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MATHEMATICS

On Generalized Orlicz Spaces

by

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1. We call a *φ-function* (and denote by Greek letters $\varphi, \psi, \ldots$) every continuous function $\varphi(u)$ defined for $u \geqslant 0$, non decreasing, vanishing only for $u = 0$, and such that $\varphi(u) \to \infty$ for $u \to \infty$.

For $x(t)$ measurable in $\langle a, b\rangle$ (where $-\infty \leqslant a < b \leqslant +\infty$) we write $\mathcal{I}_\varphi(x) = \int_a^b \varphi(|x(t)|)\,dt$. If $\mathcal{I}_\varphi(x) < \infty$, then x is called *φ-integrable,* we then write $x \in L^\varphi\langle a, b\rangle$. If $\lambda x \in L^\varphi\langle a, b\rangle$ for a constant $\lambda = \lambda_x > 0$, we write $x \in L^{*\varphi}\langle a, b\rangle$. The set $L^\varphi\langle a, b\rangle$ is convex and the set $L^{*\varphi}\langle a, b\rangle$ is a linear space (called the Orlicz space, if φ is convex, [2]).

In this paper I give some results pertaining to the question, which theorems known for the space $L^{*\varphi}\langle a, b\rangle$ in case of a convex φ can be generalized to arbitrary φ-functions.

We say that φ is *non-weaker than ψ for large u* (in symbols $\psi \overset{l}{\prec} \varphi$), if there are constant $c, d, l\ k > 0$ such that

$$(+)\qquad c\psi(lu) \leqslant d\varphi(ku) \quad \text{for} \quad u \geqslant u_0.$$

If $\varphi \overset{l}{\prec} \psi$ and $\psi \overset{l}{\prec} \varphi$, we write $\varphi \overset{l}{\sim} \psi$ and call this relation the *equivalence for large u*$\cdot \varphi \overset{l}{\sim} \psi$ if and only if there are constants $a, b, k_1, k_2 > 0$ such that for sufficiently large u

$$a\varphi(k_1 u) \leqslant \psi(u) \leqslant b\varphi(k_2 u),$$

$\overset{l}{\sim}$ is an equivalence relation and $\overset{l}{\prec}$ is transitive. If φ is convex, then our definitions of $\overset{l}{\prec}$ and $\overset{l}{\sim}$ are equivalent to those of [2]; for arbitrary φ they are more general that those given in [1] and [2]. We say that φ satisfies *the condition* (Δ_α) for large u, if

$$\varphi(\alpha u) \leqslant d_\alpha \varphi(u) \quad \text{for} \quad u \geqslant u_0(\alpha),$$

where $d_\alpha > 1$ is a constant. Moreover, φ is said to satisfy *the condition* (Λ_α), $a > 1$, for large u, if

$$\varphi(u) c_\alpha \leqslant \varphi(au) \quad \text{for} \quad u \geqslant u_0(a),$$

$c_\alpha > 1$ being a constant.

If $\varphi \overset{l}{\sim} \psi$ and if φ satisfies the condition (Δ_α) or (Λ_α), then so does ψ.

If the inequality $(+)$ is satisfied for $u \geqslant u_0 = 0$, we say that φ *is non-weaker than* ψ for $u \geqslant 0$ (or: for all u), in symbols $\psi \overset{a}{\prec} \varphi$. In a similar way we introduce the relation of equivalence for all u (writing $\varphi \overset{a}{\sim} \psi$) and the conditions (Δ_α) and (Λ_α) for all u.

2.1. The following properties are equivalent: (a) φ satisfies (Δ_2) for large u; (b) φ satisfies (Δ_α), $a > 1$, for large u; (c) $\varphi \overset{l}{\sim} \chi$, where $\chi(u) = \psi(u^r)$ and ψ is a concave φ-function. The exponent r may be chosen equal to $r = \lg d_\alpha / \lg a$, where, in case (b), d_α is the constant defined in (Δ_α) and in case (c), χ satisfies (Δ_α) with the constant d_α.

2.2. The following properties are equivalent: (a) φ satisfies (Λ_α), $a > 1$, for large u, (b) $\varphi \overset{l}{\sim} \chi$, where $\chi(u) = \psi(u^s)$ and ψ is a convex φ-function. The exponent s may be taken $s = \lg c_\alpha / \lg a$, where, assuming (a), c_α is the constant defined in (Λ_α), and, assuming (b), χ satisfies (Λ_α) with the constant c_α.

2.3. A necessary and sufficient condition for $\varphi \overset{l}{\sim} \psi$, where ψ is a convex function, is

$$\frac{\varphi(u_2)}{u_2} \geqslant m \frac{\varphi(n u_1)}{u_1} \quad \text{for} \quad u_2 \geqslant u_1 \geqslant u_0,$$

m, n being positive constants. Replacing in the last inequality $\geqslant$ by $\leqslant$ we obtain a necessary and sufficient condition for equivalence of φ to a concave function for large u. Theorems analogous to 2.1 — 2.3 hold also in the case $\overset{a}{\sim}$ and for the conditions (Δ_α), (Λ_α) for all u.

3. A necessary and sufficient condition for the inclusion

$$\bigcap_{\nu=1}^{\infty} L^{\varphi_\nu} \langle a, b \rangle \subset L^{\psi} \langle a, b \rangle$$

is the existence of a positive integer m and of a constant $d > 0$ such that

$$(*) \qquad \psi(u) \leqslant d \sup [\varphi_1(u), \varphi_2(u), \ldots, \varphi_m(u)],$$

where $(*)$ is to be satisfied for $u \geqslant u_0$, if $\langle a, b \rangle$ is finite and for $u \geqslant 0$, if $\langle a, b \rangle$ is infinite.

3.1. The inclusion

$$L^{\psi} \langle a, b \rangle \subset \bigcup_{\nu=1}^{\infty} L^{\varphi_\nu} \langle a, b \rangle$$

holds if and only if there are: a positive integer m and a constant $c_m > 0$ such that the inequality

$$(**) \qquad \varphi_m(u)\, c_m \leqslant \psi(u)$$

is satisfied for all $u \geqslant u_0$, if $\langle a, b\rangle$ is finite, and for $u \geqslant 0$, if $\langle a, b\rangle$ is infinite.

As corollaries to 3 and 3.1 we note

3.2. (a) If $L^{\varphi_n}\langle a, b\rangle \subset L^{\varphi_{n+1}}\langle a, b\rangle$, $L^{\varphi_{n+1}}\langle a, b\rangle \neq L^{\varphi_n}\langle a, b\rangle$ for $n = 1, 2, \ldots$, then for every ψ, $L^{\psi}\langle a, b\rangle \neq \bigcup_{\nu=1}^{\infty} L^{\varphi_\nu}\langle a, b\rangle$;

(b) If $L^{\varphi_n}\langle a, b\rangle \supset L^{\varphi_{n+1}}\langle a, b\rangle$, $L^{\varphi_{n+1}}\langle a, b\rangle \neq L^{\varphi_n}\langle a, b\rangle$ for $n = 1, 2 \ldots$, then for every ψ, $L^{\psi}\langle a, b\rangle \neq \bigcap_{\nu=1}^{\infty} L^{\varphi_\nu}\langle a, b\rangle$.

3.3. A necessary and sufficient condition for $L^{\varphi}\langle a, b\rangle = L^{*\varphi}\langle a, b\rangle$ is that φ should satisfy (Δ_2) for large u, or for $u \geqslant 0$, depending whether $\langle a, b\rangle$ is finite or infinite.

3.4. If φ does not satisfy (Δ_2) for large u and if $\langle a, b\rangle$ is finite, then there exist functions x and y such that

$$\mathcal{I}_\varphi(x) < \infty, \quad \mathcal{I}_\varphi(\lambda x) = \infty \quad \text{for} \quad \lambda > 1,$$

$$\mathcal{I}_\varphi(\lambda y) < \infty \quad \text{for} \quad 0 < \lambda < 1, \quad \mathcal{I}_\varphi(y) = \infty.$$

An analogous theorem remains true for infinite $\langle a, b\rangle$; the condition (Δ_2) must then be satisfied for all u.

3.5. A necessary and sufficient condition for the inclusion $L^{*\varphi_2}\langle a, b\rangle \subset L^{*\varphi_1}\langle a, b\rangle$ (resp. the equation $L^{*\varphi_2}\langle a, b\rangle = L^{*\varphi_1}\langle a, b\rangle$, to hold is $\varphi_1 \overset{l}{\prec} \varphi_2$ (resp. $\varphi_1 \overset{l}{\sim} \varphi_2$), if $\langle a, b\rangle$ is finite and $\varphi_1 \overset{a}{\prec} \varphi_2$ (resp. $\varphi_1 \overset{a}{\sim} \varphi_2$), if $\langle a, b\rangle$ is infinite.

4. In this section we shall introduce a kind of convergence (called φ-convergence) in the space $L^{*\varphi}\langle a, b\rangle$. The φ-convergence is a special case of the modular convergence in the sense of Musielak and Orlicz (cf. [4]). A sequence of elements x_n of $L^{*\varphi}\langle a, b\rangle$ is called *φ-convergent* to x_0 (in symbols $x_n \xrightarrow{\varphi} x_0$), if $\mathcal{I}_\varphi(\lambda(x_n - x_0)) \to 0$ with a certain $\lambda > 0$ (depending on the sequence x_n). In $L^{*\varphi}\langle a, b\rangle$, an F-norm may be introduced in such a way that the convergence of a sequence x_n to 0 with respect to this norm implies $\mathcal{I}_\varphi(x_n) \to 0$ and so $x_n \xrightarrow{\varphi} 0$, too. This norm defined by the formula $\|x\|_\varphi = \inf\{\varepsilon > 0 : \mathcal{I}_\varphi(x/\varepsilon) \leqslant \varepsilon\}$ (cf. [3], [4]) will be called the *norm generated by* φ. By the symbol $[L^{*\varphi}\langle a, b\rangle, \|\,\|_\varphi]$ we denote the Fréchet space $L^{*\varphi}\langle a, b\rangle$ with the norm $\|\,\|_\varphi$.

4.1. Let the interval $\langle a, b\rangle$ be finite; the following properties are equivalent: (a) φ satisfies the condition (Δ_2) for large u; (b) $L^{*\varphi}\langle a, b\rangle$ is separable; (c) $\mathcal{I}_\varphi(x_n) \to 0$ implies $\|x_n\|_\varphi \to 0$. An analogous theorem is true for infinite $\langle a, b\rangle$, we have only to replace in (a) the expression "for large u" by "or $u \geqslant 0$". As regards (a) and (b) in the case o a convex φ, (cf. e.g. [5]).

4.2. An element $x \in L^{*\varphi}\langle a, b\rangle$ is called *finite*, if $\mathcal{I}_\varphi(\lambda x) < \infty$ for an arbitrary $\lambda > 0$. We denote the set of the finite elements by $M^\varphi\langle a, b\rangle$.

4.3. Let $\langle a, b\rangle$ be finite; the element x is finite if and only if $\|x\|_\varphi$ is absolutely continuous, i.e. $\|x\chi_{E_n}\|_\varphi \to 0$ for $E_n \subset \langle a, b\rangle$, $|E_n| \to 0$. Here χ_E denotes the characteristic function of the set E. An analogous theorem is true for infinite $\langle a, b\rangle$, assuming φ to satisfy the condition (Δ_2) for small u *). If (Δ_2) does not hold for small u, then there exist elements with absolutely continuous norm which are not finite.

4.4. The set $M^\varphi\langle a, b\rangle$ is identical with the smallest linear subspace closed in $[L^{*\varphi}\langle a, b\rangle, \|\,\|_\varphi]$ containing the bounded functions in the case of finite $\langle a, b\rangle$ and containing the bounded functions equal to zero outside a certain finite subinterval of $\langle a, b\rangle$ in the case of infinite $\langle a, b\rangle$. Hence, $M^\varphi\langle a, b\rangle$ is separable with respect to the norm $\|\,\|_\varphi$.

4.5. A necessary and sufficient condition for $M^\varphi\langle a, b\rangle = L^{*\varphi}\langle a, b\rangle$ is that φ should satisfy (Δ_2) for large u, if $\langle a, b\rangle$ is finite and for $u \geqslant 0$, if $\langle a, b\rangle$ is infinite.

4.6. The following three relations are equivalent: (a) $L^{*\varphi}\langle a, b\rangle = L^{*\psi}\langle a, b\rangle$; (b) $M^\varphi\langle a, b\rangle = M^\psi\langle a, b\rangle$; (c) $\|x_n\|_\psi \to 0$ implies $\|x_n\|_\varphi \to 0$ and conversely, where in the case of finite $\langle a, b\rangle$ we restrict x_n to bounded measurable functions and for infinite $\langle a, b\rangle$ to bounded measurable functions vanishing outside a certain finite subinterval of $\langle a, b\rangle$, respectively; (d) $x_n \xrightarrow{\psi} 0$ implies $x_n \xrightarrow{\varphi} 0$ and conversely, where x_n are taken as in (c).

4.61. If $\varphi \overset{l}{\sim} \psi$, $\langle a, b\rangle$ is finite, the relations $\|x_n\|_\psi \to 0$ and $\|x_n\|_\varphi \to 0$ are equivalent and analogously in case $\overset{a}{\sim}$ and $\langle a, b\rangle$ infinite.

4.7. If $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and if ψ is a convex function, then in $L^{*\varphi}\langle a, b\rangle$ norm

$$\|x\|_\varphi^* = \inf\left\{\varepsilon > 0 : \mathcal{I}_\varphi\left(\frac{x}{\varepsilon^{1/s}}\right) \leqslant 1\right\}$$

may be defined. This norm is s-homogenous, i. e. $\|\lambda x\|_\varphi^* = |\lambda|^s \|x\|_\varphi^*$ and equivalent to the norm generated by φ; consequently, $\|x_n\|_\varphi^* \to 0$ implies $\mathcal{I}_\varphi(x_n) \to 0$. For $s = 1$, this theorem is known.

4.71. If an s-homogenous, complete norm $\|\,\|^0$ is defined in $L^{*\varphi}\langle a, b\rangle$, $0 < s \leqslant 1$ and if the convergence to 0 with respect to this norm implies

*) Condition (Δ_α) for small u means that $\varphi(u)\, c_\alpha \leqslant \varphi(\alpha u)$ holds for $u \leqslant u_0(\alpha)$, where $c_\alpha > 1$ is a constant.

the modular convergence to 0, then $\varphi \overset{l}{\sim} \chi$, if $\langle a, b\rangle$ is finite and $\varphi \overset{a}{\sim} \chi$, if $\langle a, b\rangle$ is infinite, $\chi(u) = \psi(u^s)$; ψ is a convex function. Special cases of 4.71 were proved in [3] and [6]. Let us note that if φ satisfies the condition (Λ_α) and the constant c_α is known, it is possible to introduce the s-homogenous norm and the degree of homogeneity of the s-norm in 4.7 is completely determined; $s = \lg c_\alpha / \lg \alpha$.

4.8. A set $X_0 \subset L^{*\varphi}\langle a, b\rangle$ is called *φ-bounded*, if $t_n x_n \overset{\varphi}{\to} 0$ whenever $x_n \in X_0$ and $t_n \to 0$. A set X_0 is φ-bounded if and only if it is bounded in $[L^{*\varphi}\langle a, b\rangle. \|\|_\varphi]$.

4.81. (*a*) If φ satisfies the condition (Λ_α) for large u in the case of $\langle a, b\rangle$ finite, and for $u \geqslant 0$ in the case of $\langle a, b\rangle$ infinite, then every set $X_r = \{x : \mathfrak{I}_\varphi(x) \leqslant r\}$ is φ-bounded; (b) if a certain set $X_r = \{x : \mathfrak{I}_\varphi(x) \leqslant r\}$ is φ-bounded, then φ satisfies (Λ_α) for large u, if $\langle a, b\rangle$ is finite, and for $u > 0$, if $\langle a, b\rangle$ is infinite.

The case of φ satisfying the condition (Δ_2), $\langle a, b\rangle$ finite, was considered in [6].

Full proofs of the above theorems will appear in Studia Mathematica. The author wishes to express sincere thanks to Professor. W. Orlicz for his helpful criticism and valuable remarks.

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