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MATHEMATICS

ON SOME CRITERIA FOR THE REGULARITY OF SPACES OF THE TYPE (B)

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I. In the present note we accept the terminology and the denotations used by Banach in his well-known book (1).

We shall moreover introduce the following definitions:

The space E of the type (B) is called a regular space, if for every $x \in \overline{E}$ there exists an $X \in \overline{\overline{E}}$ such that

$$f(x) = X(f) \text{ for all } f \in E.$$

A set of elements $\mathfrak{M}$ is called transfinitely closed, if for every transfinite bounded sequence $x_1, x_2, \ldots, x_\eta, \ldots (\eta < \vartheta)$ there exists at least one element x_0 (called the transfinite limit of the given sequence) such that

$$\underline{\lim}_{\eta<\vartheta} f(x_\eta) \leqslant f(x_0) \leqslant \overline{\lim}_{\eta<\vartheta} f(x_\eta) \text{ for all } f \in \overline{E}.$$

A. Plessner has given the following criterion for the regularity of the space (2):

A) In order that a space* should be regular, it is necessary and sufficient that its unite sphere $\|x\| \leqslant 1$ should be transfinitely closed.

On the other hand, V. Šmulian has kindly informed me that the following proposition is true:

B) In order that a convex, closed set K should be transfinitely closed, it is necessary and sufficient that for every sequence of convex, closed and non-empty sets

$$K_1 \supset K_2 \supset \ldots \supset K_\eta \supset \ldots \quad (\eta < \vartheta)$$

of which each but the first is contained in the preceding one, there should exist an element x_0 belonging to all $K_\eta (\eta < \vartheta)$.

We give briefly the proof of this proposition.

Let $x_1, x_2, \ldots x_\eta, \ldots (\eta < \vartheta)$ be a transfinite bounded sequence of elements and let K_η denote the smallest convex, closed set containing $x_\eta, x_{\eta+1}, \ldots x_\xi, \ldots (\xi < \vartheta)$. It is easily seen that

$$\underline{\lim}_{\eta<\vartheta} f(x_\eta) = \sup_{\eta<\vartheta} \inf_{x \in K_\eta} f(x); \quad \overline{\lim}_{\eta<\vartheta} f(x_\eta) = \inf_{\eta<\vartheta} \sup_{x \in K_\eta} f(x).$$

* We consider here and in what follows only spaces of the type (B).

On the other hand, an arbitrary element x_0 belongs to all K_η if and only if

$$\sup_{\eta<\vartheta} \inf_{x\in K_\eta} f(x) \leqslant f(x_0) \leqslant \inf_{\eta<\vartheta} \sup_{x\in K_\eta} f(x) \text{ for all } f\in\overline{E},$$

and the statement of Šmulian is proved.

Banach has proved that

C) A space $\overline{E}$ is regular, if it is separable and its unite sphere $(\|x\|\leqslant 1)$ is weakly compact [see (1), pp. 189—190].

Gantmacher and Šmulian (2) have proved that

D) The unite sphere of every regular space is weakly compact.

II. In connection with the above propositions we prove the following.

Theorem 1. *In order that a space should be regular, it is necessary and sufficient that its unite sphere should be weakly compact and that every face of this sphere should be transfinitely closed.*

Hereby under a face of the sphere $\|x\|\leqslant r$ we understand a convex set of boundary points of this sphere which satisfy the equation $f(x)=\|f\|$ for a certain $f\in\overline{E}\,(f\neq 0)$.

Proof. The necessity of the conditions of the theorem follows directly from propositions A) and D).

In order to prove that if these conditions are satisfied the space will be regular, it is in virtue of the propositions A) and B) sufficient to show that in the given space *there exists for every transfinite sequence of convex, closed and non-empty sets* $\{K_\eta\}_{\eta<\vartheta}$, *containing each the following one, an element* x_0 *belonging to all* K_η.

Let a be an arbitrary point outside K_1. Applying Mazur's theorem that every convex, closed set is weakly closed, it is easily verified that the distance between any point and a convex, closed and weakly compact set is always attained. Let $x_\eta\,(\eta<\vartheta)$ be the point at which the distance from a to K_η is attained. Let

$$\|x_\eta-a\|=d_\eta\,(\eta<\vartheta);\ \sup_{\eta<\vartheta} d_\eta=d.$$

Obviously we have $d_{\eta_1}\leqslant d_{\eta_2}$ when $\eta_1\leqslant\eta_2$.

We consider the following two cases:

1) $d_\eta<d$ for every $\eta<\vartheta$.

Let $d_{\eta_1}<d_{\eta_2}<\ldots\longrightarrow d$. It is obvious that the sequence $\eta_1<\eta_2<\eta_3<\ldots$ defines a transfinite index ϑ.

Let x_0 be a weak condensation point of the sequence $\{x_{\eta_n}\}$: we can assume that

$$x_{\eta_n}\longrightarrow x_0 \quad \text{when} \quad n\longrightarrow\infty.$$

By Mazur's theorem we conclude that $x_0\in K_{\eta_p}\,(p=1, 2, 3\ldots)$.

Let η be an arbitrary index $<\vartheta$. There exists an integer N such that $d_\eta<d_{\eta_N}$; consequently $\eta<\eta_N$ and $x_0\in K_{\eta_N}\subset K_\eta$. Thus, x_0 is the required element.

2) $d_{\eta_0}=d$ for a certain $\eta_0<\vartheta$.

It is obvious that in this case $d=d_\eta$ for $\eta_0\leqslant\eta<\vartheta$. Let S be a sphere of radius $d=d_{\eta_0}$ with the centre a and $S_\eta=K_\eta\cdot S$ for all $\eta_0\leqslant\eta<\vartheta$. Evidently S_{η_0} is a part of a certain face of S. Consequently, by the condition of the theorem, S_{η_0} is transfinitely closed. But then there exists, according to B), an element x_0 belonging to all $S_\eta\,(\eta_0\leqslant\eta<\vartheta)$ and consequently to all $K_\eta\,(\eta<\vartheta)$ which proves the statement.

Corollary. *If the unite sphere of a given space is weakly compact and all the faces of this sphere are separable, then the space is regular.*

In fact every transfinite sequence of separable closed sets containing each the following one contains only an enumerable number of different sets.

Hence, by B) it is easily seen that every convex, closed, weakly compact and separable set is transfinitely closed. Consequently, under the conditions stated all the faces of the unite sphere are transfinitely closed.

Theorem 1 can be generalized in the following way.

Theorem 1a. *In order that a space should be regular, it is necessary and sufficient that its unite sphere should be weakly compact and that on the boundary of this sphere there should be at least one point* y_0, *in a certain neighbourhood of which every face of the sphere is transfinitely closed; in particular, if in the neighbourhood of a point on the boundary of the unite sphere every face is separable and the unite sphere itself is weakly compact, then the space is regular.*

In the present note we confine ourselves to the statement that the theorem can be proved in the same way provided that the point a is chosen in the following special way.

Let T_0 be a sphere of radius R_0 enclosing K_1, and let T_1 be a concentric sphere of radius $3R$.

Then we choose on the ray $y=-ty_0\,(t>0)$ a point $y_1=-t_0y_0$ such that the boundary of the sphere with the centre at y_1 and radius t_0+3R_0 should contain in the neighbourhood of the point $3Ry_0$ enclosing the sphere T_1 only transfinitely closed faces. It is easily seen that such a choice is always possible.

III. Clarkson (3) has introduced the conception of uniform convexity of the space. A space is called uniformly convex, if for every $\varepsilon>0$ such $\delta>0$ can be found that from the inequality

$$\|x-y\|>\delta\,(\|x\|=\|y\|=1)$$

follows $\left\|\frac{x+y}{2}\right\|\leqslant 1-\varepsilon$.

We shall call a unite sphere uniformly convex in the r-neighbourhood of the point $x_0\,(\|x_0\|=1)$, if for every $\varepsilon>0$ such $\delta>0$ can be found that from

$$\|x-y\|>\delta;\ \|x\|=\|y\|=1;\ \|x-x_0\|\leqslant r;\ \|y-y_0\|\leqslant r$$

there follows

$$\left\|\frac{x+y}{2}\right\|\leqslant 1-\varepsilon.$$

Applying the same method as in the proof of Theorem 1, we shall now prove the following.

Theorem 2. *Every uniformly convex space is regular.*

Proof. Let $K_1, K_2, \ldots K_\eta\ldots(\eta<\vartheta)$ be a transfinite sequence of convex, closed and non-empty sets containing each the following one. According to the remarke made above, the theorem will be proved if we shall show that whatever be the sequence, there will always exist a point belonging to all $K_\eta\,(\eta<\vartheta)$.

It is easily verified that *in a uniformly convex space the distance between a point and a convex closed set is always attained and moreover only in one point.*

Let a be an arbitrary point outside K_1; let the distance from a to $K5$ be attained in x_η. Let $\|a-x_\eta\|=d_\eta;\ \sup_{\eta<\vartheta} d_\eta=d.$

Two cases are now possible:

1) $d_{\eta_0} = d$ $(\eta_0 < \vartheta)$. Then in virtue of uniform convexity of the unite sphere it follows that x_{η_0} is the required element.

2) $d_{\eta_1} < d_{\eta_2} < \ldots \to d$ and there does not exist an $\eta < \vartheta$ for which $d_\eta = d$.

The sequence $x_{\eta_1}, x_{\eta_2}, \ldots$ has a strong limit x_0. In fact, let us assume that at the contrary

$$\| x_{\eta_{m_p}} - x_{\eta_{n_p}} \| \geqslant \delta > 0,$$

where $p = 1, 2, 3 \ldots, \eta_{n_1} < \eta_{m_1} < \eta_{n_2} < \eta_{m_2} \ldots$

Since $\lim\limits_{p\to\infty} \| x_{\eta_{m_p}} \| = \lim\limits_{p\to\infty} \| x_{\eta_{n_p}} \| = d$ and the space is uniformly convex, there exists such $\varepsilon > 0$ that

$$\left\| \frac{x_{\eta_{m_p}} + x_{\eta_{n_p}}}{2} \right\| \leqslant (1 - \varepsilon)\, d \qquad (p \geqslant N).$$

But $\frac{x_{\eta_{m_p}} + x_{\eta_{n_p}}}{2} \in K_{\eta_{n_p}}$; therefore

$$d_{\eta_{n_p}} \leqslant \left\| \frac{x_{\eta_{m_p}} + x_{\eta_{n_p}}}{2} \right\| \leqslant (1 - \varepsilon)\, d$$

and we have come to a contradiction. Thus, there exists $\lim\limits_{n\to\infty} x_{\eta_n} = x_0$. Since the sequence $\eta_1, \eta_2, \ldots$ defines ϑ, we conclude, applying Mazur's theorem mentioned above, that $x_0 \in K_\eta$ for $\eta \quad \vartheta$. This proves the theorem.

From the last theorem in particular it follows that a part of Clarkson's (3) results on the differentiation of abstract functions form a corollary of the results obtained by I. Gelfand in his dissertation (1935) (4).

Theorem 2 admits the following generalization which can be proved in the same way as Theorem 1a.

Theorem 2a. *If on the boundary of the unite sphere of a space there exists at least one point y_0 in a certain neighbourhood $\| x - y_0 \| \leqslant r$ of which the unite sphere is uniformly convex, then the space is regular.*

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