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BOGDAN MIELNIK

QUANTUM LOGIC: IS IT NECESSARILY ORTHOCOMPLEMENTED?

1. Introduction

One of the intriguing problems of the present day theory is the lack of similarity between general relativity and quantum mechanics. General relativity is a product of a long evolution line of classical theories leading toward structural flexibility. The most characteristic steps of that evolution were: (1) the discovery of space-time geometry (stage of Minkowski space), (2) the generalization of the geometry (introduction of the pseudo-Riemannian manifolds), and (3) the discovery that geometry depends on matter. In spite of its classical character general relativity is an example of an evolved theory: its fundamental structure is not given a priori (apart from generalities concerning the category) but is conditioned by physics.

The development which led to quantum theories was not similar to that. Here, there was only one decisive step: the abandoning of causal schemes and the transition to the probabilistic wave mechanics. Subsequent progress consisted in improving the symbolic language of states and observables sufficiently to include probabilistic information of increasing complexity. In spite of its rapid development the quantum theory did not undergo any further intrinsic changes of fundamental character and has not achieved a structural flexibility analogous to that of general relativity. Similarly, as in the twenties, present day quantum mechanics represents the variety of possible physical situations (pure states) by the same standard mathematical structure which is the unit sphere in a separable Hilbert space. Unlike the Riemannian manifolds the quantum mechanical unit spheres do not differ one from another: they are all isomorphic. The worlds of the present-day quantum mechanics thus present a picture of structural monotony: they are all 'painted' on that same standard ideally symmetric surface. The formalism of the quantum theory of infinite systems and quantum field theory is not very different from that. In spite of several mathematical refinements

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(the introduction of C*-algebras, the Gelfand-Segal construction) the basic structural framework of the theory is conserved at the cost of quantitative multiplication: when meeting a new level of physical reality the quantum theory responds by simply producing infinite tensor products of its basic structure. The resulting development is more similar to an expansion than to an intrinsic evolution: one just submits other branches of physical theory to the standard language of states and observables which has almost become the only acceptable way of thinking in quantum theories. A still unfinished stage of that expansion process is the programme of gravity quantization. A somewhat disquieting question arises, however: is the structure of the present day quantum theory indeed general enough to assure that further progress may be achieved just by continuing the techniques of operators in Hilbert spaces? Or, perhaps, the situation is different. It may be that present day quantum theory still represents a relatively primitive stage of development and lacks some essential evolutionary steps leading towards structural flexibility [2]. If this were so, further development would involve a programme opposite to the 'quantization of gravity': instead of modifying general relativity to fit quantum mechanics one should rather modify quantum mechanics to fit general relativity. The way toward flexible quantum structures was recently opened in the convex set theoretical approach to quantum mechanics [1, 3, 5, 6]. On the other hand, there exist conservative arguments supporting the necessity of the present day form of quantum theory, which are found in the axiomatics of quantum logic [4, 7]. This is why the axioms of quantum logic should be critically reexamined.

2. Lattice of Macroscopic Measurements

According to a generally accepted philosophy the 'quantum logic' is the set of all 'questions' which may be put to micro-object. By a *question* (also: *proposition, yes-no measurement*) one usually understands any physical arrangement which, when interacting with a microobject, may or may not produce a certain macroscopic effect interpreted as the answer 'yes'. Though the 'question' may be put to any single micro-object, the answer becomes conclusive only if obtained for a great number of its independent replies. This leads to an abstract scheme where 'questions' idealize the macroscopic devices used to test the statistical ensembles of microsystems. Let now Q denote the set of all 'questions' for certain definite physical objects. For completeness it will be assumed that Q contains two trivial questions: 'I' to which the answer is always 'yes' and '0' to which the answer 'yes' is never given. The existence of statistical ensembles as the counterpart of Q allows us to introduce a certain structure in Q which is the most recognized element of geometry in quantum theory.

DEFINITIONS. Given an ensemble x and a question $a\in Q$, one says that the answer 'yes' to the question a is certain for the individuals of x if 'yes' is obtained for the average fraction 1 of the individuals of that ensemble. The ensembles x for which the answer 'yes' to the question a is certain will be told to form the 'certainly yes domain' of a. Given two questions $a, b\in Q$, we say that a is more restrictive than $b (a\leqslant b)$ if the certainty of the answer 'yes' to the question a implies the certainty of 'yes' to the question b. Thus, $a\leqslant b$ if the 'certainly yes' domain of a is contained in the 'certainly yes' domain of b.

The relation $\leqslant$ is reflexive and transitive. The further properties of $\leqslant$ are associated with the 'logical' interpretation. According to that interpretation the questions $a\in Q$ represent the elements of an abstract 'logic' which reflects the nature of the microsystems: the relation $\leqslant$ is the implication of the logic. Since in any logical system the pair of implications $a\Rightarrow b$ and $b\Rightarrow a$ means that 'a is equivalent to b', one generally assumes that a similar property should hold in Q.

AXIOM I. Two questions $a, b\in Q$ with identical 'certainly yes' domains are physically equivalent (i.e., cannot be distinguished by observing how they select any statistical ensemble). Formally:

$$a\leqslant b \quad\text{and}\quad b\leqslant a\Rightarrow a=b. \tag{2.1}$$

In consequence, the relation $\leqslant$ introduces a partial order in Q with upper and the lower bounds I and 0. Because of common experience of classical and quantum phenomenology one also assumes that the partial ordering $\leqslant$ makes Q a lattice.

AXIOM II. For every pair $a, b\in Q$ the partial order $\leqslant$ determines the unique lowest upper bound $a\vee b\in Q$ called the *union* of a and b. Similarly, for every $a, b \in Q$ there exists in Q the greatest lower bound $a \wedge b$ called the *intersection* of a and b.

The physical interpretation given to the union $a \vee b$ is that of an experimental arrangement which yields the answer 'yes' with certainty for those systems for which either a or b give certainly the answer 'yes'. Similarly, $a \wedge b$ is interpreted as an arrangement which yields the answer 'yes' with certainty if both a and b yield the answer 'yes' certainly. The 'logical' interpretation given to the operations $\wedge$ and $\vee$ is that of conjunction and alternative. Since Q is a 'logic', and the logical systems admit negation, one generally assumes the following axiom about an orthocomplemented nature of Q:

AXIOM III. There exists in Q a mapping $a \to a'$ which to every $a \in Q$ assigns precisely one $a' \in Q$ called a *negative* of a, such that:

$$a \leqslant b \Rightarrow b' \leqslant a' \tag{2.2}$$

$$a \wedge a' = 0; \; a \vee a' = \mathrm{I} \tag{2.3}$$

$$(a \vee b)' = a' \wedge b'; \; (a \wedge b)' = a' \vee b' \tag{2.4}$$

$$(a')' = a. \tag{2.5}$$

The physical interpretation given to the mapping $a \to a'$ is consistent with the general idea that the question is an arbitrary macroscopic arrangement producing certain macroscopic alternative effects of which one is called 'yes' and the other is 'no'. Now, if a is an arrangement of that kind, the a' is interpreted as essentially the same arrangement with an opposite convention determining what is 'yes' and what is 'no'.

The set of questions Q with the lattice operations $\wedge$, $\vee$ and orthocomplementation $a \to a'$ is sometimes considered the fundamental structure of quantum theory reflecting the nature of the corresponding physical objects. In case of classical objects the questions $a \in Q$ correspond to the subsets of a classical phase space. The symbols $\leqslant$, $\vee$ and $\wedge$ then have the sense of theoretical inclusion, union, and product respectively, while the mapping $a \to a'$ is the operation of taking the set theoretical complement. In that case the logic Q, apart of properties listed in Axioms I, II, III fulfills the distributive law:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{2.6}$$

One thus infers that the distributive property of the logic Q is a manifestation of the classical nature of the corresponding physical objects. A different case of a 'logic' is obtained by analyzing the structure of orthodox quantum mechanics. Here the 'questions' are the self adjoint operators with two-point spectrum $\{0, 1\}$ in a certain Hilbert space $\mathscr{H}$ (orthogonal projectors). Hence, there is one-to-one correspondence between the elements $a \in Q$ and closed vector subspaces of $\mathscr{H}$. The closed vector subspaces in $\mathscr{H}$ form an orthocomplemented lattice which is not distributive. Hence one infers that in the micro-world classical logic is no longer valid, but a new type of logic becomes relevant in which the alternative is not distributive with respect to the conjunction. One consistently interprets the non-distributive property of Q as the main sign of a non-classical character of the corresponding objects.

3. Motivation of Hilbert Space Formalism

For a certain time the 'quantum logic' Q was considered to be the fundamental structure of quantum theory and has been studied to provide information concerning the most general form possible of quantum mechanics. According to a general belief, the answer should be obtained in the framework of some universal axioms which should reflect the nature of the macroscopic 'yes-no' effects and thus should be valid for any quantum system. The problem of an axiom which would replace the distributive law of classical logic was studied by Piron [7]. He postulated the following property of weak modularity as the one which holds for both orthodox classical and orthodox quantum systems:

$$a \leqslant b \Rightarrow a \vee (a' \wedge b) = b. \tag{3.1}$$

Piron's axiom has no immediate physical interpretation. However, it has been additionally clarified by Pool [8] who has shown that (3.1) is a necessary condition which allows us to associate uniquely the elements $a \in Q$ with some idempotent operations upon the statistical ensembles which represent the selection processes performed by the corresponding measuring devices. In Piron's scheme the weak modularity has been completed by axioms of atomicity and covering [7]. On that basis an important theorem was proved [7]: every irreducible 'quantum logic' must be isomorphic to the lattice of closed vector subspaces in a Hilbert space over one of three basic number fields (real numbers, complex numbers or quaternions). Every reducible quantum logic is a simple product of Hilbert space lattices and thus, corresponds to the orthodox theory with superselection rules.

The above results have a certain unexpected feature. They provide a good structural description of the existing theory. However, they seem to exclude the possibility of generalizations: we return here to the well known scheme of states and observables with the Hilbert space at the bottom [7, 8]. Moreover, the scheme of Piron and Pool is so compact that it is difficult to see in which point it could be relaxed without denying something very fundamental. This is sometimes taken as an argument against the possibility of further generalizations of the present day quantum scheme. However, the conclusion from the lattice theoretical results [7, 8] might be just the opposite. After all, most of the essential progress in physics has been achieved by denying something apparently obvious. Thus, general relativity denied the axioms of Euclid. Present day quantum mechanics has denied the even more obvious distributive law. There is no reason to think that this process is ended. The theorems of Piron and Pool exhibit a conservative quality of quantum logical axioms: it may thus be, that these axioms are the next 'obvious thing' to be negated in the future. Is such a step possible?

4. Critique of Axiomatic Approach

It is a specific status of quantum axiomatics that it should reflect phenomenology. In order to verify the phenomenological background of quantum logical axioms a careful identification must be made in order to specify the elements of physical reality which correspond to the abstract 'questions'. At this point axiomatic theory is elusively elegant. A 'question' ('proposition'), we say, is an arbitrary macroscopic arrangement which, when interacting with a micro-object, may or may not produce a certain definite macroscopic effect: the presence of the effect is conventionally taken as the answer 'yes' whereas its absence is 'no' (or vice versa). Now, it is argued, the validity of the basic axioms of quantum logic (apart from weak modularity) is almost a matter of tautology. For instance, two 'yes-no measurements' with the identical 'certainly yes' domains are obviously testing for the same feature, and so the difference between them is not

essential; this motivates the identity law (2.1). Similarly, the existence of a unique orthocomplement a' for an arbitrary 'yes-no' arrangement a is beyond discussion: a' is simply that some measuring arrangement with the roles of 'yes' and 'no' interchanged. An apparently more involved problem concerns the existence of the union $a \vee b$ and intersection $a \wedge b$ for any $a, b \in Q$. Here, some plausible existence arguments can also be given, through the constructive prescription is not clear. The above arguments would be indeed difficult to reject if not for the circumstance that the underlying definition is oversimplified. In spite of its elegant generality, the idea of a 'question' as a quite arbitrary macroscopic arrangement which produces a certain macroscopic alternative effect is wrong. To illustrate this, consider a statistical ensemble of any objects and a macroscopic device which yields the answer 'yes' for an average of $\frac{1}{2}$ of them in a completely random way. A good approximation is a semitransparent mirror in the path of a photon beam (Figure 1).

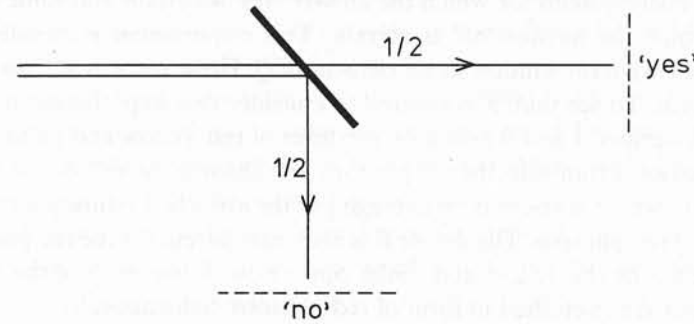

Fig. 1.

No doubt, this is a certain macroscopic arrangement producing a macroscopic alternative effect: either the photon reaches the screen 'yes' or it does not. However, the arrangement on Figure 1 cannot be considered one of 'questions'. If it were, it would produce a sequence of catastrophes in the structure of 'quantum logic'. First of all, it would not be clear which device is the 'negative' of the semitransparent mirror a. By insisting on the purely verbal solution (just the interchange of 'yes' and 'no') one would conclude that a' is acting, in fact, identically as a: for it too gives the answer 'yes' in a completely random way for an average of $\frac{1}{2}$ of the beam photons. Thus, $a' = a$. This would further imply: $0 = a \wedge a' =$

$=a \wedge a = a = a \vee a = a \vee a' = \mathrm{I}$ and so, the whole structure of Q would collapse.

One might reply, that the axioms of quantum logic are exact, but they must be properly understood. The semi-transparent mirror in Figure 1 is not a good example of a 'question' since it is not at all a measuring device: it does not verify any physical property of the transmitted photons. This is a good answer, but it means that the whole approach of 'quantum logic' should start from an information which is inverse to the usually given. *Not every arrangement producing a macroscopic alternative effect is a question (yes-no measurement)* – the right information should read. Indeed, one feels, that in order to be a quantum mechanical measuring device, the macroscopic arrangement should do something more specific than merely produce the 'yes' and 'no' effects in an arbitrary way. In some axiomatic approaches this is assured by requiring that the 'yes-no measurement' should have the non-trivial certainty domains: there should be some microsystems for which the answer 'yes' is certain and some other for which the answer 'no' is certain. This requirement eliminates the semi-transparent window as an element of Q. However, it is still far from sufficient. To see that, it is enough to consider two hypothetical macroscopic devices A and B acting on mixtures of red, yellow and violet light. The device A transmits the red photons and absorbs the yellow and violet ones: however, it re-emits on average $\frac{1}{2}$ of the absorbed yellow photons in form of red photons. The device B is also transparent for the red photons and absorbs the yellow and violet ones: now, however, $\frac{1}{2}$ of the violet photons are re-emitted in form of red photons. Schematically:

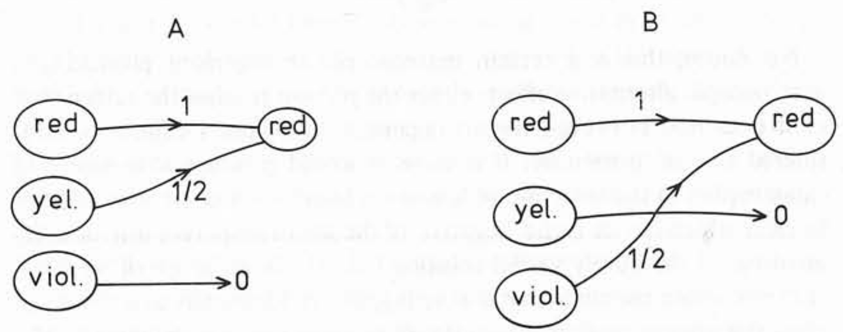

Fig. 2.

Both devices A and B have a common 'certainly yes' domain: they are completely transparent only to the red photons. They also have the property of performing idempotent operations on photon mixtures which in some treatments is considered a fundamental quality of the 'yes-no measurements'. However, A and B have different domains of 'certainly no' and so, they are not physically equivalent. This difference, in spite of Axiom I, cannot be considered non-essential and absorbed into the identity relation (2.1). Indeed, if we insisted that A and B are merely two different physical realizations of that same abstract question $a \in Q$, we would have two essentially different prescriptions for production of the negative a': once by taking A' ('certainly yes' for the violet) and once by taking B' ('certainly yes' for the yellow). Formally: $A = B$ but $A' \neq B'$. In consequence, something would be broken in the assumed structure of Q: either the identity axiom (2.1) or the uniqueness of the orthocomplement. We therefore reach the conclusion that the macroscopic devices A and B are still not 'good enough' to represent the abstract 'questions'. Indeed, the most essential condition is still missing. According to Ludwig's thermodynamical condition [5] the macroscopic 'yes-no measuring device' apart from possessing non-trivial certainty domains must also have the property of minimizing the randomness of the 'yes' and 'no' answers. A generalized version of this idea was employed in [6] by requiring that, for a given 'certainly yes' domain, the yes-no measurement should have a maximal possible 'certainly not' domain. This requirement is, finally, the sufficient condition which allows one to distinguish the subclass of those macroscopic devices which correspond to the abstract 'questions'. An essential problem now arises: is it necessarily so, that the counter-examples against the quantum logical axioms must automatically vanish when the class of the macroscopic 'yes-no' arrangements is restricted to the subclass of proper random-minimizing 'yes-no measurements'?

If the orthodox theory is not a priori assumed, the answer to this question must remain conditional. It depends essentially on the validity of a certain intuitive image which we usually associate with the phenomenology of physical objects and which, in general, may or may not be true. According to this image, each 'question' $a \in Q$ determines a certain specific property of micro-objects: the objects having that property are those for which the answer 'yes' is certain. Now, we intuitively assume that for each domain of micro-objects which possess a certain 'property' there is a unique complementing domain of micro-objects with an 'opposite property': so that, once it is known for which objects the answer of the 'yes-no measurement' is 'certainly yes' it is also uniquely determined for which ones it should be 'certainly no'. This image is true in orthodox quantum mechanics because of the orthogonal structure of the closed vector subspaces in a Hilbert space. However, it may be not of universal validity. In fact, it is not a logical impossibility to imagine a hypothetical physical world where to every domain of micro-objects with a certain special property there would be many possible 'complementing domains' corresponding to many possible ways of being 'opposite' to that property. If that were so, there could exist many random minimizing 'yes-no measurements' with a common domain of 'certainly yes' and different 'certainly no' domains. A hypothetical sequence of such devices is represented in Figure 3.

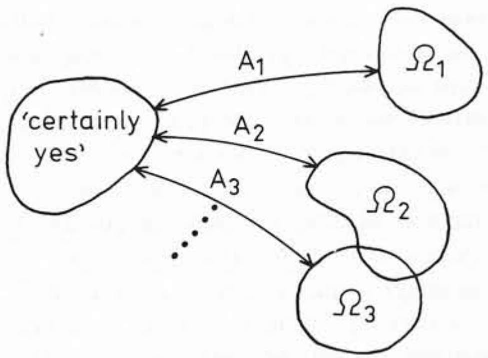

Fig. 3.

The devices $A_1, A_2, \ldots$ schematically represented in Figure 3 choose the same domain of micro-objects on which the answer should be 'certainly yes' but they minimize the randomness in favor of various 'certainly no' domains $\Omega_1, \Omega_2, \ldots$ For each of those devices the verbal negation operation (yes $\leftrightarrows$ no) could be easily performed leading to a sequence of devices $A'_1, A'_2, \ldots$ with different 'certainly yes' domains $\Omega_1, \Omega_2, \ldots$ Contrary to Axiom I, the devices $A_1, A_2, \ldots$ would be physically different, and even if we tried to neglect the difference by insisting that (2.1) defines the right physical equivalence, the negatives $A'_1, A'_2, \ldots$ could no longer be identified on that same principle. It thus becomes clear that the axioms of 'quantum logic' are not so absolute as they seem at the first sight. Even the apparently obvious laws of identification (2.1) and orthocomplementation (2.2–5) are not logically inevitable. Similarly, like the distributive law of classical logic, they are conditioned by the physical properties of the corresponding micro-objects. This suggests that before deciding what the 'quantum logic' is and which axioms it must fulfill, the theory should go deeper and look for the justification for the axiomatic structures in physics of the statistical ensembles themselves. The steps taken in this direction lead to the recently formulated convex scheme of quantum mechanics.

5. Convex Scheme of Q.M.

In the orthodox approach to quantum logic the statistical ensembles are an implicit counterpart. Their fundamental role has been rediscovered in the convex scheme of quantum mechanics [1, 5, 6]. The basic concept of this scheme is that of a *quantum state*. Given a statistical ensemble of certain micro-objects the *state* stands for an averaged quality of a random ensemble individual. Formally, states are equivalence classes of statistical ensembles. Given micro-objects of certain definite kind (e.g. electrons) the fundamental structure of the scheme is the set S of all states. If the micro-objects obey orthodox quantum mechanics, then S is the set of all positive operators with unit trace in a certain Hilbert space $\mathscr{H}$ ('density matrices'). If it is not a priori assumed that the orthodox theory holds, it may only be granted that S has a structure of a convex set: the convex combinations $p_1x_1 + p_2x_2$ for $x_1, x_2 \in S$ and $p_1, p_2 \geqslant 0, p_1 + p_2 = 1$ mean the state mixtures and the extremal points of S represent the pure states. In principle, S might be considered a convex set 'in itself' with the convex combination axiomatically introduced [3]. However, for the sake of illustrative qualities, one usually represents S as being embedded in a certain affine topological space E which can be constructed by a formal extension of S: the points in S then represent the pure and mixed states of the system, whereas the points of E out of S have no physical interpretation [6]. For the reason of physical completeness it is assumed that S is a closed convex subset of E.

Though the structure of S reflects a relatively simple phenomenology (it only shows which states are the mixtures of which other states) there is an extensive physical information contained in the geometry of S. In particular, shape of S determines the structure of the macroscopic alternative measurements which is so fundamental in other axiomatic approaches. This is due to the following concept of a normal functional.

DEFINITION. Given an affine space E with an affine linear combination $\lambda_1 x_1 + \lambda_2 x_2 (x_1, x_2 \in E, \lambda_1, \lambda_2 \in R, \lambda_1 + \lambda_2 = 1)$ a function $\phi: E \to R$ is called *linear* if $\phi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \phi(x_1) + \lambda_2 \phi(x_2)$ for every $x_1, x_2 \in E, \lambda_1, \lambda_2 \in R$, $\lambda_1 + \lambda_2 = 1$. Given an affine topological space E and a closed convex subset $S \subset E$, a continuous linear functional $\phi: E \to R$ is called *normal* on S iff $0 \leqslant \phi(x) \leqslant 1$ for every $x \in S$.

The normal functional admit a simple geometric representation. Any non-trivial linear functional in E can be represented by a pair of parallel hyperplanes on which it takes the values 0 and 1. Now, the functional ϕ is normal on S if the subset S is contained in the closed region of E limited by the hyperplanes $\phi = 0$ and $\phi = 1$.

The normal functionals have a natural physical interpretation. Let $x \in S$ be a statistical ensemble and suppose, that there is a macroscopic device which produces a certain macroscopic alternative effect 'yes-no'. If one translates the 'yes' and 'no' into the numbers: 'yes' $= 1$ and 'no' $= 0$, the action of the device is completely characterized by a number $\phi(x)$ $(0 \leqslant \phi \times \times (x) \leqslant 1)$ which represents the statistically averaged answer to the individuals of the ensemble x. Since it is implicit in the definition of the statistical ensemble that it is composed of independent individuals, the process of testing any mixed ensemble is equivalent to testing independently each of the mixture components. Consistently, $\phi(p_1 x_1 + p_2 x_2) = = p_1 \phi(x_1) + p_2 \phi(x_2)$ and so, every 'yes-no' arrangement defines a certain normal functional on S. Here, no limitations are present which are essential for the 'quantum logic'. Every macroscopic alternative arrangement is included in the scheme and is mathematically represented by a normal functional, no matter whether or not it minimalizes the random element in the 'yes' and 'no' answers. We thus reach a generalized scheme of quantum theory based on the theory of convex sets ('convex scheme' [6]). In that scheme the collection of all states of a physical system is represented by a closed convex set S in an affine topological space. The set of all macroscopic 'yes-no' devices corresponds to the collection of all normal func-

tionals on S mathematically represented by all possible ordered pairs of closed hyperplanes enclosing the set S and labelled by numbers 0 and 1 (see Figure 4).

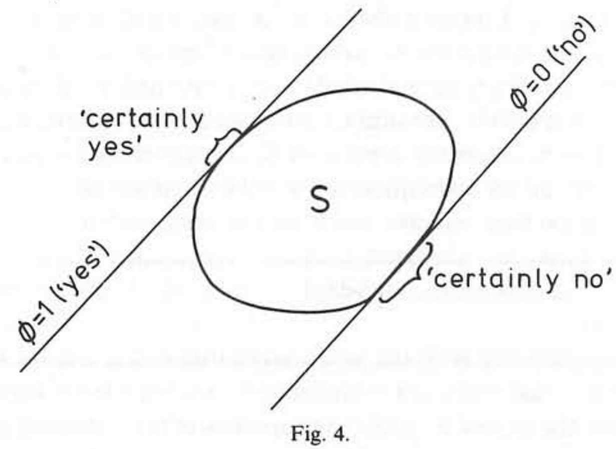

Fig. 4.

Since the set of the normal functionals is determined by the shape of the convex set S, so is also the collection of macroscopic 'yes-no devices'. Consequently it is a feature of the convex scheme that in it the structure of the 'yes-no measurements' is not decreed a priori but is determined by the more fundamental structure of the statistical ensembles. By analyzing the precise mechanism of this determination we reach a certain new structure which is a natural candidate for a replacement of traditional 'quantum logic'.

6. Logic of Properties

It is a controversial problem, whether the formalism of quantum theory can be used to describe the properties of the single micro-object 'as it is, in all its complexity' (Piron [7]). The single act of measurement in quantum mechanics is not conclusive, and therefore, the direct interpretation of quantum mechanical formalism is that of a statistical scheme. The notion of property of a single system can however be introduced as a next abstraction stage of the theory. In the axiomatic approach of Jauch and Piron [4, 6] this is done by analyzing the structure of Q. Below, another method will be employed which departs directly from the properties of statistical ensembles.

Statistical ensembles are, in a way, macroscopic entities: though it might be impossible to predict the behaviour of a single micro-individual in a given physical situation, one can predict the behaviour of the ensemble as a whole. Therefore, there is no difficulty in defining the physical properties of the ensembles. By saying that a certain ensemble has a certain property we simply have in mind that the ensemble behaves in a specified way in some definite physical circumstances. If now the ensembles are represented by points of the convex set S, the properties are just the subsets of S. It is still an open question, whether a subset of S should fulfill some regularity requirements (such as the measurability) in order to represent a physically verifiable property. As pointed out by Giles [9] the answer must depend upon the degree of idealization which is permitted by the theory.

The main difficulty with the single individuals in a statistical theory lies in the fact that there is no immediate correspondence between the properties of the ensembles and the properties of the individuals. In fact, not every property of the ensemble is of such a nature that it may be attributed to each single ensemble individual. A strictly macroscopic example is obtained by considering a human ensemble composed half of men and half of women: the fifty-fifty composition then is a property of the ensemble which, however, cannot be attributed to each single ensemble individual. Quite similarly, one can have a beam of photons of which the average fraction $\frac{1}{2}$ penetrates through a certain Nicol prism. However, it may be that the ability of penetrating through the prism with the probability $\frac{1}{2}$ cannot be attributed to each single beam photon, for the beam is just a mixture of two types of photons one of which is certainly transmitted and the other certainly absorbed by the prism. In general, a property P of statistical ensemble is a proper starting point for a definition of a certain property of the single micro-objects if two conditions hold: (1) whenever two ensembles have the property P their mixtures must also have it, and (2) whenever a mixture has the property P, each of the mixture components must also have it. These requirements mean that the properties of the single microsystems are represented only by special subsets $P \subset S$ which fulfill the following definition [6].

DEFINITION. Given a convex S, a *wall* (also: *face*) of S is any subset $P \subset S$ such that: (1) $x_1, x_2 \in P$, $p_1, p_2 \geqslant 0$, $p_1 + p_2 = 1 \Rightarrow p_1 x_1 + p_2 x_2 \in P$, and (2) $p_1 x_1 + p_2 x_2 \in P$ with $x_1, x_2 \in S$, $p_1, p_2 > 0$, $p_1 + p_2 = 1 \Rightarrow x_1, x_2 \in P$. Geometrically, a *wall* is any convex subset of S which possesses the property of 'absorbing intervals': whenever P contains any internal point of a certain straight line interval $I \subset S$ it must also contain the whole interval I.

The concept of a wall generalizes that of an extreme point: the extreme points are just one-point walls of S. Any non-empty convex set S has two improper walls: the whole of S and the empty set $\emptyset$. For any convex set the walls form a partially ordered set with the ordering relation $\leqslant$ being the set theoretical inclusion $\subset$. As seen from the definition, the common part of any family of walls is also a wall. This implies that the walls form a lattice: for any two walls $P, R \subset S$ the greatest lower bound $P \wedge R$ is just the common part $P \cap R$ whereas the lowest upper bound $P \vee R$ is obtained by taking the common part of all walls containing both P and R. If the points of S represent the pure and mixed states of a certain hypothetical system, the walls of S represent the possible properties of the system ordered according to their generality. In particular, the whole of S represents the most general property possible (no property) whereas the empty wall $\emptyset$ stands for the impossible property (no system with that property). The extreme points of S (if they exist) are atoms in the lattice of walls: they correspond to maximally specified properties, in agreement with the Dirac idea of pure states as being the maximum sets of non-contradictory information which one can have about the microsystem. It still remains an open question what sort of regularity requirements a wall should fulfill in order to be an operationally verifiable property. The standard quantum mechanical convention is to consider the lattice of closed walls of S as representing the physically essential properties of the system; this lattice will in future be denoted by $\mathbb{P}$.

The existence of normal functions on S allows one to define a natural notion of orthogonality in $\mathbb{P}$.

DEFINITION. Two properties P, R are called *excluding* or *orthogonal* ($P \perp R$) if there is at least one macroscopic 'yes-no' arrangement which answers certainly 'yes' for systems with the property P and certainly 'no' for the systems with property R (or vice versa). Thus, $P \perp R$ if there is a macroscopic device able to distinguish the property P from the property R without an element of probabilistic uncertainty.

The set of properties $\mathbb{P}$ with the relations of inclusion ($\leqslant$) and exclusion

($\perp$) is that structure of quantum theory which most directly reflects the nature of micro-objects. It has been thus proposed that the lattice $\mathbb{P}$ should be considered the 'logic' of a quantum system instead of the lattice of macroscopic measurements Q [6]. The above idea of quantum logic is wider than the orthodox one. The 'propositions' (properties) here are not necessarily in one-to-one correspondence with some 'yes-no measurements'. The 'property' is an abstracted quality of statistical ensembles and therefore, it should be verifiable: however, it is not a priori supposed that the verification might be always reduced to a single act of measurement. In spite of a more abstract sense of $\mathbb{P}$, the problem of the validity of the standard lattice theoretical axioms becomes much simpler for the 'logic of properties'. In fact, the identity axiom (2.1) is automatically fulfilled, since the partial ordering $\leqslant$ is now the set 'theoretical inclusion'. The lattice axiom, too, automatically holds because the closed walls of S must form a lattice. It is not so with the orthocomplementation law which has now a quite different status. The notion of 'negation' is not immanent for the properties. What becomes natural here is the more primitive relation of the exclusion $\perp$. Depending on the structure of that relation the operation of negation can or cannot be constructed on $\mathbb{P}$. The following definition seems to express the physical idea of what the negation is.

DEFINITION. Let P be a property and let $P^{\perp}$ denote the subset of all properties which are orthogonal to P. If in $P^{\perp}$ a greatest element exists,

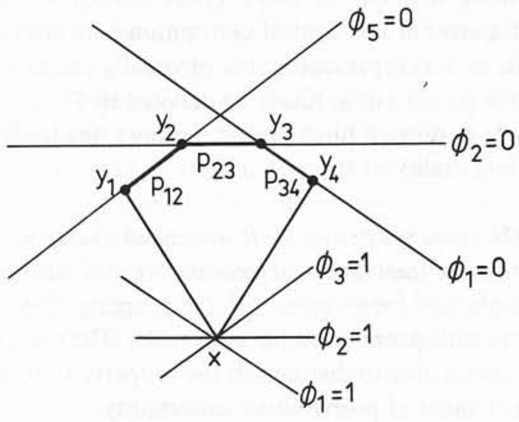

Fig. 5.

this element is called the *negative* of P and denoted P'. If for every $P \in \mathbb{P}$ the negative P' exists, we say that the logic $\mathbb{P}$ admits negation.

As is easily seen, the existence of negation, in general, is not ensured by the structure of the walls. A hypothetical case where negation could not be constructed because of the geometry of S is shown on Figure 5.

For the convex set represented here the subset composed of one point x is a wall and the family $\{x\}^{\perp}$ contains the seven non-trivial walls: the four one-point walls $\{y_1\}$, $\{y_2\}$, $\{y_3\}$, $\{y_4\}$ and the three straight line segments P_{12}, P_{23}, P_{34}. The family $\{x\}^{\perp}$ thus contains three maximal walls P_{12}, P_{23}, P_{34} but it does not contain the greatest one: the convex set S does not possess a wall which would contain the three segments P_{12}, P_{23}, P_{34} and be orthogonal to x. As a consequence, no unique orthogonal complement can be defined for the 'property' $P=\{x\}$. The above situation has not very much to do with the possibility of interchanging the 'yes' and 'no' answers in the 'yes-no measurements' and cannot be excluded by considering the nature of the macroscopic measuring devices. Inversely, this is the absence of the situations like that represented in Figure 5 which must be first granted to explain the origin of the usually assumed structure of Q. In fact, if the convex set in Figure 5 represented the collection of all pure and mixed states of a certain physical system, the structure of the 'properties' would make possible the existence of three different 'yes-no' measurements with the same domain of 'certainly yes' (the pure state x) which would, however, minimize the random element in different ways, by choosing three different 'certainly not' domains P_{12}, P_{23}, P_{34}. This would lead to a non-orthodox structure of Q with the identity law broken (see also [6]). This shows that the logic of properties $\mathbb{P}$, in a sense, lies one level deeper than the phenomenology of 'yes-no measurements'. The existence or non-existence of negation in $\mathbb{P}$ is one simple fact which justifies or disproves the whole system of axioms which are traditionally employed to describe the structure of 'questions'. An essential problem now arises: have we indeed some universal reasons to believe in the orthocomplemented structure of $\mathbb{P}$?

If we dismiss some verbal arguments, we are really left with one intuitive picture which can be of importance. This is the picture of matter and of a certain selection process which subtracts a component of matter with a certain definite property. Now, if the subtraction is done with enough care, the component which remains ('the rest') depends only on *what* was subtracted but it does not depend on *how* it was subtracted. This simple picture is, in fact, one of the deepest constructional principles of present day theory and is the true origin of the subsequent image of $\mathbb{P}$ as being an orthocomplemented lattice. Indeed, one usually takes for granted that each 'property' distinguishes a certain component of matter: 'the rest' then uniquely defines the 'complementing property'. This idea finds a particular realization in classical theory, where the properties correspond to subsets in a classical phase space and the 'subtraction' is the operation of taking the set theoretical complement. A different mechanism stands for the same in orthodox quantum theory. Here the states of matter are described by vectors in a linear space (wave vectors) which obey a linear evolution equation. Now, if a wave vector is selected by subtracting a certain component, the uniqueness of 'the rest' is due to the existence of the linear operations. This explains the strong position of the orthocomplementation axiom in the present day theory: whenever one deals with some quanta which are well described by a linear wave equation, the orthocomplemented structure of 'properties' will naturally appear. On the other hand, this also indicates that the orthocomplemented structure of $\mathbb{P}$ might not be universal: it does not express the nature of *any* theory, it just expresses the essence of linearity. Is linearity a necessary attribute of quantum mechanics? In spite of the traditional philosophy of the superposition principle, schemes based on non-linear wave mechanics have always been a tempting alternative for quantum theory. One might expect them to contribute something to the understanding of the measurement problem: for a possibility is open that the Schrödinger evolution equation and the measurement axioms are just two opposite approximations to a still undiscovered theory. Thus, the quantum axiomatics should not be too quickly closed. This may become of special importance in problems which involve the quantization of gravity.

In fact, if the gravitational field has a quantum character, an intriguing problem concerns the behaviour of a hypothetical single graviton. Is this behaviour similar to the dynamics of the macroscopic gravitational universe governed by Einstein's equations? In principle, it must not be so. It is possible that the single graviton in vacuum (if such an entity exists) is well described by a certain linear law, in agreement with the spirit of orthodox quantum mechanics, and that the non-linear behaviour of the macroscopic gravitational field is a secondary phenomenon due to interactions in a cloud of many gravitons. In this case the formalism of operator fields in Hilbert spaces would be sufficient to describe the quantum gravitation. However, this hypothesis has some disadvantages. In fact, if the graviton were described by a linear wave, a question would arise, as to in what space-time this wave propagates? Is it just the flat Minkowski space-time? If so, general relativity would be a theory with a background of Minkowski metric masked by clouds of gravitons. This is, however, against an innate aesthetics of general relativity, where the background metric which is not seen under the cover of the macroscopic field is unphysical and should not enter into the formulation of the theory. Thus, it may be that the situation is different. Perhaps, even a single graviton in vacuum modifies the space time in which it exists and so, creates a nonlinearity in its own propagation law. This would lead to a new picture of quantum theory where the selection processes could no longer be associated with linear decomposition operations of certain 'state vectors' and the mechanism which had accounted for the uniqueness of the orthocomplement in the orthodox theory would no longer be valid. The resulting properties of 'non-linear quanta' would not have to imitate the orthogonality structure of the closed vector subspaces in $\mathscr{H}$, but they could form a generalized type of logic where no unique negation operation can be constructed.

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