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A NOTE ON LORENTZ SPACES WITH ORLICZ-TYPE METRICS

There are considered some generalizations of Orlicz spaces with Orlicz metrics generated by φ – functions depending on a parameter.

Key words: measure preserving function, a monotone rearrangement of a measurable function, Lorentz space, Orlicz space, modular, Luxemburg norm, φ – function with a parameter, φ – function.

The theory of spaces L^p was generalized by G.G. Lorentz ([4]) to a more general case of spaces (see also [1], 1.3, 1.7, and [3], p. 145). B. Kotkowski ([2]) considered also Lorentz-Orlicz space with a concave function. In this paper there will be investigated some other generalizations of Lorentz space with Orlicz metrics generated by a φ-function depending on parameter.

Definition 1. *Let* φ: $[0,\infty)\times[0,\infty) \to [0,\infty)$ *be a* φ *–function depending on parameter, i.e.*

1. $\varphi(t,u)$ *is a nondecreasing, leftcontinuous function of* u, *with* $\varphi(t,0) = 0$, $\varphi(t,u) > 0$ *for* $u > 0$, $\varphi(t,u) \to 0$ *as* $u \to \infty$ for a.e. $t \in [0,\infty)$.
2. $\varphi(t,u)$ *is Lebesgue-measurable with respect to* t *for every* $u \geq 0$.

Let μ be a δ –additive measure on $[0,\infty)$, and let us denote by x^* the monotone rearrangement of a function x with respect to the measure μ (see [2], p. 871, compare also [3], p. 83). Then from the assumtions of Definition 1 it follows that $\varphi(t,x^*(t))$ is measurable. Let us consider for an arbitrary μ –measurable function x on $[0,\infty)$ the functionals

$$\eta_{\varphi,\mu}(x) = \int_0^\infty \varphi(t,x^*(t))\,dt,$$

$$\|x\|_{\Lambda_{\varphi,\mu}} = \inf\left\{\epsilon > 0: \eta_{\varphi,\mu}\left(\frac{x}{\epsilon}\right) \leqslant \epsilon\right\}$$

and sets

$$\Lambda_{\varphi,\mu} = \left\{x: \exists\ \lambda > 0 \quad \eta_{\varphi,\mu}(\lambda x) < \infty\right\},$$

$$\Lambda^*_{\varphi,\mu} = \left\{x: \quad \forall\ \lambda > 0 \ \ \eta_{\varphi,\mu}(\lambda x) < \infty\right\},$$

$$\Lambda^{o}_{\varphi,\mu} = \{x: \quad \eta_{\varphi,\mu}(x) < \infty\}.$$

In the case of Lebesgue measure $m = \mu$, we shall write $\eta_{\varphi} = \eta_{\varphi,\mu}$ $\| \|_{\Lambda_{\varphi}} = \| \|_{\Lambda_{\varphi,\mu}}$, $\Lambda_{\varphi} = \Lambda_{\varphi,\mu}$, $\Lambda^{*}_{\varphi} = \Lambda^{*}_{\varphi,\mu}$, $\Lambda^{0}_{\varphi} = \Lambda^{0}_{\varphi,\mu}$.

Let us remark that taking $\varphi(t,u) = t^{r/p-1}u^{r}$ where $p \geqslant 1$, $r \geqslant 1$, sets Λ_{φ}, Λ^{*}_{φ}, Λ^{o}_{φ} are all equal to Lorentz spaces $L_{p,r}$ and $\| \|_{\Lambda_{\varphi}}$ is equivalent to the norm $\| \|_{p,r}$ in $L_{p,r}$ ([1], 1.3).

In general $\eta_{\varphi,\mu}$, $\| \|_{\Lambda_{\varphi,\mu}}$ do not need to be a modular and an F-norm.

As an example we may take $\mu = m$, $\varphi(t,u) = e^{t}u$. Let $A, B \subset [0,\infty)$ be disjoint sets of Lebesgue measure $\ln 5$, $x = \chi_A$, $y = \chi_B$ – characteristic functions of the sets A, B, respectively. Then

$$\eta_{\varphi}\left(\frac{1}{3}x + \frac{2}{3}y\right) > \eta_{\varphi}(x) + \eta_{\varphi}(y) \quad \text{and} \quad \|x+y\|_{\Lambda_{\varphi}} > \|x\|_{\Lambda_{\varphi}} + \|y\|_{\Lambda_{\varphi}}$$

Lemma 1. *Let φ be a leftcontinuous φ–function without parameter, i.e. $\varphi(t,u) = \varphi(u)$ and let $\psi(u) = \inf\{s \geqslant 0: \ \varphi(s) > u\}$. Then $\psi(u) < t$ if and only if $u < \varphi(t)$.*

Proof. Since $\lim\limits_{s\to\infty} \varphi(s) = \infty$, so $\psi(u) < \infty$. Moreover,

(1) $$(\psi(u), \infty) \subset \{s \geqslant 0: \ \varphi(s) > u\} \subset [\psi(u), \infty).$$

Then $\psi(u) < t \Rightarrow t \in \{s \geqslant 0: \ \varphi(s) > u\} \Rightarrow \varphi(t) > u$.

By (1) and the definition of ψ,

$$\varphi(t) > u \Rightarrow \underset{\epsilon>0}{\exists} \ \varphi(t-\epsilon) > u \Rightarrow t - \epsilon \in \{s \geqslant 0: \ \varphi(s) > u\} \Rightarrow t-\epsilon \geqslant \psi(u) \Rightarrow$$
$$\Rightarrow t > \psi(u) \quad \text{for} \quad t > 0.$$

For $t = 0$ and $u \geqslant 0$ the implication is true.

Lemma 2. *If φ is a leftcontinuous φ–function without parameter then $\varphi(x^{*}(t)) = [\varphi|x(t)|]^{*}$.*

Proof. By Lemma 1, we obtain for a function $\psi(u) = \inf.\{s > 0: \ \varphi(s) > u\}$

(1) $$\varphi(|x(s)|) > u \Leftrightarrow \psi(u) < |x(s)|$$

and

(2) $$\psi(u) < x^{*}(s) \Leftrightarrow u < \varphi(x^{*}(s)).$$

There holds also

$$p(x,t) > s \qquad x^{*}(s) > t, \quad \text{where} \quad p(x,t) = \mu\{s > 0: \quad |x(s)| > t\}.$$

Hence for $t = \psi(u)$ we obtain

(3) $$p(x, \psi(u)) > s \Leftrightarrow x^{*}(s) > \psi(u).$$

By definition, we have, applying (1) and denoting by $\circ$ the composition operator,

(4) $p(\varphi\circ|x|, u) = \mu\{s>0: \ \varphi|x(s)|>u\} = \mu\{s>0: \ |x(s)|>\psi(u)\} = p(x,\psi(u))$.

Next, by (4), (3) and (2), we get

$$[\varphi(|x(s)|]^* = \sup\{u\geqslant 0: \ p(\varphi\circ|x|, u)>s\} = \sup\{u\geqslant 0: \ p(x,\psi(u))>s\} =$$
$$= \sup\{u\geqslant 0: \ x^*(s)>\psi(u)\} = \sup\{u\geqslant 0: \ \varphi(x^*(s))>u\} = \varphi(x^*(s)).$$

Lemma 3. *Let* u, v *be two Lebesgue measurable functions on* $[0,\infty)$ *and let* g *be a nondecreasing, nonnegative function on* $[0,\infty)$.

Then $\int_0^\infty g(t)\,[u(t)+v(t)]^*\,dt \leqslant \int_0^\infty g(t)\,u^*(t)\,dt + \int_0^\infty g(t)\,v^*(t)\,dt$, *where the operator* * *is defined with respect to the Lebesgue measure.*

Proof. Since $g=g^*$ a.e. the proof follows from [3] p. 97, 2.2.8.

Theorem 1. *Let* g *be a positive, nondecreasing function on* $[0,\infty)$ *and let* φ *be a left-continuous* φ*–function without parameter. Let* $\phi(t,u) = g(t)\cdot\varphi(u)$. *Then* η_ϕ *is a modular,* $\|\ \|_{\Lambda_\phi}$ *is an* F*–norm and* Λ_ϕ, Λ^*_ϕ *are linear spaces.*

Proof. By Lemma 2, we have $\varphi(x^*(t)) = [\varphi(|x(t)|)]^*$, whence

$$(1) \qquad \eta_\phi(z) = \int_0^\infty g(t)\,[\varphi(|z(t)|)]^*\,dt,$$

for an arbitrary Lebesgue measurable z.

Let us fix $\alpha\geqslant 0$, $\beta\geqslant 0$ such that $\alpha+\beta=1$ and let x, y be two Lebesgue measurable functions on $[0,\infty)$. Then

$$\varphi(|\alpha x(t)+\beta y(t)|) \leqslant \varphi(|x(t)|) + \varphi(|y(t)|),$$

whence

$$(2) \qquad [\varphi(|\alpha x(t)+\beta y(t)|)]^* \leqslant [\varphi(|x(t)|)+\varphi(|y(t)|)]^*$$

(see [2] p. 871).

Applying (1), (2) and Lemma 3 to $u(t)=\varphi(|x(t)|)$, $v(t)=\varphi(|y(t)|)$, we obtain

$$\eta_\phi(\alpha x+\beta y) = \int_0^\infty g(t)\,[\varphi(|\alpha x(t)+\beta y(t)|)]^*\,dt \leqslant$$
$$\leqslant \int_0^\infty g(t)\,[\varphi(|x(t)|)+\varphi(|y(t)|)]^*\,dt \leqslant \int_0^\infty g(t)\,[\varphi(|x(t)|)]^*\,dt +$$
$$+ \int_0^\infty g(t)\,[\varphi(|y(t))|)]^*\,dt = \eta_\phi(x)+\eta_\phi(y).$$

The remaining part of the proof follows from the fact that η_ϕ is a modular, and from definitions of $\| \|_{\Lambda_\phi}$, Λ_ϕ, Λ^*_ϕ.

Theorem 2. *Let* φ *be a* φ–function with parameter satisfying the inequality $\varphi(2t,u) \leqslant C\varphi(t,u)$ *for all* $u \geqslant 0$ *and a.e.* $t \in [0, \infty)$, *with a constant* $C > 0$. *Then* $\Lambda_{\varphi,\mu}$, $\Lambda^*_{\varphi,\mu}$ *are linear spaces and*

$$\eta_{\varphi,\mu}(\alpha x + \beta y) \leqslant 2C(\eta_{\varphi,\mu}(x) + \eta_{\varphi,\mu}(y)),$$

$$\|x + y\|_{\Lambda_{\varphi,\mu}} \leqslant C_0(\|x\|_{\Lambda_{\varphi,\mu}} + \|y\|_{\Lambda_{\varphi,\mu}}),$$

where $C_0 = \max\{1,2C\}$, $\alpha, \beta \geqslant 0$ *and* $\alpha + \beta \leqslant 1$.

Proof. We prove the inequality for $\eta_{\varphi,\mu}$. There holds the inequality

(1) $$(x + y)^*(t) \leqslant x^*(t/2) + y^*(t/2)$$

and the equality

(2) $$(ax)^*(t) = ax^*(t)$$

for every $t \geqslant 0$ and $a > 0$.

Hence for arbitrary $\alpha \geqslant 0$, $\beta \geqslant 0$ with $\alpha + \beta \leqslant 1$ and arbitrary μ-measurable x, y we have

$$\varphi(t,(\alpha x + \beta y)^*(t)) \leqslant \varphi(t, \alpha x^*(t/2) + \beta y^*(t/2)) \leqslant \varphi(t,x^*(t/2)) + \varphi(t,y^*(t/2)).$$

Applying there inequalities, we obtain

$$\eta_{\varphi,\mu}(\alpha x + \beta y) \leqslant \int_0^\infty \varphi(t,x^*(t/2))\,dt + \int_0^\infty \varphi(t,y^*(t/2))\,dt =$$

$$= 2\int_0^\infty \varphi(2t,x^*(t))\,dt + 2\int_0^\infty \varphi(2t,y^*(t))\,dt.$$

From the assumed inequality for φ we get

$$\eta_{\varphi,\mu}(\alpha x + \beta y) \leqslant 2C(\eta_{\varphi,\mu}(x) + \eta_{\varphi,\mu}(y)).$$

The inequality for $\| \|_{\Lambda_{\varphi,\mu}}$ we obtain in the same manner as for the norm generated by an arbitrary modular.

Let $\|x\|_{\Lambda_{\varphi,\mu}} < \infty$, $\|y\|_{\Lambda_{\varphi,\mu}} < \infty$ and let $\epsilon > 0$ be arbitrary. We take $a = \|x\|_{\Lambda_{\varphi,\mu}} + \frac{\epsilon}{2C_0}$, $b = \|y\|_{\Lambda_{\varphi,\mu}} + \frac{\epsilon}{2C_0}$. Since $a > \|x\|_{\Lambda_{\varphi,\mu}}$, $b > \|y\|_{\Lambda}$ and sets $\{\epsilon > 0:$

$\eta_{\varphi,\mu}(\frac{z}{\epsilon}) \leqslant \epsilon \}$ are of the form $[\, \|z\|_{\Lambda_{\varphi,\mu}}, \infty)$ or $(\, \|z\|_{\Lambda_{\varphi,\mu}}, \infty)$, there hold the inequalities $\eta_{\varphi,\mu}(\frac{x}{a}) \leqslant a$ and $\eta_{\varphi,\mu}(\frac{y}{b}) \leqslant b$. Hence $\eta_{\varphi,\mu}(\frac{x+y}{C_0\,(a+b)}) = \eta_{\varphi,\mu}(\frac{a}{a+b}\,\frac{x}{C_0 a} +$

$+ \frac{b}{a+b}\,\frac{y}{C_0\, b}) \leqslant 2C\, \eta_{\varphi,\mu}(\frac{x}{C_0 a}) + 2\,C\, \eta_{\varphi,\mu}(\frac{y}{C_0 b}) \leqslant C_0\, \eta_{\varphi,\mu}(\frac{x}{a}) + \eta_{\varphi,\mu}(\frac{y}{b}) \leqslant$

$\leqslant C_0(a+b)$, because $\eta_{\varphi,\mu}(\alpha\, x)$ is nonincreasing with respect to α for fixed x.

By the definition of $\|\ \|_{\Lambda_{\varphi,\mu}}$, we get $\|x+y\|_{\Lambda_{\varphi,\mu}} \leqslant C_0(a+b) = C_0\,(\|x\|_{\Lambda_{\varphi,\mu}} +$

$+ \|y\|_{\Lambda_{\varphi,\mu}}) + \epsilon$.

Consequently,

$$\|x+y\|_{\Lambda_{\varphi,\mu}} \leqslant C_0(\|x\|_{\Lambda_{\varphi,\mu}} + \|y\|_{\Lambda_{\varphi,\mu}}).$$

Definition 2. (*see* [5] p. 43). *Let* φ_1, φ_2 *be* φ*-functions with parameter. We say that* $\varphi_1 \mapsto\!\!3\, \varphi_2$ *if there exist constans* $K_1, K_2 > 0$ *and a nonnegative, integrable function* h *such that* (*) $\varphi_1(t,u) \leqslant K_1\, \varphi_2(t,K_2 u) + h(t)$ *for all* $u \geqslant 0$ *and a.e.* $t \in [0,\infty)$. *If* (*) *holds with* $K_2 = 1$, *we shall say that* $\varphi_1 \overset{1}{\mapsto\!\!3}\, \varphi_2$.

Theorem 3. *Let the* φ*-functions* φ_1, φ_2 *with parameter satisfy* $\varphi_2 \overset{1}{\mapsto\!\!3}\, \varphi_1$. *Then* $\Lambda^o_{\varphi_1,\mu} \subset \Lambda^o_{\varphi,\mu}$.

Proof. Since $x \in \Lambda^o_{\varphi,\mu}$ is equivalent to $x^* \in L^\varphi_0$, where L^φ_0 is a generalized Orlicz class, so the theorem follows from [5] theorem 8.4a), p. 45.

In order to prove a converse theorem, we shall need some auxiliary definitions and lemmas. First, we recall the following

Definition 3. *Let* $A, B \subset [0,\infty)$ *satisfy the condition* $m(A) = m(B) > 0$, *where* m *is the Lebesgue measure. A function* $h : A \to B$ *is called measure preserving, if for an arbitrary* m*-measurable set* $E \subset B$ *there holds* $m(h^{-1}(E)) = m(E)$ (see e.g. [3]).

Lemma 4. *Let* x *be a simple function on* $[0,\infty)$. *Then there exists a measure preserving function* $h : [0,\infty) \to [0,\infty)$ *such that* $x(t) = x^*(h(t))$ *a.e.*

Proof. Let $x = \sum_{n=1}^{k} a_n \cdot \chi_{A_n}$, where $m(A_n) < \infty$, $A_i \cap A_j = \phi$; for $i \neq j$, $n = 1,\dots,k$.

We may always suppose that the sequence (a_n) is decreasing. Then

$x^* = \sum_{n=1}^{k} a_n \cdot \chi_{B_n}$, where $B_n = [\sum_{i=1}^{n} m(A_i), \sum_{i=1}^{n} m(A_i)]$. By [3] p. 96, there exist measure preserving functions $h_n : A_n \to B_n$ for $n = 1, \ldots, k$ and $h_0 : A_0 \to B_0$, where $A_0 = [0, \infty) \setminus \sum_{n=1}^{k} A_n$, $B_0 = [0, \infty) \setminus \sum_{n=1}^{k} B_n$. We define $h = \sum_{n=0}^{k} h_n \cdot \chi_{A_n}$. The function h satisfies the conditions of the lemma.

Lemma 5. *If the function* $h : [0, \infty) \to [0, \infty)$ *is measure preserving, then* $g \circ h$ *is integrable in* $[0, \infty)$ *if and only if* g *is integrable in* $[0, \infty)$. *Moreover, for* g *integrable in* $[0, \infty)$ *we have*

$$\int_0^\infty g(h(t))\, dt = \int_0^\infty g(t)\, dt.$$

Proof. First, suppose g to be a simple function, i.e. $g = \sum_{n=1}^{k} a_n \cdot \chi_{A_n}$, where $A_i \cap \cap A_j = \phi$ for $i \neq j$. Then $g \circ h = \sum_{n=1}^{k} a_n \cdot \chi_{B_n}$, where $B_n = h^{-1}(A_n)$. Since $m(A_n) = m(B_n)$, so

$$\int_0^\infty g(t)dt = \sum_{n=1}^{k} a_n\, m(A_n) = \sum_{n=1}^{k} m(B_n) = \int_0^\infty g(h(t))\, dt.$$

If g is an arbitrary measurable nonnegative function, then we apply the above result to simple functions $0 \leqslant g_n \leqslant g$ with $g_n \uparrow g$ and the thesis follows, by Beppo-Levi theorem. For an arbitrary integrable g, it is sufficient to write g as the difference of its positive and negative part.

Theorem 4. *Let the following conditions be satisfied:*

1. $\Lambda^o_{\varphi_1} \subset \Lambda^o_{\varphi_2}$,

2. φ_2 *is a convex* φ*-function without parameter,*

3. *there exist a constant* $C > 0$ *and a nonnegative function* g *such that for arbitrary sequence* (h_n) *of measure preserving functions* $h_n : [0, \infty) \to [0, \infty)$ *the sequence* $\left(\int_0^\infty g(h_n(t), t)dt \right)$ *is bounded. Than* $\varphi_2 \overset{1}{\dashv} \varphi_1$.

Proof. Let us suppose the theorem to be not true. Arguing as in the proof of theorem 8.4 in [5], p. 45, we construct a sequence $(\tilde{x}_{n_k})$ of nonnegative, simple functions such that

(1) $$\int_0^\infty \varphi_2(\tilde{x}_{n_k}(t))\, dt = 1,$$

where (A_k) is a sequence of pairwise disjoint sets, and

(2) $$b_{n_k}(t) = \varphi_2(\tilde{x}_{n_k}(t)) - 2^{n_k} \varphi_1(t, \tilde{x}_{n_k}(t)) \geqslant 0.$$

We define

$$x = \sum_{k=1}^{\infty} \widetilde{x}_{n_k} \cdot \chi_{A_k}, \quad y_l = \sum_{k=1}^{l} \widetilde{x}_{n_k} \cdot \chi_{A_k}.$$

We shall show that $\eta_{\varphi_2}(x) = \infty$ and $\eta_{\varphi_1}(x) < \infty$.

Let $h_l : [0, \infty) \to [0, \infty)$ be measure preserving functions such that $y_l^*(h_l(t)) = y_l(t)$ (existence of such h_l follows from Lemma 4). Since $x \geqslant y_l \geqslant 0$, we have $x^* \geqslant y_l^*$ for arbitrary $l \in N$ (see [2]). Hence, by Lemma 5, convexity of φ_2 and (1), we obtain

$$\eta_{\varphi_2}(x) = \int_0^{\infty} \varphi_2(x^*(t))\, dt \geqslant \int_0^{\infty} \varphi_2(y_l^*(t))\, dt = \int_0^{\infty} \varphi_2(y_l^*(h_l(t)))\, dt =$$

$$= \int_0^{\infty} \varphi_2(y_l(t))\, dt = \sum_{k=1}^{l} \int_{A_k} \varphi_2(\widetilde{x}_{n_k}(t))\, dt = l.$$

Since l is arbitrary, we thus have $\eta_{\varphi_2}(x) = \infty$. Moreover, $y_l \uparrow x$ implies $y_l^* \uparrow x^*$ ([2]), whence $\varphi_1(t, y_l^*(t)) \uparrow \varphi_1(t, x^*(t))$. Consequently,

$$\eta_{\varphi_1}(x) = \lim_{l \to \infty} \int_0^{\infty} \varphi_1(t, y_l^*(t))\, dt = \lim_{l \to \infty} \eta_{\varphi_1}(y_l). \tag{3}$$

Applying the assumptions of φ_1, y_l and Lemma 5, we get

$$\eta_{\varphi_1}(y_l) = \int_0^{\infty} \varphi_1(t, y_l^*(t))\, dt = \int_0^{\infty} \varphi_1(h_l(t), y_l(h_l(t)))\, dt = \tag{4}$$

$$= \int_0^{\infty} \varphi_1(h_l(t), y_l(t))\, dt \leqslant C \int_0^{\infty} \varphi_1(t, y_l(t))\, dt + \int_0^{\infty} g(h_l(t), t)\, dt \leqslant$$

$$\leqslant C \int_0^{\infty} \varphi_1(t, y_l(t))\, dt + A,$$

where $\int_0^{\infty} g(h_l(t), t)\, dt \leqslant A/C$ for every $l \in N$.

From the definition of y_l and from (2), (1) we have

$$\int_0^{\infty} \varphi_1(t, y_l(t))\, dt = \sum_{k=1}^{l} \int_{A_k} \varphi_1(t, \widetilde{x}_{n_k}(t))\, dt \leqslant \tag{5}$$

$$\leqslant \sum_{k=1}^{n} \left[1/2^{n_k} \int_{A_k} \varphi_2(\widetilde{x}_{n_k}(t))\, dt - \int_{A_k} b_{n_k}(t)\, dt \right] \leqslant$$

$$\leq \sum_{k=1}^{1} 1/2^{n_k} \int_{A_k} \varphi_2(\widetilde{x}_{n_k}(t))\,dt = \sum_{k=1}^{1} 1/2^{n_k} < 1.$$

By (3), (4) and (5), we finally obtain

$$\eta_{\varphi_1}(x) \leq C + A.$$

Let us still remark that the function $g(t,s) = g_1(t)\, g_2(s)$ with $g_1, g_2 \in L^2$ satisfies the first of the requirements in condition 3 of Theorem 4.

Definition 4. *Let* x_n, x *be μ-measurable functions,* $x \in \Lambda_{\varphi,\mu}$ *and* $x_n \in \Lambda_{\varphi,\mu}$ *for large* n. *If* $\eta_{\varphi,\mu}(a(x_n - x)) \to 0$ *for some* $a > 0$, *then the sequence* (x_n) *will be called convergent to* x *in* $\Lambda_{\varphi,\mu}$ *in the weak sense. If* $\|x_n - x\|_{\Lambda_{\varphi,\mu}} \to 0$, *then* (x_n) *will be called convergent to* x in $\Lambda_{\varphi,\mu}$ *in the strong sense.*

Theorem 5. The following conditions are equivalent:

$$\|x_n - x\|_{\Lambda_{\varphi,\mu}} \to 0,$$

$$\eta_{\varphi,\mu}(a(x_n - x)) \to 0 \quad \textit{for every} \quad a > 0.$$

The easy proof is omitted.

Theorem 6. *Let* μ *be a fixed measure on* $[0,\infty)$. *Let* φ_1, φ_2 *be* φ*–functions with parameter and* $\varphi_2 \prec \varphi_1$. *Then* $\Lambda_{\varphi_1,\mu} \subset \Lambda_{\varphi_2,\mu}$ *and* $\Lambda^*_{\varphi_1,\mu} \subset \Lambda^*_{\varphi_2,\mu}$. *If moreover,* φ_2 *is locally integrable with respect Lebesgue measure (see* [5], p. 47), *then convergence in weak sense (strong sense) in* $\Lambda_{\varphi_1,\mu}$ *is stronger then the convergence in the weak sense (strong sense) in* $\Lambda_{\varphi_2,\mu}$.

Proof. By theorem 8.5a) of [5], p. 47, we have $L^{\varphi_1} \subset L^{\varphi_2}$ and the required inclusions follow from the definition of the spaces under consideration. If $\eta_{\varphi_1,\mu}(a(x_n - x)) \to 0$, then $\rho_{\varphi_1}(a(x_n - x)^*) \to 0$, where ρ_{φ_1} is the modular in L^{φ_1}. Applying theorem 8.5a, we obtain $\rho_{\varphi_2}(a(x_n - x)^*) \to 0$, i.e. $\eta_{\varphi_2}(a(x_n - x)) \to 0$.

Theorem 7. *If* $\Lambda_{\varphi_1} \subset \Lambda_{\varphi_2}$ *and there are satisfied conditions* 2. *and* 3. *from Theorem* 4, *then* $\varphi_2 \prec \varphi_1$.

The proof is ommited.

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