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MATHEMATICS

Some Further Properties of φ-Functions

by

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1. Denote by S the class of continuous positive functions defined for $u > 0$, and define the following functions (which may assume also the value ∞):

$$\underline{h}_\varphi(\lambda) = \varliminf_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)}, \qquad \overline{h}_\varphi(\lambda) = \varlimsup_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)} \qquad \text{for } \lambda > 0.$$

A function $\varphi \in S$ will be called a quasi φ-function (briefly: $q\varphi$-function), if there exist the limits

$$(*) \quad s_\varphi = \lim_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda}, \qquad (**)\ \sigma_\varphi = \lim_{\lambda\to 0+} \frac{\lg \overline{h}_\varphi(\lambda)}{-\lg \lambda},$$

finite or infinite. A function φ, continuous and nondecreasing for $u \geqslant 0$, vanishing at zero only and tending to ∞ as $u \to \infty$ is a $q\varphi$-function, and $\sigma_\varphi \geqslant s_\varphi \geqslant 0$ (cf. [5]). Such $q\varphi$-functions are called φ-functions according to the terminology of [4].

Nonincreasing functions of the class S are also $q\varphi$-functions, and $s_\varphi \leqslant \sigma_\varphi \leqslant 0$. To denote $q\varphi$-functions, we shall use Greek letters $\varphi, \psi, \chi, \varrho, \ldots$ In some cases the same symbols are to denote $\varphi \in S$.

Generalizing the definition from [4], functions $\varphi, \psi \in S$ will be said equivalent for large u (l-equivalent), in symbols $\varphi \overset{l}{\sim} \psi$, if

$$a\varphi(k_1 u) \leqslant \psi(u) \leqslant b\varphi(k_2 u) \quad \text{for } u \geqslant u_0,$$

a, b, k_1, k_2 being positive constants. It is easily seen that $\overset{l}{\sim}$ is an equivalence relation.

We shall give now some simple properties of the l-equivalence and of the indices s_φ, σ_φ.

1.1. If $\varphi \overset{l}{\sim} \psi$, where φ is a $q\varphi$-function, $\psi \in S$, then ψ is a $q\varphi$-function, and $s_\varphi = s_\psi$, $\sigma_\varphi = \sigma_\psi$.

1.2. (a) Let $\varphi_r(u) = (\varphi(u))^r$, $r \neq 0$; then $s_{\varphi r} = rs_\varphi$, $\sigma_{\varphi r} = r\sigma_\varphi$, and if $\varphi \overset{l}{\sim} \varphi_1$, then $\varphi_r \overset{l}{\sim} \varphi_{1r}$.

(b) If $\varphi(u) = u^r \psi(u)$, then $s_\psi = r + s_r$, $\sigma_\varphi = r + \sigma_\psi$ (the existence of indices on one side of these equations implies existence of the indices on the other side).

1.3. If $s_\varphi = \sigma_\varphi = r \neq 0$, then such a $q\varphi$-function will be called *quasiregularly increasing*; if $s_\varphi = \sigma_\varphi = 0$, *quasislowly varying* φ-functions of a regular increase in the sense of Karamata [2], i.e. $q\varphi$-functions satisfying the condition

(1.3.1) $$\varphi(u)/\varphi(\lambda u) \to g(\lambda) \text{ for } \lambda > 0,$$

where $g(\lambda)$ is finite and $\neq 0$ for every λ, $g(\lambda)$ not identically equal to 1, are quasiregularly increasing.

If $\lambda > 0$, from (1.3.1) it follows $g(\lambda) = \lambda^{-r}$, $r \neq 0$, for $g(\lambda_1 \lambda_2) = g(\lambda_1) g(\lambda_2)$ and $g(\lambda)$ is obviously of the first class of Baire. Hence, $s_\varphi = \sigma_\varphi = r$.

Assuming $g(\lambda) = 1$ for $\lambda > 0$, we obtain $q\varphi$-functions slowly varying in the sense of Karamata; then $s_\varphi = \sigma_\varphi = 0$. **1**.2 implies immediately

1.4. φ is quasiregularly increasing if and only if $\varphi(u) = u^r \psi(u)$, $r \neq 0$, where ψ is quasislowly varying.

1.5. If φ is a convex (concave) φ-function, then $\varphi \overset{l}{\sim} \varphi_1$, where φ_1 is a convex (concave) φ-function possessing a continuous, strictly increasing (decreasing) derivative for $u \geqslant 0$.

Let φ be a convex φ-function and let $\varphi(u) u^{-1} \to \infty$ as $u \to \infty$. Then $\varphi(u) u^{-1}$ is increasing for sufficiently large u and replacing φ by an equivalent function we may assume $\varphi(u) u^{-1}$ to be increasing for $u > 0$. Let $\varphi_1(u) = \int_0^u \varphi(t) t^{-1} dt$; since $\varphi(\frac{1}{2} u) \leqslant \varphi_1(u) \leqslant \varphi(u)$ for $u \geqslant 0$, φ_1 possesses the required properties. If $\varphi(u) u^{-1} \to d$, $d < \infty$, then $\varphi(u) \overset{l}{\sim} du$ and the function $\varphi_1(u) = du - d \lg(1+u)$ is l-equivalent to φ and satisfies the required conditions. If φ is concave, then it is strictly increasing. Since φ^{-1} is a convex φ-function, $\varphi^{-1} \overset{l}{\sim} (\varphi^{-1})_1$, where $(\varphi^{-1})_1$ denotes the function defined for φ^{-1} in the same way as φ_1 was defined for φ. Obviously, $((\varphi^{-1})_1)^{-1}$ possesses the required properties and by **1**.61, $\varphi \overset{l}{\sim} ((\varphi^{-1})_1)^{-1}$.

1.6. If $\varphi \overset{l}{\sim} \varphi_1$, $\varrho(u) = \int_0^u \varphi(t)\, dt$, $\varrho_1(u) = \int_0^u \varphi_1(t)\, dt$, $\varrho(u) \to \infty$, $\varrho_1(u) \to \infty$ as $u \to \infty$ for two $q\varphi$-functions φ, φ_1, then $\varrho \overset{l}{\sim} \varrho_1$.

The inequality

(**1**.6.1) $$a\varphi(k_1 u) \leqslant \varphi_1(u) \leqslant b\varphi(k_2 u) \text{ for } u \geqslant u_0$$

is satisfied for some positive constants a, b, k_1, k_2; hence,

$$ak_1^{-1} \int_{u_0 k_1}^{uk_1} \varphi(t)\, dt \leqslant \int_{u_0}^{u} \varphi_1(t)\, dt \leqslant bk_2^{-1} \int_{u_0 k_2}^{uk_2} \varphi(t)\, dt \quad \text{for } u \geqslant u_0$$

and

$$\frac{1}{2}\, ak_1^{-1} \int_0^{uk_1} \varphi(t)\, dt \leqslant \int_0^{u} \varphi_1(t)\, dt \leqslant 2\, bk_2^{-1} \int_0^{uk_2} \varphi(t)\, dt$$

for $u \geqslant \bar{u}_0 \geqslant u_0$, where $\bar{u}_0$ is sufficiently large, i.e.

$$a' \varrho(k_1 u) \leqslant \varrho_1(u) \leqslant b' \varrho(k_2 u).$$

1.61. If $\varphi \overset{l}{\sim} \varphi_1$ and φ, φ_1 are strictly increasing φ-functions, then $\varphi^{-1} \overset{l}{\sim} \varphi^{-1}$.

φ, φ_1 satisfy the inequality (**1**.6.1); if $\varphi_1(u) = v$, $u = \varphi_1^{-1}(v)$, then the inequalities $k_1 u \leqslant \varphi^{-1}(a^{-1} v)$, $\varphi^{-1}(b^{-1} v) \leqslant k_2 u$, i. e. $k_2^{-1} \varphi^{-1}(b^{-1} v) \leqslant \varphi_1^{-1}(v) \leqslant \leqslant k_1^{-1} \varphi^{-1}(a^{-1} v)$ hold for $v \geqslant v_0 = \varphi_1(u_0)$.

1.7. If $\psi(u) = \int_0^u \varphi(t)\, dt$ is finite for $u > 0$, $\psi(u) \to \infty$, where φ is a $q\varphi$-function, then

(1.7.1) $$1+s_\varphi \leqslant s_\psi, \qquad \sigma_\psi \leqslant 1+\sigma_\varphi.$$

L'Hopital's rule yields $\bar{h}_\psi(\lambda) \leqslant \lambda^{-1} \bar{h}_\varphi(\lambda)$, $\underline{h}_\psi(\lambda) \geqslant \lambda^{-1} \underline{h}_\varphi(\lambda)$, and it suffices to apply the definition of indices s and σ.

Remark. If φ is quasiregularly increasing, i.e. if $s_\varphi = s = r$, ψ is also quasiregularly increasing and $\leqslant$ in (1.7.1) may be replaced by $=$.

1.71. (a) If φ is a φ-function, ψ has the same meaning as in **1**.7, then

(**1**.71.1) $$1+s_\varphi = s_\psi, \qquad 1+\sigma_\varphi = \sigma_\varphi.$$

(b) If φ is a convex φ-function having a continuous derivative for $u \geqslant 0$, then

(**1**.71.2) $$s_\varphi = 1+s_{\varphi'}, \qquad \sigma_\varphi = 1+\sigma_{\varphi'}.$$

For a nondecreasing φ, the inequality $u\varphi(\frac{1}{2}u)/2 \leqslant \psi(u) \leqslant u\varphi(u)$ is satisfied for $u \geqslant 0$, whence $\psi \overset{l}{\sim} u\varphi$. Now, it is sufficient to apply **1**.1, and **1**.2 (b). The part (b) is a trivial consequence of (a).

Since, according to **1**.7 and [5], **2**.3 (b), $s'_\varphi > 0$ implies φ to be l-equivalent to a convex φ-function, owing to **1**.7 we obtain:

Formula (**1**.71.1) holds if φ' is a continuous at 0 $q\varphi$-function such that $s_{\varphi'} > 0$ or $s_{\varphi'} = \sigma_{\varphi'} = 0$ (in particular, if φ' is slowly varying).

1.8. Let φ be a strictly increasing φ-function; then $s_\varphi = 1/\sigma_{\varphi-1}$.

Let $\infty > s_\varphi > 0$ and $0 < s < s_\varphi$. By [5], **2**.3 (b), $\varphi \overset{l}{\sim} \chi_s$, $\chi_s = \psi(u^s)$, where ψ is a convex function. According to **1**.5 we may assume ψ to be strictly increasing. If $\chi_s(u) = v$, then $\psi^{-1}(v) = (\chi_s^{-1}(v))^s$, whence $\sigma_{\psi-1} = s\sigma_{\chi_s^{-1}}$, by **1**.2 (a). Since ψ^{-1} is a concave function, we have $\sigma_{\psi-1} \leqslant 1$ and by **1**.61, $\sigma_{\varphi-1} = \sigma_{\chi_s^{-1}}$, whence $\sigma_{\varphi-1} \leqslant \leqslant 1/s$. Thus $\sigma_{\varphi-1} \leqslant 1/s_\varphi$. If $s_\varphi = \infty$, the inequality $\sigma_{\varphi-1} \leqslant 1/s$ is satisfied for an arbitrary $s > 0$; hence $\sigma_{\varphi-1} = 0$. Let $0 < \sigma_\varphi < \infty$; assuming $\sigma_\varphi < \sigma$, we have $\varphi \overset{l}{\sim} \chi_\sigma$, $\chi_\sigma = \psi(u^\sigma)$, where ψ is concave. Arguing as above we state $s_{\varphi-1} \geqslant 1/\sigma$, whence $s_{\varphi-1} \geqslant 1/\sigma_\varphi$. If $\sigma_\varphi = 0$, σ may be taken arbitrarily small; consequently, $s_{\varphi-1} = \infty$. Applying the above proved inequality to φ^{-1}, we obtain $s_\varphi \geqslant 1/\sigma_{\varphi-1}$.

2. The following conditions play a role when investigating properties of $q\varphi$-functions:

$$(\infty_s) \lim_{u\to\infty} \varphi(u)\, u^{-s} = \infty, \qquad (\infty_\sigma^0) \lim_{u\to\infty} \varphi(u)\, u^{-\sigma} = 0,$$

$$(0_s) \lim_{u\to 0+} \varphi(u)\, u^{-s} = 0.$$

Denote $s_\varphi^* = \sup s$, where the sup is taken over exponents s such that (∞_s) holds, $\sigma_\varphi^* = \inf \sigma$, where σ are exponents satisfying (∞_σ^0).

2.1. The following inequalities hold for any φ-function: $s_\varphi \leqslant s_\varphi^* \leqslant \sigma_\varphi^* \leqslant \sigma_\varphi$.

Let $s_\varphi > 0$, $s < s' < s_\varphi$. By [5], **2**.3 (a), $\varphi \overset{l}{\sim} \chi_{s'}$, $\chi_{s'} = \psi(u^{s'})$, where ψ is a convex φ-function. Hence $\varphi(u) \geqslant a\psi(k_1^{s'} u^{s'})$ for $u \geqslant u_0$, whence $\varphi(u)\, u^{-s} = \varphi(u)\, u^{-s'} \cdot u^{s'-s} \geqslant a\psi(k_1^{s'} u^{s'})\, u^{-s'} \cdot u^{s'-s} \geqslant a\psi(k_1^{s'} u_0)\, u_0^{-s'}\, u^{s'-s}$; thus (∞_s) is satisfied and $s_\varphi \leqslant s_\varphi^*$. Analogously we show $\sigma_\varphi^* \leqslant \sigma_\varphi$.

2.2. In this section we always assume φ to be a φ-function satisfying the conditions (0_1), (∞_1). Then a complementary function φ^* may be defined as follows:

$$\varphi^*(v) = \sup_{u \geqslant 0} (uv - \varphi(u)).$$

It is easily proved that φ^* is a φ-function satisfying (0_1), (∞_1) and that to every $v \geqslant 0$ there exists a u_v such that $\varphi^*(v) = u_v v - \varphi(u_v)$, [1], [7].

In the following we shall prove some theorems on complementary functions.

2.3. (a) *If* $\varphi_1(u) = a\varphi(bu)$, $a, b > 0$, *then* $\varphi_1^*(u) = a\varphi^*(u/ab)$.

(b) *If* $\varphi(u) \geqslant \varphi_1(u)$ *for* $u \geqslant u_0$, *then* $\varphi_1^*(u) \geqslant \varphi^*(u)$ *for* $u \geqslant u_0^*$.

(c) *If* $\varphi \overset{l}{\sim} \varphi_1$, *then* $\varphi^* \overset{l}{\sim} \varphi_1^*$.

To prove (a) note that $uv - a\varphi(bu) = a\left(bu\dfrac{v}{ab} - \varphi(bu)\right)$; hence $\varphi_1^*(v) = \sup\limits_{u \geqslant 0}(uv - a\varphi(bu)) = a\sup\limits_{u' \geqslant 0}\left(u'\dfrac{v}{ab} - \varphi(u')\right) = a\varphi^*(v/ab)$. As regards the proof of (b), cf. [6]. (a) and (b) imply (c) immediately.

2.4. The following formulae are satisfied for any convex φ-function:

$$\text{(o)}\ \frac{1}{s_{\varphi^*}} + \frac{1}{\sigma_\varphi}, \qquad \text{(oo)}\ \frac{1}{\sigma_{\varphi^*}} + \frac{1}{s_\varphi} = 1.$$

These inequalities are valid also in the limit cases, when the indices assume values 1, ∞, by usual conventions as regards the indeterminate expressions under consideration.

By **1**.5, taking into account **2**.3 (c) and the fact that s_φ and σ_φ are invariants of the relation $\overset{l}{\sim}$, we may assume φ to possess a derivative strictly increasing to ∞. Since $\varphi^*(u) = \int\limits_0^u (\varphi')^{-1}(t)\, dt$ is finite for convex φ-functions, we obtain the required formulae applying **1**.71 and **1**.8 successively.

The theorem may be proved also by applying [5], **1**.41 and **2**.3 (a) directly.

Formulae **2**.4 (o), (oo) are satisfied always, if $s_\varphi > 1$, for this condition implies $\varphi \overset{l}{\sim} \psi$, ψ is convex. Let $s_\varphi \leqslant 1$; the function $(\varphi^*)^* = \overline{\varphi}$ is called associated with the function φ (φ itself need not be convex). Obviously, $\overline{\varphi}$ is a convex φ-function satisfying (0_1), (∞_1); moreover [7], [5],

$$\overline{\varphi}(u) \leqslant \varphi(u) \text{ for } u \geqslant 0.$$

2.5. Inequalities $s_{\overline{\varphi}} \geqslant s_\varphi$, $\sigma_{\overline{\varphi}} \leqslant \sigma_\varphi^*$ are satisfied.

The first inequality is trivial; in fact, for $s_\varphi > 1$, $\varphi \overset{l}{\sim} \overline{\varphi}$, and always $s_{\overline{\varphi}} \geqslant 1$, $\overline{\varphi}$ being convex. To prove the second inequality suppose $\sigma_\varphi < \infty$ and note that $\sigma_\varphi = \inf \lg d_\alpha / \lg \alpha$ for an arbitrary φ-function, where inf is taken over all constants d_α, $\alpha > 1$, satisfying the inequality $\varphi(\alpha u) \leqslant d_\alpha \varphi(u)$ for $u \geqslant u(\alpha)$. Let $\varphi_1(u) =$

$= d_\alpha^{-1}\varphi(\alpha u)$, where d_α, $\alpha > 1$. If $\varphi_1(u) \leqslant \varphi(u)$ for $u \geqslant u(\alpha)$, then by **2**.3 (b), $\varphi_1^*(u) = d_\alpha^{-1}\varphi^*\left(\frac{d_\alpha}{\alpha}u\right) \geqslant \varphi^*(u)$ for $u \geqslant u^*(\alpha)$, and for $(\varphi^*)^* = \bar{\varphi}$ and $\left(d_\alpha^{-1}\varphi^*\left(\frac{d_\alpha}{\alpha}u\right)\right)^* = d_\alpha^{-1}\bar{\varphi}(\alpha u)$ there holds $d_\alpha^{-1}\bar{\varphi}(\alpha u) \leqslant \bar{\varphi}(u)$ for $u \geqslant \bar{u}_\alpha$, i.e. $\sigma_{\bar{\varphi}} \leqslant \leqslant \lg d_\alpha/\lg\alpha$, $\sigma_{\bar{\varphi}} \leqslant \sigma_\varphi$.

2.6. The following inequality holds for an arbitrary φ-function (satisfying (0_1), (∞_1)):

$$\frac{1}{s_{\varphi^*}} + \frac{1}{\sigma_\varphi} \leqslant 1.$$

By the definition of $\bar{\varphi}$ and by 2.4, there holds $\frac{1}{s_{\varphi^*}} + \frac{1}{\sigma_{\bar{\varphi}}} = 1$ and it is sufficient to apply **2**.5.

It follows from **2**.4 and from [4] that the following properties are equivalent for convex φ [3]:

$$(\alpha) \qquad \varphi(2u) \leqslant d\varphi(u) \qquad \text{for } u \geqslant u_0,$$

$$(\beta) \qquad \varphi^*(\alpha u) \geqslant c_\alpha\varphi^*(u) \qquad \text{for } u \geqslant u^*(\alpha), \quad \text{where } c_\alpha > \alpha > 1.$$

If the convexity of φ is not assumed, (α) implies (β). Another trivial consequence of **2**.4 is: if φ is a convex pseudoregularly increasing φ-function, then φ^* has the same property.

3. Let a φ-function φ satisfy the conditions (0_1), (∞_1). Denote

$$(3.0.1) \qquad g(v) = \int_0^\infty e^{-\varphi(t)}e^{tv}\,dt \qquad \text{for } v \geqslant 0.$$

The function $\chi(v) = \lg g(v)/g(0)$ is a strictly increasing, convex φ-function satisfying the condition (∞_1) and $\varphi^* \overset{l}{\sim} \chi$. (cf. [8], where a theorem on analytic representation of a convex φ-function by means of a sum of a series of exponential functions is proved.).

Assuming t to be sufficiently large, we have $e^{-\varphi(t)}e^{tv} < e^{-vt-t}$; hence $g(v) < \infty$ for $v \geqslant 0$. Convexity of $\chi(v)$ is verified in a usual manner, e.g. applying Schwarz's inequality to $g(v)$. Let $0 < \lambda < 1$; the following inequality is satisfied for $v > -\lg\lambda$:

$$(*) \qquad g(v+\lg\lambda) = \int_0^\infty e^{-\varphi(t)+tv}e^{t\lg\lambda}\,dt \leqslant e^{\varphi^*(v)}\frac{1}{-\lg\lambda}.$$

Given $v \geqslant v_0$, choose u_v so that $\varphi^*(v) = u_v v - \varphi(u_v)$. If $v \geqslant v_0$, we have $u_v \geqslant 1$ and there holds the inequality

$$(**) \qquad g(v) \geqslant \int_{u_v-1}^{u_v} e^{-\varphi(t)}e^{tv}\,dt \geqslant e^{u_v v-\varphi(u_v)}e^{-v} = e^{\varphi^*(v)}e^{-v}.$$

If $v \geqslant v_1$, where v_1 is sufficiently large, we have $2v + \lg\lambda > v$ and by (*) and (**),

$$-v+\varphi^*(v) \leqslant \lg g(v) \leqslant \varphi^*(2v) + \lg(-\lg\lambda).$$

Taking into account that φ^* fulfills the condition (∞_1), the last inequality yields $\chi \overset{l}{\sim} \varphi^*$.

3.1. Let φ be an arbitrary φ-function. Suppose $0 < s_\varphi^{\to} \leqslant 1$, $0 < s < s_\varphi$, then φ is l-equivalent to $\psi(u^s)$, where ψ is a convex φ-function. The function $\varphi_1(u) = \varphi(u^{1/s})$ is l-equivalent to ψ and satisfies (∞_1), for φ satisfies (∞_s), by 2. Moreover, one may suppose that φ_1 satisfies (0_1), replacing φ_1 by an l-equivalent function. Let $\bar{g}_1(v)$ denote the integral (3.0.1), where φ is replaced by φ_1^*. By 3, $\bar{\varphi}_1 \overset{l}{\sim} \lg \bar{g}_1(u)/\bar{g}_1(0) = \chi_1(u)$, where $\bar{\varphi}_1 = (\varphi_1^*)^*$. Since $\varphi_1 \overset{l}{\sim} \psi$, by 2.3 (c) $\bar{\varphi}_1 \overset{l}{\sim} \varphi_1$.

Since $\varphi(u) = \varphi_1(u^s)$, $\bar{g}_1$ is an integral function of the variable v, hence $\lg \bar{g}_1(u^s)/\bar{g}_1(0)$ is l-equivalent to $\varphi(u)$ and it is a locally analytic function for $u > 0$. If $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$, then taking into account that χ_1 increases monotonically, we obtain $\chi_1((\alpha v_1 + \beta v_2)^s) \leqslant \chi_1(\alpha^s v_1^s + \beta^s v_2^s) \leqslant \alpha^s \chi_1(v_1^s) + \beta^s \chi_1(v_2^s)$. Hence:

Let φ be an arbitrary φ-function satisfying (0_1) and let $s_\varphi > 0$, $s < s_\varphi$ when $s_\varphi \leqslant 1$, $s = 1$ when $s_\varphi > 1$ or $s_\varphi = 1$ and φ is equivalent to a convex φ-function satisfying (∞_1). Let $\varrho_s(v) = \sup_{u \geqslant 0} (uv - \varphi(u^{1/s}))$. By these assumptions

(a) $\varphi \overset{l}{\sim} \chi_\varphi$, where

$$\chi_\varphi(v) = \lg \left(\int_0^\infty e^{-\varrho_s(t)} e^{tv^s} dt \right) \Big/ \int_0^\infty e^{-\varrho_s(t)} dt;$$

(b) χ_φ is an s-convex function, i.e. $\chi_\varphi(\alpha v_1 + \beta v_2) \leqslant \alpha^s \chi_\varphi(v_1) + \beta^s \chi_\varphi(v_2)$ for $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$, and χ_φ is locally analytic for $v > 0$.

Let us yet note that if $s_\varphi = 0$, then $u\varphi \overset{l}{\sim} \psi$, where $\psi(u) = \int_0^u \varphi(t)\, dt$, whence ψ is convex, and by the previous theorem we obtain also in this case existence of locally analytic functions l-equivalent to φ, defined by integrals of the above type with a factor u^{-1}.

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