REARRANGEMENT-INVARIANT BANACH FUNCTION SPACES

by

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1. Introduction

The present paper is largely concerned with the theory of normed spaces of measurable functions which are rearrangement-invariant. Special examples of r.i. spaces are the classical L^p-spaces, the Orlicz spaces and the spaces which were recently introduced by Halperin [5] and Lorentz [9]. The importance of such spaces in analysis was recently shown by Boyd [2] and Shimogaki [21]. In that they showed that r.i. spaces are well suited for studying problems in Fourier analysis and related subjects.

We have restricted the discussion to the case where the underlying measure space is a totally finite measure space largely for the reason that for such spaces the decreasing

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rearrangement exists for every measurable function. Thus the first part of the paper is devoted to a discussion of the properties of the decreasing rearrangement of an arbitrary measurable function. In these sections the reader may find several improvements of older results as are presented in [7].

Several new results for rearrangement-invariant spaces are given. In particular, a general representation theorem for such spaces is given. Furthermore, it is shown that Banach function spaces which are rearrangement invariant can be renormed with a norm which is rearrangement invariant.

Some aspects of the theory of doubly stochastic transformations are discussed. It is shown that such transformations are contractions on spaces which possess a rearrangement-invariant norm. This result, in particular, implies that rearrangement invariant norms have the levelling length property in the sense of Ellis and Halperin [6].

A separate section is devoted to show that the classical inequality of Hardy, Littlewood and Pólya for convex functions has a natural extension to rearrangement-invariant spaces, and a new proof for this result is given.

In the last three sections of the paper the properties of certain compact convex sets are discussed. In particular, a complete answer is given to the question: Which is the smallest closed convex subset of L^1 determined by the characteristic functions of measurable sets of equal measure? A partial answer to this question was recently given by Nehari in [16]. A number of problems concerning those compact convex sets are given in the final section.

Most of the material collected in this paper was obtained by the author during the last three years and was presented at various occasions in courses on integration theory at the California Institute of Technology and lectures at other institutions. The author is most grateful for the Organizing Committee of the Symposium to have the author presented with the opportunity to publish part of this material on this occasion.

For the sake of convenience of the reader we include the following list of the section headings.

2. Spectral measures.

Let (X, Λ, μ) be a finite measure space, i.e., X is a non-empty point set, Λ is a σ-algebra of subsets of X, and μ is a countably additive non-negative measure defined on Λ such that $\mu(X) < \infty$. The notation $\int d\mu$ will always denote integration (with respect to μ) over the whole set, and c_E will denote the characteristic function of the set $E \subset X$. By $M = M(X,\mu)$ we shall denote the set of all extended real-valued μ-measurable functions defined on X.

Let R denote the real number system and let $S = S(R)$ be the Riesz space of all real step functions on R, where a step function on R is a function on R which takes on only a finite number of different values and each non-zero value is taken on on an arbitrary interval of finite length. Then for every $f \in M(X,\mu)$ and $s \in S(R)$ we have that $s(f) \in M$, and, in fact, $s(f)$ is a simple measurable function. This simple observation justifies the following notation,

For every $f \in M$, the mapping $s \to \int s(f)d\mu$ of S into R will be denoted by I_f.

It is easy to see that I_f is a positive linear functional on S. Furthermore I_f is <u>continuous</u> in the following sense.

If $s_n(t) \downarrow 0$ for all $t \in R$ (the sequence s_n of step functions decreases to zero everywhere), then $I_f(s_n) \downarrow 0$, i.e., I_f is an <u>integral</u>.

By μ_f we shall denote the Radon measure on R which is determined by I_f. Then the following formula holds

For all $f \in M$ and for all $s \in S$ we have

$$\int s(f)\,d\mu = \int_R s\,d\mu_f \ . \tag{2.1}$$

The measure μ_f will be called the <u>spectral measure of</u> f with respect to μ or shortly the μ-spectral measure of f. It can be represented by the right-continuous increasing function $e_f(t) = \mu(\{x: f(x) \leq t\})$, $t \in R$. In addition, we shall also introduce the <u>distribution function</u> $d_f(t) = \mu(\{x: f(x) > t\})$, $t \in R$, of f which is a right-continuous decreasing function. Then $e_f(t) + d_f(t) = \mu(X)$ for all $t \in R$. It is easy to see that $d_f(t) \to \mu(\{x: f(x) = + \infty\})$ as $t \to + \infty$ and $e_f(t) \to \mu(\{x: f(x) = - \infty\})$ as $t \to - \infty$. The functions e_f and d_f are discontinuous at $t \in R$ if and only if $\mu(\{x: f(x) = t\}) > 0$. Furthermore, the mapping $f \to e_f$ is decreasing and the mapping $f \to d_f$ is increasing, and if $f_n \uparrow f$ everywhere, then $e_{f_n} \downarrow e_f$ and $d_{f_n} \uparrow d_f$.

3. Spectral equivalency.

Let (X, Λ, μ) and (X', Λ', μ') be two finite measure spaces. A measurable function $f \in M(X,\mu)$ and a measurable function $f' \in M'(X',\mu')$ are called spectrally equivalent or equimeasurable whenever $\mu_f = \mu'_{f'}$, and in that case we shall write $f \sim f'$.

It is easy to see that the binary relation $\sim$ is an equivalence relation between the classes of the almost everywhere equal measurable functions.

In the following lemma we shall collect some simple results concerning this equivalence relation for future reference. The proof is omitted.

(3.1) LEMMA. i) $f \sim g$ *if and only if* $f^+ \sim g^+$ *and* $f^- \sim g^-$.

ii) *If* $\inf(|f_1|,|f_2|) = 0$ *and* $\inf(|g_1|,|g_2|) = 0$ *and* $f_1 \sim g_1$ *and* $f_2 \sim g_2$, *then* $f_1 + f_2 \sim g_1 + g_2$. *In particular, if* $f \sim g$, *then* $|f| \sim |g|$.

iii) *If* $f \sim af$, *for some* $0 \leq a \neq 1$, *then* $f = 0$ *almost everywhere*.

iv) *If* $f \sim g$ *and if* $s \in S(R)$, *then* $s(f) \sim s(g)$

v) *If* $f_n \uparrow f$ *everywhere and* $g_n \uparrow g$ *everywhere and* $f_n \sim g_n$ *for all* n, *then* $f \sim g$.

vi) *If* T *is a measure preserving transformation of* (X, Λ, μ), *then* $Tf \sim f$ *for all* $f \in M(X,\mu)$.

vii) *If* $f \in L^1(X,\mu)$ *and* $f' \in M(X',\mu')$, *then* $f \sim f'$ *implies that* $f' \in L^1(X',\mu')$ *and* $\int f d\mu = \int f' d\mu'$, *where* (X,Λ,μ) *and* (X',Λ',μ') *are finite measure spaces.*

If E is a μ-measurable subset of X and f is a measurable function of some finite measure space such that $f \approx c_E$, then f is almost everywhere equal to the characteristic function of a measurable set whose measure is equal to the measure of E. If we apply this to simple functions which are measurable functions which take on only a finite number of different values we obtain the following simple result.

(3.2) LEMMA. *Two simple functions are spectrally equivalent if and only if they take on the same value on sets of equal positive measure.*

4. *Decreasing rearrangements of measurable functions.*

If the values of a measurable function f are rearranged in such a way that the measures of the sets of the spectral family $\{x: f(x) > t\}$, $t \in R$, is preserved, then the resulting measurable function is obviously spectrally equivalent with f. It is natural to ask whether it is possible to rearrange the values of a function in decreasing order in an equimeasurable fashion? The purpose of this section is to discuss the well-known affirmative answer to this question for measurable functions on finite measure spaces (see [7]).

We shall first recall the definition of the right and left inverses of a monotone function.

Let $p(t)$, $t \in R$, be a decreasing function. Then the <u>right-continuous inverse</u> q of p is defined as follows:

$$(4.1) \qquad q(u) = \inf(t : p(t) \le u) \ , \ u \in R \ ,$$

where we set $q(u) = +\infty$ whenever $p(t) > u$ for all $t \in R$ and $q(u) = -\infty$ whenever $p(t) \le u$ for all $t \in R$. Then q is right-continuous and decreasing and it satisfies the following relations

$$(4.2) \qquad q(p(t)) \ge t \quad \text{for all} \quad t \in R \quad \text{and} \quad q(p(t)-\varepsilon) \ge t$$

for all $t \in R$ and all $0 < \varepsilon \in R$. Observe also that $q(u) = \sup(t : p(t) \ge u) = \sup(t : p(t-) \ge u)$, and $q(u-) = \inf(t : p(t) < u)$. Furthermore, $q(p(t)-) \ge t$ and $q((p(t)+\varepsilon)-) \le t$ for all $t \in R$. The right inverse of q is equal to p.

The following simple observation is fundamental for what follows. If p is a decreasing function defined on a finite interval $0 \le t \le a$, then

$$(4.3) \qquad q(u) = \inf(t : p(t) \le u) = m(\{t : p(t) > u\}) \ ,$$

where m denotes the Lebesgue measure.

Thus we have the following lemma.

(4.4) LEMMA. *If* p *is a decreasing function defined on the interval* $0 \leq t \leq a$, *then its m-spectral measure* m_p *of* p *is determined by its right-inverse* q.

We are now in a position to answer the following question. For a given measurable function $f \in M(X,\mu)$ of a finite measure space does there exist a right-continuous decreasing function on the Lebesgue measure space $\{t: 0 \leq t \leq \mu(X)\}$ which is spectrally equivalent with f? The answer is contained in Lemma 4.4. Indeed, this lemma states that if such a decreasing function exists, then its right-inverse must be equal to the distribution function d_f of f, and so the right-inverse of d_f satisfies the required conditions. This justifies the following definition.

(4.5) DEFINITION. *Let* $f \in M(X,\mu)$ *be a measurable function defined on a finite measure space. Then the right-inverse* δ_f *of its distribution function* d_f, *which satisfies* $f \sim \delta_f$, *is called the* μ-*decreasing rearrangement of* f, *or shortly, the decreasing rearrangement of* f.

In symbols

(4.6) $$\delta_f(t) = \inf(u:\mu(\{x:f(x) > u\}) \leq t) = \inf(u:d_f(u) \leq t) \;, \quad 0 \leq t \leq \mu(X) \;.$$

It is easy to see that f is integrable if and only if δ_f is Lebesgue integrable over the interval $0 \leq t \leq a$, where $a = \mu(X)$, and in that case $\int f d\mu = \int_0^a \delta_f(t)dt$. Furthermore, $f \sim g$ if and only if $\delta_f = \delta_g$.

In the following lemma we shall collect some simple properties of δ_f. The proof of the lemma is left to the reader.

(4.7) LEMMA. i) $f_1 \leq f_2 \Rightarrow \delta_{f_1} \leq \delta_{f_2}$

ii) *If* $a \geq 0$, *then* $\delta_{af} = a\delta_f$.

iii) *For any right-continuous increasing function* p *on* R *we have* $\delta_{p(f)} = p(\delta_f)$

iv) *For any* $s \in S(R)$ *we have* $s(f) \sim s(\delta_f)$.

v) *If* $f_n \to 0$ *in measure, then* $\delta_{f_n} \to 0$ *uniformly on every closed subinterval of* $[0,\mu(X))$.

We recall that if f is a measurable function, then $\text{ess. sup } f = \inf(k:\mu(\{x:f(x) > k\}) = 0)$. The values of δ_f can then be expressed in terms of the values of f in the following way.

(4.8) THEOREM. *For all* $f \in M(X,\mu)$ *and for all* $0 < t < \mu(X) < \infty$, we have

$\delta_f(t) = \inf(\text{ess.sup}(f-fc_E)\colon \mu(E) \leq t)$, *and*

$\delta_f(t-) = \inf(\text{ess.sup}(f-fc_E)\colon \mu(E) < t)$.

PROOF. If $u \in R$ is such that $d_f(u) \leq t$, then $E_u = \{x: f(x) > u\}$ satisfies $\mu(E_u) \leq t$, and so ess.sup$(f-fc_E) \leq u$ implies that $\inf(\text{ess.sup}(f-fc_E): \mu(E) \leq t) \leq \delta_f(t)$. Conversely, if $\mu(E) \leq t$ and, if we set ess.sup$(f-fc_E) = u$, then $\{x: f(x) > u\} \subset E$, and so $d_f(u) \leq t$. Hence, $\delta_f(t) \leq \inf(\text{ess.sup}(f-fc_E): \mu(E) \leq t)$, and the proof is finished.

The following theorem will be used repeatedly in what follows.

(4.9) THEOREM. _Let_ $f \in M(X,\mu)$, $a = \mu(X) < \infty$. _Then_

i) $\delta_f(t) = \delta_{f+}(t) - \delta_{f^-}(a-t)$ _for all_ $0 \leq t \leq a$;

and $\inf(\delta_{f+}(t), \delta_{f^-}(a-t)) = 0$ _for all_ $0 \leq t \leq a$.

ii) $(\delta_f)^+ = \delta_{f+}$ _and_ $(\delta_f)^-(t) = \delta_{f^-}(a-t)$, $0 \leq t \leq a$.

iii) $-\delta_{f^-}(t) = \delta_f(a-t)$, $0 \leq t \leq a$.

iv) $|\delta_f|(t) = \delta_{f+}(t) + \delta_{f^-}(a-t)$, _and if_ $|\delta_f| = g$, _then_ $\delta_g = \delta_{|f|}$.

Furthermore, $\frac{1}{2}(\delta_{f+} + \delta_{f^-}) \leq \delta_{|f|}$.

PROOF. We shall only prove i) If $t < 0$, then $d_f(t) = a - d_{f^-}((-t)-)$, and if $t > 0$, then $d_f(t) = d_{f+}(t)$. Then it follows from the definition of δ_f that if $d_f(u) \leq t$ implies $u \geq 0$, then $\delta_f(t) = \delta_{f+}(t)$. On the other hand if $d_f(u) \leq t$ for some $u < 0$, then $\delta_f(t) = \inf(u: a-d_{f^-}((-u)-) \leq t)$ $= \inf(u: -d_{f^-}((-u)-) \leq t-a) = \inf(u: d_{f^-}((-u)-) \geq a-t) =$

$= -\sup(v: d_{f^-}(v-) \geq a-t) = -\delta_{f^-}(a-t)$; and the proof is finished.

Before we shall continue with a discussion of the theory of decreasing rearrangements of measurable functions we shall first present a well-known inequality of Hardy and one of its consequences. Furthermore, Hardy's inequality may also serve as a motivation for what is to follow.

5. An inequality of Hardy.

We shall use the following simple inequality of Hardy repeatedly. The proof of it follows immediately by integration by parts and is left to the reader to verify.

(5.1) THEOREM (*Hardy*). i) *Let* f_1, f_2 *be Lebesgue measurable functions defined on the interval* $a \leq t \leq b$ *and which are integrable over all the intervals* $[a,t)$, $a \leq t < b$. *If* $F_1(t) = \int_a^t f_1(u)du \leq \int_a^t f_2(u)du = F_2(t)$ *for all* $a \leq t < b$, *then for all* $a \leq t < b$ $\int_a^t f_1(u)g(u)du \leq \int_a^t f_2(u)g(u)du$ *for all positive decreasing functions* g *on* $a \leq t \leq b$ *such that* $gf_i \in L^1[a,t]$ $(i = 1,2)$ *for all* $a \leq t < b$.

ii) *If* $f_1, f_2 \in L^1[a,b]$ *and if* $F_1(t) \leq F_2(t)$ *for all*

$a \leq t \leq b$ *and* $F_1(b) = F_2(b)$, *then*
$\int_a^b f_1(u)g(u)du \leq \int_a^b f_2(u)g(u)du$ *for all decreasing functions* g *on* $a \leq t \leq b$ *such that* $gf_i \in L^1[a,b]$ $(i = 1,2)$.

The following theorem is an easy consequence of ii).

(5.2) THEOREM. *If* $f \in L^1(X,\mu)$, $a = \mu(X) < \infty$, *then* $\frac{1}{t}\int_o^t \delta_f(u)du$ *is decreasing in* t *for all* $0 \leq t \leq a$.

PROOF. Let $0 < t_1 < t_2 \leq a$, and let $f_1 = \frac{t_1}{t_2} c_{[0,t_2)}$, and let $f_2 = c_{[0,t_1)}$. Then $F_1(t) \leq F_2(t)$ for all $0 \leq t \leq a$ and $F_1(a) = t_1 = F_2(a)$. Then by ii) of (5.1) we have $\frac{t_1}{t_2}\int_o^a \delta_f(u)\, c_{[0,t_2)}(u)du = \frac{t_1}{t_2}\int_o^{t_2} \delta_f(u)du \leq \int_o^{t_1} \delta_f(u)du$, and so $\frac{1}{t_2}\int_o^{t_2} \delta_f(u)du \leq \frac{1}{t_1}\int_o^{t_1} \delta_f(u)du$ completes the proof.

6. A preorder relation for $L^1(X,\mu)$.

Let (X,Λ,μ) be again a finite measure space. In [6], Hardy, Littlewood and Pólya introduced for the first time a very important preorder relation for n-tuples of real numbers and later for Lebesgue integrable functions. We shall present this preorder relation here by means of the following definition.

(6.1) DEFINITION. *If* $f,g \in L^1(X,\mu)$, *then we shall write* $f \prec g$ *whenever* $\int_o^t \delta_f(u)du \leq \int_o^t \delta_g(u)du$ *for all* $0 \leq t \leq a = \mu(X)$ *and* $\int f d\mu = \int g d\mu$. *If only* $\int_o^t \delta_f(u)du \leq \int_o^t \delta_g(u)du$ *holds for all* $0 \leq t \leq a$, *then we shall write* $f \prec\prec g$.

The reader should compare the present definition with the conditions imposed on f_1, f_2 in Theorem 5.1.

If $f \leq g$, then obviously $f \prec\prec g$. Furthermore, $f \leq g$ and $f \prec g$ implies $f = g$ μ - a.e.

It is easy to see that $f \sim g$ if and only if $f \prec g$ and $g \prec f$ if and only if $f \prec\prec g$ and $g \prec\prec f$. The relations $\prec$ and $\prec\prec$ are preorder relations of which the latter is finer than the former.

The preorder relations $\prec$ and $\prec\prec$ are in no way compatible with the linear space structure of L^1 . This will be particularl illustrated in i) of the following result.

(6.2) LEMMA. i) *If* $f,g \in L^1(X,\mu)$ *and if* $f \prec g$, *then* $rf \prec rg$ *for all* $r \in R$. ii) $f,g \in L^1(X,\mu)$, $0 \leq g$ *and* $f \prec g$ *implies* $f \geq 0$. iii) *If* $f \in L^1$, *then* $(\frac{1}{\mu(X)} \int f d\mu) c_X \prec f$, *and if* $g = c_X$, *then* $f \prec g$ implies $f = c_X$, μ - a.e.

PROOF. i) We have only to show that $f \prec g$ implies $-f \prec -g$. From iii) of (4.9) it follows that $\delta_{-f}(t) = -\delta_f(a-t)$ for all $0 \leq t \leq a = \mu(X)$, and so

$\int_o^t \delta_{-f}(u)du = -\int_o^t \delta_f(a-u)du = \int_a^{a-t} \delta_f(u)du$ implies using $f \prec g$ that $\int_o^t \delta_{-f}(u)du + \int_o^a \delta_f(u)du = \int_o^{a-t} \delta_f(u)du \leq \int_o^{a-t} \delta_g(u)du =$

$= \int_o^a \delta_g(u)du + \int_a^{a-t} \delta_g(u)du = \int_o^t \delta_{-g}(u)du + \int_o^a \delta_g du$. Hence, $-f \prec -g$.

ii) Follows immediately from i).

iii) From Theorem 5.2 it follows that

$\frac{1}{\mu(X)} (\int f d\mu) = \frac{1}{a} \int_o^a \delta_f(u)du \leq \frac{1}{t} \int_o^t \delta_f(u)du$, and so, if

$g = \frac{1}{\mu(X)} (\int f d\mu) c_X$, then $\int_o^t \delta_g(u)du = \frac{1}{\mu(X)} (\int f d\mu) t \leq \int_o^t \delta_f(u)du$

and equality for $t = a$.

Finally, if $\int_o^t \delta_f(u)du \leq t$ for all $0 \leq t \leq a$ with equality for $t = a$, then $t(\delta_f(t)-1) \leq \int_o^t ud(\delta_f) \leq 0$, and so $\delta_f(u) \leq 1$ for all $0 \leq u \leq a$. Hence $\int_o^a \delta_f(u)du = a$ implies that $\delta_f = 1$.

We shall now make another pause before proceeding with the general theory in order to show in which sense a finite measure space can be imbedded in a non-atomic measure space, a fact which will be used later to show which results in a non-atomic space carry over to the general case.

7. Imbedding of a finite measure space in a non-atomic finite measure space.

Let (X,Λ,μ) be a finite measure space. An element $A \in \Lambda$ is called a μ-atom whenever $\mu(A) > 0$ and $B \in \Lambda$, $B \subset A$ implies $\mu(B) = 0$ or $\mu(B) = \mu(A)$. It is evident that any two atoms are disjoint μ-a.e. and that any measurable function is constant a.e. on an atom.

If a measure space has finite total measure (or is totally σ-finite), then it can possess at most a countable number of atoms. Thus $X = X_o \cup (\bigcup_{i=1}^{\infty} A_i)$, where A_i $(i = 1,2,\ldots)$ are the μ-atoms and the restriction of μ to X_o is atomless.

We shall now show that every measure space can be imbedded in a non-atomic measure space. To this end, let $X_1 = X_o \cup (\bigcup_{i=1}^{\infty} I[a_i,b_i])$, where $I[a_i,b_i]$ $(i = 1,2,\ldots)$ are disjoint intervals of the real line such that $b_i - a_i = \mu(A_i)$ $(i = 1,2,\ldots)$. Then (X_1,Λ_1,μ_1) shall be the direct sum of the measure space $(X_o, \Lambda \cap X_o, \mu)$ and the Lebesgue measure spaces $(I[a_i,b_i],m)$ $(i = 1,2,\ldots)$. Then (X_1, Λ_1, μ_1) is a non-atomic finite measure space of total measure $\mu(X)$. Furthermore, $M(X,\Lambda,\mu)$ can be imbedded, i.e., identified with the subset of all elements of $M(X_1,\Lambda_1,\mu_1)$ which are constant on the intervals $I[a_i,b_i]$ $(i = 1,2,\ldots)$. More important, however, is the fact that $L^1(X,\Lambda,\mu)$ is a retract of $L^1(X_1,\Lambda_1,\mu_1)$ in the following sense.

(7.1) THEOREM. *For every* $f \in L^1(X_1,\mu_1)$ *we set*

(7.2) $$T_\mu f = fc_{X_o} + \sum_{i=1}^{\infty} \frac{1}{b_i - a_i} \left(\int_{a_i}^{b_i} f(t)dt\right) c_{A_i} .$$

Then T_μ *is a positive linear transformation of* $L^1(X_1,\mu_1)$ *onto* $L^1(X,\mu)$ *which leaves* $L^1(X,\mu)$ *invariant and which in addition has the following properties*:

i) *For all* $f \in L^1(X_1,\mu_1)$ *we have* $\int (T_\mu f) d\mu = \int f d\mu_1$ *and* $\|T_\mu f\|_1 \le \|f\|_1$.

ii) *For all* $f \in L^\infty(X_1,\mu_1)$ *we have* $\|T_\mu f\|_\infty \le \|f\|_\infty$.

iii) *For all* $f \in L^1(X_1,\mu_1)$ *and* $g \in M(X,\mu)$ *such that* $fg \in L^1(X_1,\mu_1)$ *we have* $\int fg d\mu_1 = \int (T_\mu f) g d\mu$.

iv) $T_\mu f_1 \prec f$ *for all* $f, f_1 \in L^1(X_1,\mu_1)$ *satisfying* $f \sim f_1$ *with respect to* μ_1 .

We shall omit the proof since i), ii) and iii) are trivial and iv) is a consequence of iii) of 6.2.

In what follows T_μ will always denote the transformation defined in (7.2).

The reader who is familiar with the notion of a conditional expectation will immediately recognize that $T_\mu f$ is the conditional expectation of f with respect to the σ-ring generated by $X_o \cap A$ and the intervals $I[a_i,b_i]$ $(i = 1,2,\ldots)$. We shall return to transformations of this type in section 14

Finally, the reader should observe that if $f \in M(X,\mu)$ is considered to be an element of $M(X_1,\mu_1)$, then its μ-decreasing rearrangement is equal to its μ_1-decreasing rearrangement.

8. An inequality of Hardy and Littlewood.

In this section we shall prove an inequality due to Hardy and Littlewood, which will play a fundamental role in what is to follow. The inequality given below represents a departure from the customary inequality (see [7], Theorem 378) in that the functions are no longer supposed to be non-negative.

We shall begin with a lemma. It will be convenient from now on to write δ_E in place of δ_{c_E}, where E is a measurable set.

(8.1) LEMMA. i) *Let* $f = \sum_{i=1}^{n} f_i c_{E_i} \in M(X,\mu)$ *be a simple function and suppose that* $0 < f_1 < f_2 < \ldots < f_n$. *Then* $\delta_f = \sum_{i=1}^{n} f_i' \delta_{F_i}$, *where* $f_1' = f_1, f_2' = f_2 - f_1, \ldots, f_n' = f_n - f_{n-1}$ *and* $F_i = \bigcup_{j=i}^{n} E_j$.

ii) *If* $f = \sum_{i=1}^{n} f_i c_{E_i}$, *where* $f_i > 0$

$(i = 1,2,\dots,n)$ *and* $E_1 \supset E_2 \supset \dots \supset E_n$, *then*

$$\delta_f = \sum_{i=1}^{n} f_i \, \delta_{E_i} .$$

iii) *For all* $E_1, E_2 \in \Lambda$ *we have*

$$\int_o^a \delta_{E_1}(t) \, \delta_{E_2}(a-t)dt \leq \int c_{E_1} c_{E_2} d\mu \leq \int_o^a \delta_{E_1}(t) \, \delta_{E_2}(t)dt ,$$

where $a = \mu(X)$.

PROOF. i) If $0 < t < f_1$, then $d_f(t) = \mu(\bigcup_{j=1}^{n} E_j)$; and if $f_k \leq t < f_{k+1}$, $k = 1,2,\dots,n$, then $d_f(t) = \mu(\bigcup_{j=k+1}^{n} E_j)$. Furthermore, observing that $\delta_E = c_{[0,\mu(E))}$, we obtain that i) holds.

ii) Follows immediately from i).

iii) $\int c_{E_1} c_{E_2} d\mu = \mu(E_1 \cap E_2)$ and $\int_o^a \delta_{E_1} \delta_{E_2} dt =$ $= \min(\mu(E_1),\mu(E_2))$, and so $\int c_{E_1} c_{E_2} d\mu \leq \int_o^a \delta_{E_1} \delta_{E_2} dt$. From $\int_o^a \delta_{E_1}(t) \, \delta_{E_2}(a-t)dt = (\mu(E_1) - a + \mu(E_2))^+ = (\mu(E_1 \cup E_2) +$ $+ \mu(E_1 \cap E_2) - a)^+ \leq \mu(E_1 \cap E_2) = \int c_{E_1} c_{E_2} d\mu$, the result follows.

REMARK. Statement i) of (8.1) appears to have been first used systematically by F. Riesz ([17], p. 164) .

We shall now prove the following inequality.

(8.2) THEOREM (<u>Hardy and Littlewood</u>). *If* $f,g \in M(X,\mu)$, $a = \mu(X) < \infty$, *and if* $\delta_{|f|}\,\delta_{|g|}$ *is Lebesgue integrable over* $[0,a]$, *then* fg *is* μ-*integrable and*

$$\int_0^a \delta_f(a-t)\delta_g(t)dt = \int_0^a \delta_f(t)\delta_g(a-t)dt \le \int_X fgd\mu \le \int_0^a \delta_f(t)\delta_g(t)dt .$$

PROOF. We shall first prove the result for non-negative functions. Since a non-negative measurable function is the everywhere limit of an increasing sequence of simple functions we have only to prove it for non-negative simple functions. To this end, let $f = \sum_{i=1}^{m} f_i c_{E_i}$, $f_i > 0$ $(i = 1,2,\ldots,n)$ and $E_1 \supset E_2 \supset \ldots \supset E_n$; and let $g = \sum_{j=1}^{m} g_j c_{F_j}$, $0 < g_j$ and $F_1 \supset F_2 \supset \ldots \supset F_m$. Then by ii) of (8.1) we have

$\delta_f = \sum_{i=1}^{n} f_i c_{[0,\mu(E_i))}$ and $\delta_g = \sum_{j=1}^{m} g_j c_{[0,\mu(F_j))}$, and so

$\delta_f\delta_g = \sum_{i,j} f_i g_j c_{[0,\min(\mu(E_i),\mu(F_j)))}$ implies that

$\int_0^a \delta_f\delta_g dt = \sum_{i,j} f_i g_j \min(\mu(E_i),\mu(F_j))$. Furthermore,

$\int_0^a \delta_f(t)\,\delta_g(a-t)dt = \sum_{i,j} f_i g_j(\mu(E_i)-a+\mu(F_j))^+$. Thus the result follows from iii) of (8.1).

For arbitrary $f,g \in M(X,\mu)$ we first observe that $\delta_{|f|}\,\delta_{|g|}$ integrable implies that $\int|fg|d\mu < \infty$. Then observing that $\int fgd\mu = \int f^+g^+d\mu - \int f^+g^-d\mu - \int f^-g^+d\mu$

$+ \int f^-g^-d\mu \le \int_0^a \delta_{f^+}\delta_{g^+}dt - \int_0^a \delta_{f^+}(t)\delta_{g^-}(a-t)dt - \int_0^a \delta_{f^-}(a-t)\delta_{g^+}(t)dt$

$+ \int_0^a \delta_{f^-} \delta_{g^-} dt = \int_0^a \delta_f \delta_g dt$, and similarly for the left hand side inequality.

This completes the proof.

REMARK. It is good to observe that, in general, $\delta_{fg} \le \delta_f \delta_g$ does not hold. Indeed, $f \le 0$, then not $(\delta_{|f|} \le - \delta_f)$ may hold. On the other hand, however, it is not difficult to see that if $0 \le f \in M(X,\mu)$, then $\delta_{fc_E} \le \delta_f \delta_E$ for all $E \in \Lambda$.

9. The values of an integral.

Let $f,g \in M(X,\mu)$. If $\delta_{|f|} \delta_{|g|} \in L^1[0,a]$, $a = \mu(X)$, then by (8.2) it follows that $\int |fg| d\mu \le \int_0^a \delta_{|f|} \delta_{|g|} dt$, and so, if $f' \sim f$ and $g' \sim g$, then also $|f'g'|$ is integrable. In this section we shall determine the set of values taken on by the integrals $\int fg' d\mu$ if g' runs through all the functions which are equimeasurable with g .

(9.1) THEOREM. Let $f,g \in M(X,\mu)$ and let $\delta_{|f|}\delta_{|g|} \in L^1[0,a]$,

<u>where</u> $a = \mu(X)$. <u>Then, if</u> μ <u>has no atoms, then the set of values</u> $\{\int fg'd\mu : g' \sim g\}$ <u>is the entire closed interval</u> $\left[\int_0^a \delta_f(t)\delta_g(a-t)dt, \int_0^a \delta_f(t)\delta_g(t)dt\right]$.

PROOF. The proof is divided into several parts. First we shall assume that g is a characteristic function c_E, with $\mu(E) = \alpha > 0$. Let $\beta = \inf_t d_f(t)$. Then if for some t, $d_f(t) = \alpha$, then it is easy to see that $\int fc_{F_t} = \int_0^\alpha \delta_f du$, where $F_t = \{x: f(x) > t\}$. If $\mu(F_t) \neq \alpha$ for all t , then either $\mu(F_t) < \alpha$ for all t or there exists a number t_o such that $\mu(F_{t_o}) > \alpha$ and $\mu(F_t) < \alpha$ for all $t > t_o$. In the first case we may take any set of measure α containing the set $\{x: f(x) > -\infty\}$. In the second case the situation is somewhat more involved. Let $t_n \downarrow t_o$. Then, if $\mu(\cup F_{t_n}) = \alpha$ we may take $E = \cup F_{t_n}$, if, however, $\mu(\cup F_{t_n}) < \alpha$, then by adding a set of $\{x: f(x) = t_o\}$ such that the total measure equals α will determine the required set. Working with e_f will produce the inequality on the left-hand side. Then the extension to non-negative simple functions is obtained by using ii) of (8.1). Finally for g non-negative approximation by simple functions gives the required result. Finally, considering positive and negative parts as in the proof of (8.2) the result will follow. The details are left to the reader. In order to show that all the values are taken on we write

$$\gamma(u) = \int_0^u \delta_f(t)\delta_g(u-t)dt + \int_u^a \delta_f(t)\delta_g(t)dt, \quad 0 \leq u \leq a .$$

Then γ is a continuous function in u and takes on all the values in the interval, and it is not difficult to see that there is a function $g' \sim g$ such that $\int fg'd\mu = \gamma(u)$.

(9.2) COROLLARY. <u>If</u> μ <u>has no atoms and if</u> $0 \leq f, g \in M(X,\mu)$ <u>satisfy</u> $f'g$ <u>is integrable for all</u> $f' \sim f$, <u>or</u> fg' <u>is integrable for all</u> $g' \sim g$, <u>then</u> $\delta_f \delta_g \in L^1[0,a]$ <u>and</u>

$$\max(\int f'g'd\mu : f \sim f', g \sim g') = \max(\int f'gd\mu : f' \sim f) =$$

$$= \max(\int fg'd\mu : g' \sim g) = \int_0^a \delta_f \delta_g \, dt .$$

REMARKS. 1. If μ has atoms, then the above result may be false. Indeed, if $X = X_o \cup A_1$, where A_1 is an atom and X_o is atomless, then $\int c_{X_o} c_{A_1} d\mu = 0$ and $\int_o^a \delta_{X_o} \delta_{A_1} dt =$ $= \min(\mu(X_o), \mu(A_1))$. But any $g \sim c_{A_1}$ is equal to c_{A_1} whenever $\mu(A_1) > \mu(X_o)$, and so the set of values of the integral is always $\{0\}$.

2. If X is finite and μ is the discrete measure, then the set of values of the corresponding sums do not fill the whole interval but the endpoints can be attained. (See [7], Theorem 368).

Using Theorem 7.1 we are in a position to prove the following result for general measures

(9.3) THEOREM. i) *Let* $f, g \in M(X,\mu)$ *and let* $\delta_{|f|}\delta_{|g|} \in$ $\in L^1[0,a]$, *where* $a = \mu(X)$. *Then the set of values* $\{\int fg'd\mu:\ g' \prec g\}$ *is the entire closed interval* $\left[\int_0^a \delta_f(t)\ \delta_g(a-t)dt,\ \int_0^a \delta_f \delta_g dt\right]$.

ii) *If* $0 \leq f, g \in M(X,\mu)$ *and* $\int fg'd\mu$ *is finite for all* $g' \prec g$, *then* $\delta_f \delta_g \in L^1[0,a]$, , $a = \mu(X)$, *and in that case* $\max(\int fg'd\mu : g' \prec g) =$ $= \int_0^a \delta_f \delta_g dt$.

PROOF. i) We have only to observe that by (7.1) $\int f(T_\mu g')d\mu = \int fg'd\mu_1$, $g' \sim g$ with respect to μ_1 and $T_\mu g' \prec g$ the result follows immediately from (9.1). The proof of ii) is similar.

The preceding discussion seems to justify the following definition

(9.4) DEFINITION. *A finite measure space* (X,Λ,μ) *is called adequate whenever for all* $0 \leq f, g \in M(X,\mu)$ *we have* $\max(\int fg'd\mu : g' \sim g) = \int_0^a \delta_f \delta_g dt$.

The problem of characterizing adequate measures seems to be open. We have shown that the non-atomic measures and the discrete measures are adequate.

We conclude this section with the following theorem.

(9.5) THEOREM. *If* $f, g \in L^1(X,\mu)$ *and* $f \prec g$, *then* $|f| \prec\prec |g|$.

PROOF. It is evident from Theorem 7.1 that we have to show this only for the case that μ has no atoms. Then it follows from (9.1) that $\int_0^t \delta_{|f|}(u)du = \int |f| c_E d\mu$, where E is some set of measure t . From $\int |f| c_E d\mu = \int f \operatorname{sgn}(f) c_E d\mu$ it follows using (8.2) that $\int_0^t \delta_{|f|}(u)du \leq \int_0^a \delta_f(u)\delta_h(u)du$, where $h = \operatorname{sgn}(f)c_E$. Then $f \prec g$ implies, using ii) of (5.1), that $\int_0^t \delta_{|f|}(u)du \leq \int_0^a \delta_f(u)\delta_h(u)du \leq \int_0^a \delta_g(u)\delta_h(u)du =$ $= \int gh' d\mu$ for some $h' \sim h$. Since h is a simple function which takes on only the values $+1$, 0 and -1 and the measure of the set where it is different from zero is $\leq t$ the function h' has the same property, and so $|h'|$ is the characteristic function of a set F of measure $\leq t$. Then $\int gh' d\mu \leq \int |g| c_F \leq \int_0^t \delta_{|g|}(u)du$ completes the proof.

10. The decreasing rearrangements of sums and products.

If $f,g \in L^1(X,\mu)$, then in general $\delta_{f+g} \not\leq \delta_f+\delta_g$. Indeed, assume that E_1,E_2 are two disjoint sets of equal positive measure. Then $\delta_{E_1} = \delta_{E_2} = c_{[0,\mu(E_1))}$ and $\delta_{E_1\cup E_2} = c_{[0,2\mu(E_1))}$, and so $\delta_{E_1\cup E_2} \not\leq \delta_{E_1} + \delta_{E_2}$. Concerning sums we can still show that the following result holds.

(10.1) THEOREM. *If* $f_1,\dots,f_n \in L^1(X,\mu)$, *then for all* $0 \leq t \leq a$ *we have* $\int_0^t \delta_{f_i}(u)du + \sum_{j\neq i} \int_0^t \delta_{f_j}(a-u)du \leq$

$$\leq \int_0^t \delta_{f_1+\dots+f_n}(u)du \leq \sum_{j=1}^{n} \int_0^t \delta_{f_j}(u)du \;,\; i = 1,2,\dots,n \;,$$

and with equality on both sides for $t = a$.

PROOF. From (7.1) it follows that we need to show this only for non-atomic spaces. Furthermore we may restrict the discussion to two functions. Then the result follows easily from (9.1) in the following way.

$$\int_0^t \delta_{f_1+f_2}(u)du = \max(\int_E (f+g)d\mu : \mu(E)=t) \leq \int_0^t \delta_{f_1}(u)du +$$

$$+ \int_0^t \delta_{f_2}(u)du \;,\text{ and } \int_0^t \delta_{f_1+f_2}(u)du = \max(\int_E (f+g)d\mu : \mu(E)=t) \geq$$

$$\geq \max(\int_E fd\mu : \mu(E)=t) + \int_0^t \delta_g(a-u)du = \int_0^t \delta_f(u)du + \int_0^t \delta_g(a-u)du$$

finishes the proof.

(10.2) COROLLARY. *If* $f \in L^1(X,\mu)$, *then*

$$\int_0^t |\delta_f(u)|du \leq \int_0^t \delta_{|f|}(u)du \quad \textit{for all} \quad 0 \leq t \leq a \quad \textit{and}$$

with equality for $t = a$.

PROOF. $\int_0^t |\delta_f(u)|du = \int_0^t \delta_{f^+}(u)du + \int_0^t \delta_{f^-}(a-u)du \leq$

$\leq \int_0^t \delta_{f^+ + f^-}(u)du = \int_0^t \delta_{|f|}(u)du$, by (10.1). Since $|f| \sim |\delta_f|$ equality for $t = a$ follows.

(10.3) THEOREM. *Let* g *be a bounded decreasing function on* $[0,a]$ *and let* (X,Λ,μ) *be a finite measure space with* $a = \mu(X)$. *Then the function* $p(f) = \int_0^a \delta_f(u)g(u)du$ *is sublinear on* $L^1(X,\mu)$.

PROOF. Combine (10.1) with ii) of (5.1).

For products of non-negative measurable functions the following result holds.

(10.4) THEOREM. i) *If* $0 \leq f_1,\ldots,f_n \in M(X,\mu)$, *then for all* $0 \leq t \leq a$ *we have* $\int_0^t \delta_{f_1\ldots f_n}(u)du \leq$

$\leq \int_0^t \delta_{f_1}(u)\delta_{f_2}(u) \ldots \delta_{f_n}(u)du$, *and*

i) $\int(f_1 \ldots f_n)d\mu \leq \int_0^a \delta_{f_1}(u)\delta_{f_2}(u) \ldots \delta_{f_n}(u)du$.

ii) _If_ $f_1,\dots,f_n \in M(X,\mu)$ _and if_ $\delta_{|f_1|} \cdots \delta_{|f_n|} \in L^1[0,a]$, _where_ $a = \mu(X)$, _then for all_ $0 \le t \le a$ _we have_

$$\int_o^t \delta_{|f_1 \dots f_n|}(u)du \le \int_o^t \delta_{|f_1|}(u) \dots \delta_{|f_n|}(u)du .$$

PROOF. i) We need to show the first part of i) only for two functions according to i) of (5.1). Now we may also assume that μ has no atoms. Then $\int_o^t \delta_{f_1 f_2}(u)du = \int_E f_1 f_2 d\mu$, for some E with $\mu(E) = t$, and so

$$\int_o^t \delta_{f_1 f_2}(u)du \le \int_o^t \delta_{f_1 c_E}\, \delta_{f_2 c_E} du \le \int_o^t \delta_{f_1} \delta_{f_2} du .$$

ii) Use i) and Corollary 10.2.

REMARK. Some more general inequalities concerning n-tuples of functions and their decreasing rearrangements were given by Lorentz (see [8] and [11]) .

11. Rearrangement-invariant Banach function spaces.

Banach function spaces or more generally normed Riesz spaces of measurable functions have been studied extensively (see [12] and [14]). In this paper we shall present some properties of such spaces which are in addition rearrangement-invariant and for which the underlying measure space is totally finite.

We shall recall first some of the definitions and properties of such spaces.

(11.1) DEFINITION. Let (X,Λ,μ) be a finite measure space and let $M^+(X,\mu)$ denote the set of all non-negative extended real measurable functions defined on X. A mapping ρ of M^+ into the (extended) real number system is called a function norm whenever ρ has the following properties.

i) $0 \leq \rho(f) \leq \infty$ for all $f \in M^+$; and $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.

ii) $\rho(f_1+f_2) \leq \rho(f_1) + \rho(f_2)$ for all $f_1, f_2 \in M^+$; $\rho(af) = a\rho(f)$ for all $f \in M^+$ and all $a \geq 0$; and $0 \leq f_1 \leq f_2$ implies $\rho(f_1) \leq \rho(f_2)$.

A function norm ρ is said to have the sequential Fatou property whenever

iii) $0 \leq f_n \uparrow f$ pointwise everywhere implies $\rho(f_n) \uparrow \rho(f)$.

A function norm having the sequential Fatou property is called a Fatou norm.

Given a function norm ρ we extend its domain of definition to M by defining $\rho(f) = \rho(|f|)$ for all $f \in M$, and we denote by $L^\rho = L^\rho(X,\mu)$ the set of all $f \in M$ such that $\rho(f) < \infty$. If μ-almost equal functions are identified in the usual way, L^ρ is a normed linear space with respect to the norm $\|f\| = \|f\|_\rho = \rho(|f|)$. Such spaces are obviously generalizations of the classical Lebesgue and Orlicz spaces.

From the hypothesis of ρ it does not necessarily follow that ρ has no purely-infinite sets, i.e., a set of positive measure A such that $\rho(c_B) = \infty$ for all subsets B of A of positive measure. If A is purely-infinite then every $f \in L^\rho$ vanishes on A. For the investigation of L^ρ we may therefore remove the ρ-purely-infinite sets from X. It was shown in ([14], Note IV, Theorem 8.3) that there exists a largest ρ-purely-infinite set. Removing this set from X and denoting this set again by X shows that we may assume without loss of generality that ρ is <u>saturated</u>, i.e., there are no ρ-purely-infinite sets.

Concerning completeness of L^ρ it was shown in ([14], Note I, Theorem 4.8) that L^ρ is norm-complete if and only if ρ has the <u>Riesz-Fischer property</u>, i.e., $\Sigma\rho(f_n) < \infty$ implies $\rho(\Sigma|f_n|) < \infty$. In particular, if ρ is a Fatou norm, then L^ρ is norm-complete.

Our next remark concerns the first and second associate function norms ρ' and ρ'' of ρ respectively. They are defined as follows:

(11.2) $\rho'(f) = \sup(\int|fg|d\mu : \rho(g) \leq 1)$, $f \in M^+$, and
$\rho''(f) = \sup(\int|fg|d\mu : \rho'(g) \leq 1)$, $f \in M^+$.

Since ρ is saturated ρ' is a Fatou norm. Furthermore, it was shown in ([14], Note IV, Corollary 11.6) that ρ' is also saturated which is much harder to show than the fact that ρ' is a Fatou norm. Thus ρ'' is also a saturated Fatou norm. Furthermore $\rho'' \leq \rho$ and $(\rho'')' = \rho'$. Concerning the Fatou property we have the following basic result (see [14], Note IV, Lemma 11.3).

(11.3) THEOREM (<u>Lorentz and Luxemburg</u>). <u>We have</u> $\rho = \rho''$ <u>if and only if</u> ρ <u>has the sequential Fatou property</u>.

The space $L^{\rho'}$ and $L^{\rho''}$ are norm-complete and are called the first and second associate space of L^{ρ} respectively.

We also have the following Hölder type inequality.

(11.4) <u>For all</u> $f \in L^{\rho''}$ <u>and</u> $g \in L^{\rho'}$ <u>we have</u>

$$|\int fg d\mu| \leq \int|fg|d\mu \leq \rho''(f)\rho'(g) \leq \rho(f)\rho'(g) .$$

The associate space $L^{\rho'}$ is a closed normal subspace of the Banach dual $(L^{\rho})^*$ of L^{ρ} and $L^{\rho'} = (L^{\rho})^*$ if and only if ρ satisfies the following condition: $0 \leq f_n \in L^{\rho}$ and $f_n \downarrow 0$ implies $\rho(f_n) \downarrow 0$.

It is easy to see that since the measure μ is finite that $L^{\infty} \subset L^{\rho} \subset L^1$ if and only if $\rho(c_X) < \infty$ and $\rho'(c_X) < \infty$.

In the remainder of this paper we shall only consider function norms ρ _which satisfy_ $L^{\infty} \subset L^{\rho} \subset L^1$, i.e., such that the L^{ρ}-space is between L^{∞} and L^1.

(11.5) DEFINITION. _A function norm_ ρ _is called rearrangement-invariant whenever_ $f_1 \sim f_2$ _implies_ $\rho(f_1) = \rho(f_2)$.

L^{ρ} _is called rearrangement-invariant whenever_ $f_1 \in L^{\rho}$ _and_ $f_1 \sim f_2$ _implies_ $f_2 \in L^{\rho}$.

If ρ is rearrangement-invariant then also L^{ρ} is rearrangement-invariant. The converse need not hold. Indeed, consider the function norm $\rho(f) = \int_0^1 |f(t)|dt + 2\int_1^2 |f(t)|dt$, then $\rho(c_{[0,1]}) = 1$ and $\rho(c_{[1,2]}) = 2$ and $c_{[0,1]} \sim c_{[1,2]}$.

We shall now prove the following fundamental lemma.

(11.6) LEMMA. _If_ μ _is adequate, in particular, if_ μ _has no atoms or if_ μ _is discrete and if_ ρ _is a Fatou norm, then_ L^{ρ} _is rearrangement-invariant if and only if_ $\delta_f \delta_g \in L^1[0,a]$, $a = \mu(X)$ _for all_ $0 \leq f \in L^{\rho}$ _and_ $0 \leq g \in L^{\rho'}$, _and in that case_ $L^{\rho'}$ _and_ $L^{\rho''}$ _are rearrangement-invariant._

PROOF. Assume that L^{ρ} is rearrangement invariant and that $0 \leq f \in L^{\rho}$ and $0 \leq g \in L^{\rho'}$. Then $\int f' g d\mu < \infty$ for all $f' \sim f$, and so since μ is adequate we obtain that $\int_0^a \delta_f \delta_g dt < \infty$. Conversely, if the latter condition holds and $0 \leq f' \sim f \in L^{\rho}$, then $\int f' g d\mu \leq \int_0^a \delta_f \delta_g dt < \infty$ for all $0 \leq g \in L^{\rho'}$ implies $f' \in L^{\rho}$ since ρ has the Fatou property. Finally, if $g' \sim g \in L^{\rho'}$, then $\int f g' d\mu \leq \int_0^a \delta_f \delta_g d\mu < \infty$ implies $g' \in L^{\rho'}$.

(11.7) THEOREM. Let μ be an adequate measure.

i) If L^{ρ} is rearrangement-invariant, then $g_1 \prec g_2 \in L^{\rho'}$ implies $g_1 \in L^{\rho'}$ and if $0 \leq g_1 \prec\prec g_2 \in L^{\rho'}$, then $g_1 \in L^{\rho'}$, and similarly for $L^{\rho''}$.

ii) If L^{ρ} is rearrangement-invariant and if ρ is a Fatou norm, then $f_1 \prec f_2 \in L^{\rho}$ implies $f_1 \in L^{\rho}$ and if $0 \leq f_1 \prec\prec f_2 \in L^{\rho}$, then $f_1 \in L^{\rho}$.

PROOF. We need only to prove i) the proof of ii) is similar. From (9.5) it follows that if $g_1 \prec g_2$ then $|g_1| \prec\prec |g_2|$. Hence, if $g_2 \in L^{\rho'}$ and $0 \leq f \in L^{\rho}$, then by (11.6), i) of (5.1) and (8.2) we have $\int |g_1| f d\mu \leq \int_0^a \delta_{|g_1|} \delta_f dt \leq \int_0^a \delta_{|g_2|} \delta_f dt < \infty$, and so $|g_1| \in L^{\rho'}$, i.e., $g_1 \in L^{\rho'}$. The proof of the second part of i) is similar.

There is a good generalization of the present theorem for the case that μ is arbitrary. In that case, however, we have to rely on the discussion given in section 7.

For this purpose we shall say that L^ρ is <u>universally rearrangement-invariant</u> (u.r.i) whenever $f \in L^\rho$ implies $T_\mu f_1 \in L^\rho$ for all $f_1 \in M(X_1,\mu_1)$ satisfying $f_1 \sim f$. Using then the fact that $0 \leq f,g \in M(X,\mu)$, then $\delta_f \delta_g \in L^1 - 0,a]$ if and only if $\int (T_\mu f_1) g \, d\mu < \infty$ for all $f_1 \in M(X_1,\mu_1)$ satisfying $f_1 \sim f$, the following result can be shown to hold in the same way as (11.6) and (11.7).

The reader should observe that if μ is adequate, then L^ρ is rearrangement-invariant if and only if L^ρ has the (u.r.i)-property.

(11.8) THEOREM. i) *If* L^ρ *has the* (u.r.i)-*property, then* $0 \leq f \in L^\rho$ *implies* $\delta_f \delta_g \in L^1[0,a]$, $a = \mu(X)$, *and if* ρ *is a Fatou norm then the converse holds.*

ii) *If* L^ρ *has the* (u.r.i)-*property, then* $g_1 \prec g_2 \in L^{\rho'}$ *implies* $g_1 \in L^{\rho'}$, *and* $0 \leq g_1 \prec\prec g_2 \in L^{\rho'}$ *implies* $g_1 \in L^{\rho'}$. *Similarly for* L^ρ *if* ρ *is a Fatou norm.*

We shall now turn our attention to rearrangement-invariant norms.

(11.9) THEOREM. *If* μ *is adequate, in particular if* μ *has no atoms or if* μ *is discrete, and if* ρ *is* *rearrangement-invariant*, *then for all* $0 \le f \in M(X,\mu)$ *we have* $\rho'(f) = \sup(\int_0^a \delta_f \delta_{|g|} dt : \rho(g) \le 1)$, $a = \mu(X)$,

and $\rho''(f) = \sup(\int_0^a \delta_f \delta_{|g|} dt : \rho'(g) \le 1)$.

In particular, ρ' *and* ρ'' *are rearrangement-invariant. If, in addition* ρ *is a Fatou norm, then for all* $0 \le f \in M(X,\mu)$ *we have*

$$\rho(f) = \sup(\int_0^a \delta_f \delta_{|g|} dt : \rho'(g) \le 1).$$

PROOF. We need to prove this result only for ρ'. From the definition of ρ' and from the fact that μ is adequate and ρ is rearrangement invariant we obtain immediately that if $0 \le f \in M(X,\mu)$, then

$$\rho'(f) = \sup(\int f|g|d\mu : \rho(g) \le 1) = \sup(\int_0^a \delta_f \delta_{|g|} dt . \rho(g) \le 1),$$

and the proof is finished.

(11.10) THEOREM. *If* μ *is adequate and if* ρ *is a Fatou norm which is rearrangement-invariant, then*

i) $f_1 \prec f_2 \in L^1(X,\mu)$ *implies* $\rho(f_1) \le \rho(f_2)$.

ii) $0 \le f_1 \prec\prec f_2$ *implies* $\rho(f_1) \le \rho(f_2)$.

In particular, if ρ *is rearrangement-invariant, then* ρ' *and* ρ'' *satisfy* i) *and* ii).

PROOF. We shall only prove i). From (9.5) it follows that $|f_1| \prec\prec |f_2|$, and so by (11.9) we have

$\rho(f_1) = \sup(\int_o^a \delta|f_1|^\delta|g|dt : \rho(g) \leq 1)$. Then i) of (5.1) shows that $\int_o^a \delta|f_1|^\delta|g|dt \leq \int_o^a \delta|f_2|^\delta|g|dt$, and so $\rho(f_1) \leq \rho(f_2)$.

If μ is not adequate, then (11.10) may be false as the following example will show.

Let (X, Λ, μ) be a measure space with one atom A . Thus $X = X_o \cup A$. Furthermore, assume that $\mu(X_o) = 1$ and $\mu(A) = 2$. Let $f_1 = c_{X_o} + \frac{1}{2} c_A$ and $f_2 = c_{X_o}$. Then $f_1 \prec f_2$. Let $\rho(f) = \int_{X_o} |f| d\mu + |f(c_A)|$. Then ρ is rearrangement-invariant. But $\rho(f_1) = \frac{3}{2}$ and $\rho(f_2) = 1$, and so $\rho(f_1) \not\leq \rho(f_2)$.

If μ is arbitrary, then ρ has to satisfy a stronger condition for (11.9) and (11.10) to hold.

For this purpose we shall say that ρ is <u>universally rearrangement-invariant</u> whenever $\rho(T_\mu f_1) \leq \rho(f)$ for all $0 \leq f \in M(X,\mu)$ and all $f_1 \in M(X_1,\mu_1)$ satisfying $f_1 \sim f$. (Observe that $T_\mu f_1 \prec f$ implies that $T_\mu f_1 \geq 0$ by ii) of (6.2)).

If μ is adequate, then ρ is universally rearrangement-invariant if and only if ρ is rearrangement invariant.

The following result can be shown to hold. The proof is left to the reader.

(11.11) THEOREM. *Let* ρ *be universally rearrangement invariant. Then the following statements hold.*

i) ρ' *and* ρ'' *are universally rearrangement-invariant.*

ii) *For all* $0 \leq f \in M(X,\mu)$ *we have*

$$\rho'(f) = \sup(\int_0^a \delta_f \delta_{|g|} dt: \ \rho(g) \leq 1) \ , \text{ and}$$

$$\rho''(f) = \sup(\int_0^a \delta_f \delta_{|g|} dt: \ \rho'(g) \leq 1) \ , \text{ where}$$

$a = \mu(X)$, *and if* ρ *is a Fatou norm, then for all* $f \in M(X,\mu)$ *we have also*

$$\rho(f) = \sup(\int_0^a \delta_f \delta_{|g|} dt: \ \rho'(g) \leq 1) \ .$$

iii) $f_1 \prec f_2 \in L^1(X,\mu)$ *implies* $\rho'(f_1) \leq \rho'(f_2)$ *and* $\rho''(f_1) \leq \rho''(f_2)$, *and similarly for* ρ *provided* ρ *is a Fatou norm.*

iv) $0 \leq f_1 \prec\prec f_2$ *implies* $\rho'(f_1) \leq \rho'(f_2)$ *and* $\rho''(f_1) \leq \rho''(f_2)$, *and similarly for* ρ *provided* ρ *is a Fatou norm.*

12. A representation theorem.

The rearrangement-invariant spaces such as the classical Lebesgue spaces, the Orlicz spaces and the spaces introduced by Halperin and Lorentz (see [5], [9], [10]), Boyd [2] and

Shimogaki [21] are all of the following type.

Let (X,Λ,μ) be a finite measure space and let $a = \mu(X)$. Let λ be a function norm defined on the space M^+ of all non-negative real Lebesgue measurable functions defined on the interval $[0,a]$. Furthermore, assume that λ is a Fatou norm which is rearrangement-invariant. Then the following result holds.

(12.1) LEMMA. <u>The mapping</u> $0 \leq f \to \rho(f) = \lambda(\delta_f)$ of $M^+(X,\mu)$ <u>into the extended real number system is a Fatou norm which is universally rearrangement-invariant</u>. <u>Furthermore</u>, $\rho'(f) = \lambda'(\delta_f)$ <u>for all</u> $0 \leq f \in M(X,\mu)$.

PROOF. Since Lebesgue measure has no atoms it follows immediately from (11.9) that for all $0 \leq h \in M[0,a]$ we have

$$\lambda(h) = \sup(\int_o^a \delta_h \delta_{|g|} : \lambda'(g) \leq 1) \text{ , and so}$$

$$\rho(f) = \sup(\int_o^a \delta_f \delta_{|g|} : \lambda'(g) \leq 1) .$$

Then it is easy to see that ρ is a function norm-only the subadditivity property needs a proof. This will follow from (10.1) and i) of (5.1) in the following way. Let $0 \leq f_1, f_2 \in M(X,\mu)$. Then $\rho(f_1+f_2) = \sup(\int_o^a \delta_{f_1+f_2}\delta_{|g|}dt : \lambda'(g) \leq 1) \leq$ $\leq \sup(\int_o^a \delta_{f_1}\delta_{|g|}dt + \int_o^a \delta_{f_2}\delta_{|g|}dt : \lambda'(g) \leq 1) \leq \rho(f_1) + \rho(f_2)$.

It also follows immediately that ρ is a Fatou norm. In order to prove that ρ is universally rearrangement-invariant it is sufficient to show that $0 \leq f_1 \ll f_2$ implies $\rho(f_1) \leq \rho(f_2)$.

This follows from i) of (5.1) as follows: $\rho(f_1) = \sup(\int_o^a \delta_{f_1}\delta_{|g|}dt : \lambda'(g) \leq 1) \leq \sup(\int_o^a \delta_{f_2}\delta_{|g|}dt: \lambda'(g) \leq 1) = \rho(f_2)$. Finally, the definition of ρ' immediately implies that $\rho'(f) = \lambda'(\delta_f)$ for all $0 \leq f \in M(X,\mu)$, and the proof is finished.

Taking special norms λ then gives us the examples we referred to above. For instance in the case of the classical Lebesgue spaces, $\lambda(f)=(\int_o^a \delta_f{}^p(t)dt)^{\frac{1}{p}}$, $0 \leq f \in M[0,a]$.

We shall now prove a converse namely that all universally rearrangement-invariant norms are of this type.

(12.2) THEOREM. _Let_ (X,Λ,μ) _be a finite measure space, and let_ ρ _be a Fatou norm. Then_ ρ _is universally rearrangement-invariant_ (_rearrangement invariant if_ μ _is adequate_) _if and only if there exists a rearrangement-invariant norm_ λ _defined on_ $M^+[0,a]$, $a = \mu(X)$ _such that_ $\rho(f) = \lambda(\delta_f)$ _for all_ $0 \leq f \in M(X,\mu)$.

PROOF. From (12.1) it follows that we only need to prove the existence of λ . To this end, we observe that by (11.11) we have that for all $0 \leq f \in M(X,\mu)$ $\rho(f) = \sup(\int_o^a \delta_f\delta_{|g|}: \rho'(g) \leq 1)$. Then for every $0 \leq h \in M[0,a]$ we set

$\lambda(h) = \sup(\int_o^a \delta_h\delta_{|g|}: \rho'(g) \leq 1)$. It is easy to see that λ is a Fatou norm which is rearrangement invariant which satisfies $\rho(f) = \lambda(\delta_f)$ for all $0 \leq f \in M(X,\mu)$. This completes the proof.

13. An inequality of Hardy, Littlewood and Pólya.

Let $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ be two n-vectors of real numbers, and let $x^* = (x^*_1,\dots,x^*_n)$ and $y^* = (y^*_1,\dots,y^*_n)$ be the n-vectors obtained from x and y respectively by rearranging their respective components in non increasing order. As in section 6 we shall set $x \prec y$ whenever $\sum_{i=1}^{k} x^*_i \leq \sum_{i=1}^{k} y^*_i$, $k = 1,2,\dots,n$, with equality for $k = n$.

We recall that a matrix $A = (a_{ij})$, $i,j = 1,2,\dots,n$, is said to be doubly stochastic (d.s.) whenever $a_{ij} \geq 0$, $i,j = 1,2,\dots,n$, and $\sum_{j=1}^{n} a_{ij} = 1$ for $i = 1,2,\dots,n$ and $\sum_{i=1}^{n} a_{ij} = 1$ for $j = 1,2,\dots,n$. Furthermore, it is not hard to show that a matrix A is d.s. whenever $Ax \prec x$ for all x .

If π is a permutation of the set $\{1,2,\dots,n\}$, then by x_π we shall denote the n-vector $(x_{\pi(1)},\dots,x_{\pi(n)})$, and by $\Pi(x)$ we shall denote the convex hull of all the $n!$ vectors x_π .

With these notations a fundamental theorem of Hardy, Littlewood and Pólya (see [6]) can be formulated as follows.

(13.1) THEOREM (Hardy, Littlewood and Pólya). *Let x and y be two n-vectors of real numbers. Then the following statements are equivalent.*

i) $y \prec x$.

ii) *There exists a doubly stochastic matrix* A *such that* $y = Ax$.

iii) $y \in \Pi(x)$.

iv) $\sum_{i=1}^{n} \varphi(y_i) \leq \sum_{i=1}^{n} \varphi(x_i)$ *for every continuous convex function defined on* R .

For an interesting discussion of the Hardy, Littlewood and Pólya result and the theory of doubly stochastic transformations we refer the reader to the important paper [15] of L. Mirsky.

It will be our purpose to investigate in which sense (13.1) can be extended to include measurable functions defined on finite measure spaces. In this section we shall present a result which is an extension of this kind of the statement of (13.1) that i) and iv) are equivalent.

For this purpose we shall use the following setting. Let (X, Λ, μ) be a finite measure space and let ρ be a Fatou norm such that $L^{\infty} \subset L^{\rho} \subset L^{1}$. Furthermore, we shall assume that L^{ρ} is universally rearrangement invariant, and so by (11.11) we have $f_1 \prec f_2 \in L^{\rho}$, then $f_1 \in L^{\rho}$.

We shall now introduce the following definition the terminology of which is borrowed from [15].

(13.2) DEFINITION. *A mapping* Φ *of* L^{ρ} *into the extended real number system will be called Schur-convex whenever* Φ *has the following properties*

i) $-\infty < \Phi(f) \leq +\infty$ <u>for all</u> $f \in L^{\rho}$ <u>and for some</u> $f \in L^{\rho}$, $\Phi(f) < \infty$.

ii) Φ <u>is convex</u>, i.e., $\Phi(rf_1+(1-r)f_2) \leq r\Phi(f_1)+(1-r)\Phi(f_2)$ <u>for all</u> $f_1, f_2 \in L^{\rho}$ <u>and</u> $0 \leq r \leq 1$.

iii) Φ <u>is</u> $\sigma(L^{\rho}, L^{\rho'})$-<u>lower semicontinous</u>.

iv) Φ <u>is rearrangement-invariant</u>, i.e., $f_1 \sim f_2 \in L^{\rho}$ <u>implies</u> $\Phi(f_1) = \Phi(f_2)$.

The reader should observe that a function Φ satisfying i) and ii) satisfies iii) if and only if $\int |f_n - f|\,|g|\,d\mu \to 0$ as $n \to \infty$ for all $g \in L^{\rho'}$ implies $\Phi(f) \leq \liminf_{n \to \infty} \Phi(f_n)$. In particular, such a function Φ satisfies iii) whenever $|f_n| \leq f_1 \in L^{\rho}$ and $f_n \to f$ μ-a.e. implies $\Phi(f) \leq \liminf \Phi(f_n)$.

We shall now present some examples of Schur-convex functions.

EXAMPLES. i) Let φ be a real continuous convex function defined on R such that $\liminf_{u \to -\infty} \varphi(u)/u$ is finite. Then $\Phi(f) = \int \varphi(f)\,d\mu$, $f \in L^{\rho}$ is Schur-convex. This is the type of functions considered by Hardy, Littlewood and Pólya in (13.1).

ii) If in addition φ is increasing, then

$\Phi(f) = \int_0^t \varphi(\delta_f(u))\,du$, $f \in L^{\rho}$, $0 \leq t \leq \mu(X)$ is also Schur-convex.

We shall now prove the following extension of the Hardy, Littlewood and Pólya inequality.

(13.3) THEOREM. Let μ be an adequate measure and let L^ρ be a universally rearrangement-invariant space. Then the following conditions are equivalent.

i) $f_1, f_2 \in L^\rho$ and $f_1 \prec f_2$.

ii) For every Schur-convex function Φ on L^ρ we have $\Phi(f_1) \leq \Phi(f_2)$.

PROOF. We shall first show that ii) $\Rightarrow$ i). Since for all $0 \leq t \leq \mu(X) = a$ we have that $\Phi_t(f) = \int_0^t \delta_f(u)du$ is Schur-convex we obtain immediately that $\int_0^t \delta_{f_1}(u)du \leq \int_0^t \delta_{f_2}(u)du$ for all $0 \leq t \leq a$. In order to prove the equality for $t = a$ apply the Schur-convex functions $\int f d\mu$ and $\int(-f)d\mu$ respectively.

i) $\Rightarrow$ ii). The proof is based on the following well-known result concerning lower semicontinuous functions on locally convex spaces, namely that every such a function is the supremum of its subgradients (see [1]). This means in our case that if we set for every $g \in L^{\rho'}$, $\Psi(g) = \sup(\int fg d\mu - \Phi(f): f \in L^\rho)$, then $\Phi(f) = \sup(\int fg d\mu - \Psi(g): g \in L^{\rho'})$.
Since Φ is Schur-convex and since μ is adequate we obtain immediately that

$$(13.4) \qquad \Phi(f) = \sup(\int_0^a \delta_f \delta_g dt - \Psi(g): g \in L^{\rho'}) .$$

If now $f_1 \prec f_2$, then by ii) of (5.1) we obtain

$\int_0^a \delta_{f_1} \delta_g dt \leq \int_0^a \delta_{f_2} \delta_g dt$ for all $g \in L^{\rho'}$, and so $\Phi(f_1) \leq \Phi(f_2)$ which finishes the proof.

If μ is not adequate, then the result does not hold any longer as was shown in Section 12. In order to include all the measure spaces we have to extend the notion of a Schur-convex function as follows.

A function Φ on L^ρ satisfying i), ii) and iii) is called a <u>universal Schur-convex function</u> whenever $\Phi(T_\mu f_1) \leq \Phi(f)$ for all $f \in L^\rho$ and $f_1 \in M(X_1, \mu_1)$ satisfying $f_1 \sim f$.

It is clear from the preceding theorem that if μ is adequate, then every Schur-convex function on L^ρ is a universal Schur-convex function. The function $\Phi_t(f) = \int_0^t \delta_f(u)du$, $f \in L^\rho$ are of course universal Schur-convex.

We have the following theorem. The proof is left to the reader.

(13.5) THEOREM. <u>For all</u> $f_1, f_2 \in L^\rho$, $f_1 \prec f_2$ <u>if and only if</u> $\Phi(f_1) \leq \Phi(f_2)$ <u>for all universal Schur-convex functions</u> Φ <u>of</u> L^ρ.

For increasing Schur-convex functions the following result holds.

(13.6) THEOREM. i) <u>If</u> μ <u>is adequate, then for all</u> $f_1, f_2 \in L^\rho$, $f_1 \ll f_2$ <u>if and only if</u> $\Phi(f_1) \leq \Phi(f_2)$ <u>for all increasing Schur-convex functions on</u> L^ρ.

ii) <u>For all</u> $f_1, f_2 \in L^\rho$ <u>we have</u> $f_1 \ll f_2$ <u>if and only if</u> $\Phi(f_1) \leq \Phi(f_2)$ <u>for all increasing universal Schur-convex functions on</u> L^ρ .

PROOF. We shall only prove i) since the proof of ii) is similar. Using the increasing universal Schur-convex functions $\Phi_t(f) = \int_0^t \delta_f(u)du$, $f \in L^\rho$ we see that $\Phi(f_1) \leq \Phi(f_2)$ for all such functions implies $f_1 \ll f_2$. In order to prove the converse let $f_o \in L^\rho$ be such that $\Phi(f_o) < \infty$ and let $\Phi_o(f) = \Phi(f+f_o) - \Phi(f_o)$, $f \in L^\rho$. Then $\Phi_o(0) = 0$, Φ_o is non-negative and Φ_o satisfies conditions i), ii) and iii) of (13.2). Hence, $\Phi_o(f) = \sup(\int fg d\mu - I_o(g) : g \in L^{\rho'})$. Since Φ_o is increasing it follows immediately that $\Phi_o(f) = \sup(\int fg^+ d\mu - I_o(g) : g \in L^{\rho'})$. Then finally, $\Phi(f) = \sup(\int fg^+ d\mu - I_o(g) + \Phi(f_o) - \int f_o g^+ d\mu : g \in L^{\rho'}) = \sup(\int fg^+ d\mu + a_g : g \in L^{\rho'})$. Since Φ is Schur-convex we then obtain that $\Phi(f) = \sup(\int_0^a \delta_f \delta_{g^+} dt + a_g : g \in L^{\rho'})$, and so the result follows from i) of (5.1). This completes the proof of the theorem.

We have shown in (12.2) that if ρ is u.i.r., then there exists a r.i - Fatou norm λ on $M[0,\mu(X)]$ such that $\rho(f) = \lambda(\delta_f)$. The same holds for the universal Schur convex function Φ on such an L^ρ . Namely, we have the following result

(13.7) THEOREM. *If Φ is a universal Schur-convex function on L^{ρ}, where $\rho(f) = \lambda(\delta_f)$, then there exists a Schur-convex function Φ_o on $L^{\lambda}[0,a]$ such that $\Phi(f) = \Phi_o(\delta_f)$, and conversely.*

PROOF. Indeed, $\Phi(f) = \sup(\int_o^a \delta_f \delta_g dt - I(g): g \in L^{\rho'})$, and so if we define $\Phi_o(h) = \sup(\int_o^a \delta_h \delta_g dt - I(g): g \in L^{\rho'})$ for all $h \in L^{\lambda}$, then $\Phi(f) = \Phi_o(\delta_f)$. The rest is easy.

Some years ago L. Fuchs (see [4]) generalized the Hardy, Littlewood and Pólya inequality as follows.

Let $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$ be two n-vectors and let φ be a real continuous convex function defined on R. The following result holds.

If $\sum_{i=1}^{k} y^*_i \nu_i \leq \sum_{i=1}^{k} x^*_i \nu_i$ $(k = 1,2,\ldots,n)$ with equality for $k = n$, then $\sum_{i=1}^{n} \varphi(y^*_i)\nu_i \leq \sum_{i=1}^{n} \varphi(x^*_i)\nu_i$, where $\nu = (\nu_1,\ldots,\nu_n)$ is an arbitrary n-tuple of real numbers.

This result can be immediately generalized in the following manner. Let ρ be an u.i.r. Fatou norm and let Φ be a universal Schur-convex function on $L^{\rho}(X,\mu)$. If ν is any function of bounded variation on $[0,a]$, $a = \mu(X)$ such that $L^{\rho} \subset L^1(\nu)$, then we set $f_1 \prec_\nu f_2$ whenever $\int_o^t \delta_{f_1}(u)d\nu(u) \leq \int_o^t \delta_{f_2}(u)d\nu(u)$ for all $0 \leq t \leq a$ with equality for $t = a$. Furthermore, let $\Phi_\nu(f) = \sup(\int_o^a \delta_f \delta_g d\nu - \Phi^*(g): g \in L^{\rho'})$. Then the following result holds.

(13.8) THEOREM (<u>L. Fuchs</u>). *If* $f_1, f_2 \in L^\rho$, *then* $f_1 \prec_\nu f_2$ *if and only if* $\Phi_\nu(f_1) \leq \Phi_\nu(f_2)$ *for all universal Schur-convex function* Φ *on* L^ρ.

14. Doubly stochastic transformations

The extension of the notion of a doubly stochastic matrix (see section 13 for a definition) to L^1-spaces appears to have been first given by Rota (see [18]). The theory was further developed by Ryff in [19] and [20]. We shall recall first some of the definitions and results of this theory before applying it to the theory of rearrangement-invariant spaces.

Let (X, Λ, μ) be a finite measure space. A linear transformation T of $L^1(X,\mu)$ into $L^1(X,\mu)$ is called *doubly stochastic* whenever $Tf \prec f$ for all $f \in L^1(X,\mu)$. In particular, it follows from iii) of (6.2) that if T is d.s., then $T \;\; = 1$. It is also easy to see that ii) of (6.2) implies that every d.s. transformation is non-negative. Furthermore, a d.s. transformation is a contraction in both L^1 and L^∞. Its associate transformation T' defined by the relations $\int (Tf) g \, d\mu = \int f (T'g) \, d\mu$, $f \in L^1$, $g \in L^\infty$, is a transformation which can be uniquely extended to L^1 by limits to a d.s. transformation.

(14.1) LEMMA. i) <u>A linear transformation</u> T <u>of</u> $L^1(X,\mu)$ <u>into</u> $L^1(X,\mu)$ <u>is d.s. if and only if</u> $Tf \prec f$ <u>for all</u> $0 \leq f \in L^1$.

ii) (<u>J.V. Ryff</u>). <u>A linear transformation</u> T <u>of</u> $L^\infty(X,\mu)$ <u>into</u> $L^\infty(X,\mu)$ <u>can be extended uniquely to a d.s. transformation on</u> $L^1(X,\mu)$ <u>if and only if</u> T <u>satisfies the following conditions</u>: $0 \leq Tc_E \leq 1$ <u>and</u> $\int Tc_E d\mu = \mu(E)$ <u>for all</u> $E \in \Lambda$.

PROOF. i) From $Tf \prec f$ for all $0 \leq f \in L^1$ it follows using i) of (6.2) that $Tf \prec f$ for all $f \leq 0$ in L^1. Now let $f \in L^1$. Then setting $Tf = Tf^+ + T(-f^-) = g_1 + g_2$ we obtain, using (4.9) and (10.1), that for all $0 \leq t \leq a = \mu(X)$

$$\int_0^t \delta_g(u)du = \int_0^t \delta_{g_1+g_2}(u)du \leq \int_0^t \delta_{g_1}(u)du + \int_0^t \delta_{g_2}(u)du =$$

$$= \int_0^t \delta_{f^+}(u)du + \int_0^t \delta_{-f^-}(u)du = \int_0^t \delta_{f^+}(u)du - \int_0^t \delta_{f^-}(a-u)du = \int_0^t \delta_f(u)du .$$

For $t = a$ we have $\int_0^a \delta_g(u)du = \int (Tf)d\mu = \int Tf^+ d\mu - \int Tf^- d\mu$ $= \int f^+ d\mu - \int f^- d\mu = \int f d\mu = \int_0^a \delta_f(u)du$.

ii) Following the proof of Ryff in [19], we see immediately by using (8.1) that $Tf \prec f$ for every simple function f which is non-negative and so by i) of (14.1) we see that $Tf \prec f$ for all simple f. Thus T acts as a contraction in both L^1-norm and the L^∞-norm on the simple functions. Thus T extends uniquely to a positive linear transformation T on L^1. In order to show that this transformation is d.s. we have to show by i) that for every $0 \leq f \in L^1$,

$Tf \prec f$. Let $\{f_n\}$ be an increasing sequence of simple functions such that $f_n \uparrow f$ everywhere, then $Tf_n \uparrow Tf$ and $Tf_n \prec f_n$ $(n = 1,2,\ldots)$ implies $Tf \prec f$ and the proof is complete.

Of course any linear transformation T of $L^1(X,\mu)$ into $L^1(X,\mu)$ is d.s. whenever $Tf \sim f$ for all $f \in L^1$. For instance if τ is a <u>measure preserving</u> transformation of the given measure space, then the transformation $T_\tau f = f(\tau)$ satisfies $Tf \sim f$ for all $f \in L^1$, and so is d.s. Another important class of such transformations are furnished by the so-called <u>conditional expectations</u>. Let Λ_1 be a σ-subalgebra of Λ and let $f \in L^1(X,\mu)$. Then it follows from the Radon-Nikodym theorem that there exists a unique Λ_1 -measurable function Tf such that $\int_E f d\mu = \int_E Tf d\mu$ for all $E \in \Lambda_1$. It is then easy to see that T satisfies the conditions ii) of (14.1), and so T is d.s.

A special case of this process is furnished by the following example. Let $X = X_o \cup (\bigcup_{i=1}^{\infty} X_i)$ be decomposed into a disjoint system of measurable sets of positive measure. Then, if Λ_1 is the σ-algebra generated by the family of sets $\Lambda \cap X_o$ and $\{X_i : i = 1,2,\ldots\}$, the Λ_1-conditional expectation T can be expressed as follows:

$$(14.2) \qquad Tf = fc_{X_o} + \sum_{i=1}^{\infty} \frac{1}{\mu(X_i)} \left(\int_{X_i} f d\mu\right) c_{X_i}$$

The reader should observe that we have already met this special case in section 7.

(14.3) THEOREM. <u>Let</u> ρ <u>be a Fatou norm which is u.r.i. or r.i. in case the measure is</u> μ-<u>adequate, and let as always</u> $L^\infty \subset L^\rho \subset L^1$. <u>Then every linear transformation</u> T <u>of</u> L^ρ <u>into</u> L^ρ <u>which satisfies</u> $Tf \prec f$ <u>for all</u> $0 \leq f \in L^\rho$ <u>is the restriction of some d.s. transformation of</u> L^1 <u>into</u> L^1, <u>and its associate</u> T' <u>maps</u> $L^{\rho'}$ <u>into</u> $L^{\rho'}$. <u>Furthermore</u>, $\rho(Tf) \leq \rho(f)$ <u>for all</u> $f \in L^\rho$ <u>and</u> $\rho'(T'f) \leq \rho'(f)$ <u>for all</u> $f \in L^{\rho'}$.

PROOF. Using (14.1) we see that we have only to show that $\rho(Tf) \leq \rho(f)$ for all $f \in L^\rho$. But this follows from (11.11).

In [9], Ellis and Halperin introduced the so-called <u>levelling length</u> property for function norms. In our terminology this condition reads as follows. Let ρ be a function norm. Then ρ has the levelling length property whenever for every set E of positive measure $\rho(f_1) \leq \rho(f)$, where $f_1 = c_{X-E} + (\frac{1}{\mu(E)} \int_E f d\mu) c_E$. Hence, using (14.2) we have the following result for r.i.-function norms.

(14.4) THEOREM. <u>Every Fatou norm which is</u> u.r.i. <u>or</u> r.i. <u>if</u> μ <u>is adequate has the levelling length property.</u>

REMARK. The equivalence of i) and ii) in Theorem 13.1 of Hardy, Littlewood and Pólya for general measure spaces, i.e., $f \prec g \in L^1(X,\mu)$ if and only if there exists a d.s. transformation T such that $f = Tg$, seems to be an open problem. For the case of the Lebesgue measure space of a finite interval the affirmative answer was given by Ryff in [20].

15. Some properties of the sets $\Omega(f_o) = \{f : f \prec f_o\}$.

Let (X,Λ,μ) be again a finite measure space and let ρ be a Fatou norm on $M^+(X,\mu)$ such that $L^\rho \subset L^1$ and L^ρ is universally rearrangement-invariant, i.e., $f_1 \prec f_2 \in L^\rho$ implies $f_1 \in L^\rho$ (see (11.8)). Then for every $f \in L^\rho$ and $g \in L^{\rho'}$ we have that $\delta_f\delta_g \in L^1[0,a]$, where $a = \mu(X)$. According to (10.3) the function $p_g(f) = \int_0^a \delta_f\delta_g dt$ is sublinear on L^ρ for all $g \in L^{\rho'}$. Furthermore p_g has the following properties.

(15.1) i) $-p_g(-f) = \int_0^a \delta_f(a-u)\delta_g(u)du \leq \int f_g du \leq$

$\leq \int_0^a \delta_f(u)\delta_g(u)du$ for all $f \in L^\rho$ and $g \in L^{\rho'}$.

ii) $|p_g(f)| \leq \int_0^a \delta_{|f|}\delta_{|g|}dt$ for all $f \in L^\rho$ and $g \in L^{\rho'}$.

iii) If $f_1 \prec f_2 \in L^\rho$, then $p_g(f_1) \leq p_g(f_2)$ for all $g \in L^{\rho'}$.

iv) If $f_1 \prec\prec f_2$ and $f_1, f_2 \in L^\rho$, then $p_g(f_1) \leq$ $\leq p_g(f_2)$ for all $0 \leq g \in L^{\rho'}$.

v) If F is a linear functional on L^ρ which is p_g-continuous for some $g \in L^{\rho'}$, then there exists an element $h \in L^{\rho'}$ such that $F(f) = \int fh d\mu$ for all $f \in L^\rho$.

Of course the same results hold by interchanging the role of f and g .

(15.2) DEFINITION. *If* $f_o \in L^\rho$ *, then* $\Omega_o = \Omega(f_o)$ *denotes the set of all* $f_1 \in L^1(X,\mu)$ *such that* $f_1 \prec f$.

(15.3) LEMMA. i) *For all* $f \in L^\rho$, $\Omega(f) \subset L^\rho$.

ii) $f_m = \frac{1}{\mu(X)} (\int_X f d\mu) c_X \in \Omega(f)$ *for all* $f \in L^\rho$ *and* $f_1 \in \Omega(f)$ *implies that* $f_m \prec f_1$, i.e., f_m *is the smallest element of* $\Omega(f)$ *with respect to* $\prec$.

For the following result we shall need the following compactness criterium. Let ρ be a saturated Fatou norm. Then a subset $A \subset L^\rho$ is relatively $\sigma(L^\rho, L^{\rho'})$-compact (= relatively $\sigma(L^\rho, L^{\rho'})$-sequentially compact) if and only if $N(g) = \sup (\int |fg| d\mu : f \in A)$ is finite for every $g \in L^{\rho'}$ and $0 \le g_n \in L^{\rho'}$ such that $g_n \downarrow 0$ implies $N(g_n) \downarrow 0$. For a discussion of this result and related results we refer the reader to section 5 of [13].

The locally convex topology on L^ρ generated by the family of Riesz seminorms $\int |fg| d\mu$, $g \in L^{\rho'}$ will be denoted by $|\sigma|(L^\rho, L^{\rho'})$. It is easy to see that $L^{\rho'}$ is the $|\sigma|(L^\rho, L^{\rho'})$-dual of L^ρ .

We shall now prove the following important result.

(15.3) THEOREM. *For every* $f \in L^\rho$, $\Omega(f)$ *is a* $\sigma(L^\rho, L^{\rho'})$-*compact and convex subset of* L^ρ

PROOF. We shall first show that $\Omega(f)$ is convex. To this end, let $f_1, f_2 \in \Omega(f)$ and let $g = f_1 + (1-r)f_2$, $0 \leq r \leq 1$. Then it follows from (10.1) that $\int_0^t \delta_g(u)du \leq$
$\leq r \int_0^t \delta_{f_1}(u)du + (1-r)\int_0^t \delta_{f_2}(u)du \leq \int_0^t \delta_f(u)du$ for all $0 \leq t < a$ and equality for $t = a$ follows easily. In order to show that $\Omega(f)$ is $\sigma(L^\rho, L^{\rho'})$-closed we have only to show that it is $|\sigma|(L^\rho, L^{\rho'})$-closed since the $|\sigma|(L^\rho, L^{\rho'})$-dual of L^ρ is $L^{\rho'}$. To this end, it is easy to see that we have only to show that if $f_n \in \Omega(f)$ $(n = 1,2,\ldots)$ and $f_n \to f_o$ in the $|\sigma|(L^\rho, L^{\rho'})$-topology, then $f_o \in \Omega(f)$. Then $|f_n - f| \to 0$ in measure as $n \to \infty$, and so $\int_0^t \delta_{|f_n - f_o|}(u)du \to 0$ as $n \to \infty$ for all $0 \leq t \leq a$. Using (10.1) and (9.5) we have

$|\int_0^t \delta_{f_n} du - \int_0^t \delta_{f_o} du| \leq \int_0^t \delta_{|f_n - f|} du$ for all n and for all

$0 \leq t \leq a$, and so $\int_0^t \delta_{f_n} du \to \int_0^t \delta_{f_o} du$ as $n \to \infty$ and for all $0 \leq t \leq a$. Hence, $\int_0^t \delta_{f_o} du \leq \int_0^t \delta_f du$ for all $0 \leq t \leq a$, with equality for $t = a$, i.e., $f_o \in \Omega(f)$.

Observing that $N(g) = \sup(\int |f'g| d\mu : f' \in \Omega(f)) \leq \int_0^a \delta_{|f|}\delta_{|g|}dt$ we see immediately from the result quoted above that $\Omega(f)$ is $\sigma(L^\rho, L^{\rho'})$-compact, and the proof is finished.

REMARK. For the spaces $L^1[0,1]$, the present result is due to Ryff (see [20] Theorem 2 of section 3). The above proof is entirely different from the proof given by Ryff for the space $L^1[0,1]$.

(15.4) THEOREM. <u>For every</u> $f \in L^{\rho}$, $\Omega(f)$ <u>is</u> ρ-<u>bounded</u>.

PROOF. If $\Omega(f)$ is not ρ-bounded, then there exists a sequence $f_n \in \Omega(f)$ $(n = 1,2,\ldots)$ such that $\rho(f_n) \to \infty$. Since $\Omega(f)$ is $\sigma(L^{\rho}, L^{\rho'})$-compact we conclude that $\sup_n \int f_n g d\mu$ is finite for every $g \in L^{\rho'}$, and so by the Banach-Steinhaus theorem ($L^{\rho'}$ is ρ'-complete) we obtain that $\sup_n(\sup(\int f_n g d\mu : \rho'(g) \leq 1) = \sup_n \rho(f_n) < \infty$ since $\rho = \rho''$ by (11.3). Contradiction and the proof is finished.

The sets $\Omega(f)$ can be characterized also as follows.

(15.5) THEOREM. $f_1 \in \Omega(f_o)$ <u>if and only if</u> $\int f_1 g d\mu \leq p_{f_o}(g)$ <u>for all</u> $g \in L^{\rho'}$.

PROOF. If $f_1 \in \Omega(f_o)$, then $\int f_1 g d\mu \leq \int_o^a \delta_{f_1} \delta_g dt \leq$ $\leq \int_o^a \delta_{f_o} \delta_g \, dt = p_g(f_o)$ (by ii) of (5.1)). In order to prove the converse we set $H = \{h : h \in L^{\rho}$ and $\int gh d\mu \leq p_{f_o}(g)$ for all $g \in L^{\rho'}\}$. Then $\Omega(f_o) \subset H$. If $\Omega(f_o) \neq H$, then there is an element $h_o \in H$ such that $h_o \notin \Omega(f_o)$. Hence, from the separation theorem for closed convex sets it follows that there exists an element $g_o \in L^{\rho'}$ such that $\sup(\int g_o h d\mu : h \in \Omega(f_o)) <$ $< \int g_o h_o d\mu$. Then using (9.3) and (8.2) we obtain that $\int_o^a \delta_{f_o} \delta_{g_o} dt < \int g_o h_o d\mu < \int \delta_{g_o} \delta_{h_o} < p_{f_o}(g_o)$ and a contradiction is obtained.

Using the separation theorem for closed convex sets the following result can easily be established.

(15.6) THEOREM. i) _If_ μ _is adequate, and so if_ μ _is non-atomic, then for every_ $f \in L^{\rho}$, $\Omega(f)$ _is the_ $\sigma(L^{\rho}, L^{\rho'})$-_closed convex hull of the subset of_ $\Omega(f)$ _of all functions_ f' _satisfying_ $f' \sim f$.

ii) _For every_ $f \in L^{\rho}$, $\Omega(f)$ _is the_ $\sigma(L^{\rho}, L^{\rho'})$-_closed convex hull of the subset of_ $\Omega(f)$ _of all functions_ $T_{\mu} f_1$ _of all_ $f_1 \in M(X_1, \mu_1)$ _such that_ $f_1 \sim f$.

PROOF. We shall only sketch the proof of i). Let A be the $\sigma(L^{\rho}, L^{\rho'})$-closed convex hull of the set of all $f' \sim f$. Then $A \subset \Omega(f)$. If $A \neq \Omega(f)$, then there is an element $f_o \in \Omega(f)$ such that $f_o \notin A$. From the separation theorem for closed convex sets it follows that there exists an element $g_o \in L^{\rho'}$ such that $\alpha = \sup(\int f' g_o d\mu : f' \in A) < \int f_o g_o d\mu \leq$ $\leq \int_o^a \delta_{f_o} \delta_{g_o} d\mu \leq \int_o^a \delta_f \delta_{g_o} d\mu$ by ii) of (5.1). Since μ is adequate we have that $\alpha = \int_o^a \delta_f \delta_{g_o} dt$, and so $\alpha < \int_o^a \delta_f \delta_{g_o} dt$ is a contradiction and the proof is finished.

REMARK. Recently Z. Nehari considered in [16] the following problem. Let (X, Λ, μ) be a finite non-atomic measure space and let $E \in \Lambda$ be a set of positive measure. Determine the smallest closed convex subset A in $L^1(X, \mu)$ which contains all the c_F, $F \in \Lambda$ and $\mu(F) = \mu(E)$. It was shown in [16] that if $f \in L^1[0,1]$ satisfies $0 \leq f \leq 1$, ess.sup $f = 1$, ess. inf $f = 0$ and $\int f d\mu = \mu(E)$, then $f \in A$. Since every $f \in L^1$ which is equimeasurable with c_E is of the form c_F where $\mu(F) = \mu(E)$ we see that the preceding result provides a complete answer to Nehari's question namely $A = \Omega(c_E)$, and so $f \in A$ if and only if $0 \leq f \leq 1$ and $\int f d\mu = \mu(E)$.

16. Every u.r.i.-L^ρ-space has an equivalent u.r.i.-Fatou norm.

The purpose of this section is to prove the following interesting result.

(16.1) THEOREM. *If ρ is a saturated Fatou norm such that L^ρ is universally rearrangement-invariant, then there exists a universal rearrangement invariant saturated Fatou norm ρ_1 such that ρ and ρ_1 are equivalent, i.e., there exists a constant $\gamma > 0$ such that $\gamma^{-1}\rho \leq \rho_1 \leq \gamma\rho$.*

PROOF. For every $0 \leq f \in M(X,\mu)$ we set

$$\rho_1(f) = \sup(\int f'|g|d\mu : f' \in \Omega(f) \text{ and } \rho'(g) \leq 1) =$$

$$= \sup(\int_0^a \delta_f \delta_{|g|} dt : \rho'(g) \leq 1) .$$

It is easy to see that ρ_1 is a saturated Fatou norm and that ρ_1 is u.r.i.

Furthermore, $\rho \leq \rho_1$. Since $\Omega(f)$ is ρ-bounded by (15.5) we see that $\rho(f) < \infty$ implies $\rho_1(f) < \infty$. Thus $L^{\rho_1} = L^{\rho}$. Then since both spaces are complete and norm convergence implies convergence in measure it follows from the closed-graph theorem that ρ and ρ_1 are equivalent, and the proof is complete.

17. Extremal properties of $\Omega(f)$ and some related problems.

It was shown in [20] that if $f' \sim f$ and $f \in L^1[0,1]$, then f' is an extreme point of $\Omega(f)$. The proof of this result carries over immediately for general finite measure spaces, and so we have the following result. For the proof we refer the reader to [20], Theorem 4 of section 4.

(17.1) THEOREM. *If* $f \in L^{\rho}(X,\mu)$ *and* $f' \sim f$, *then* f' *is an extreme point of* $\Omega(f)$.

The following problem seems to be open.

PROBLEM 1. Determine all the extreme points of $\Omega(f)$?

It seems reasonable to conjecture that at least in the case that μ is adequate $f' \in \Omega(f)$ is extreme if and only if $f' \sim f$.

We shall now present as a first attempt to a solution of the problem a criteron which is necessary and sufficient for an element of $f' \in \Omega(f)$ to be extreme. Since $\Omega(f) \subset L^1$ it is sufficient to consider this problem for L^1-spaces only.

To this end, let $f_o \in L^1(X,\mu)$ and let $\Omega_o = \Omega(f_o)$. For every $f \in \Omega_o$ we define the following function p^*_f on L^∞ by means of the following definition.

$$(17.2) \qquad p^*_f(g) = \int_o^a \delta_{f_o} \delta_g \, du - \int fg du \ , \ g \in L^\infty .$$

Then by (10.3) and (8.2) we have that p^*_f is non-negative and sublinear. Furthermore, $p^*_f(-g) = \int fg du - \int_o^a \delta_{f_o}(u) \, \delta_g(a-u) du =$ $q^*_f(g)$ is also sub linear and non-negative on L^∞.

The following result can now be shown to hold.

(17.3) THEOREM. f <u>is not an extreme point of</u> $\Omega_o = \Omega(f_o)$ <u>if and only if there exists an element</u> $f_1 \in L^1(X,\mu)$ <u>such that</u> $f_1 \neq 0$ <u>and</u> $|\int f_1 g du| \leq p^*_f(g)$ <u>for all</u> $g \in L^\infty$.

PROOF. Assume first that for some $f_1 \neq 0$ in L^1 we have $|\int f_1 g du| \leq p^*_f(g)$ for all $g \in L^\infty$. Then by (15.5) we have $f + f_1 \in \Omega_o$ and also $f - f_1 \in \Omega_o$, i.e., $f = \frac{1}{2}((f+f_1)+(f-f_1))$. Thus f is not an extreme point.

If f is not an extreme point, then $f = \frac{1}{2}(h_1+h_2)$ with $h_1, h_2 \in \Omega_o$ and $h_1 \neq h_2$. Let $f_1 = \frac{h_2-h_1}{2}$. Then $f + f_1 = h_2 \in \Omega_o$ and $f - f_1 = h_1 \in \Omega_o$, and so by (15.5) we obtain that $|\int f_1 g d\mu| \leq p^*_f(g)$ for all $g \in L^\infty$ and the proof if finished.

(17.4) COROLLARY. f _is not an extreme point of_ Ω_o _if and only if there exists an element_ $0 \neq f_1 \in L^1$ _such that_ $\int f_1 g d\mu \leq p^*_f(g)$ _and_ $\int f_1 g d\mu \leq q^*_f(g)$ _for all_ $g \in L^\infty$.

For every finite system of non-negative linear functionals there exists a largest non-negative sub linear functional which is majorized by each of the given systems. For every $f \in \Omega(f)$ we shall denote the $\inf(p^*_f, q^*_f)$ by r_f; and it can be given in the following form.

(17.5) For all $g \in L^\infty$, $r_f(g) = \inf(p^*_f(g_1)+q^*_f(g_2):$ $g_1+g_2 = g$, $g_1, g_2 \in L^\infty)$. Then (17.4) implies immediately the following result.

(17.6) THEOREM. $f \in \Omega_o = \Omega(f_o)$ _is extreme if and only if_ $r_f(g) = 0$ _for all_ $g \in L^\infty$.

If $f_o = c_X$, then $\Omega_o = \{c_X\}$, and so its set of extreme points is closed. In this connection the following problem seems to be open.

PROBLEM 2. For every $f \in L^1(X,\mu)$ determine the $\sigma(L^1, L^\infty)$-closure of the set $\{f' : f' \sim f\}$?

The preorder relation $f_1 \prec f_2$ in L^1 can be extended in the following manner to include certain measures. To this end, let $f_o \in L^1(X,\mu)$ and let $K(f_o)$ denote the cone of all universal Schur-convex functions which are finite on $\Omega_o = \Omega(f_o)$. Furthermore, $WL_o = WL(\Omega_o)$ shall denote the set of all positive Radon measure defined on the $\sigma(L^1,L^\infty)$-compact set Ω_o .

Let $\nu_1, \nu_2 \in WL_o$. Then we set $\nu_1 \prec \nu_2$ whenever $\nu_1(\Phi) \leq \nu_2(\Phi)$ for all $\Phi \in K(f_o)$. If $\nu_1 \prec \nu_2$, then $\nu_1(\Omega_o) = \nu_2(\Omega_o)$. Furthermore, it is easy to see that if $f_1, f_2 \in \Omega_o$, then $f_1 \prec f_2$ if and only if $\nu_1 \prec \nu_2$, where $\nu_1 = \epsilon_{f_1}$ and $\nu_2 = \epsilon_{f_2}$ are the discrete measures of total measures 1 supported by $\{f_1\}$ and $\{f_2\}$ respectively.

PROBLEM 3. Find all the maximal measures of $WL(\Omega_o)$ with respect to the preorder relation $\prec$ introduced above? Is an element $f \in \Omega_o$ extreme if and only if ϵ_f is maximal?

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