MOTION AND FORM*

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There were so many interesting points raised during this workshop, that I wonder whether I can still say anything new on quanuum logic. Therefore, I would like to make a confession. For thirteen years I have been fighting <u>against</u> quantum logic. Due to some paradoxes of sociology, my activities were finally noticed by somebody who said: "Mielnik? Ah, yes, he is doing something <u>in</u> quantum logic".

Being so frustrated in my efforts, I am close to recognizing that after all, (in either a positive or a negative sense), I do belong to "quantum logic". However, I would like to understand this concept broadly. For me a quantum logic is not necessarily an orthocomplemented lattice [1, 2, 3, 4], nor even a lattice without orthocomplementation.[5] It may be any sort of information sufficient to reconstruct the structure of states in a physical theory (as, for example, the algebra of observables[6, 7], geometry of states[8 - 11], geometry of convex sets[12 - 20], manuals[21], formal dialogues[22], etc.) With this in mind, I shall try to explain why quantum logic might become increasingly important for the rest of physics.

At first sight, the situation does not look encouraging. The "logic" seems far away from the main trends in theoretical physics. Let me just remind you of our colleagues computing graphs in perturbative expansions, with a determination to read physical sense in their infinities. Then, do you still remember the groups of scientists moved by the good news that there are dispersion relations which can be iterated? Or, the career of the B-ta function (Weneziano model),

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the Regge poles, the solitons, the instantons, etc.? Who cares about yes-no measurements? Yet, it turns out that quantum logic is not separate from the main stream of theoretical physics. If it was not popular enough, this was for two reasons. First of all, the "great trends" of theoretical physics are created by mass sociology, like the hits of pop-music, and have not yet achieved their maturity. Next, "quantum logic" itself, enclosed in its castle of propositions and corollaries, has not properly recognized its links with the rest of physics. To justify this, I shall discuss the general inter-relation between the two elements quoted in the title of my talk, "MOTION" and "FORM".

By *form* I shall understand the collection of all time-independent aspects of *shape*, *structure*, *geometry* or *"logic"* present in a physical theory. (Forgive me for this sequence of almost-synonyms.) Below, when speaking about the *form*, I shall have in mind the geometry of the phase space of a physical system (though the other meanings are still admissible). The idea of *motion* is common for everybody; for the moment I shall leave it unspecified.

The relation between *motion* and *form* is of a fundamental character. It seems as relevant for physics as the relation between *spirit* and *matter* was for philosophy. Note that *motion* shows some affinity to *spirit*, (at least, I can hardly imagine spirit being static). You may wonder whether mathematical physics should enter into such transcendental analogies. The truth is that it can hardly avoid them. If you look at the textbooks on foundations you see that, in fact, they represent the modern book of Genesis; for they tell you what was at the beginning (in either a chronological or a logical sense). And if you read papers on axiomatic quantum theory, they tell you that at the beginning there was a *form*.

The dynamics appeared afterwards, to suit the already existing form. Let me give you examples. In axiomatic quantum mechanics, the *form* is introduced in many ways, e.g. as an orthocomplemented lattice, convex geometry, or algebra of observables, but it always yields the Hilbert space structure of quantum states. Once this is assumed, the dynamical evolution must be represented by a one-parameter group of unitary transformations. Due to Hilbert space geometry, any such group has the form $e^{-itH} : t \in \mathbb{R}$, where H is a self-adjunct operator. Hence, one arrives at the quantum "Hamiltonians". If, moreover, the pure states correspond to one-particle "wave functions", some plausible arguments indicate that

$$H = \frac{\bar{p}^2}{2m} + V(\bar{x})$$

are the simplest Hamiltonians; hence, Schrödinger quantum mechanics follow. Let me also recall the axiomatic treatment of classical mechanics. Here, the basic structure is that of a symplectic manifold. Once this structure is postulated, the motions are represented by one-parameter groups of symplectomorphisms. Their generators are the classical Hamiltonians and the corresponding equations of motion coincide with the canonical equations. In both cases (as well as in almost all axiomatic schemes) the "arrow of creation" leads from the form to the motion:

"FORM $\longrightarrow$ MOTION"

This scheme pleases us since we are formalists. However, is it indeed convenient that the form should be decided at the beginning? Trying to understand the difficulties of quantum logic in new areas, (such as non-linear fields), I noticed that the axiomatic approaches have a certain defect. They read the book of creation backwards. Indeed, when one traces the natural development of a physical theory, one sees that it is opposite to the one described by the axiomatic approaches: the *motion* is given at the beginning, and the *form* emerges later. To illustrate this, let me re-examine the example of Schrodinger's quantum mechanics.

Here, the entity present at the beginning was the complex "wave function" ψ. Initially, little was known about the nature of ψ. Hilbert spaces were not yet discovered by physicists, and the set of all ψ's did not yet have any recognized geometry. It was amorphous. However, it moved. The evolution equations were one of the early guesses of the theory:

$$i \frac{\partial\psi}{\partial t} = \frac{1}{2} \Delta \psi + V(\bar{x}, t)\psi. \quad (\hbar = m = 1) \qquad (1)$$

Now, a structure started to emerge. First of all, wave functions which are proportional remain proportional after any time evolution: this makes it possible to assume that the classes of proportional solutions ("rays") possess some physical meaning. The evolution equations (1) are *linear* and conserve linear dependence: this suggests that the linear subspaces of the wave functions too possess a physical sense. So, the first elements of the "logic"[1-5] start to appear. Next, for any wave ψ the integral

$$\int_{\mathbb{R}^3} |\psi(x, t)|^2 \, d_3x$$

is conserved by the time evolution: this integral defines a natural norm of the wave functions and suggests the choice of the statistical interpretation (Born). Moreover, for any two solutions ψ, φ, the quantity

$$(\psi, \varphi) = \int_{\mathbb{R}^3} \bar{\psi} \varphi \, d_3x$$

is conserved by the time evolution: this quantity becomes a basic geometric element in the space of the wave functions ψ. In this way one arrives at the Hilbert space geometry of quantum states and the rest of the scheme follows. The role of the dynamical equations here is principal: were it not for the dynamical equations (1), the theory would not arrive at the Hilbert space structure[23,24]. It is not otherwise in the case of classical mechanics. Today, this theory is described in terms of symplectic manifolds, fiber bundles, etc. However, this geometric language was absent when the theory was created. What were known then, were just the equations of motion. It was due to the particular character of these equations that the "Poisson bracket" turned out to be conserved and the symplectic manifolds emerged. The above development is not accidental; it reflects a natural hierarchy of elements within a physical theory: The motion creates the form.

"The MOTION creates the FORM"

This hierarchy seems so essential, that while fully recognizing the importance of form, I want to propose an abstract description of a dynamical theory inverse to the one given by the axiomatic approaches. This description follows.[23,25]

Suppose, one has a set Φ of elements φ, ψ, ... denoting the pure states of a hypothetical system. Assume that Φ is a topological space with a physically meaningful topology. Though the physical motivation is not yet complete, it seems reasonable to assume that Φ possesses also a structure of a generalized differential manifold (in general, of infinite dimension, except for some special spaces of states like those of spin or polarization). No other structure on Φ will be assumed for the moment. The manifold Φ is a phase space of a hypothetical system. However, Φ is still devoid of any geometry (like that of a symplectic manifold or that of a projective Hilbert space): it is left open whether it will represent an orthodox quantum system or a classical system or any other entity. The next element to be introduced is the dynamics. However, in order to be informative the concept of dynamics must be understood globally.

In most physical theories the dynamics are introduced by distinguishing a certain one-parameter group of transformations of the phase space which represents the time evolution of the system. This description, though elegant, is not sufficient. A one-parameter group of transformations can only describe one particular evolution of the system on a certain fixed external surrounding (e.g. a vacuum). However, the same system can be submerged in many external surroundings in which it may evolve according to different one-parameter groups. To understand the dynamical nature of the system it is essential to know all of them; just as to understand the nature of an individual one has to know all his abilities and not only his actual behavior. This leads to the idea that the dynamics, in order to be informative, should be global[26,23,25], i.e., it should tell about the open possibilities. The simplest global description is obtained by introducing a counterpart of Φ: a certain set Ξ whose elements ξ, η, ... denote the external conditions in which the system can be submerged. Below, I shall assume that these conditions are dissipation-free, and therefore, to every $\xi \in \Xi$ there corresponds a unique two-parameter family of transformations $g(\xi; t, t') : \Phi \to \Phi$

$t \leq t'$, mapping the pure state onto the pure states and representing the evolution process which the system undergoes under the influence of the external condition ξ. It seems only a minor restriction to assume that $g(\xi; t, t')$ are diffeomorphisms of Φ. Now, the ways of the two approaches part. In the traditional description of the dynamics one chooses a certain fixed $\xi_o \in \Xi$ and one considers a single one- or two-parameter family of transformations: $g_o(t, t') = g(\xi_o; t, t')$ ("fixed Hamiltonian"). In the "global" description one has to consider a wider set of all operations $g(\xi; t, t')$ corresponding to all time intervals and all external conditions:

$$G = \{g(\xi; t, t') : \xi \in \Xi, t \leq t'\} \qquad (2)$$

Below, I shall assume that G contains also limiting transformations corresponding to "asymptotic" external conditions; therefore it should be closed in some natural topology. Following Lubkin,[27] I shall call the elements of G the "dynamically achievable" transformations. It seems admissible to assume that the achievable transformations can be repeated with a time delay and superposed one after another. Hence, G has a natural structure of a semigroup. This semigroup does not represent any particular evolution process but is rather the semigroup of open possibilities. Therefore, I call it the semigroup of mobility.

In spite of the popularity of the "fixed Hamiltonian approach" the global view of dynamics seems to be gaining ground. Its ideas are seen in the algebraic approaches dealing with sets of many operations[6] and in semigroup approaches where the Hamiltonian is left unspecified.[28-31] The variety of external conditions have been introduced to the quantum field theory by Schwinger[32] in his S-matrix formalism. The advantages of the dynamics of open systems have been recognized[33]. The semigroup G has been introduced[26,25] to represent the element of motion in non-relativistic dynamical theories; it seems informative enough to allow the reconstruction of the form. To justify this, let me check what G is in the case of Schrödinger's quantum mechanics.

Consider Schrödinger's wave packet in one dimension. Here, the system is described in terms of functions of two operators q and p, $q, p = i$ (I put $\hbar = 1$). The "external conditions" are the external potentials in Schrodinger's wave equation, $V = V(q, t)$. The simplest evolution operations are

$$e^{-i\tau(\frac{p^2}{2} + V(q))}, \quad \tau = t' - t \geq 0,$$

and represent the dynamical evolution of Schrodinger's wave packet in the presence of the time independent external potentials $V = V(q)$. The other achievable operations are their products and limits. In particular, the unitary operations $e^{-iV(q)}$ are dynamically achievable.

They are limiting cases of the transformations caused by strong external fields acting within a short time ("shock transformations"):

$$e^{-iv(q)} = \lim_{\tau\to o} e^{-i\tau(\frac{p}{2}^{2} + \frac{1}{\tau} V(q))} \qquad (3)$$

The entire mobility of the system is not immediately obvious. Thus, the operations

$$e^{-i\tau\frac{p^2}{2}} \quad (\tau \geq 0)$$

are achievable as they represent free evolution; however, can their inverses

$$e^{+i\tau\frac{p^2}{2}}$$

be dynamically achieved? The answer to this question follows from the formula[26]:

$$\underbrace{e^{-i\tau\frac{p^2}{2}}\, e^{-\frac{1}{\tau}\frac{q^2}{2}} \cdots\cdots e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}}}_{12 \text{ terms}} = 1 \qquad (4)$$

Note that all signs in the exponents on the left side of (4) are identical. For $\tau \geq 0$ this formula represents a sequence of evolution operations in Schrödinger's wave mechanics.

The $e^{-i\tau\frac{p^2}{2}}$ are free evolution operations.

The $e^{-i\frac{1}{\tau}\frac{q^2}{2}}$ are "shock transformations" corresponding to oscillator shaped pulses of the external potential. The whole formula represents a "closed circuit" of the evolution: after the sequence of 12 operations all the wave packets must return to their initial shape. As a consequence, one has also:

$$\underbrace{e^{-i\frac{1}{\tau}\frac{q^2}{2}}\, e^{-i\tau\frac{p^2}{2}} \cdots\cdots e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}}}_{11 \text{ terms}} = e^{+i\tau\frac{p^2}{2}}$$

Note that all terms on the left side are achievable. Hence,

$e^{+i\tau\frac{p^2}{2}}$ $(\tau > 0)$ is achievable too. Formula (5) yields a prescription for how to manoeuvre the system into an operation inverse to its free evolution, so that it will reproduce its past states. What is curious is that prescription (5) give the operation

$$e^{+i\tau\frac{p^2}{2}}$$

as a whole; so the past state can be recovered without actually worrying what this state was and what the present state of the system is. The assumption about one space dimension here is not essential; the same is true for Schrodinger's particle in $\mathbb{R}^3$. What is still more curious is that an analogous "retrospection formula" exists for finite particle systems.[34] Given any number of Schrodinger particles evolving under the influence of a certain fixed interaction potential, there are manoeuvres with the external field which can force the system to perform an operation inverse to its natural evolution, thus recovering its past state, whatever this state was.[34] This operation can be effected with any desired accuracy for each fixed number of N particles, but it disappears in the thermodynamical limit $N \to \infty$. These results, besides telling a story about the "resurrection of finite particle systems" happen to be essential for the general mobility problem. It turns out that the operations inverse to the natural evolution occupy a key position among the unitary transformations. Once they are achievable, all other unitary operations can be effectively approximated with the help of Trotter formulae.[26,34] As a result, the mobility semigroup of Schrodinger's quantum mechanics of many particles coincides with the unitary group in the corresponding Hilbert space of states. This explains the true origin of the unitary group in orthodox quantum mechanics: this group would unavoidably arise from the very dynamics of the theory, even if no other arguments concerning the Hilbert space geometry were available. This also throws some light on the detailed state structure of that theory. Let me recall the question about the operational sense of all pure states raised by Lamb.[35] In agreement with its superposition principle, quantum mechanics predicts that, given any two state vectors, any linear combination of them represents a physically admissible state. However, in quantum mechanical experiments only a small fraction of these states has been created; for usually one deals with the eigenstates of energy, momentum, angular momentum, spin, etc. The question is: are the other states represented in nature, or are they redundant? As noticed by Lamb the answer involves the dynamics. By taking adequate external potentials in Schrodinger's evolution equation, Lamb shows that, starting from a certain initial one-particle state, any other state can be generated by a sequence of dynamical operations.[35] Hence, any "wave packet" is a physically realistic state. The same conclusion follows from the results about mobility.[26,34] These results, moreover, yield some insight into the structure of many particle states, so important for the interpretational controversies of quantum theory.[36] According to the doctrine, the states of composite systems are to be constructed by taking the antisymmetric (or symmetric) tensor products; and it is an essential question, whether all the elements of the tensor product space are indeed physically achievable. Again, the answer depends on the dynamics. If one assumes the validity of the many particle Schrodinger equation, then every unitary operation in the antisymmetric tensor product space can be dynamically achieved[34] and therefore, any tensor-product state can be generated out of any other state of the same type. Hence, the structure of states assumed in many particle quantum mechanics is not accidental. Of course, this cannot substitute for experimental evidence, such as that based on Bell's inequality.[37] However, it shows consistency links: one cannot disbelieve the existence of any particular tensor product state without disbelieving simultaneously the linear Schrodinger evolution equation for many particles. Apart from these concrete results, the mobility theorems suggest some conclusions concerning general dynamical theories.

1. When trying to generalize quantum theory by introducing non-linearities, one confronts a certain technical problem. The formalism of the present day theory is so adapted to the existence of the unitary group that it is hard to imagine any "quantum theory" in which this element would be missing. This is the source of a criticism which is often raised: "What about unitarity?". Now, it seems that the answer to that question can be given. In orthodox quantum mechanics the unitary group appears because it represents the dynamical mobility. This indicates that in the non-linear theories the semigroup G is the right substitute for unitarity.

2. In current dynamical theories the phase space is usually introduced with a certain a priori postulated geometry: it is either a symplectic manifold or a (projective) Hilbert space or a mathematically sublimated variant of these structures. The dynamics is introduced afterwards, on the already prepared ground of the phase space. This approach, though founded on a good tradition, is not necessary. The geometry of the phase should not be postulated, because it follows from the dynamics. This is seen in orthodox quantum mechanics, where the "scalar products" arise because they are invariants of the unitary group. In turn, the unitary group arises because it represents the dynamical mobility. It is not otherwise in the classical theory, where the geometry of the symplectic manifold, even if not postulated from the beginning, would nevertheless arise from the classical mobility group. This shows that the geometry of the phase space is not "fundamental", but "dynamical": it emerges after applying the Klein programme to the transformation semigroup G acting on the manifold Φ. Having this semigroup, one has to find out whether the manifold Φ is a Hilbert space, a symplectic manifold or any other structure.

3. Similar conclusions concern all other aspects of form. Thus, given the phase space Φ and the semigroup G one can infer something about the class of functional observables of the theory: it should be one of the invariant function subspaces of G on Φ.[23,25] In turn, the observables determine the convex set S of all pure and mixed states, (the statistical figure of the theory[23]). Given the statistical figure, the propositions of quantum logic can be found as closed faces of S.[5]

What one has here is a scheme for physical theories, different from that of the axiomatic approaches. In this scheme the form is secondary and is to be read off from the dynamics; quite similarly to the way in which the shape of an animal's bones can be determined by observing the animal in motion.

The above link between dynamics and structure might be of interest for new domains where the dynamics has already been introduced but where the geometry is still missing. To illustrate this, consider a hypothetical theory of real two-component "wave functions" in one space dimension:

$$\psi = \begin{Vmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{Vmatrix} .$$

As the dynamical equations assume:

$$\frac{\partial}{\partial t} \begin{Vmatrix} \psi_1 \\ \psi_2 \end{Vmatrix} = \begin{Vmatrix} & 1 \\ -1 & \end{Vmatrix} \frac{\partial}{\partial x} \begin{Vmatrix} \psi_1 \\ \psi_2 \end{Vmatrix} + V(x, t) \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \begin{Vmatrix} \psi_1 \\ \psi_2 \end{Vmatrix} \quad (6)$$

where V(x, t) are arbitrary functions representing the "external potentials". If $V(x, t) = 0$, the free evolution equations are immediately integrable and allow the class of motion integrals of the form:

$$\int_{-\infty}^{+\infty} [g(\psi_1) + h(\psi_2)] \, dx,$$

where g and h are arbitrary functions. However, for the general motions with $V(x, t) \neq 0$ the only functional still conserved is:

$$I(\psi) = k \int_{-\infty}^{+\infty} (\psi_1^{\,2} + \psi_2^{\,2}) \, dx, \quad K \in \mathbb{R} . \quad (8)$$

Therefore, (8) becomes the basic functional of the theory suggesting the normalization of the pure states and the statistical interpretation with $\psi_1{}^2 + \psi_2{}^2$ defining the "localization probability". Note that the resulting structure is typically quantum mechanical, with ψ_1 and ψ_2 behaving like the real and imaginary parts of a complex wave packet. This structure, however, would no longer arise in the case of a different dynamics. As an example, consider the non-linear equations:

$$\frac{\partial}{\partial t} \begin{Vmatrix} \psi_1 \\ \psi_2 \end{Vmatrix} = \begin{Vmatrix} & 1 \\ -1 & \end{Vmatrix} \frac{\partial}{\partial x} \begin{Vmatrix} \psi_1 \\ \psi_2 \end{Vmatrix} + V(x,t) \begin{Vmatrix} 0 & f(\psi_2{}^2) \\ -f(\psi_1{}^2) & 0 \end{Vmatrix} \begin{Vmatrix} \psi_1 \\ \psi_2 \end{Vmatrix} \quad (9)$$

where the function f is given. Here again the quantities (s) are the integrals of the free evolution. Now, however, other functionals are distinguished as the invariants of arbitrary motions :

$$I(\psi) = k\int_{-\infty}^{+\infty} \left[\phi(\psi_1^2) + \phi(\psi_2^2)\right] dx, \qquad (10)$$

where $\phi(u) = \int f(u)du$. Were the non-linear waves (9) to represent the pure states of a hypothetical system, the quantity (10) and not (8) would become the basic functional of the theory suggesting the norm and the statistical interpretation. Thus, for example, taking

$$f(u) = \frac{r}{2} u^{\frac{r}{2} - 1} \qquad (r > 0)$$

one would obtain a dynamical theory whose norm functional would be:

$$I(\psi) = \|\psi\|^r = k\int_{-\infty}^{+\infty} (\psi_1^r + \psi_2^r)\, dx \qquad (11)$$

and therefore, the pure states would admit a natural representation in the normed linear space L^r instead of the Hilbert space L^2. This is how the motion determines the form of the theory.

The above example is naive but it illustrates the genesis of the form as something deduced from the dynamics. This should not be taken against the status of the form in the physical theory. On the contrary, it shows that the problem of form cannot be avoided in mature dynamical theories. As a matter of fact, the hypothesis that "logic" might be conditioned by physics was formulated almost at the birth of "quantum logic".[39] Because of a certain rigidity of the orthocomplemented lattices it was not pursued later, which led to an isolation of the "logic" and most probably to some losses in other domains, which often undertake new dynamical problems but treat them in terms of the old formalism, (this might be one of the difficulties with "quantization" of non-linear fields). Recently, however, some developments were initiated in which the contingent character of logic can no longer be ignored. The idea about non-linear quanta has emerged.[23, 24, 40, 41, 42] The structure of logic seems more flexible at present,[5] (see also our dialogue with Piron[43]). A non-linear variant of quantum field theory with the non-linearity affecting the states has been considered.[44] One of the most interesting situations exists around the gap between quantum theories and general relativity.

This gap deserves some comments. It reminds me of the opinion that the fundamental problems of physics belong to the past. Einstein, Planck, de Broglie, Schrödinger, Heisenberg, Dirac and others were happy because they were working in a fortunate time when there were clear inconsistencies within theoretical physics (such as the failure of Maxwell electrodynamics to be Galileo invariant or black body radiation). However, according to this criterion, we are happier today. The gap between quantum physics and general relativity is deeper and more evident than just a little inconsistency such as black body radiation. It is an abyss. Shall we be able to cross it? Attempts to fill the gap with the already known Hilbert space structure of quantum theory continue, but have not yet brought a decisive solution. A basically different attempt has been proposed by Penrose[45] who assumes that the hypothetical gravitation, like the macroscopic gravity field, is non-linear and should be represented by a complex left flat solution of the Einstein equations ("heaven"). Recently an algebraic theory of "complex heavens" has been developed by Plebanski and his co-workers.[46] However, the statistical interpretation of "non-linear gravitation" is still missing and the geometry of the theory has not yet emerged. One feels the need for some more structural elements and it seems doubtful whether they should be Hilbert spaces and orthocomplemented lattices. Here, there is a challenge for form if we understand it widely enough.

I would like to derive some conclusions from these developments, though they are basically unconcluded. It seems to me now that the elements of form and motion can be given an almost human sense. We, who work on foundations, are mostly formalists and we worship form. Our colleagues working on graphs, scattering amplitudes etc., are the followers of motion. In the scheme which I presented, motion is at the beginning and form emerges later. However, if you dislike being merely the end of a creation instead of representing its origin, please do not treat this scheme literally. Perhaps it is superfluous to decide what came first, the chicken or the egg? What matters is the link. I feel we have arrived at the stage when neither the pragmatic trends nor the foundations can achieve much by developing alone. Therefore, if I were to propose a program, it would be a program to investigate the links. What kind of motions are basically possible? What forms are associated with them?

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