ON VON NEUMANN'S THEORY OF MEASUREMENTS IN QUANTUM STATISTICS

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1. Introduction

It is well-known, that the theory of von Neumann algebras, founded by J. von Neumann himself, has its origin as a mathematical foundation of quantum mechanics, and that von Neumann wrote his monumental classic book [Q] basing on his earlier studies of von Neumann algebras. In recent years, however, the modern theory of von Neumann algebras, which is developed by Dixmier, Kadison, Kaplansky, Segal and the others and summarized beautifully in Dixmier [1], almost loses touch with quantum mechanics.

Very recently, Segal [4] observed that the modern theory of von Neumann algebras has an oppotunity to rewrite partly [Q], especially he noticed that the concavity of the entropy in quantum statistics has an alternative proof. However, he did not explicitly point out that his point of view gives an another perspective on quantum mechanics especially on von Neumann's theory of measurements.

The present note is, in essence, an explanatory note which may give an exposition on some facts mainly contained in [Q; Chap. V] under the point of view which is based on the recent developments of the modern theory of von Neumann algebras. Especially, Proposition 1 is the center of the note which may clarify in some sense hitherto mysterious character of von Neumann's theory of measurements in quantum statistics.

Throughout the note, for the sake of convenience, the technical terms and the symbols used in [Q] are employed. Also, the terminology on von Neumann algebras in Dixmier [1] is used without any explanation.

2. Statistical operator

The *statistical operator* U of an ensemble in quantum statistics is introduced in [Q; p. 319] as an hermitean positive-definite completely continuous operator satisfying

(Tr)
$$\mathrm{Exp}(R) = \mathrm{Tr}(UR),$$

where R is an observable corresponding to a physical quantity, and where Tr means the trace of operators introduced in [Q; p. 181]. The condition (Tr) therefore assumes that the statistical operator U is of the trace class. Von Neumann [Q; p. 328] taught us that the most general statistical ensemble which is compatible with our basic assumptions is characterized, according to the law (Tr), by a definite statistical operator U.

It is interesting to observe, in the point of view belonging to the modern theory of von Neumann algebras, that *the statistical operator is nothing but the Radon-Nikodym derivertive of* Exp-*functional with respect to the trace* Tr which is developed by H. A. Dye [2] and the others in 1950s. This view is employed recently by I. E. Segal [4] to exploit his theory of entropy of states defined on an operator algebra. The same view is also emplyed in the present note to develop von Neumann's theory of measurements under a light of the modern theory of von Neumann algebras.

Since a statistical operator U is positive, hermitean and completely continuous, it is expressed in

$$U=\sum_{n=1}^{\infty}w_n P_n, \tag{1}$$

where $w_1, w_2, \cdots$ are positive numbers and $P_1, P_2, \cdots$ are mutually orthogonal one-dimensional projections. Since U belongs to the trace class, $\mathrm{Tr}(U)$ is finite. Conveniently, it will be normalized in the sense that $\mathrm{Tr}(U)=1$. Since the trace is additive, this convention and (1) imply

$$\sum_{n=1}^{\infty} w_n=1. \tag{2}$$

Conversely, the statistical operator U can be defined by (1) and (2). Von Neumann [Q; pp. 295–296] introduced the statistical operator using (1) and (2) previously to his definition described in the above.

Let φ_n be the unit vector belonging to the range of P_n, then (Tr) becomes

$$\mathrm{Exp}(R)=\sum_{n=1}^{\infty} w_n(R\varphi_n, \varphi_n). \tag{3}$$

The expression gives, according to [Q; pp. 328–329], an intuitive description of the statistical operator: If several states $\varphi_1, \varphi_2, \cdots$ (which are mutually orthogonal and normalized) have the respective probabilities $w_1, w_2, \cdots$ then the statistical operator of the mixture of these states is the weighted mean (1) of the projections belonging to the states.

It may be interesting to note that a statistical operator corresponds in a certain sense to a trial [partition] in the theory of probability. Apparently, the statistical operator divides the Hilbert space into mutually orthogonal subspaces with certain probabilities, and similarly a trial divides the sample space into mutually exclusive subsets with certain probabilities. Moreover, there is a further affinity between them: If P_n is the characteristic function of a set belonging the trial and if w_n is the probability of the set, then (1) gives a bounded measurable function which acts a role of the statistical operator on L^2-space over the sample space. Therefore, the satistical operator can be considered as a non-commutative extension of the concept of trial.

3. Measurement and conditional expectation

To avoid the complication, let us assume hereafter that R, an observable corresponding to a physical quantity, is bounded and has a pure discrete simple sectrum. Let $\varphi_1, \varphi_2, \cdots$ be the complete orthnormal basis corresponding to the propervalues $\lambda_1, \lambda_2, \cdots$ of R respectively. Von Neumann [Q; p. 347] observed that R has the value λ_n in the fraction $(U\varphi_n, \varphi_n)$ after the measurement, and that we obtain a mixture with the statistical operator

$$U'=\sum_{n=1}^{\infty}(U\varphi_n, \varphi_n)P_{[\varphi_n]}, \tag{4}$$

after the measurement. This change, given by the process

1. $U\to U'$,

is the statistical development of a state by measurements, and it differs essentially from the causal development of a state given by the process

2. $U\to U_t=e^{-(2\pi i/h)tH}Ue^{(2\pi i/h)tH}$,

where H is the Hamiltonean. Just as in classical mechanics, von Neumann [Q; p. 358] taught us, Process **2.** does not reproduce the most important property of the real world, namely its irreversibility, the fundamental difference between the time direction, "future" and "past", whereas **1.** is certainly not prima facie reversible.

The present note is interested in the process **1.** under the light of the modern theory of von Neumann algebras. It seems that the secret of von Neumann's theory of measurements in quantum statistics is condensed in the following proposition:

PROPOSITION 1. *The process* **1.** *is nothing but the conditional expectation of U in the sense of* [5] *conditioned by the von Neumann subalgebra $\mathfrak{A}$ generated by the observable R:*

(5) $$U'=E[U|\mathfrak{A}],$$

or alternatively, in the convention of probabilists, the conditional expectation of U *conditioned by* R;

(5′) $$U'=E[U|R].$$

It will be recalled that the conditional expectation of U conditioned by $\mathfrak{A}$ or R is defined by the equality

(6) $$\mathrm{Tr}(U^{E}A)=\mathrm{Tr}(UA),$$

for all $A\in\mathfrak{A}$, where $U^{E}=E[U|R]$ is the unique operator of $\mathfrak{A}$ satisfying (6). Therefore, it suffices to show that

(6′) $$\mathrm{Tr}(U'A)=\mathrm{Tr}(UA),$$

for egery $A\in\mathfrak{A}$. Since it is assumed that R has the pure point simple spectrum, and since $\mathfrak{A}$ is generated by $P_{[\varphi_1]}$, $P_{[\varphi_2]}$, $\cdots$, it needs to show that

(6″) $$\mathrm{Tr}(U'P_{[\varphi_m]})=\mathrm{Tr}(UP_{[\varphi_m]}),$$

for $m=1, 2, \cdots$. It is easy to show that (4) implies (6″):

$$\begin{aligned}\mathrm{Tr}(U'P_{[\varphi_m]})&=\mathrm{Tr}([\Sigma(U\varphi_n, \varphi_n)P_{[\varphi_n]}]P_{[\varphi_m]})\\&=\mathrm{Tr}[(U\varphi_m, \varphi_m)P_{[\varphi_m]}]\\&=(U\varphi_m, \varphi_m)\mathrm{Tr}(P_{[\varphi_m]})\\&=(U\varphi_m, \varphi_m)\\&=\mathrm{Tr}(UP_{[\varphi_m]});\end{aligned}$$

this completes the proof of Proposition 1.

Proposition 1 tells us that the conditional expectation is a mathematical model of the measurement in quantum statistics. Therefore, it may be possible to consider that the conditional expectation in general von Neumann algebra corresponds to an extension of the concept of measurement. Even in quantum mechanics, it is easy to give that the measurement for an operator having not necessarily discrete spectrum is defined by (5). Moreover, using (5), it is also possible that the measurement of a set of not necessarily simultaneously observable quantities is introduced. It seems to the authors that the formulation of measurements gives a role of non-commutative martingales (discussed in [5] as M-nets) in quantum statistics.

It may be interesting to observe that [Q; Chap. VI, §2] gives an example of "non-commutative measurement". If $\mathfrak{H}_I$ and $\mathfrak{H}_{II}$ are the Hilbert spaces corresponding to two systems I and II, von Neumann [Q; p. 422] taught us that the composite system $I+II$ corresponds to the direct (tensor) product $\mathfrak{H}_I\otimes\mathfrak{H}_{II}$. If U_I and U_{II} are the statistical operators of I and II respectively, which are the restrictions of a statistical operator U of $I+II$, von Neumann [Q; p. 425] taught us, that their interrelations are given by

(7) $$U_I=E[U|\mathfrak{A}_I],\quad U_{II}=E[U|\mathfrak{A}_{II}],$$

where $\mathfrak{A}_I$ and $\mathfrak{A}_{II}$ are the algebras of all (bounded) operators on $\mathfrak{A}_I$ and $\mathfrak{A}_{II}$ respectively. According to Proposition 1, it is possible to describe these relations (7) in terms of measurements, that the restriction U_I of U in I is nothing but the mixture after the measurement of $\mathfrak{A}_I$ in a statistics U.

4. Entropy

After the thermodynamical considerations through [Q; Chap. V, §2], von Neumann [Q; p. 379] determined the *entropy* of a mixture U expressed in (1) by

(8) $$H(U)=-\mathrm{Tr}(U\log U),$$

up to a multiplicative constant, or equivaelntly by

(8′) $$H(U)=-\sum_{n=1}^{\infty} w_n\log w_n\,.$$

Clearly,$H(U)$ is non-negative. By (2), $H(U)$ vanishes if and only if exactly one $w_n=1$ and $w_m=0$ for $m\neq n$, that is, $H(U)$ vanishes if and only if $U=P_{[\varphi]}$ for a certain vector φ. Otherwise, $H(U)>0$.

It is interesting to note, comparing with a remark at the end of §2, that the entropy of a statistical operator is completely same with the entropy of a trial in the theory of information introduced by Shannon. The entropy of a trial in the sense of Shannon is equal to the entropy of the statistical operator corresponding to the trial on L^2-space over the sample space. It may be surprised to know that von Neumann [Q; p. 400, footnote 202] wrote as follows: In general, $k\log 2$ is the "thermodynamical value" of knowledge, which consists of an alternative of two cases.

Since Process **2.** defines an automorphism of the factor of all (bounded) operators which preserves the trace, it is evident that the entropy is constant under process **2.** [Q; p. 388], whereas process **1.** increases the entropy:

PROPOSITION 2. [Q; pp. 380–387]. *The measurement increases the entropy:*

(9) $$H(U)\leqq H(U^{E}).$$

If U is simultaneously observable with R, then $H(U)=H(U^E)$.

The first half of the proposition is proved recently by Segal [4; Thm. 2] and by us [3; Thm. 3] in different ways under more general settings. The second half follows from the fact that the hypothesis implies $U\in\mathfrak{A}$ or $U=U^E$ according to a property of the conditional expectation, cf. [5], since $\mathfrak{A}$ is maximally abelian.

It is to be remarked that the converse of the second half of Proposition 2 is true [Q; p. 387]. However, the detailed proof will be omited here since it is somewhat complicated. We hope to discuss it in future under general circumstances.

By (9), there is a non-negative quantity:

$$I[U\,|\,R]=H(U^E)-H(U), \tag{10}$$

which is called in [3] the *information of U with respect to R*. From Proposition 2 and a remark in the above, it is obvious that the information of U with respect to R vanishes if and only if U is simultaneously observable with R. Therefore, it is possible to consider that the information of U indicates the degree of the unsimultaneous observability of U with R. However, it is more natural and reasonable to consider, as in the theory of information, that $I[U\,|\,R]$ is a measure of the knowledge which is obtainable by the measurement. Using this measure, the increment of the entropy by the measurement is compensated and balanced by the increment of the knowledge. One of the authors wishes to discuss an extension of $I[U\,|\,R]$ in [6] under suitable general circumstances.

It seems remarkable that Proposition 2 presents a paradox of the quantum statistical entropy: Doubtlessly, one enforces a measurement to reduce the ambiguity of a statistical state, whereas the quantum statistical measurement increases the entropy (the measure of the ambiguity). However, the paradox is possible to explain away if one remembers Proposition 1: Since the conditional expectation is a kind of averaging, quantum statistical measurement is really an observation of the average of the state, and it is natural that an averaging increases the ambiguity.

Finally, it will be noticed that the entropy satisfies the following concavity:

PROPOSITION 3. [Q; p. 390]. *If U and V are two statistical operators, then the mixing of U and V is not entropy-diminishing*:

$$H(\alpha U+\beta V)\geqq\alpha H(U)+\beta H(V), \tag{11}$$

where α *and* β *are positive tumbers with* $\alpha+\beta=1$.

Proposition 3 is proved by Segal [4; Thm. 1] and is given an alternative proof by [3; Thm. 2] basing on the theory of operatorconvex functions. The authors awared by a letter of C. Davis, that he obtained an another proof also basing on the theory of operator convex functions.

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Added in Proof (April 19, 1962). Very recently, C.Davis published
C.Davis, Operator-valued entropy of a quantum mechanical measurent, Proc. Jap. Acad., 37 (1961), 533-538.
In his paper, Davis employed a similar point of view. According to his formulation, a quantum statistical measurement corresponds to a "pinching". Furthermore he gave a simplfied proof of the operator-concavity of the operator entropy as mentioned in the end of the present note.