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MATHEMATICS

ON CONVERGENCE IN MEAN OF FOURIER SERIES

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Let the function $M(u)$ be defined for $0 \leqslant u < +\infty$ and satisfy the following conditions:

1) $M(0)=0$; $M(u)$ is $\geqslant 0$, convex and steadily increasing for $0 \leqslant u < +\infty$.

2) $\lim\limits_{u\to+0} \frac{M(u)}{u}=0$, $\lim\limits_{u\to+\infty} \frac{M(u)}{u}=+\infty$.

3) There exist two constants $a \geqslant 0$ and $K>0$ such that for $u \geqslant a$ we have

$$M(2u) \leqslant KM(u) \tag{1}$$

Then we say (cf. [1]) that the function $M(u)$ belongs to the class Ω and write $M \in \Omega$. It is known [2,3] that every function $M(u)$ belonging to the class Ω can be represented in the form

$$M(u)=\int_0^u \varphi(t)\,dt \quad (0 \leqslant u < +\infty) \tag{2}$$

where the function $\varphi(t)$ satisfies the following conditions:

α) $\varphi(0)=0$, $\varphi(t)>0$ for $0<t+\infty$;

β) $\varphi(t)$ is non-decreasing and continuous from the right for $0 \leqslant t < +\infty$;

γ) $\lim\limits_{t\to+\infty} \varphi(t)=+\infty$.

Besides, if $M \in \Omega$, there exists only one function $\varphi(t)$ satisfying formula (2) and conditions α), β), γ). If $M \in \Omega$, we denote by L^M the class of all functions $f(x)$ measurable on $[0, 2\pi]$ and satisfying the condition

$$\int_0^{2\pi} M[|f(x)\ dx < +\infty \tag{3}$$

Theorem. *Let $M \in \Omega$. Then, in order that for every function $f \in L^M$ we should have*

$$\lim_{n\to\infty} \int_0^{2\pi} M(|f-s_n(f)|)\,dx=0 \tag{4}$$

where $s_n(f)$ denote the partial sums of the Fourier series of $f(x)$, it is necessary that condition

$$\underline{\lim}_{u\to+\infty} \frac{M(2u)}{M(u)} > 2 \tag{5}$$

hold. If the function $\varphi(t)$ *from formula* (2) *satisfies, besides* α), β), γ), *the condition that*

δ) $\varphi'(t)$ *exist and be non-increasing for* $0<t<+\infty$ *and* $\lim_{t\to+\infty}\varphi'(t)=0$,

then the fulfilment of condition (5) *is also sufficient in order that for every function* $f\in L^M$ *we should have* (4).

Proof. We need the following lemma.

Lemma 1. If $M\in\Omega$ and

$$\varliminf_{u\to+\infty}\frac{M(2u)}{M(u)}=2 \tag{6}$$

then for every λ, $1\leqslant\lambda<+\infty$, we have

$$\varliminf_{u\to+\infty}\frac{\varphi(\lambda u)}{\varphi(u)}=1 \tag{7}$$

We omit the proof for lack of space.

It is known [2] that in L^M a norm can be defined, which makes L^M a space of type (B). Let us assume that this norm is introduced as in [2], and denote the norm of $f\in L^M$ by $\|f\|_M$. Let $D_n(t)=\dfrac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}}$ be the Dirichlet kernel. It is proved in [1], pp. 196—200, that for every function $M\in\Omega$ we have

$$\|D_n(t)\|_M\geqslant\frac{1}{250}\max_{z\varphi(z)\geqslant1}\varphi(z)\log\frac{n}{z\varphi(z)}\quad(n=1,2,\dots) \tag{8}$$

Let us now suppose that the function $M\in\Omega$ is such that for every $f\in L^M$ the relation (4) holds. Then ([1], p. 197, Lemma 5) there exists a constant $A>0$ such that

$$\|D_n(t)\|_M\leqslant A\min_{0<\zeta<\infty}\frac{n+M(\zeta)}{\zeta}\quad(n=1,2,\dots) \tag{9}$$

We define for $n=1,2,\dots$ the number ζ_n by the relation

$$M(\zeta_n)=n \tag{10}$$

Then, by (9)

$$\|D_n(t)\|_M\leqslant 2A\frac{n}{\zeta_n}\quad(n=1,2,\dots) \tag{11}$$

Suppose now that (5) does not hold. Then we have (6). For every λ, $1\leqslant\lambda<+\infty$, we can define a sequence $\{u_k\}_{k=1}^{\infty}$ of positive numbers, depending on λ, such that $\lim_{k\to\infty}u_k=+\infty$

$$\lim_{k\to\infty}\frac{\varphi(\lambda u_k)}{\varphi(u_k)}=1 \tag{12}$$

and

$$\frac{1}{4}u_k\varphi(u_k)\leqslant M(u_k)\leqslant u_k\varphi(u_k)\quad(k=1,2,\dots) \tag{13}$$

For, if λ, $1\leqslant\lambda<+\infty$, is fixed, there exists, by Lemma 1, a sequence $\{u_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}u_k=+\infty$, $\varphi(\lambda u_k)\leqslant 2\varphi(u_k/2)$ and

$$\lim_{k\to\infty}\frac{\varphi(\lambda u_k)}{\varphi\left(\frac{u_k}{2}\right)}=1 \tag{14}$$

Clearly

$$M(u_k)\geqslant M(u_l)-M\left(\frac{u_k}{2}\right)\geqslant\frac{u_k}{2}\varphi\left(\frac{u_k}{2}\right)\geqslant\frac{1}{4}u_k\varphi(\lambda u_k)\geqslant\frac{1}{4}u_k\varphi(u_k) \quad (15)$$

so that $\{u_k\}_{k=1}^{\infty}$ satisfies the first of the inequalities (13).

The second one of the inequalities (13) follows from the inequality $M(u)\leqslant u\varphi(u)$ which is true for every u, $0\leqslant u<+\infty$, and (12) follows from (14). We now put

$$B=16e^{1000A}, \quad \lambda=2B \quad (16)$$

where A is the constant from (11). Let us further so define a sequence $\{u_k\}_{k=1}^{\infty}$ that $1\leqslant u_1<u_2<\ldots\lim_{k\to\infty} u_k=+\infty$ and (12) and (13) be satisfied. Let n_k $(k=1, 2,\ldots)$ be the least positive integer such that $\zeta_{n_k}\geqslant Bu_k$. We can evidently suppose that u_1 is so large that $u_1\varphi(u_1)\geqslant 1$ and $\varphi(\zeta_{n_1-1})\geqslant 1$. Then, for $k=1, 2,\ldots$ we obtain

$$\zeta_{n_k}-\zeta_{n_1-1}\leqslant(\zeta_{n_k}-\zeta_{n_k-1})\varphi(\zeta_{n_k-1})\leqslant\int_{\zeta_{n_k-1}}^{\zeta_{n_k}}\varphi(t)\,dt=$$
$$=M(\zeta_{n_k})-M(\zeta_{n_k-1})=n_k-(n_k-1)=1$$

and hence $\zeta_{n_k}\leqslant\zeta_{n_k-1}+1\leqslant Bu_k+1\leqslant 2Bu_k$. We obtain the inequality

$$Bu_k\leqslant\zeta_{n_k}\leqslant 2Bu_k=\lambda u_k \quad (k=1, 2,\ldots) \quad (17)$$

Now we have $\lambda>2$, so that by (12) for $k\geqslant k_0$ the inequality $\varphi(2u_k)/\varphi(u_k)\leqslant 2$ holds. It follows that for $k\geqslant k_0$

$$\frac{\zeta_{n_k}}{B}\varphi\left(\frac{\zeta_{n_k}}{B}\right)\leqslant 2u_k\varphi(2u_k)=2u_k\varphi(u_k)\frac{\varphi(2u_k)}{\varphi(u_k)}\leqslant 4u_k\varphi(u_k)\leqslant$$
$$\leqslant 16M(u_k)\leqslant 16M\left(\frac{\zeta_{n_k}}{B}\right)\leqslant\frac{16}{B}M(\zeta_{n_k})=\frac{16}{B}n_k \quad (18)$$

Clearly

$$\frac{\zeta_{n_k}}{B}\varphi\left(\frac{\zeta_{n_k}}{B}\right)\geqslant u_k\varphi(u_k)\geqslant u_1\varphi(u_1)\geqslant 1 \quad (k=1, 2,\ldots) \quad (19)$$

By (8), (18) and (19) we obtain for $k\geqslant k_0$

$$\|D_{n_k}(t)\|_M\geqslant\frac{1}{250}\max_{z\varphi(z)\geqslant 1}\varphi(z)\log\frac{n_k}{z\varphi(z)}\geqslant$$
$$\geqslant\frac{1}{250}\varphi\left(\frac{\zeta_{n_k}}{B}\right)\log\frac{n_k}{\frac{\zeta_{n_k}}{B}\varphi\left(\frac{\zeta_{n_k}}{B}\right)}\geqslant\frac{1}{250}\varphi\left(\frac{\zeta_{n_k}}{B}\right)\log\frac{n_k}{\frac{16}{B}n_k}\geqslant$$
$$\geqslant\frac{1}{250}\varphi(u_k)\log\frac{B}{16}=4A\varphi(u_k) \quad (20)$$

In virtue of (11) and (17)

$$\|D_{n_k}(t)\|_M\leqslant 2A\frac{M(\zeta_{n_k})}{\zeta_{n_k}}\leqslant 2A\frac{\zeta_{n_k}\varphi(\zeta_{n_k})}{\zeta_{n_k}}\leqslant 2A\varphi(\lambda u_k) \quad (21)$$

Combining (20) and (21) we obtain

$$4A\varphi(u_k)\leqslant 2A\varphi(\lambda u_k) \quad (k\geqslant k_0)$$

and hence

$$\lim_{k\to\infty}\frac{\varphi(\lambda u_k)}{\varphi(u_k)}\geqslant 2$$

which contradicts (12). The necessity of condition (5) is thus proved.

In order to prove the sufficiency of the conditions of the theorem, we shall need the

Lemma 2. If $H \in \Omega$ and (5) holds, then

$$0 < \underline{\lim}_{u \to +\infty} \frac{M(u)}{u \int_1^u \frac{\varphi(t)}{t}\, dt} \leqslant \overline{\lim}_{u \to +\infty} \frac{M(u)}{u \int_1^u \frac{\varphi(t)}{t}\, dt} < +\infty \tag{22}$$

The proof of the lemma is omitted.

Now, from ((1), p. 192, Theorem 9) it follows easily that, if $M \in \Omega$, φ satisfies the condition δ) and (22) holds, then for every $f \in L^M$ we have (4). Our theorem is proved.

Remark. Let $M \in \Omega$ and let $N(v)$ be the function conjugate to $M(u)$ in the sense of W. H. Young (see (2) or (1), pp. 186—191). It can be proved that if $M(u)$ satisfies (5) then $N \in \Omega$, and if, furthermore, φ satisfies δ), then for every $f \in L^N$ we have

$$\lim_{n \to \infty} \int_0^{2\pi} N(|f - s_n(f)|)\, dx = 0.$$

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