $\operatorname{dom}\Psi(\cdot,t)=X$.

Indeed, replacing in the proof of Theorem 1.4 the sequence (x_i) from Lemma 1.1 by the family (w_i) from Lemma 1.7 and taking $B=\{1\}$ we obtain this implication easily.

We say that Φ satisfies the Δ_2-condition [3], if there are a constant k and a non-negative function $h: T\to R$ such that $\sup_{\mu\in M}\int_T h(t)\,d\mu<\infty$ and $\Phi(2x,t)\leqslant k\Phi(x,t)+h(t)$ for all $x\in X$ and a.e. $t\in T$.

Taking $\Psi(x,t)$ as $\Phi(2x,t)$ in Corollary 1.6 and Remark 1.8 we obtain immediately

COROLLARY 1.9. Assume X be separable, ν_M be a nonatomic submeasure and Φ be $\mathcal{N}'$-function continuous at zero. Then the Δ_2-condition is necessary and sufficient for linearity of the Orlicz class $\operatorname{dom} I_\Phi$, that is for equality $\operatorname{dom} I_\Phi=L_\Phi$.

2. Let now the submeasure ν_M be purely atomic, with countably many atoms of finite ν_M-submeasure. As easily seen, we may simply assume that $T=N$, $Q=2^N$ and each singleton $\{n\}$ is an atom of ν_M. By assumption, $0<p_n=\nu_M\{n\}<\infty$ for all $n\in N$. If $\mu\in M$, we write $\mu(n)$ instead of $\mu(\{n\})$. We shall identify every function $\Phi: X\times N\to[0,\infty]$ with the sequence $(\Phi_n)=\{\Phi_n\}_{n\in N}$, where $\Phi_n: X\to[0,\infty]$ is defined by $\Phi_n(x)=\Phi(x,n)$. If $x: N\to X$ then $x=(x_n)$ and

$$I_\Phi(x)=\sup_{\mu\in M}\sum_{n=1}^{\infty}\Phi_n(x_n)\,\mu(n).$$

If Φ is an $\mathcal{N}'$-function then

$$(\Phi)_\alpha^0=\{x=(x_n):\ \sup_{\mu\in M}\sum_{n=k}^{\infty}\Phi_n(\alpha x_n)\,\mu(n)<\infty\ \text{for some } k\in N\}.$$

Now, let Φ and Ψ be $\mathcal{N}'$-functions. Denote

$$E_n(\alpha,\beta,\gamma,\delta)=\{x\in X:\ \Phi_n(\alpha x)\leqslant\min(\delta/p_n,\ \Psi_n(\beta x)/\gamma)\},\tag{2.1}$$

$$F_n(\alpha,\beta,\gamma,\delta)=\sup\{\Psi_n(\beta x):\ x\in E_n(\alpha,\beta,\gamma,\delta)\},\tag{2.2}$$

for arbitrary $\alpha,\beta,\gamma,\delta>0$.

From (2.1) and (2.2) we get immediately

$$\Psi_n(\beta x) \leqslant \gamma \Phi_n(\alpha x) + F_n(\alpha, \beta, \gamma, \delta) \tag{2.3}$$

if $\Phi_n(\alpha x) p_n \leqslant \delta$.

We say that the sequence (x_l), where $x_l : N \to X$ for all $l \in N$, is I_Φ-modular convergent to $x : N \to X$ if $I_\Phi(\lambda(x - x_l)) \to 0$ for some $\lambda \in R$, as $l \to \infty$.

THEOREM 2.1. I. [7] Let A and B be nonempty subsets of R. If $\bigcap_{\alpha \in A} (\Phi)_\alpha^0 \subset \bigcup_{\beta \in B} (\Psi)_\beta^0$ then there exist numbers $\alpha \in A$, $\beta \in B$, γ and δ such that $\sup_{\mu \in M} \sum_{n=m}^{\infty} F_n(\alpha, \beta, \gamma, \delta) \mu(n) < \infty$ for some $m \in N$.

II. Let I_Φ-modular convergence imply I_Ψ-modular convergence. Then there are constants γ, δ such that

$$\sup_{\mu \in M} \sum_{n=m}^{\infty} F_n(1, 1, \gamma, \delta) \mu(n) < \infty$$

for some $m \in N$.

PROOF. I. Let the thesis be not fulfilled. We choose sequences $(\alpha_k) \subset A$, $(\beta_k) \subset B$ such that

$$\forall_{\alpha \in A} \; \exists_{k'} \; \forall_{k \geqslant k'} \; \alpha_k \geqslant \alpha \quad \text{and} \quad \forall_{\beta \in B} \; \exists_{k''} \; \forall_{k \geqslant k''} \; \beta_k \leqslant \beta .$$

Denoting $b_{nk} = F_n(\alpha_k, \beta_k, 2^k, k/2^k)$ we have

$$\sup_{\mu \in M} \sum_{n=m}^{\infty} b_{nk} \mu(n) = \infty$$

for all $k, m \in N$. Using this fact, we can find a sequence (N_k) of subsets of N with the following properties: $N_1 = \{1, 2, \ldots, n_1\}, \ldots, N_k = \{n_{k-1} + 1, \ldots, n_k\} \ldots$ and

$$\sup_{\mu \in M} \sum_{n \in N_k} b_{nk} \mu(n) > k, \tag{2.4}$$

$$\sup_{\mu \in M} \sum_{n \in N_k \setminus \{n_k\}} b_{nk} \mu(n) \leqslant k, \tag{2.5}$$

where $\sup_{\mu \in M} \sum_{n \in \varnothing} b_{nk} \mu(n) = 0$, by definition.

By (2.2) and (2.4), we will choose $\bar{x}_n \in E_n(\alpha_k, \beta_k, 2^k, k/2^k)$ such that

$$\sup_{\mu \in M} \sum_{n \in N_k} \Psi_n(\beta_k \bar{x}_n) \mu(n) > k . \tag{2.6}$$

Moreover, we have

$$\Phi_n(\alpha_k \bar{x}_n) \leqslant \Psi_n(\beta_k \bar{x}_n)/2^k \quad \text{and} \quad \Phi_n(\alpha_k \bar{x}_n) \leqslant k/(2^k p_n) .$$

Hence we obtain

$$\sup_{\mu \in M} \sum_{n \in N_k} \Phi_n(\alpha_k \bar{x}_n) \mu(n) \leqslant \sup_{\mu \in M} \sum_{n \in N_k \setminus \{n_k\}} (\Psi_n(\beta_k \bar{x}_n)/2^k) \mu(n) + k/2^k \leqslant$$

$$\leqslant \sup_{\mu \in M} \sum_{n \in N_k \setminus \{n_k\}} (b_{nk}/2^k) \mu(n) + k/2^k \leqslant k/2^{k-1} . \tag{2.7}$$

We define $\bar{x}$ as $\bar{x}(n)=\bar{x}_n$ for each $n \in N$. Let $\alpha \in A$ be an arbitrary number. There exists $k' \in N$ such that $\alpha_k \geqslant \alpha$ for all $k \geqslant k'$. Then we have

$$\sup_{\mu \in M} \sum_{k=k'}^{\infty} \sum_{n \in N_k} \Phi_n(\alpha \bar{x}_n)\,\mu(n) \leqslant \sum_{k=k'}^{\infty} \sup_{\mu \in M} \sum_{n \in N_k} \Phi_n(\alpha_k \bar{x}_n)\,\mu(n) \leqslant \sum_{k=k'}^{\infty} k/2^{k-1} < \infty ,$$

by (2.7). Hence $\bar{x} \in \bigcup_{\alpha \in A} (\Phi)^0_\alpha$.

Let now $\beta \in B$, $m \in N$. We will choose $k'' \in N$ such that $\beta \geqslant \beta_k$ and $n_{k-1}+1 \geqslant m$ for each $k \geqslant k''$. Then

$$\sup_{\mu \in M} \sum_{n=m}^{\infty} \Psi_n(\beta \bar{x}_n)\,\mu(n) \geqslant \sup_{\mu \in M} \sum_{n \in N_k} \Psi_n(\beta_k \bar{x}_n) > k$$

for all $k \geqslant k'$, by (2.6). Thus $\bar{x} \notin \bigcup_{\beta \in B} (\Psi)^0_\beta$. This contradiction finishes the proof of I.

II. Contrary, assume that

$$\sup_{\mu \in M} \sum_{n=m}^{\infty} b_{nk}\,\mu(n) = \infty$$

for all $m, k \in N$, where $b_{nk}=F_n(1, 1, 2^k, k/2^k)$.

Analogically as in part I we find a sequence (N_k) of subsets of N and a sequence $(\bar{x}_n)$ of elements of X such that

$$\sup_{\mu \in M} \sum_{n \in N_k} \Psi_n(\bar{x}_n)\,\mu(n) > k , \tag{2.8}$$

$$\sup_{\mu \in M} \sum_{n \in N_k} \Phi_n(\bar{x}_n)\,\mu(n) \leqslant k/2^{k-1} \tag{2.9}$$

holds for all $k \in N$. Put

$$\bar{x}_{ln} = \begin{cases} 0 & \text{for} \quad n < l \\ \bar{x}_n & \text{for} \quad n \geqslant l , \end{cases}$$

where $l, n \in N$. Let us denote $x_l=(\bar{x}_{ln})_{n \in N}$. Choosing for every $l \in N$ an integer $m(l)$ such that $l \in N_{m(l)}$ and using (2.9) we get

$$I_\Phi(x_l) = \sup_{\mu \in M} \sum_{n=l}^{\infty} \Phi_n(\bar{x}_n)\,\mu(n) \leqslant \sum_{k=n_{m(l)}}^{\infty} \sup_{\mu \in M} \sum_{n \in N_k} \Phi_n(\bar{x}_n)\,\mu(n) \leqslant$$

$$\leqslant \sum_{k=n_{m(l)}}^{\infty} k/2^{k-1} \to 0 \text{ as } l \to \infty .$$

Now, by (2.8) we obtain

$$I_\Psi(x_l) = \sup_{\mu \in M} \sum_{k=n_{m(l)}}^{\infty} \sum_{n \in N_k} \Psi_n(\bar{x}_n)\,\mu(n) \geqslant \sup_{\mu \in M} \sum_{n \in N_k} \Psi_n(\bar{x}_n)\,\mu(n) > k$$

for all $k \geqslant n_{m(l)}$. So $I_\Psi(x_l)=\infty$ for every $l \in N$.

Thus we found a sequence (x_l) such that it is I_Φ-modular convergent but it is not I_Ψ-modular convergent, a contradiction.

COROLLARY 2.2. If $(\Phi)_1^0 \subset (\Psi)_1^0$ then there exist a nonnegative sequence (c_n) and constants γ, δ such that $\sup_{\mu \in M} \sum_{n=1}^{\infty} c_n \mu(n) < \infty$ for some $m \in N$ and

$$\Psi_n(x) \leqslant \gamma \Phi_n(x) + c_n$$

holds for every $n \in N$ and $x \in X$ satisfying $\Phi_n(x) p_n \leqslant \delta$.

PROOF. This follows immediately from the inequality (2.3) and the above theorem, taking $A = \{1\}$, $B = \{1\}$ and $c_n = F_n(1, 1, \gamma, \delta)$ for all $n \in N$.

THEOREM 2.3. Let Φ satisfy the condition (c) and let Ψ fulfil the condition (e). If $\operatorname{dom} I_\Phi \subset \operatorname{dom} I_\Psi$ then there exist constants γ, δ and a nonnegative sequence (c_n) such that $\sup_{\mu \in M} \sum_{n=1}^{\infty} c_n \mu(n) < \infty$ and

(Δ) $\Psi_n(x) \leqslant \gamma \Phi_n(x) + c_n$

is satisfied as $n \in N$ and $\Phi_n(x) p_n \leqslant \delta$.

PROOF. The inclusion $\operatorname{dom} I_\Phi \subset \operatorname{dom} I_\Psi$ implies $(\Phi)_1^0 \subset (\Psi)_1^0$. Then, by Corollary 2.2, it is enough to show that $c_n < \infty$ for each $n \in N$. We have

$$c_n = F_n(1, 1, \gamma, \delta) \leqslant \sup\{\Psi_n(x) : \Phi_n(x) p_n \leqslant \delta\}.$$

Since the set $\{x \in X : \Phi_n(x) p_n \leqslant \delta\}$ is bounded and Ψ fulfils the condition (e), so $c_n < \infty$ for all $n \in N$.

THEOREM 2.4. Let Φ and Ψ be $\mathcal{N}'$-functions and let the set $\operatorname{dom} I_\Phi$ include only the elements with the property $\sup_{\mu \in M} \sum_{n=m}^{\infty} \Phi_n(x_n) \mu(n) \to 0$ as $m \to \infty$. Under these assumptions, if the condition (Δ) from the above theorem is satisfied and $\operatorname{dom} \Phi_n(\cdot) \subset \operatorname{dom} \Psi_n(\cdot)$, $n \in N$, then $\operatorname{dom} I_\Phi \subset \operatorname{dom} I_\Psi$.

PROOF. Let $x = (x_n) \in \operatorname{dom} I_\Phi$. Then there exists an integer l such that $\Phi_n(x_n) p_n \leqslant \delta$ for each $n \geqslant l$. Now, by the (Δ) condition, we obtain

$$I_\Psi(x) \leqslant \sup_{\mu \in M} \sum_{n=1}^{l-1} \Psi_n(x_n) \mu(n) + \gamma I_\Phi(x) + \sup_{\mu \in M} \sum_{n=1}^{\infty} c_n \mu(n) < \infty.$$

So $x \in \operatorname{dom} I_\Psi$.

By 2.2 and 2.4, we get immediately

COROLLARY 2.5. Let $M = \{\mu\}$ where $\mu(n) = 1$ for all $n \in N$ and let Φ, Ψ be arbitrary $\mathcal{N}'$-functions without parameter, i.e. $\Phi_n(x) = \varphi(x)$ and $\Psi_n(x) = \psi(x)$ for all $n \in N$, $x \in X$. Then there holds $\operatorname{dom} I_\varphi \subset \operatorname{dom} I_\psi$ iff $\operatorname{dom} \varphi(\cdot) \subset \operatorname{dom} \psi(\cdot)$ and there exist constants γ, δ such that

$$\psi(x) \leqslant \gamma \varphi(x)$$

is fulfilled as $\varphi(x) \leqslant \delta$.

REMARK. Suppose $\sup_n p_n < \infty$ and $\inf_n \mu(n) > 0$ for some $\mu \in M$. Then for any $\mathcal{N}'$-function Φ, we have

$$\sup_{\mu \in M} \sum_{n=m}^{\infty} \Phi_n(x_n) \mu(n) < \infty \text{ iff } \sum_{n=m}^{\infty} \Phi_n(x_n) \mu(n) < \infty.$$

Under these assumptions, if Φ satisfies the condition (c) and Ψ satisfies the condition (e), then the conditions (Δ) and $\operatorname{dom} \Phi_n(\cdot) \subset \operatorname{dom} \Psi_n(\cdot)$ are necessary and sufficient for the inclusion $\operatorname{dom} I_\Phi \subset \operatorname{dom} I_\Psi$.

We say that $\Psi \mathring{<} \Phi$ if there exist constants β, δ and a nonnegative sequence (c_n) such that $\sup_{\mu \in M} \sum_{n=1}^{\infty} c_n \mu(n) < \infty$ and

$$\Psi_n(\beta x) \leqslant \Phi_n(x) + c_n \tag{2.10}$$

holds for all $n \in N$ and $x \in X$ with $\Phi_n(x) p_n \leqslant \delta$.

THEOREM 2.6. Let Φ and Ψ be $\mathcal{N}'$-functions such that Φ satisfies the condition (c). Then the following statements are equivalent:

(i) $\Psi \mathring{<} \Phi$,

(ii) for all $x = (x_n) : N \to X$ and some $k > 0$ we have

$$\|x\|_\Psi \leqslant k \|x\|_\Phi ,$$

(iii) $L_\Phi \subset L_\Psi$.

PROOF. (i)⇒(ii). Suppose $d = \sup_{\mu \in M} \sum_{n=1}^{\infty} c_n \mu(n) > 1$. Let $x = (x_n)$ be an arbitrary element from L_Φ. Then $I_\Phi(\alpha x) \leqslant \delta$ for some $\alpha > 0$. Hence $\Phi_n(\alpha x_n) p_n \leqslant \delta$ for each $n \in N$. Now, using the inequality (2.10), we obtain

$$I_\Phi(\beta \alpha x_n) \leqslant I_\Phi(\alpha x_n) + d .$$

Therefore, we get

$$\|x\|_\Psi \leqslant (2d/\beta) \|x\|_\Phi$$

and putting $k = 2d/\beta$ we have (ii). The implication (ii)⇒(iii) is evident. (iii)⇒(i). First, we show that the inclusion $L_\Phi \subset L_\Psi$ implies $(\Phi)_1^0 \subset \bigcup_{\beta > 0} (\Psi)_\beta^0$. Indeed, let $x = (x_n) \in (\Phi)_1^0$. So $\sup_{\mu \in M} \sum_{n=m}^{\infty} \Phi_n(x_n) \mu(n) < \infty$ for some $m \in N$. If we put

$$\bar{x}_n = \begin{cases} x_n & \text{for} \quad n \geqslant m \\ 0 & \text{for} \quad n < m , \end{cases}$$

then $\bar{x} = (\bar{x}_n) \in L_\Phi$. Therefore $\bar{x} \in L_\Psi$. However, $L_\Psi = \bigcup_{\beta > 0} (\Psi)_\beta \subset \bigcup_{\beta > 0} (\Psi)_\beta^0$, so $\bar{x} \in \bigcup_{\beta > 0} (\Psi)_\beta^0$, too.

Now, by the above and by Theorem 2.1 part I, we get the existence of constants β_1, δ and of a sequence $(d_n) \subset [0, \infty)$ such that $\sup_{\mu \in M} \sum_{n=m}^{\infty} d_n \mu(n) < \infty$ for some $m \in N$ and that

$$\Psi_n(\beta_1 x) \leqslant \Phi_n(x) + d_n \tag{2.11}$$

is satisfied for all $n \in N$ and $x \in X$ with $\Phi_n(x) p_n \leqslant \delta$. Fix $n < m$ and denote

$$C_n = \{x \in X : \Phi_n(x) \, p_n \leqslant \delta\} ,$$

$$D_\beta = \{x \in X : \Psi_n(\beta x) \, p_n \leqslant 1\} .$$

We have $C_n \subset \bigcup_{\beta>0} D_\beta$, beacause $L_\Phi \subset L_\Psi$. Therefore

$$C_n = \bigcup_{\beta>0} (D_\beta \cap C_n). \tag{2.12}$$

Since C_n is a closed subset of X then it can be treated as a complete metric space. So in virtue of (2.12) and Baire theorem we find an open ball $K(x_0, r)$ such that $K(x_0, r) \cap C_n \subset D_{\beta_2} \cap C_n$ for some $\beta_2 > 0$. By convexity and evenness of Φ and Ψ we get

$$K(0,r) \cap C_n \subset D_{\beta_2} \cap C_n.$$

Since Φ satisfies the condition (c), so the set C_n is bounded in X. Thus $\alpha_n C_n \subset \subset K(0, r) \cap C_n$ for some $0 < \alpha_n \leqslant 1$. Hence

$$C_n \subset (1/\alpha_n) D_{\beta_2} = \{x \in X:\ \Psi_n(\beta_2 \alpha_n x) p_n \leqslant 1\}. \tag{2.13}$$

Now, putting

$$\beta = \min_{1 \leqslant n < m} (\beta_1, \beta_2 \alpha_n) \text{ and } c_n = \begin{cases} d_n & \text{for} \quad n \geqslant m \\ 1/p_n & \text{for} \quad n < m, \end{cases}$$

we obtain the inequality (2.10), by (2.11) and (2.13).

REMARK. The implications (i)⇒(ii), (ii)⇒(iii) need not the assumption of the condition (c) about Φ.

We say that the function Φ satisfies the δ_2^0-condition if there exist constants a, k and a sequence $(c_n) \subset [0, \infty)$ such that $\sup_{\mu \in M} \sum_{n=m}^{\infty} c_n \mu(n) < \infty$ for some $m \in N$ and that

$$\Phi_n(2x) \leqslant k\Phi_n(x) + c_n$$

holds for all $n \in N$ and $x \in X$ with $\Phi_n(x) p_n \leqslant a$.

PROPOSITION 2.8. Let Φ be continuous at zero and let it satisfy the conditions (d) and δ_2^0 with such $c = (c_n)$ that $\sup_{\mu \in M} \sum_{n=m}^{\infty} c_n \mu(n) \to 0$ as $m \to \infty$. Under these assumptions the norm and modular convergence are equivalent in L_Φ, i.e. if $x_l = (x_{ln}) \in L_\Phi$ then $\|x_l\|_\Phi \to 0 \Leftrightarrow I_\Phi(kx_l) \to 0$ for some $k \in R$.

PROOF. It is known that $\|x_l\|_\Phi \to 0 \Leftrightarrow I_\Phi(kx_l) \to 0$ for every $k \in R$. So, it is enough to show that $I_\Phi(x_l) \to 0$ implies $I_\Phi(2x_l) \to 0$. Let $I_\Phi(x_l) \to 0$ as $l \to \infty$. Hence we have $\Phi_n(x_{ln})\mu(n) \to 0$ for all $\mu \in M$, $n \in N$, as $l \to \infty$. From this fact and by (d), we get $x_{ln} \to 0$ for all $n \in N$, as $l \to \infty$. Then $\Phi_n(2x_{ln}) \to 0$, by continuity of Φ at zero. Let now $\varepsilon > 0$ be given. First we choose $j \in N$ such that

$$\sup_{\mu \in M} \sum_{n=j}^{\infty} c_n \mu(n) < \varepsilon/3,$$

then we find $N > 0$ such that

$$I_\Phi(x_l) < \varepsilon/3k \quad \text{and} \quad \Phi_n(2x_{ln}) < \varepsilon/3(j-1) p_n$$

for all $l>N$ and $n<j$. Hence and by the δ_2^0-condition, we get

$$I_\Phi(2x_l)\leqslant \sum_{n=1}^{j-1} \Phi_n(2x_{ln})\, p_n + kI_\Phi(x_l) + \sup_{\mu\in M} \sum_{n=j}^{\infty} c_n\mu(n) < \varepsilon$$

for $l>N$. So, the thesis of our proposition is obtained.

The above proposition has a partial converse.

PROPOSITION 2.9. If the modular convergence is equivalent to the norm convergence in L_Φ, then the condition δ_2^0 holds. Moreover, if we additionaly suppose that Φ satisfies the condition (c), then it satisfies the condition (d), too.

PROOF. The condition δ_2^0 follows immediately from the second part of Theorem 2.1 in which we put $\Psi_n(x)=\Phi_n(2x)$.

Now, assume the condition (d) is not satisfied, i.e.

$$\inf_{\|x\|=r} \Phi_n(x)=0$$

for some $r>0$ and $n\in N$. Then we find a sequence $(x_l)\subset X$ such that $\|x_l\|=r$ for every $l\in N$ and $\lim_{l\to\infty} \Phi_n(x_l)=0$. Hence and by the assumption of equivalence of modular and norm convergence, we have $\lim_{l\to\infty} \Phi_n(kx_l)=0$ for every $k\in N$. Thus, an increasing sequence (l_k) of integers can be choosen such that

$$\lim_{k\to\infty} \Phi_n(kx_{l_k})=0 .$$

However, $\|kx_{l_k}\|=kr$, then by the condition (c) it holds

$$\lim_{k\to\infty} \Phi_n(kx_{l_k})=\infty .$$

This contradiction concludes the proof.

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