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TOPOLOGY

AND

LINEAR TOPOLOGICAL SPACES

BY

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The purpose of this book is to pick up theoretical points in the book of S. Banach:

Théorie des opérations linéaire,

and to arrange them by modern method. I made a course of lectures on Banach spaces at Tokyo University during 1947-48 and had a great mind to write this book. I finished the manuscript in 1947 by the valuable help of Messrs T. Shibata, A. Saito, F. Shirai, H. Kuroda and O. Takenouchi to take notes of my lectures, but I had to correct some points after by kind remarks of Messrs I. Amemiya and O. Takenouchi. To these I express my warmest thanks.

The reader need only be aquainted with elementary properties of real numbers, which are to be found in most books on elementary analysis. I want this book will be a good introduction to modern analysis.

Tokyo, November, 1951 Hidegorô Nakano

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For a set A we define $a \in A$ or $A \ni a$ to mean that a is an <u>element</u> of A, and we shall write $a \bar{\in} A$ or $A \bar{\ni} a$ if a is not contained in A. For two sets A and B we define $A \supset B$ or $B \subset A$ to mean that B is a <u>subset</u> of A, that is, A includes B

We shall make use of the notation

$$\{x : \ C(x)\}$$

to denote the set consisting of all x satisfying the condition $C(x)$. We denote by $\{a_1, a_2, \ldots\}$ the set composed of elements $a_1, a_2, \ldots$, and the <u>empty set</u> will be denoted by O.

For a system of sets $A_\lambda (\lambda \in \Lambda)$ we denote by $\sum_{\lambda \in \Lambda} A_\lambda$ the <u>union</u> of all A_λ, and by $\prod_{\lambda \in \Lambda} A_\lambda$ their <u>intersection</u>, that is,

$$\prod_{\lambda \in \Lambda} A_\lambda = \{x : \ x \in A_\lambda \ \text{ for all } \lambda \in \Lambda\},$$

$$\sum_{\lambda \in \Lambda} A_\lambda = \{x : \ x \in A_\lambda \ \text{ for some } \lambda \in \Lambda\}.$$

For a sequence of sets $A_\nu (\nu = 1, 2, \ldots)$ we may write

$$\sum_{\nu=1}^{\infty} A_\nu = A_1 \dot{+} A_2 \dot{+} \cdots, \qquad \prod_{\nu=1}^{\infty} A_\nu = A_1 A_2 \cdots .$$

A set R may be called a <u>space</u>, if we shall be concerned only with subsets of R, and then every subset of R may be called a <u>point set</u>, while every element of R may be called a <u>point</u>. Let R be a space. For every point set A in R we denote the <u>complement</u> of A by A', that is,

$$A' = \{x : \ x \bar{\in} A\}.$$

Then we have obviously

$$(\sum_{\lambda \in \Lambda} A_\lambda)' = \prod_{\lambda \in \Lambda} A'_\lambda, \qquad (\prod_{\lambda \in \Lambda} A_\lambda)' = \sum_{\lambda \in \Lambda} A'_\lambda$$

for any system of point sets $A_\lambda (\lambda \in \Lambda)$. For two point sets A and B, if $A \supset B$, then we denote AB' by $A - B$, that is,

$$A - B = \{x : \ x \in A, \ x \bar{\in} B\}.$$

As a method for infinite process we are permitted to make use of the following axiom due to Zermelo:

<u>Choice Axiom</u>. <u>For any space</u> R <u>we can find a correspondence of a point</u> $a_A \in R$ <u>to every point set</u> $A \neq O$ <u>such that</u> $a_A \in A$.

By virtue of Choice Axiom we will prove the following theorem due to Zorn, which will be applied often in this book instead of the transfinite induction.

Maximal Theorem. *Let $\mathfrak{C}$ be a condition for a finite number of points in a space S. If a point set A_0 satisfies the condition $\mathfrak{C}$, that is, if the condition $\mathfrak{C}(x_1, x_2, \dots, x_\varkappa)$ is satisfied for every finite number of points $x_1, x_2, \dots, x_\varkappa \in A_0$, then there exists a maximal point set A which include A_0 and satisfies the condition $\mathfrak{C}$, that is, there is no other than A which includes A and satisfies the condition $\mathfrak{C}$.*

Proof. We can assume by Choice Axiom that to every point set $X \neq 0$ there is determined a point $x \in X$ corresponding to X. Let $\mathfrak{K}_0$ be the totality of point sets which include A_0 and satisfy the condition $\mathfrak{C}$. For a point set $A \in \mathfrak{K}_0$, if there is a point $x \bar{\in} A$ for which $\{A, x\} \in \mathfrak{K}_0$, then we obtain a point a_A corresponding to the totality of such points x and we have

$$\{A, a_A\} \in \mathfrak{K}_0, \qquad a_A \bar{\in} A.$$

We need only prove that there is a point set $A \in \mathfrak{K}_0$ for which there is no such corresponding point a_A.

We suppose that corresponding to every point set $A \in \mathfrak{K}_0$ there is determined such a point a_A. We shall consider subsets $\mathfrak{K} \subset \mathfrak{K}_0$ which satisfy the conditions:

*) $\quad \mathfrak{K} \ni A_0$,

**) $\quad \mathfrak{K} \ni A$ implies $\mathfrak{K} \ni \{A, a_A\}$,

***) $\quad \mathfrak{K} \ni A_\lambda \ (\lambda \in \Lambda)$ implies $\mathfrak{K} \ni \sum_{\lambda \in \Lambda} A_\lambda$ if $A_\lambda \ (\lambda \in \Lambda)$ are mutually comparable, that is, if for every two elements $\lambda_1, \lambda_2 \in \Lambda$ we have $A_{\lambda_1} \supset A_{\lambda_2}$ or $A_{\lambda_1} \subset A_{\lambda_2}$.

$\mathfrak{K}_0$ satisfies obviously these conditions. Let $\mathfrak{K}_1$ be the intersection of all subsets $\mathfrak{K} \subset \mathfrak{K}_0$ satisfying these conditions. Then we see easily that $\mathfrak{K}_1$ also satisfies these conditions, that is, $\mathfrak{K}_1$ is the least subset of $\mathfrak{K}_0$ satisfying these conditions.

Let $\mathfrak{K}_2$ be the totality of point sets in $\mathfrak{K}_1$ which are comparable

with every point set in $\mathfrak{K}_1$. Then we have obviously $\mathfrak{K}_2 \ni A_0$ and we see easily that $\mathfrak{K}_2$ satisfies the condition ***). Now we shall prove that $\mathfrak{K}_2$ satisfies the condition **). For any $A_2 \in \mathfrak{K}_2$, if we put

$$\mathfrak{K}_3 = \{A : A_2 \supset A \in \mathfrak{K}_1\} + \{A : \{A_2, a_{A_2}\} \subset A \in \mathfrak{K}_1\},$$

then we have obviously $\mathfrak{K}_3 \ni A_0$, and we see easily that $\mathfrak{K}_3$ satisfies the condition ***). Furthermore $\mathfrak{K}_3$ satisfies the condition **). Because, if $\mathfrak{K}_3 \ni A$ but $\mathfrak{K}_3 \bar{\ni} \{A, a_A\}$, then, since $A \in \mathfrak{K}_1$ implies $\{A, a_A\} \in \mathfrak{K}_1$, and since A_2 is comparable with every point set in $\mathfrak{K}_1$, we must have

$$A \subset A_2 \subset \{A, a_A\},$$

and consequently $A = A_2$ or $A_2 = \{A, a_A\}$, contradicting the assumption $\{A, a_A\} \bar{\in} \mathfrak{K}_3$. Therefore $\mathfrak{K}_3$ satisfies the conditions *), **), ***), and hence we obtain $\mathfrak{K}_3 = \mathfrak{K}_1$, since $\mathfrak{K}_1$ is the least subset of $\mathfrak{K}_0$ satisfying the conditions *), **), ***). Consequently $\{A_2, a_{A_2}\}$ is comparable with every point set $A \in \mathfrak{K}_1$, that is, $\{A_2, a_{A_2}\} \in \mathfrak{K}_2$. Thus $\mathfrak{K}_2$ also satisfies the conditions *), **), ***), and hence we obtain likewise $\mathfrak{K}_2 = \mathfrak{K}_1$. Accordingly $\mathfrak{K}_1$ is a system of mutually comparable point sets, and hence putting

$$A_1 = \sum_{A \in \mathfrak{K}_1} A,$$

we obtain $A_1 \in \mathfrak{K}_1$ by the condition ***). For such A_1 we have $\{A_1, a_{A_1}\} \in \mathfrak{K}_1$ by the condition **), and consequently we obtain $\{A_1, a_{A_1}\} \subset A_1$, contradicting the assumption $a_{A_1} \bar{\in} A_1$. Therefore there exists a point set $A \in \mathfrak{K}_0$ for which there is no point $x \bar{\in} A$ such that $\{A, x\} \in \mathfrak{K}_0$. Such a point set A is obviously a maximal point set which includes A_0 and satisfies the condition $\mathfrak{C}$.

CHAPTER I

TOPOLOGICAL SETS

§1 Topology conditions

Let R be a space. A collection of point sets $\mathfrak{T}$ in R is said to be a topology, if $\mathfrak{T}$ satisfies the topology conditions:

1) $\mathfrak{T} \ni 0, R$;

2) $\mathfrak{T} \ni A, B$ implies $\mathfrak{T} \ni AB$;

3) $\mathfrak{T} \ni A_\lambda \ (\lambda \in \Lambda)$ implies $\mathfrak{T} \ni \sum_{\lambda \in \Lambda} A_\lambda$.

For instance, all point sets of R constitutes obviously a topology. This topology is called the discrete topology of R. Two point sets R and 0 also constitutes a topology on R. This topology is called the trivial topology of R.

A space associated with a topology is called a topological space. On the other hand, a space without topology may be called an abstract space.

Let R be a topological space with a topology $\mathfrak{T}$. Every point set of $\mathfrak{T}$ is said to be open, i.e., $\mathfrak{T}$ is the collection of all open sets. A point set A is said to be closed, if the complement A' is open. If we denote the totality of closed sets by $\mathfrak{T}'$, then we have by definition

$$\mathfrak{T}' = \{ X : X' \in \mathfrak{T} \}$$

and we see easily by the topology conditions that $\mathfrak{T}'$ satisfies the conditions:

1') $\mathfrak{T}' \ni 0, R$;

2') $\mathfrak{T}' \ni A, B$ implies $\mathfrak{T}' \ni A \dotplus B$;

3') $\mathfrak{T}' \ni A_\lambda \ (\lambda \in \Lambda)$ implies $\mathfrak{T}' \ni \prod_{\lambda \in \Lambda} A_\lambda$.

Because we have

$$(A \dotplus B)' = A'B', \qquad (\prod_{\lambda \in \Lambda} A_\lambda)' = \sum_{\lambda \in \Lambda} A_\lambda'.$$

Furthermore, we see easily that if a collection of point sets $\mathfrak{T}'$ in a space R satisfies the conditions 1'), 2'), and 3'), then there exists

uniquely a topology $\mathfrak{T}$ on R such that $\mathfrak{T}'$ is the totality of closed sets by the topology $\mathfrak{T}$.

§2 Open basis, neighbourhood system

Let R be a topological space with a topology $\mathfrak{T}$. A collection of open sets $\mathfrak{U}$ is called an open basis, if we have

$$A = \sum_{A \supset X \in \mathfrak{U}} X \qquad \text{for every } A \in \mathfrak{T}.$$

For each open basis $\mathfrak{U}$ we see immediately by definition that $\mathfrak{U} \subset \mathfrak{T}$ and $\mathfrak{U}$ satisfies the basis conditions:

1) $\mathfrak{U} \ni 0$;

2) $\sum_{X \in \mathfrak{U}} X = R$;

3) $\mathfrak{U} \ni A, B$ implies $AB = \sum_{AB \supset X \in \mathfrak{U}} X$.

Conversely we have:

Theorem 1. For an abstract space R, if a collection of point sets $\mathfrak{U}$ in R satisfies the basis conditions 1), 2), and 3), then there exists uniquely a topology $\mathfrak{T}$ on R such that $\mathfrak{U}$ is an open basis of $\mathfrak{T}$.

Proof. Putting $\mathfrak{T} = \{ X : X = \sum_{X \supset Y \in \mathfrak{U}} Y \}$, we see easily that $\mathfrak{T}$ satisfies the topology conditions, that is, $\mathfrak{T}$ is a topology on R. For this topology $\mathfrak{T}$, it is obvious by definition that $\mathfrak{U}$ is an open basis of $\mathfrak{T}$. Furthermore such topology $\mathfrak{T}$ is determined uniquely. Because, if $\mathfrak{T}_0$ is another topology such that $\mathfrak{U}$ is an open basis of $\mathfrak{T}_0$, then we have obviously by definition that $\mathfrak{T}_0 \subset \mathfrak{T}$. On the other hand, from $\mathfrak{T}_0 \supset \mathfrak{U}$ we conclude $\mathfrak{T}_0 \supset \mathfrak{T}$ by the topology condition 3).

For a topological space R with a topology $\mathfrak{T}$, a collection of open sets $\mathfrak{N}$ is called a neighbourhood system, if $a \in A \in \mathfrak{T}$ implies $a \in X \subset A$ for some $X \in \mathfrak{N}$. With this definition, we see immediately that a collection of open sets $\mathfrak{N}$ is a neighbourhood system, if and only if $\{\mathfrak{N}, 0\}$ is an open basis. Consequently we see by the basis condition 3) that if $\mathfrak{N}$ is a neighbourhood system, then for any $A, B \in \mathfrak{N}$, $a \in AB$ implies $a \in X \subset AB$ for some $X \in \mathfrak{N}$.

Conversely we have:

Theorem 2. For a collection of point sets $\mathfrak{N}$ in an abstract space R, 1) if for each $a \in R$ we can find $X \in \mathfrak{N}$ such that $a \in X$, and 2) if for any $A, B \in \mathfrak{N}$, $a \in AB$ implies $a \in X \subset AB$ for some $X \in \mathfrak{N}$, then there exists uniquely a topology $\mathfrak{T}$ on R such that $\mathfrak{N}$ is a neighbourhood system of $\mathfrak{T}$.

Proof. Putting $\mathfrak{L} = \{\mathfrak{N}, O\}$, we see easily by assumption that $\mathfrak{L}$ satisfies the basis conditions. Thus, there exists by Theorem 1 uniquely a topology $\mathfrak{T}$ such that $\mathfrak{L}$ is an open basis of $\mathfrak{T}$. For such $\mathfrak{T}$, $\mathfrak{N}$ is obviously by definition a neighbourhood system of $\mathfrak{T}$. Such topology $\mathfrak{T}$ is uniquely determined. Because, if $\mathfrak{T}_0$ is another topology such that $\mathfrak{N}$ is a neighbourhood system of $\mathfrak{T}_0$, then $\mathfrak{L}$ is by definition an open basis of $\mathfrak{T}_0$, and hence we obtain $\mathfrak{T}_0 = \mathfrak{T}$ by the uniqueness of the topology for which $\mathfrak{L}$ is an open basis.

§3 Opener, closure

Let R be a topological space with a topology $\mathfrak{T}$. Corresponding to every point set A we define the opener A° to mean

$$A^\circ = \sum_{A \supset X \in \mathfrak{T}} X .$$

The opener A° may be also denoted by $A^{\mathfrak{T}\circ}$, if we need indicate the topology $\mathfrak{T}$.

We see immediately by the topology condition 3) that the opener A° is open, that is, $A^\circ \in \mathfrak{T}$ for every point set A. Therefore we can say that the opener A° is the greatest open set included in A. For an open basis $\mathfrak{L}$ we have obviously by definition also

$$A^\circ = \sum_{A \supset X \in \mathfrak{L}} X .$$

For an arbitrary point set A, every point of the opener A° is called an inner point of A. With this definition we have obviously:

Theorem 1. For a neighbourhood system $\mathfrak{N}$, a point a is an inner point of a point set A if and only if $a \in X \subset A$ for some $X \in \mathfrak{N}$.

Corresponding to every point set A we define the closure A^- to mean

$$A^- = \prod_{A \subset X \in \mathfrak{T}'} X$$

for the totality of closed sets $\mathfrak{T}'$. The closure A^- may be also denoted by $A^{\mathfrak{T}-}$, if we need indicate the topology $\mathfrak{T}$.

By virtue of the condition 3') in §1, we see that the closure A^- is closed, that is, $A^- \in \mathfrak{T}'$ for every point set A. Thus we can say that the closure A^- is the least closed set including A.

For an arbitrary point set A, every point of the closure A^- is called a contact point of A. With this definition we have:

Theorem 2. For a neighbourhood system $\mathfrak{N}$, a point a is a contact point of a point set A, if and only if $a \in X \in \mathfrak{N}$ implies $AX \neq O$.

Proof. If there is an open set X such that $a \in X$ but $AX = O$, then we have naturally $a \overline{\in} X'$ and $A \subset X'$. As X' is closed, we have then by definition $A^- \subset X'$, and hence a is not a contact point of A.

Conversely, if a is not a contact point of A, that is, if $a \overline{\in} A^-$, then we have naturally $a \in A^{-\prime}$. As $A^{-\prime}$ is open, we can find $X \in \mathfrak{N}$ such that $a \in X \subset A^{-\prime}$, that is, $a \in X \in \mathfrak{N}$, $AX \subset A^- X = O$.

A point a of a point set A is called an isolated point of A, if there is an open set U such that $AU = \{a\}$. A contact point of a point set A is called a limiting point of A, if it is not an isolated point of A. With this definition we see easily that a point a is a limiting point of a point set A if and only if for every open set $X \ni a$ the intersection AX contains a point different from the point a.

From the definition of opener and closure, we conclude immediately that a point set A is open if and only if $A^\circ = A$; and that a point set A is closed if and only if $A^- = A$. Thus we see that a point set A is closed if and only if A contains all limiting points of A.

§4 Calculus of topological notations

From the definition of opener and closure we conclude immediately

(1) $\quad A^\circ \subset A \subset A^-$

and furthermore

(2) $\quad O^\circ = O, \quad O^- = O,$
$\quad\quad R^\circ = R, \quad R^- = R.$

Since we have for the topology $\mathfrak{T}$ of R

$$\Big(\sum_{A \supset X \in \mathfrak{T}} X\Big)' = \prod_{A \supset X \in \mathfrak{T}} X' = \prod_{A' \subset Y \in \mathfrak{T}'} Y ,$$

$$\Big(\prod_{A\subset X\in\mathfrak{Z}'} X\Big)' = \sum_{A\subset X\in\mathfrak{Z}'} X' = \sum_{A'\supset Y\in\mathfrak{Z}} Y,$$

we obtain by definition

(3) $A^{\circ\prime} = A^{\prime -}, \quad A^{-\prime} = A^{\prime\circ}.$

As A° is open and A^- is closed, we have naturally

(4) $A^{\circ\circ} = A^\circ, \quad A^{--} = A^-,$

and further we obtain by the topology condition 3) and 3') in §1

(5) $\Big(\sum_{\lambda\in\Lambda} A_\lambda^\circ\Big)^\circ = \sum_{\lambda\in\Lambda} A_\lambda^\circ, \quad \Big(\prod_{\lambda\in\Lambda} A_\lambda^-\Big)^- = \prod_{\lambda\in\Lambda} A_\lambda^-.$

We see easily by definition

(6) $A\supset B$ implies $A^\circ\supset B^\circ$ and $A^-\supset B^-$.

For two point sets A and B we have by the formula (6)

$$(AB)^\circ \subset A^\circ B^\circ.$$

On the other hand we have $AB\supset A^\circ B^\circ$ by the formula (1), and hence we obtain by the formulas (4) and (6)

$$(AB)^\circ \supset (A^\circ B^\circ)^\circ = A^\circ B^\circ$$

Therefore we have $(AB)^\circ = A^\circ B^\circ$ for every point sets A and B.

From this relation we conclude $(A\dotplus B)^- = A^-\dotplus B^-$ by duality, i.e.

$$(A\dotplus B)^- = (A\dotplus B)^{\prime\circ\prime} = (A'B')^{\circ\prime}$$
$$= (A'^{\circ} B'^{\circ})' = A'^{\circ\prime}\dotplus B'^{\circ\prime} = A^-\dotplus B^-.$$

Thus we have for every point sets A and B

(7) $(AB)^\circ = A^\circ B^\circ, \quad (A\dotplus B)^- = A^-\dotplus B^-.$

Since we have obviously by the formula (1)

$$AB\subset A^-B^-,$$

we obtain by the formulas (5) and (6)

$$(AB)^-\subset (A^-B^-)^- = A^-B^-.$$

As $A = AB\dotplus AB' \subset AB\dotplus B'$, we have by the formulas (6), (7), and (3)

$$A^-\subset (AB\dotplus B')^- = (AB)^-\dotplus B'^- = (AB)^-\dotplus B^{\circ\prime}.$$

Therefore we obtain

(8) $A^-B^\circ \subset (AB)^- \subset A^-B^-.$

From this relation we conclude by duality

(9) $A^-\dotplus B^\circ \supset (A\dotplus B)^\circ \supset A^\circ\dotplus B^\circ.$

For an arbitrary point set A we have

(10) $A^{-\circ-\circ} = A^{-\circ}, \quad A^{\circ-\circ-} = A^{\circ-}$

Because, we have $A^{-\circ-}\supset A^{-\circ}$ by the formula (1), and hence by the formulas (6) and (4)

$$A^{-\circ-\circ}\supset A^{-\circ\circ} = A^{-\circ}.$$

On the other hand we have $A^{-\circ}\subset A^-$ by the formula (1), and hence by the formulas (6) and (4)

$$A^{-\circ-\circ}\subset A^{--\circ} = A^{-\circ}$$

Therefore we obtain the first relation. We can conclude the second from the first by duality.

For every point sets A and B we have

(11) $(A^\circ B)^{-\circ} = A^{\circ-\circ}B^{-\circ}, \quad (A^-B)^{-\circ} = A^{-\circ}B^{-\circ}.$

Because, we have by the formulas (8) and (4)

$$A^{\circ-}B^- \supset (A^\circ B)^- \supset (A^{\circ\circ}B^-)^- \supset A^{\circ-}B^{-\circ}$$

and hence by the formulas (4) and (7)

$$A^{\circ-\circ}B^{-\circ}\supset (A^\circ B)^{-\circ}\supset A^{\circ-\circ}B^{-\circ}.$$

Therefore the first relation is proved. Similarly, we have by the formulas (4) and (8)

$$A^-B^-\supset (A^-B)^-\supset A^{-\circ}B^-$$

and hence by the formulas (6), (4), and (7)

$$A^{-\circ}B^{-\circ}\supset (A^-B)^{-\circ}\supset A^{-\circ}B^{-\circ}.$$

Thus the second relation also is proved.

From the formula (11) we conclude by duality

(12) $(A^-\dotplus B)^{\circ-} = A^{-\circ-}\dotplus B^{\circ-}, \quad (A^\circ\dotplus B)^{\circ-} = A^{\circ-}\dotplus B^{\circ-}$

For two point sets A and B, we define $A\succ B$ to mean $A^\circ\supset B^-$

With this definition we see easily by the formula (6)

$A\succ B\supset C$ implies $A\succ C$;

$C\supset A\succ B$ implies $C\succ B$.

Furthermore we see easily by the formula (4) that $A\succ B$ is equivalent to each one of the relations

$$A\succ B^-, \quad A^\circ\succ B, \quad A^\circ\succ B^-$$

Recalling the formulas (7), (8), and (9), we can conclude easily further that $A_1\succ B_1$, $A_2\succ B_2$ implies

$A_1A_2\succ B_1B_2$ and $A_1\dotplus A_2\succ B_1\dotplus B_2$.

A point set $\mathfrak{U}$ is said to be *regularly open*, if $\mathfrak{U}^{-\circ} = \mathfrak{U}$. By virtue of the formula (10), $A^{-\circ}$ is regularly open for every point set A. For every pair of regularly open sets $\mathfrak{U}$ and $\mathfrak{V}$, the intersection $\mathfrak{U}\mathfrak{V}$ also is regularly open by the formula (11). But the union $\mathfrak{U} \dotplus \mathfrak{V}$ is not necessarily regularly open. Thus we define the *regular sum* $\mathfrak{U} \oplus \mathfrak{V}$ to mean $\mathfrak{U} \oplus \mathfrak{V} = (\mathfrak{U} \dotplus \mathfrak{V})^{-\circ}$ for regularly open sets $\mathfrak{U}$ and $\mathfrak{V}$.

§5 *First category sets, second category sets*

Let R be a topological space with a topology $\mathfrak{T}$ in the sequel. For two point sets A and B we shall say that A is *dense* in B, if we have $(AB)^- \supset B$. Especially, if a point set A is dense in the space R, that is, if $A^- = R$, then A is said to be *dense*. With this definition we can say that a point set A is dense if and only if $A\mathfrak{U} \neq 0$ for every open set $\mathfrak{U} \neq 0$. Because, if $A\mathfrak{U} = 0$ for an open set $\mathfrak{U}$, then we have by §4(8) $\mathfrak{U} = R\mathfrak{U} = A^-\mathfrak{U}^\circ \subset (A\mathfrak{U})^- = 0$.

A point set A is said to be *nowhere dense*, if $A^{-\circ} = 0$. With this definition, it is evident by the formula §4(6) that if a point set A is nowhere dense, then every $B \subset A$ also is nowhere dense. Furthermore, if both A and B are nowhere dense, then $A \dotplus B$ also is nowhere dense. Because we have by §4(7), (8)

$$(A \dotplus B)^{-\circ} = (A^- \dotplus B^-)^\circ \subset A^{-\circ} \dotplus B^-.$$

As $A^{-\circ} = 0$ by assumption, we obtain hence by §4(6), (4)

$$(A \dotplus B)^{-\circ} \subset B^{-\circ} = 0,$$

and consequently $(A \dotplus B)^{-\circ} = 0$.

Theorem 1. *For any point set A, both $A^{\circ -} - A^\circ$ and $A^- - A^{-\circ}$ are nowhere dense.*

Proof. By virtue of the formulas §4(3), (4), (11) we have

$$(A^{\circ -} - A^\circ)^{-\circ} = (A^{\circ -} A^{\circ\prime})^{-\circ} = A^{\circ-\circ} A^{\circ\prime -\circ} = A^{\circ-\circ} A^{\circ-\prime} \subset A^{\circ-} A^{\circ-\prime} = 0,$$

$$(A^- - A^{-\circ})^{-\circ} = (A^- A^{-\circ\prime})^{-\circ} = A^{-\circ} A^{-\circ\prime-\circ} = A^{-\circ} A^{-\circ\prime\circ} \subset A^{-\circ} A^{-\circ\prime} = 0.$$

A point set A is said to be of the *first category* or a *first category set*, if there is a sequence of nowhere dense sets A_ν ($\nu = 1, 2, \ldots$)

such that $A = \sum_{\nu=1}^{\infty} A_\nu$. With this definition, we see easily that if a point set A is of the first category, then every $B \subset A$ also is of the first category; and for a sequence of first category sets A_ν ($\nu = 1, 2, \ldots$) the union $\sum_{\nu=1}^{\infty} A_\nu$ also is of the first category.

Theorem 2. (Banach) *For a system of open sets $\mathfrak{U}_\lambda$ ($\lambda \in \Lambda$), if $A\mathfrak{U}_\lambda$ is of the first category for every $\lambda \in \Lambda$, then $A \sum_{\lambda \in \Lambda} \mathfrak{U}_\lambda$ also is of the first category.*

Proof. By virtue of Maximal Theorem, we can find a maximal system of open sets $\mathfrak{V}_\gamma$ ($\gamma \in \Gamma$) such that $\mathfrak{V}_{\gamma_1}\mathfrak{V}_{\gamma_2} = 0$ for $\gamma_1 \neq \gamma_2$, and for any $\gamma \in \Gamma$ there is $\lambda \in \Lambda$ for which $\mathfrak{V}_\gamma \subset \mathfrak{U}_\lambda$. For such a maximal system of open sets $\mathfrak{V}_\gamma$ ($\gamma \in \Gamma$), we see easily that we have

$$\sum_{\lambda \in \Lambda} \mathfrak{U}_\lambda \subset \Big(\sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma \Big)^-.$$

As $A\mathfrak{U}_\lambda$ is by assumption of the first category for every $\lambda \in \Lambda$, $A\mathfrak{V}_\gamma$ also is of the first category for every $\gamma \in \Gamma$, and hence corresponding to every $\gamma \in \Gamma$ we can find by definition a sequence of nowhere dense $A_{\gamma,\nu}$ such that

$$A\mathfrak{V}_\gamma = \sum_{\nu=1}^{\infty} A_{\gamma,\nu}$$

For such $A_{\gamma,\nu}$ ($\gamma \in \Gamma$; $\nu = 1, 2, \ldots$), as $A_{\gamma,\nu} \subset \mathfrak{V}_\gamma$ for every $\gamma \in \Gamma$, we have $\mathfrak{V}_\rho A_{\gamma,\nu} = 0$ for $\rho \neq \gamma \in \Gamma$, and hence we have by §4(1), (4), (11)

$$\mathfrak{V}_\rho \Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ} \subset \mathfrak{V}_\rho^{-\circ} \Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ} = \Big(\mathfrak{V}_\rho \sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ} = A_{\rho,\nu}{}^{-\circ} = 0$$

for every $\rho \in \Gamma$ and $\nu = 1, 2, \ldots$. Thus we obtain for every $\nu = 1, 2, \ldots$

$$\Big(\sum_{\rho \in \Gamma} \mathfrak{V}_\rho \Big) \Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ} = 0,$$

and consequently by the formulas §4(2), (8)

$$\Big(\sum_{\rho \in \Gamma} \mathfrak{V}_\rho \Big)^- \Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ} = 0.$$

On the other hand we have by the formula §4(11)

$$\Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ} = \Big\{ \Big(\sum_{\rho \in \Gamma} \mathfrak{V}_\rho \Big) \Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big) \Big\}^{-\circ}$$
$$= \Big(\sum_{\rho \in \Gamma} \mathfrak{V}_\rho \Big)^{-\circ} \Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ} \subset \Big(\sum_{\rho \in \Gamma} \mathfrak{V}_\rho \Big)^- \Big(\sum_{\gamma \in \Gamma} A_{\gamma,\nu} \Big)^{-\circ}.$$

Therefore $\sum_{\gamma \in \Gamma} A_{\gamma,\nu}$ is nowhere dense for every $\nu = 1, 2, \ldots$. As

$$A \sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma = \sum_{\gamma \in \Gamma} \sum_{\nu=1}^{\infty} A_{\gamma,\nu} = \sum_{\nu=1}^{\infty} \sum_{\gamma \in \Gamma} A_{\gamma,\nu},$$

$A \sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma$ is hence of the first category. We have obviously

$$A \Big(\sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma \Big)^- \subset A \sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma \dotplus \Big\{ \Big(\sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma \Big)^- - \sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma \Big\},$$

and $\big(\sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma \big)^- - \sum_{\gamma \in \Gamma} \mathfrak{V}_\gamma$ is nowhere dense by Theorem 1. Therefore we see that $A\sum_{\lambda\in\Lambda} U_\lambda$ also is of the first category.

A point set A is said to be of the *second category*, if AU is not of the first category for every open set U such that $AU \neq 0$. With this definition we see easily that if all A_λ $(\lambda \in \Lambda)$ are of the second category, then $\sum_{\lambda\in\Lambda} A_\lambda$ also is of the second category. If a point set A is of the second category and $A \subset B \subset A^{-}$, then B also is of the second category. Because $BU^\circ \neq 0$ implies $AU^\circ \neq 0$, since every point of B is a contact point of A. Furthermore, if A is of the second category and B is of the first category, then AB' is of the second category. Because, if $AB'U^\circ$ is of the first category and $\neq 0$, then, as $AU^\circ \subset AB'U^\circ \dotplus B$, AU° also is of the first category and $\neq 0$, contradicting the assumption.

Recalling Theorem 2, we conclude immediately:

Theorem 3. *If a point set A is not of the first category, then there is an open set U such that AU is of the first category, and AU' is of the second category.*

We shall also say that the topology $\mathfrak{T}$ is of the *second category*, if the space R is of the second category as a point set.

§6 Baire sets, Borel sets

A point set A is said to be a *Baire set*, if we can find an open set U and two first category sets F_1 and F_2 such that $A = UF_1' \dotplus F_2$. With this definition we have:

Theorem 1. *For every Baire set A, the complement A' also is a Baire set.*

Proof. If $A = UF_1' \dotplus F_2$ for an open set U and two first category sets F_1, F_2, then we have

$$A' = (U' \dotplus F_1)F_2' = \{U'^\circ \dotplus (U' - U'^\circ) \dotplus F_1\}F_2'$$
$$= U'^\circ F_2' \dotplus \{(U' - U'^\circ) \dotplus F_1\}F_2'.$$

Here $U' - U'^\circ$ is nowhere dense by §5 Theorem 1. Therefore A' is a Baire set by definition.

Theorem 2. *For every sequence of Baire sets A_ν $(\nu = 1, 2, \dots)$, both $\sum_{\nu=1}^{\infty} A_\nu$ and $\prod_{\nu=1}^{\infty} A_\nu$ are Baire sets.*

Proof. We can put by assumption

$$A_\nu = U_\nu^\circ F_\nu' \dotplus G_\nu \qquad (\nu = 1, 2, \dots)$$

for first category sets F_ν, G_ν $(\nu = 1, 2, \dots)$. Then we have

$$\sum_{\nu=1}^{\infty} A_\nu = \Big(\sum_{\nu=1}^{\infty} U_\nu^\circ\Big)\Big\{\sum_{\nu=1}^{\infty} U_\nu^\circ - \sum_{\nu=1}^{\infty} U_\nu^\circ F_\nu'\Big\}' \dotplus \sum_{\nu=1}^{\infty} G_\nu,$$
$$\sum_{\nu=1}^{\infty} U_\nu^\circ - \sum_{\nu=1}^{\infty} U_\nu^\circ F_\nu' \subset \sum_{\nu=1}^{\infty} U_\nu^\circ F_\nu \subset \sum_{\nu=1}^{\infty} F_\nu.$$

As $\sum_{\nu=1}^{\infty} F_\nu$ is of the first category, $\sum_{\nu=1}^{\infty} U_\nu^\circ - \sum_{\nu=1}^{\infty} U_\nu^\circ F_\nu'$ also is hence of the first category. Therefore $\sum_{\nu=1}^{\infty} A_\nu$ is a Baire set by definition. As $\prod_{\nu=1}^{\infty} A_\nu = \big(\sum_{\nu=1}^{\infty} A_\nu'\big)'$, we conclude by Theorem 1 further that $\prod_{\nu=1}^{\infty} A_\nu$ also is a Baire set.

A collection of point sets $\mathfrak{A}$ is said to be *totally additive*, if $\mathfrak{A} \ni A$ implies $\mathfrak{A} \ni A'$ and $\mathfrak{A} \ni A_\nu$ $(\nu = 1, 2, \dots)$ implies $\mathfrak{A} \ni \sum_{\nu=1}^{\infty} A_\nu$. Considering complements, we see easily that if $\mathfrak{A}$ is totally additive, then $\mathfrak{A} \ni A_\nu$ $(\nu = 1, 2, \dots)$ implies $\mathfrak{A} \ni \prod_{\nu=1}^{\infty} A_\nu$. For every system of totally additive collections $\mathfrak{A}_\lambda$ $(\lambda \in \Lambda)$, we see easily by definition that $\prod_{\lambda\in\Lambda} \mathfrak{A}_\lambda$ also is totally additive. Therefore there exists the least totally additive collection $\mathfrak{L}$ which contains all open sets. For such $\mathfrak{L}$, every point set $B \in \mathfrak{L}$ is called a *Borel set*. With this definition we have obviously by Theorems 1 and 2

Theorem 3. *Every Borel set is a Baire set.*

A collection of point sets $\mathfrak{A}$ is said to be *universally additive*, if $\mathfrak{A} \ni A$ implies $\mathfrak{A} \ni A'$ and $\mathfrak{A} \ni A_\lambda$ $(\lambda \in \Lambda)$ implies $\mathfrak{A} \ni \sum_{\lambda\in\Lambda} A_\lambda$. Considering complements, we see easily that if $\mathfrak{A}$ is universally additive, then $\mathfrak{A} \ni A_\lambda$ $(\lambda \in \Lambda)$ implies $\mathfrak{A} \ni \prod_{\lambda\in\Lambda} A_\lambda$. Similarly as Borel sets, we see easily that there exists the least universally additive collection of point sets $\mathfrak{U}$ which contains all open sets. Every point set contained in such $\mathfrak{U}$ is called a *topological set*. Since every universally additive collection is naturally totally additive, we see by definition that every Borel set is a topological set. Consequently, all open sets and closed sets are topological sets. Furthermore, it is evident by definition that if every point set composed only of a single point is a topological set, then every point set is a topological set.

§7 Compact sets

A point set A is said to be <u>compact</u>, if for each system of open sets $\mathfrak{U}_\lambda$ ($\lambda \in \Lambda$) such that $A \subset \sum_{\lambda \in \Lambda} \mathfrak{U}_\lambda$ we can find a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \varkappa$) such that $A \subset \sum_{\nu=1}^{\varkappa} \mathfrak{U}_{\lambda_\nu}$. Let $\mathfrak{N}$ be a neighbourhood system. If for each system $\mathfrak{U}_\lambda \in \mathfrak{N}$ ($\lambda \in \Lambda$) such that $A \subset \sum_{\lambda \in \Lambda} \mathfrak{U}_\lambda$ we can find a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \varkappa$) such that $A \subset \sum_{\nu=1}^{\varkappa} \mathfrak{U}_{\lambda_\nu}$, then A is compact. Because if $A \subset \sum_{\lambda \in \Lambda} A_\lambda^\circ$, then for each point $x \in A$ we can find $\mathfrak{U}_x \in \mathfrak{N}$ such that $x \in \mathfrak{U}_x \subset A_{\lambda_x}^\circ$ for some $\lambda_x \in \Lambda$. For such $\mathfrak{U}_x \in \mathfrak{N}$ ($x \in A$), as $A \subset \sum_{x \in A} \mathfrak{U}_x$, we can find by assumption a finite number of points $x_\nu \in A$ ($\nu = 1, 2, \ldots, \varkappa$) such that $A \subset \sum_{\nu=1}^{\varkappa} \mathfrak{U}_{x_\nu}$, and then we have obviously $A \subset \sum_{\nu=1}^{\varkappa} A_{\lambda_{x_\nu}}^\circ$ for such x_ν ($\nu = 1, 2, \ldots, \varkappa$). Consequently A is compact by definition.

<u>Theorem 1</u>. <u>If a point set A is compact, then AB also is compact for every closed set B.</u>

<u>Proof</u>. If $AB \subset \sum_{\lambda \in \Lambda} A_\lambda^\circ$, then we have obviously

$$A \subset AB + B' \subset \sum_{\lambda \in \Lambda} A_\lambda^\circ + B'.$$

As A is compact by assumption, we can find a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \varkappa$) such that $A \subset \sum_{\nu=1}^{\varkappa} A_{\lambda_\nu}^\circ + B'$, and hence

$$AB \subset \Big(\sum_{\nu=1}^{\varkappa} A_{\lambda_\nu}^\circ\Big)B + B'B \subset \sum_{\nu=1}^{\varkappa} A_{\lambda_\nu}^\circ ,$$

since $B' = B^{-\prime} = B'^{\circ}$ by the formula §4(3).

<u>Theorem 2</u>. <u>If a point set A is compact, then every subset $B \subset A$ composed of infinite points has a limiting point $a \in A$ such that $B\mathfrak{U}$ is composed of infinite points for every open set $\mathfrak{U} \ni a$.</u>

<u>Proof</u>. If for each point $x \in A$ we can find an open set $\mathfrak{U}_x \ni x$ such that $B\mathfrak{U}_x$ is composed only of a finite number of points, then, as A is compact by assumption, we can find a finite number of points $x_\nu \in A$ ($\nu = 1, 2, \ldots, \varkappa$) such that $A \subset \sum_{\nu=1}^{\varkappa} \mathfrak{U}_{x_\nu}$, and hence

$$B = AB \subset \sum_{\nu=1}^{\varkappa} B\,\mathfrak{U}_{x_\nu} ,$$

contradicting the assumption that B consists of infinite points.

<u>Theorem 3</u>. <u>In order that a point set A be compact, it is necessary and sufficient that $A \prod_{\lambda \in \Lambda} B_\lambda^- = 0$ implies $A \prod_{\nu=1}^{\varkappa} B_{\lambda_\nu}^- = 0$ for some finite number of elements</u> $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \varkappa$).

<u>Proof</u>. $A \prod_{\lambda \in \Lambda} B_\lambda^- = 0$ is equivalent to

$$A \subset \Big(\prod_{\lambda \in \Lambda} B_\lambda^-\Big)' = \sum_{\lambda \in \Lambda} B_\lambda^{-\prime} = \sum_{\lambda \in \Lambda} B_\lambda'^{\circ},$$

and similarly $A \prod_{\nu=1}^{\varkappa} B_{\lambda_\nu}^- = 0$ is equivalent to $A \subset \sum_{\nu=1}^{\varkappa} B_{\lambda_\nu}'^{\circ}$. Thus we see easily by definition that our Theorem is valid.

<u>Theorem 4</u>. <u>If a point set A is compact, then for any sequence of closed sets</u> $B_1 \supset B_2 \supset \ldots$ <u>such that</u> $AB_\nu \neq 0$ ($\nu = 1, 2, \ldots$) <u>we have</u>

$$A \prod_{\nu=1}^{\infty} B_\nu \neq 0.$$

<u>Proof</u>. Since $A \prod_{\mu=1}^{\nu} B_\mu = AB_\nu \neq 0$ for every $\nu = 1, 2, \ldots$, we conclude by Theorem 3

$$A \prod_{\nu=1}^{\infty} B_\nu \neq 0 .$$

CHAPTER II

CLASSIFICATION OF TOPOLOGIES

§8 Comparation of topologies

Let R be an abstract space. For two topologies $\mathfrak{T}$ and $\mathfrak{V}$ on R, if $\mathfrak{T} \supset \mathfrak{V}$, then we shall say that $\mathfrak{T}$ is stronger than $\mathfrak{V}$, or that $\mathfrak{V}$ is weaker than $\mathfrak{T}$.

Theorem 1. Let $\mathfrak{M}$ and $\mathfrak{N}$ be neighbourhood systems respectively of topologies $\mathfrak{V}$ and $\mathfrak{T}$ on R. In order that $\mathfrak{V} \subset \mathfrak{T}$, it is necessary and sufficient that $\mathfrak{M} \ni U \ni x$ implies $U \supset V \ni x$ for some $V \in \mathfrak{N}$.

Proof. If $\mathfrak{V} \subset \mathfrak{T}$, then $\mathfrak{M} \ni U \ni x$ implies obviously $\mathfrak{T} \ni U \ni x$, and hence we can find by the definition of neighbourhood systems $V \in \mathfrak{N}$ such that $U \supset V \ni x$. Conversely, if $\mathfrak{M} \ni U \ni x$ implies $U \supset V \ni x$ for some $V \in \mathfrak{N}$, then $\mathfrak{V} \ni X \ni x$ implies $X \supset U \ni x$ for some $U \in \mathfrak{M}$, and hence $U \supset V \ni x$ for some $V \in \mathfrak{N}$. Consequently $\mathfrak{V} \ni X$ implies $\mathfrak{T} \ni X$, that is, $\mathfrak{V} \subset \mathfrak{T}$.

From the definition of compact sets we conclude easily:

Theorem 2. For two topologies $\mathfrak{V} \subset \mathfrak{T}$ on a space R, if a point set A is compact by $\mathfrak{T}$, then A also is compact by $\mathfrak{V}$.

The discrete topology is stronger than every other topology, because every point set is open by the discrete topology. The trivial topology is obviously weaker than every other topology.

For a system of topologies $\mathfrak{T}_\lambda (\lambda \in \Lambda)$ on R, we see easily that the intersection $\prod_{\lambda\in\Lambda} \mathfrak{T}_\lambda$ also is a topology on R. This topology $\prod_{\lambda\in\Lambda} \mathfrak{T}_\lambda$ is obviously weaker than $\mathfrak{T}_\lambda$ for every $\lambda \in \Lambda$. Furthermore, for any topology $\mathfrak{T}$ which is weaker than $\mathfrak{T}_\lambda$ for every $\lambda \in \Lambda$ we have naturally $\mathfrak{T} \subset \prod_{\lambda\in\Lambda} \mathfrak{T}_\lambda$. Thus $\prod_{\lambda\in\Lambda} \mathfrak{T}_\lambda$ is the strongest weaker topology of a system of topologies $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$, and will be denoted by $\bigcap_{\lambda\in\Lambda} \mathfrak{T}_\lambda$. For a neighbourhood system $\mathfrak{N}_\lambda$ of $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$, we may denote the strongest weaker topology of $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$ by $\bigcap_{\lambda\in\Lambda} \mathfrak{N}_\lambda$.

We shall prove further the existence of the weakest stronger topology of $\mathfrak{T}_\lambda (\lambda \in \Lambda)$. For a neighbourhood system $\mathfrak{N}_\lambda$ of $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$, if we

denote by $\mathfrak{N}_0$ the totality of point sets $\prod_{\nu=1}^{\kappa} U_{\lambda_\nu}$ for every finite number of point sets $U_{\lambda_\nu} \in \mathfrak{N}_{\lambda_\nu}$, $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \kappa)$, then we see easily that $\mathfrak{N}_0$ satisfies the conditions in §2 Theorem 2. Therefore there exists uniquely a topology $\mathfrak{T}_0$ for which $\mathfrak{N}_0$ is a neighbourhood system. As $\mathfrak{N}_0 \supset \mathfrak{N}_\lambda$ for every $\lambda \in \Lambda$, this topology $\mathfrak{T}_0$ is stronger than $\mathfrak{T}_\lambda$ for every $\lambda \in \Lambda$. If a topology $\mathfrak{T}$ is stronger than every $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$, then we have obviously $\mathfrak{T} \supset \mathfrak{N}_0$, and hence $\mathfrak{T}$ is stronger than $\mathfrak{T}_0$. Therefore $\mathfrak{T}_0$ is the weakest stronger topology of $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$. We shall denote by $\bigcup_{\lambda\in\Lambda} \mathfrak{T}_\lambda$ or by $\bigcup_{\lambda\in\Lambda} \mathfrak{N}_\lambda$ the weakest stronger topology of a system of topologies $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$.

Now we can state:

Theorem 3. Let $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$ be a system of topologies on a sapce R. There exists the strongest weaker topology $\bigcap_{\lambda\in\Lambda} \mathfrak{T}_\lambda$ and

$$\bigcap_{\lambda\in\Lambda} \mathfrak{T}_\lambda = \prod_{\lambda\in\Lambda} \mathfrak{T}_\lambda .$$

There exists the weakest stronger topology $\bigcup_{\lambda\in\Lambda} \mathfrak{T}_\lambda$ and for a neighbourhood system $\mathfrak{N}_\lambda$ of $\mathfrak{T}_\lambda$ $(\lambda \in \Lambda)$

$$\left\{ \prod_{\nu=1}^{\kappa} U_{\lambda_\nu} : U_{\lambda_\nu} \in \mathfrak{N}_{\lambda_\nu}, \lambda_\nu \in \Lambda \ (\nu = 1, 2, \ldots, \kappa), \kappa = 1, 2, \ldots \right\}$$

is a neighbourhood system of $\bigcup_{\lambda\in\Lambda} \mathfrak{T}_\lambda$.

If a topology $\mathfrak{V}$ is weaker than a topology $\mathfrak{T}$ on R, then for any point set A we have $A^{\mathfrak{V}\circ} \in \mathfrak{V} \subset \mathfrak{T}$, and hence $A^{\mathfrak{V}\circ} \subset A^{\mathfrak{T}\circ}$. Thus we obtain further by duality

$$A^{\mathfrak{V}-} = (A'^{\mathfrak{V}\circ})' \supset (A'^{\mathfrak{T}\circ})' = A^{\mathfrak{T}-}.$$

Therefore we have that $\mathfrak{V} \subset \mathfrak{T}$ implies

$$A^{\mathfrak{V}\circ} \subset A^{\mathfrak{T}\circ} \subset A \subset A^{\mathfrak{T}-} \subset A^{\mathfrak{V}-} .$$

Theorem 4. Let $\mathfrak{T}_\lambda (\lambda \in \Lambda)$ be a system of topologies on a space R such that for each pair $\lambda_1, \lambda_2 \in \Lambda$ we can find $\lambda \in \Lambda$ for which we have $\mathfrak{T}_\lambda \supset \{\mathfrak{T}_{\lambda_1}, \mathfrak{T}_{\lambda_2}\}$. Putting $\mathfrak{T} = \bigcup_{\lambda\in\Lambda} \mathfrak{T}_\lambda$, we have then for every point set A

$$A^{\mathfrak{T}\circ} = \sum_{\lambda\in\Lambda} A^{\mathfrak{T}_\lambda\circ} , \qquad A^{\mathfrak{T}-} = \prod_{\lambda\in\Lambda} A^{\mathfrak{T}_\lambda -}$$

Proof. From assumption we can conclude easily by Theorem 3 that $\sum_{\lambda\in\Lambda} \mathfrak{T}_\lambda$ is an open basis of $\mathfrak{T}$. Therefore we have by definition

$$A^{\mathfrak{T}\circ} = \sum_{A \supset X \in \sum_{\lambda\in\Lambda} \mathfrak{T}_\lambda} X = \sum_{\lambda\in\Lambda} \Big(\sum_{A \supset X \in \mathfrak{T}_\lambda} X \Big) = \sum_{\lambda\in\Lambda} A^{\mathfrak{T}_\lambda\circ}$$

From this relation we conclude the other assertion by duality.

§9 Relative topology

Let R be an abstract space and S a subspace of R. Corresponding to every point set A in R we obtain uniquely a point set AS in the subspace S. This point set AS in the subspace S is called the induced set of A into the subsape S and denoted by A^S. The complement $A^{S'}$ means the one in the subspace S, namely $A^{S'} = S - AS$. As $S - AS = SA' = A'^S$, we have then

(1) $A^{S'} = A'^S$.

For any system of point sets $A_\lambda \ (\lambda \in \Lambda)$ we have obviously

(2) $(\sum_{\lambda\in\Lambda} A_\lambda)^S = \sum_{\lambda\in\Lambda} A_\lambda^S$,

(3) $(\prod_{\lambda\in\Lambda} A_\lambda)^S = \prod_{\lambda\in\Lambda} A_\lambda^S$.

Let R be now a topological space with a topology $\mathfrak{T}$, and S a subspace of R. Putting

(4) $\mathfrak{T}^S = \{ A^S : A \in \mathfrak{T} \}$,

we see easily that $\mathfrak{T}^S$ satisfies the topological conditions in the subspace S, and hence $\mathfrak{T}^S$ is a topology on the subspace S. This topology $\mathfrak{T}^S$ is called the relative topology of $\mathfrak{T}$ in the subspace S.

Concerning the relative topology, we have obviously by (2)

Theorem 1. For a neighbourhood system $\mathfrak{N}$ of $\mathfrak{T}$,

$$\mathfrak{N}^S = \{ X^S : X \in \mathfrak{N} \}$$

is a neighbourhood system of the relative topology $\mathfrak{T}^S$.

For the totality of the closed sets $\mathfrak{T}^{S'}$ in the subspace S by the relative topology $\mathfrak{T}^S$, we have by definition

$$\mathfrak{T}^{S'} = \{ X^{S'} : X \in \mathfrak{T} \},$$

and hence by the formula (1)

$$\mathfrak{T}^{S'} = \{ Y^S : Y \in \mathfrak{T}' \} = \mathfrak{T}'^S.$$

Thus we have

(5) $\mathfrak{T}^{S'} = \mathfrak{T}'^S$.

By virtue of (4) and (5), we obtain at once

Theorem 2. A point set $A \subset S$ is open or closed by the relative topology $\mathfrak{T}^S$, if and only if we can find an open or closed set X by $\mathfrak{T}$ respectively such that $A = SX$.

For a point set A in the subspace S, the opener or closure of A with respect to the relative topology $\mathfrak{T}^S$ will be denoted by $A^{S\circ}$ or A^{S-} respectively. But we shall write $A^{S-\circ}$, $A^{S\circ -}$, $A^{S\circ}$ instead of $A^{S-S\circ}$, $A^{S\circ S-}$, $(A^S)^{S\circ}$ respectively. On the other hand, A° and A^- should mean the opener and closure of A with respect to the topology $\mathfrak{T}$.

Recalling the definition of the closure, we have by the formulas (3) and (5)

$$A^{S-} = \prod_{A^S \subset X^S \in \mathfrak{T}^{S'}} X^S = (\prod_{AS \subset X \in \mathfrak{T}'} X)^S = (AS)^{-S}$$

for every point set A in R. Therefore we have

(6) $A^{S-} = (AS)^{-S}$.

Recalling §4(3), we conclude from (6) by (1)

$$A^{S\circ} = A^{S'-'} = A'^{S-'} = (A'S)^{-S'} = (A'S)'^{\circ S} = (A \dotplus S')^{\circ S},$$

and hence we have

(7) $A^{S\circ} = (A \dotplus S')^{\circ S}$.

From (6) and (7) we conclude immediately

(8) $A^{\circ S} \subset A^{S\circ} \subset A^S \subset A^{S-} \subset A^{-S}$.

This relation yields at once

(9) $A \supset B$ implies $A^S \supset B^S$.

By virtue of Theorem 2 we have obviously

(10) $A^{\circ S\circ} = A^{\circ S}$, $A^{-S-} = A^{-S}$.

Theorem 3. If $S \in \mathfrak{T}$, then we have for every point set A

$$A^{S\circ} = A^{\circ S}, \quad A^{S-} = A^{-S}.$$

Proof. As S is open by assumption, we have by (6) and §4(8)

$$A^{S-} = (AS)^{-S} \supset (A^- S^\circ)^S = (A^- S)^S = A^{-S}.$$

On the other hand we have obviously $A^{S-} \subset A^{-S}$ by the formula (8). Thus we obtain $A^{S-} = A^{-S}$. From this relation we conclude $A^{S\circ} = A^{\circ S}$ by duality.

Theorem 4. If S is dense by the topology $\mathfrak{T}$, then we have

$$A^{\circ - S} = A^{\circ S -}, \quad A^{-\circ S} = A^{-S\circ}.$$

Proof. By virtue of (8), we have obviously $A^{\circ S-} \subset A^{\circ - S}$. On the other hand we obtain by the formulas (6) and §4(8)

$$A^{\circ S-} = (A^{\circ} S)^{-S} \supset (A^{\circ\circ} S^{-})^{-S} = A^{\circ -S},$$

because $S^{-} = R$ by assumption. Therefore the first relation is obtained. The second will be conclude from the first by duality.

Theorem 5. Let S be dense by $\mathfrak{T}$. If a point set $\mathfrak{U}$ is regularly open by $\mathfrak{T}$, then the induced set $\mathfrak{U}^{S}$ also is regularly open by the relative topology $\mathfrak{T}^{S}$, and $(\mathfrak{U} S)^{-\circ} = \mathfrak{U}$. If a subset $\mathfrak{U} \subset S$ is regularly open by $\mathfrak{T}^{S}$, then we have $\mathfrak{U}^{-\circ} S = \mathfrak{U}$.

Proof. If $\mathfrak{U}$ is regularly open by $\mathfrak{T}$, then we have by Theorem 4

$$\mathfrak{U}^{S-\circ} = \mathfrak{U}^{-\circ S-\circ} = \mathfrak{U}^{-\circ-\circ S} = \mathfrak{U}^{S}$$

and hence $\mathfrak{U}^{S}$ is regularly open by $\mathfrak{T}^{S}$. Furthermore we have by §4(11)

$$(\mathfrak{U} S)^{-\circ} = \mathfrak{U}^{-\circ} S^{-\circ} = \mathfrak{U},$$

because $S^{-} = R$ by assumption. If a subset $\mathfrak{U} \subset S$ is regularly open by $\mathfrak{T}^{S}$, then we have by Theorem 4 and by the formula (6)

$$\mathfrak{U}^{-\circ S} = (\mathfrak{U} S)^{-S\circ} = \mathfrak{U}^{S-\circ} = \mathfrak{U}^{S},$$

and hence $\mathfrak{U}^{-\circ} S = \mathfrak{U}$.

Theorem 6. If S is dense by $\mathfrak{T}$ and two subsets $\mathfrak{U}, \mathfrak{V} \subset S$ are regularly open by $\mathfrak{T}^{S}$, then we have

$$(\mathfrak{U}\mathfrak{V})^{-\circ} = \mathfrak{U}^{-\circ}\mathfrak{V}^{-\circ}.$$

Proof. As $S\mathfrak{U}^{-\circ} = \mathfrak{U}$, $S\mathfrak{V}^{-\circ} = \mathfrak{V}$ by Theorem 5, we have by the formula §4(11)

$$(\mathfrak{U}\mathfrak{V})^{-\circ} = (S\mathfrak{U}^{-\circ}\mathfrak{V}^{-\circ})^{-\circ} = S^{-\circ}\mathfrak{U}^{-\circ-\circ}\mathfrak{V}^{-\circ-\circ} = \mathfrak{U}^{-\circ}\mathfrak{V}^{-\circ},$$

because $S^{-} = R$ by assumption.

Theorem 7. A point set A is dense in a point set S by the topology $\mathfrak{T}$, if and only if the induced set A^{S} is dense by the relative topology $\mathfrak{T}^{S}$.

Proof. If $S \subset (SA)^{-}$, then we have by the formula (6)

$$A^{S-} = (AS)^{-S} = S,$$

and hence A^{S} is dense by the relative topology $\mathfrak{T}^{S}$. Conversely, if A^{S} is dense by $\mathfrak{T}^{S}$, then we have by the formula (6)

$$S = A^{S-} = (AS)^{-S} \subset (AS)^{-},$$

and hence A is dense in S by $\mathfrak{T}$.

For two subsets $S_1 \subset S_2 \subset R$ we see easily by definition (4) that the relative topology $\mathfrak{T}^{S_1}$ of $\mathfrak{T}$ in S_1 coincides with the relative topology of $\mathfrak{T}^{S_2}$ in S_1 for the relative topology $\mathfrak{T}^{S_2}$ of $\mathfrak{T}$ in S_2, that is,

(11) $$\mathfrak{T}^{S_1} = (\mathfrak{T}^{S_2})^{S_1} \quad \text{for} \quad S_1 \subset S_2.$$

§10 Regular topologies

A topology $\mathfrak{T}$ on a space R is said to be *regular*, if

$$A = \sum_{A \succ X \in \mathfrak{T}} X \qquad \text{for every } A \in \mathfrak{T}.$$

A topological space will be said to be *regular*, if its topology is so. With this definition we see easily that if a topology $\mathfrak{T}$ is regular, then for an open basis $\mathfrak{U}$ of $\mathfrak{T}$ we have $A = \sum_{A \succ X \in \mathfrak{U}} X$. We also can say that a topology $\mathfrak{T}$ is regular, if and only if $\mathfrak{T} \ni A \ni a$ implies $A \succ X \ni a$ for some $X \in \mathfrak{T}$.

Theorem 1. If a topology $\mathfrak{T}$ is regular, then for each topological set A, $A \ni a$ implies $A \supset \{a\}^{-}$.

Proof. Let $\mathfrak{A}$ be the totality of point sets A such that $A \ni a$ implies $A \supset \{a\}^{-}$. If $\mathfrak{A} \ni A$, then for any $a \in A$ we have $\{a\}^{-} \subset A$, and hence $\{a\}^{-\prime} \ni b$ for every $b \bar{\in} A$. As $\mathfrak{T}$ is regular by assumption, we can find then $X \in \mathfrak{T}$ such that $\{a\}^{-\prime} \succ X \ni b$, and hence $\{a\}^{-\prime} \supset \{b\}^{-}$. Consequently we have $A\{b\}^{-} = 0$ for every $b \bar{\in} A$, that is, $A' \ni b$ implies $A' \supset \{b\}^{-}$. Thus $\mathfrak{A} \ni A$ implies $\mathfrak{A} \ni A'$. Furthermore it is obvious that $\mathfrak{A} \ni A_\lambda$ ($\lambda \in \Lambda$) implies $\mathfrak{A} \ni \sum_{\lambda \in \Lambda} A_\lambda$. Therefore $\mathfrak{A}$ is universally additive. As $\mathfrak{T}$ is regular, we see easily that $\mathfrak{A}$ contains all open sets. Accordingly $\mathfrak{A}$ contains all topological sets by definition.

Theorem 2. If a topology $\mathfrak{T}$ is regular, then every compact topological set is closed.

Proof. Let A be a compact topological set. As A' is a topological set too, we have by Theorem 1 that $b \in A'$ implies $\{b\}^{-} \subset A'$. Therefore, for each point $b \bar{\in} A$, corresponding to each point $x \in A$ we can find an open set X_x such that $x \in X_x \prec \{b\}^{-\prime}$. For such X_x ($x \in A$), as $A \subset \sum_{x \in A} X_x$, we can find a finite number of points $x_1, \ldots, x_n \in A$ such that $A \subset \sum_{\nu=1}^{\kappa} X_{x_\nu}$. Then we have by the formula §4(7)

$$A^- \subset \sum_{\nu=1}^{\kappa} X_{x_\nu}^- \subset \{b\}^{-\prime},$$

and hence $b \,\overline{\in}\, A^-$ for every $b \,\overline{\in}\, A$. Therefore A is closed.

Theorem 3. If a topology $\mathcal{T}$ is regular and a point set A is compact, then its closure A^- also is compact.

Proof. If $A^- \subset \sum_{\lambda\in\Lambda} X_\lambda^\circ$, then for each point $x \in A$ we can find $\lambda_x \in \Lambda$ such that $x \in X_{\lambda_x}^\circ$. For such $X_{\lambda_x}^\circ$, as $\mathcal{T}$ is regular by assumption, there is an open set Y_x for which $x \in Y_x \prec X_{\lambda_x}^\circ$. Then we have obviously $A \subset \sum_{x\in A} Y_x$, and hence we can find a finite number of points $x_\nu \in A$ ($\nu = 1, 2, \ldots, \kappa$) such that

$$A \subset \sum_{\nu=1}^{\kappa} Y_{x_\nu}$$

and hence by the formula §4(7)

$$A^- \subset \sum_{\nu=1}^{\kappa} Y_{x_\nu}^- \subset \sum_{\nu=1}^{\kappa} X_{\lambda_{x_\nu}}^\circ .$$

Therefore A^- also is compact.

Recalling the relation §9(9), we conclude easily

Theorem 4. If a topology $\mathcal{T}$ is regular, then the relative topology $\mathcal{T}^S$ also is regular for every subspace S.

Theorem 5. For a system of regular topologies $\mathcal{T}_\lambda$ ($\lambda \in \Lambda$), the weakest stronger topology $\bigcup_{\lambda\in\Lambda} \mathcal{T}_\lambda$ also is regular.

Proof. If $\bigcup_{\lambda\in\Lambda} \mathcal{T}_\lambda \ni U \ni a$, then we can find by §8 Theorem 3 a finite number of point sets $U_{\lambda_\nu} \in \mathcal{T}_{\lambda_\nu}$, $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \kappa$) such that

$$U \supset \prod_{\nu=1}^{\kappa} U_{\lambda_\nu} \ni a .$$

Since every $\mathcal{T}_\lambda$ is regular by assumption, we can find $V_{\lambda_\nu} \in \mathcal{T}_{\lambda_\nu}$ such that

$$U_{\lambda_\nu} \supset V_{\lambda_\nu}^{\mathcal{T}_{\lambda_\nu}-} \supset V_{\lambda_\nu} \ni a \qquad (\nu = 1, 2, \ldots, \kappa).$$

Then, putting $V = \prod_{\nu=1}^{\kappa} V_{\lambda_\nu}$, we have $V \in \bigcup_{\lambda\in\Lambda} \mathcal{T}_\lambda$ by §8 Theorem 3. Furthermore, putting $\mathcal{T} = \bigcup_{\lambda\in\Lambda} \mathcal{T}_\lambda$, we have

$$a \in V \subset V^{\mathcal{T}-} \subset \prod_{\nu=1}^{\kappa} V^{\mathcal{T}_{\lambda_\nu}-} \subset \prod_{\nu=1}^{\kappa} V_{\lambda_\nu}^{\mathcal{T}_{\lambda_\nu}-} \subset \prod_{\nu=1}^{\kappa} U_{\lambda_\nu} \subset U .$$

Therefore $\bigcup_{\lambda\in\Lambda} \mathcal{T}_\lambda$ is regular by definition.

§11 Normal topologies

Let R be a topological space with a topology $\mathcal{T}$. For an open

set U, if $X \prec Y \subset U$ implies $X \prec Z \prec Y$ for some point set Z, then $\mathcal{T}$ is said to be normal in an open set U. With this definition, we see at once that if $\mathcal{T}$ is normal in an open set U, then $\mathcal{T}$ also is normal in every open set $V \subset U$.

Theorem 1. If $\mathcal{T}$ is regular, then for every open set U such that the closure U^- is compact, $\mathcal{T}$ is normal in U.

Proof. If $X \prec Y \subset U$, then we have naturally $X^- \subset Y^\circ \subset U \subset U^-$. Since $\mathcal{T}$ is regular by assumption, we have by definition

$$X^- \subset Y^\circ = \sum_{Y^\circ \succ Z \in \mathcal{T}} Z .$$

As U^- is compact by assumption, X^- also is compact by §7 Theorem 1, and hence we can find a finite number of open sets $Z_\nu \prec Y^\circ$ ($\nu = 1, 2, \ldots, \kappa$) such that $X^- \subset \sum_{\nu=1}^{\kappa} Z_\nu \prec Y^\circ$, that is, $X \prec \sum_{\nu=1}^{\kappa} Z_\nu \prec Y$.

Theorem 2. If $\mathcal{T}$ is normal in an open set U, then for two sequences of closed sets $A_\nu^-, B_\nu^- \subset U$ ($\nu = 1, 2, \ldots$) such that

$$\Big(\sum_{\nu=1}^{\infty} A_\nu^-\Big)^-\Big(\sum_{\nu=1}^{\infty} B_\nu^-\Big) = \Big(\sum_{\nu=1}^{\infty} A_\nu^-\Big)\Big(\sum_{\nu=1}^{\infty} B_\nu^-\Big)^- = 0,$$

we can find two open sets $A, B \subset U$ such that

$$A \supset \sum_{\nu=1}^{\infty} A_\nu^- , \qquad B \supset \sum_{\nu=1}^{\infty} B_\nu^- , \qquad AB = 0 .$$

Proof. Since $\mathcal{T}$ is normal in U by assumption, we can find two sequences of open sets X_ν, Y_ν ($\nu = 1, 2, \ldots$) consecutively such that

$$A_\nu^- \prec X_\nu \prec U\Big(\sum_{\mu=1}^{\infty} B_\mu^-\Big)^{-\prime}\Big(\sum_{\mu=1}^{\nu-1} Y_\mu\Big)^{-\prime},$$

$$B_\nu^- \prec Y_\nu \prec U\Big(\sum_{\mu=1}^{\infty} A_\mu^-\Big)^{-\prime}\Big(\sum_{\mu=1}^{\nu} X_\mu\Big)^{-\prime}.$$

For such open sets X_ν and Y_ν ($\nu = 1, 2, \ldots$), if we put

$$A = \sum_{\nu=1}^{\infty} X_\nu , \qquad B = \sum_{\nu=1}^{\infty} Y_\nu ,$$

then it is evident that

$$U \supset A \supset \sum_{\nu=1}^{\infty} A_\nu^- , \qquad U \supset B \supset \sum_{\nu=1}^{\infty} B_\nu^- .$$

Since we have for every $\nu = 1, 2, \ldots$

$$X_\nu \sum_{\mu=1}^{\nu-1} Y_\mu = 0 , \qquad Y_\nu \sum_{\mu=1}^{\nu} X_\mu = 0 ,$$

we obtain $X_\nu Y_\mu = 0$ for every $\nu, \mu = 1, 2, \ldots$, and hence $AB = 0$.

A topology $\mathcal{T}$ is said to be locally normal, if there is a system of open sets U_λ ($\lambda \in \Lambda$) such that $\sum_{\lambda\in\Lambda} U_\lambda = R$ and $\mathcal{T}$ is normal in U_λ for every $\lambda \in \Lambda$.

Theorem 3. If $\mathcal{T}$ is locally normal and every open set A satisfies the condition that $A \ni x$ implies $A \supset \{x\}^-$, then $\mathfrak{T}$ is regular.

Proof. Let A be an arbitrary open set. For any point $x \in A$ we can find an open set $U \ni x$ such that $\mathfrak{T}$ is normal in U, because $\mathfrak{T}$ is locally normal by assumption. Then we obtain $AU \supset \{x\}^-$ by assumption, and hence there is an open set X for which we have $AU \succ X \succ \{x\}^-$. This relation yields obviously $A \succ X \ni x$. Therefore $\mathfrak{T}$ is regular.

A topology $\mathfrak{T}$ on a space R is said to be *normal*, if $\mathfrak{T}$ is normal in R. We also may say that a topological space is normal, if its topology is normal. With this definition we have

Theorem 4. In order that a topology $\mathfrak{T}$ be normal, it is necessary and sufficient that for each pair of closed sets A, B subject to $AB = 0$, we can find two open sets X, Y such that $A \subset X$, $B \subset Y$, and $XY = 0$.

Proof. If $\mathfrak{T}$ is normal and $AB = 0$ for two closed sets A, B, then we have obviously $A \prec B'$ and hence there is a point set X such that $A \prec X \prec B'$. This relation yields $A \subset X^\circ$, $B \subset X^{-\prime} = X'^\circ$.

Conversely, if the stated condition is satisfied and $A \prec B$, then $A^- B'^- = A^- B^{\circ\prime} = 0$, and hence we can find two open sets X, Y such that $A^- \subset X$, $B'^- \subset Y$, and $XY = 0$. This relation yields obviously

$$A^- \subset X \subset X^- \subset Y' \subset B'^{-\prime} = B^\circ .$$

Therefore $\mathfrak{T}$ is normal by definition.

Even if a topology $\mathfrak{T}$ on a space R is normal, the relative topology $\mathfrak{T}^S$ ($S \subset R$) is not necessarily normal. If the relative topology $\mathfrak{T}^S$ is normal for every subspace S of R, then $\mathfrak{T}$ is said to be *completely normal*.

§12 Compact topologies

A topology $\mathfrak{T}$ on a space R is said to be *compact*, if R is compact by $\mathfrak{T}$. With this definition we have

Theorem 1. In order that a point set S be compact by $\mathfrak{T}$, it is necessary and sufficient that the relative topology $\mathfrak{T}^S$ be compact.

Proof. If S is compact by $\mathfrak{T}$ and $S = \sum_{\lambda \in \Lambda} X_\lambda^S$ for a system of open sets $X_\lambda \in \mathfrak{T}$ ($\lambda \in \Lambda$), then we have $S \subset \sum_{\lambda \in \Lambda} X_\lambda$, and hence we can find a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \varkappa$) such that we have $S \subset \sum_{\nu=1}^{\varkappa} X_{\lambda_\nu}$, that is, $S = \sum_{\nu=1}^{\varkappa} X_{\lambda_\nu}^S$. Therefore the relative topology $\mathfrak{T}^S$ is compact by definition.

Conversely, if the relative topology $\mathfrak{T}^S$ is compact and $S \subset \sum_{\lambda \in \Lambda} X_\lambda$ for a system of open sets $X_\lambda \in \mathfrak{T}$ ($\lambda \in \Lambda$), then we have $S = \sum_{\lambda \in \Lambda} X_\lambda^S$, and hence we can find a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \varkappa$) such that

$$S = \sum_{\nu=1}^{\varkappa} X_{\lambda_\nu}^S \subset \sum_{\nu=1}^{\varkappa} X_{\lambda_\nu} .$$

Therefore S is compact by $\mathfrak{T}$.

Recalling the formula §9(11) we conclude immediately from Theorem 1

Theorem 2. If a point set A is compact by a relative topology $\mathfrak{T}^B$ for some point set $B \supset A$, then A also is compact by the relative topology for every point set $S \supset A$.

A topology $\mathfrak{T}$ on a space R is said to be *locally compact*, if there is a system of open sets $U_\lambda \in \mathfrak{T}$ ($\lambda \in \Lambda$) such that $\sum_{\lambda \in \Lambda} U_\lambda = R$ and the closure U_λ^- is compact for every $\lambda \in \Lambda$. We also shall say that a topological space is locally compact, if its topology is so.

A neighbourhood system $\mathfrak{N}$ of a topology $\mathfrak{T}$ is called a *compact neighbourhood system*, if the closure of every point set of $\mathfrak{N}$ is compact. With this definition, we see easily by definition that a topology $\mathfrak{T}$ is locally compact, if and only if $\mathfrak{T}$ has a compact neighbourhood system.

Recalling §11 Theorem 1, we obtain obviusly by definition

Theorem 3. If a topology $\mathfrak{T}$ is regular and locally compact, then $\mathfrak{T}$ is locally normal.

§13 Separative topologies

Let R be a topological space with a topology $\mathfrak{T}$. A point a is said to be *separated* from a point b by the topology $\mathfrak{T}$, if we can find an open set U such that $U \ni a$ but $U \bar{\ni} b$. Thus, if a is separated from b, then we have $a \bar{\in} \{b\}^-$. If each point of R is separated by the topology $\mathfrak{T}$ from every other point of R, then we shall say that R is separated by its topology $\mathfrak{T}$, or that the topology $\mathfrak{T}$ is separative.

With this definition we have obviously

Theorem 1. In order that a topological space R be separated by its topology, it is necessary and sufficient that $\{x\}^- = \{x\}$ for every point $x \in R$.

A topological space R is called a Hausdorff space, if for each pair of different points a and b we can find two open sets U and V such that $a \in U$, $b \in V$, and $UV = 0$. It is evident by definition that a Hausdorff space is separated by its topology.

Theorem 2. If a topological space R is regular and separated, then R is a Hausdorff space.

Proof. For each pair of different points a and b, we can find an open set A such that $a \in A$ but $b \bar{\in} A$, because R is separated by assumption. Then there is an open set U such that $a \in U^- < A$, because R is regular by assumption. Furthermore, putting $V = U^{-\prime}$, we have obviously $UV = 0$ and $V \supset A' \ni b$. Therefore R is a Hausdorff space.

Theorem 3. If a topology $\mathfrak{T}$ is separative and locally normal, then $\mathfrak{T}$ is regular.

Proof. Let A be an arbitrary open set. For each point $a \in A$ we can find an open set $U \ni a$ such that $\mathfrak{T}$ is normal in U, because R is locally normal by assumption. Since $\mathfrak{T}$ is furthermore separative by assumption, we have then by Theorem 1 $\{a\}^- = \{a\} \subset AU$, and hence there is an open set V such that $\{a\}^- \subset V^- < AU$. This relation yields $a \in V^- < A$. Therefore $\mathfrak{T}$ is regular by definition.

If a topological space R is separated, then every point of R is by Theorem 1 itself a closed set, and hence every point set in R is a topological set. Therefore we obtain by §10 Theorem 2

Theorem 4. If a topology $\mathfrak{T}$ is separative and regular, then every compact set is closed.

Furthermore we can prove

Theorem 5. In a Hausdorff space R, every compact set is closed.

Proof. Let A be a compact set. If $a \bar{\in} A$, then corresponding to every point $x \in A$ we can find open sets U_x and V_x such that

$$x \in U_x, \quad a \in V_x, \quad U_x V_x = 0,$$

because R is a Hausdorff space by assumption. Then we have obviously $A \subset \sum_{x \in A} U_x$. As A is compact by assumption, there is hence a finite number of points $x_\nu \in A$ ($\nu = 1, 2, \ldots, \kappa$) such that $A \subset \sum_{\nu=1}^{\kappa} U_{x_\nu}$. For such $x_\nu \in A$ ($\nu = 1, 2, \ldots, \kappa$), we have obviously

$$a \in \prod_{\nu=1}^{\kappa} V_{x_\nu}, \quad \left(\sum_{\nu=1}^{\kappa} U_{x_\nu}\right)\left(\prod_{\nu=1}^{\kappa} V_{x_\nu}\right) = 0,$$

and hence $a \in \prod_{\nu=1}^{\kappa} V_{x_\nu} \subset A'$. Therefore every point $a \bar{\in} A$ is not a contact point of A. Thus A is closed.

Theorem 6. If a Hausdorff space R is locally compact, then R is regular and locally normal.

Proof. Let $\mathfrak{N}$ be a compact neighbourhood system. If $a \in A^\circ$, then we can find $B \in \mathfrak{N}$, such that $a \in B \subset A^\circ$. As R is a Hausdorff space by assumption, corresponding to every point $x \in B^- B'$ we can find two open sets X_x and Y_x such that $a \in X_x$, $x \in Y_x$, and $X_x Y_x = 0$. Since B^- is compact by assumption, $B^- B'$ also is compact by §7 Theorem 1 and hence we can find a finite number of points x_ν ($\nu = 1, 2, \ldots, \kappa$) such that $B^- B' \subset \sum_{\nu=1}^{\kappa} Y_{x_\nu}$. For such x_ν ($\nu = 1, 2, \ldots, \kappa$), putting

$$X = B \prod_{\nu=1}^{\kappa} X_{x_\nu}, \quad Y = \sum_{\nu=1}^{\kappa} Y_{x_\nu},$$

we obtain two open sets X and Y such that

$$a \in X \subset B, \quad XY = 0, \quad B^- B' \subset Y.$$

Then, since $X^- Y \subset (XY)^- = 0$ by the formulas (8), (2) in §4, we obtain $X^- B^- B' = 0$, and hence $X^- = X^- B^- \subset B$, that is, $a \in X^- < B$. Therefore R is regular by definition. Accordingly R is locally normal by §10 Theorem 3.

Theorem 7. For two topologies $\mathfrak{T}_1 \subset \mathfrak{T}_2$ on a space R, if R is separated by $\mathfrak{T}_1$, then R also is separated by $\mathfrak{T}_2$; and if R is a Hausdorff space by $\mathfrak{T}_1$, then R also is a Hausdorff space by $\mathfrak{T}_2$.

Proof. If R is separated by $\mathfrak{T}_1$, then for any pair of different points a and b, we can find $U \in \mathfrak{T}_1$ such that $U \ni a$ but $U \not\ni b$. Then, as $\mathfrak{T}_1 \subset \mathfrak{T}_2$, we also have $U \in \mathfrak{T}_2$. Thus R also is separated by $\mathfrak{T}_2$. We also can prove likewise the other assertion.

Theorem 8. *For two topologies $\mathfrak{T}_1 \subset \mathfrak{T}_2$ on a space R, if R is compact by $\mathfrak{T}_2$ and a Hausdorff space by $\mathfrak{T}_1$, then $\mathfrak{T}_1 = \mathfrak{T}_2$.*

Proof. For every point set $A \in \mathfrak{T}_2$, recalling §7 Theorem 1, we see that the complement A' is compact by $\mathfrak{T}_2$, and hence A' also is compact by $\mathfrak{T}_1$, because $\mathfrak{T}_1 \subset \mathfrak{T}_2$. Consequently A' is closed by $\mathfrak{T}_1$ by Theorem 5, that is, $A \in \mathfrak{T}_1$. Therefore we obtain $\mathfrak{T}_1 \supset \mathfrak{T}_2$.

We have obviously by definition

Theorem 9. *If a topological space R is separated or a Hausdorff space, then every subspace of R also is separated or a Hausdorff space respectively by the relative topology.*

§14 Sequential topologies

Let R be a topological space with a topology $\mathfrak{T}$. A system of open sets U_λ ($\lambda \in \Lambda$) is called a *neighbourhood system of a point* a, if for each open set $A \ni a$ we can find $\lambda \in \Lambda$ such that $a \in U_\lambda \subset A$. For a point a, if there is a neighbourhood system of a which is composed of countable open sets, then the topology $\mathfrak{T}$ is said to be *sequential* at the point a, and such neighbourhood system is called a *countable neighbourhood system of* a. If a topology $\mathfrak{T}$ is sequential at every point, then $\mathfrak{T}$ is said to be *sequential*. We also shall say that a topological space is sequential, if its topology is sequential.

If $\mathfrak{T}$ is sequential at a point a, then we see easily by definition that there is a countable neighbourhood system $U_\nu \in \mathfrak{T}$ ($\nu = 1, 2, \ldots$) such that $U_1 \supset U_2 \supset \ldots$.

For a point sequence $a_\nu \in R$ ($\nu = 1, 2, \ldots$), if there is a point a such that for each open set $A \ni a$ we can find ν_0 such that we have $a_\nu \in A$ for every $\nu \geqq \nu_0$, then we shall say that $a_\nu \in R$ ($\nu = 1, 2, \ldots$) is *convergent* to a *limit* a and we shall write $\lim_{\nu\to\infty} a_\nu = a$.

Theorem 1. *If $\lim_{\nu\to\infty} a_{\nu,\mu} = a_\mu$ ($\mu = 1, 2, \ldots$), $\lim_{\mu\to\infty} a_\mu = a$, and $\mathfrak{T}$ is sequential at the point a, then we can find ν_μ ($\mu = 1, 2, \ldots$) such that $\lim_{\mu\to\infty} a_{\nu_\mu, \mu} = a$.*

Proof. For a countable neighbourhood system $U_1 \supset U_2 \supset \ldots$ of the point a, we can find μ_σ ($\sigma = 1, 2, \ldots$) such that $a_\mu \in U_\sigma$ for $\mu \geqq \mu_\sigma$. We see easily further that there is ν_μ ($\mu \geqq \mu_1$) such that $a_{\nu_\mu, \mu} \in U_\sigma$ for $\mu_\sigma \leqq \mu < \mu_{\sigma+1}$. Then we have obviously $\lim_{\mu\to\infty} a_{\nu_\mu,\mu} = a$, putting $\nu_1, \nu_2, \ldots, \nu_{\mu_1 - 1}$ arbitrary, because $U_1, U_2, \ldots$ is a neighbourhood system of a.

Theorem 2. *If $\mathfrak{T}$ is sequential, then for each contact point a of a point set A we can find a point sequence $a_\nu \in A$ ($\nu = 1, 2, \ldots$) such that $\lim_{\nu\to\infty} a_\nu = a$.*

Proof. For a countable neighbourhood system $U_1 \supset U_2 \supset \ldots$ of a contact point a of a point set A, as $a \in U_\nu \in \mathfrak{T}$, we can find a point sequence $a_\nu \in A U_\nu$ ($\nu = 1, 2, \ldots$), and then we have obviously $\lim_{\nu\to\infty} a_\nu = a$.

Even if a point sequence is convergent, its limit is not necessarily uniquely determined. But we see easily by definition that if R is a Hausdorff space, then for each convergent point sequence its limit is uniquely determined. Consequently we see by §13 Theorem 2 that if R is separated and regular, then for every convergent point sequence its limit is uniquely determined.

A topology $\mathfrak{T}$ is said to be *separable*, if there is a dense set which is composed of countable points. We also shall say that a topological space is separable, if its topology is separable. Even if a topology is separable, its relative topology is not necessarily separable.

A topology $\mathfrak{T}$ is said to be *completely separable*, if there is a neighbourhood system composed of countable open sets, and such a neighbourhood system is called a *countable neighbourhood system* of $\mathfrak{T}$. With this definition, it is evident that if a topology $\mathfrak{T}$ is completely separable, then $\mathfrak{T}$ is sequential and every relative topology of $\mathfrak{T}$ also is completely separable. Furthermore, if a topology $\mathfrak{T}$ is completely separable, then $\mathfrak{T}$ is separable. Because, for a countable neighbourhood system U_ν ($\nu = 1, 2, \ldots$), a point sequence $a_\nu \in U_\nu$ ($\nu = 1, 2, \ldots$)

constitutes a dense set.

<u>Theorem</u> 3.(Tychonof) <u>If a topology $\mathcal{T}$ is regular and completely separable, then $\mathcal{T}$ is normal.</u>

<u>Proof</u>. For two point sets $A \prec B$, as $\mathcal{T}$ is regular and completely separable, we can find a sequence of open sets B_ν ($\nu = 1, 2, \ldots$) such that

$$B^\circ = \sum_{\nu=1}^{\infty} B_\nu, \qquad B_\nu \prec B^\circ \qquad (\nu = 1, 2, \ldots).$$

Hence we see by duality that there is a sequence of closed sets A_ν ($\nu = 1, 2, \ldots$) such that

$$A^- = \prod_{\nu=1}^{\infty} A_\nu, \qquad A_\nu \succ A^- \qquad (\nu = 1, 2, \ldots).$$

For such closed sets A_ν and open sets B_ν ($\nu = 1, 2, \ldots$), putting

$$F_0 = A_1 \dotplus B_1^-,$$

$$G_\nu = B_\nu \prod_{\mu=0}^{\nu-1} F_\mu^\circ, \qquad F_\nu = A_\nu \dotplus \sum_{\mu=1}^{\nu} G_\mu^-,$$

we obtain open sets G_ν and closed sets F_ν ($\nu = 1, 2, \ldots$). Then we have obviously $G_\nu \subset F_\mu$ for every $\nu, \mu = 1, 2, \ldots$, and hence

$$\sum_{\nu=1}^{\infty} G_\nu \subset \prod_{\mu=1}^{\infty} F_\mu .$$

As $A^- \prec A_\nu \subset F_\nu$ ($\nu = 1, 2, \ldots$), we have

$$\sum_{\nu=1}^{\infty} G_\nu \supset \sum_{\nu=1}^{\infty} (A^- B_\nu \prod_{\mu=0}^{\nu-1} F_\mu^\circ) = \sum_{\nu=1}^{\infty} A^- B_\nu = A^- B^\circ = A^-,$$

because $A^- \subset B^\circ$ by assumption. As $G_\nu \subset B_\nu \prec B^\circ$, we have

$$\prod_{\nu=1}^{\infty} F_\nu \subset \prod_{\nu=1}^{\infty} (B^\circ \dotplus A_\nu \dotplus \sum_{\mu=1}^{\nu} G_\mu^-) = \prod_{\nu=1}^{\infty} (B^\circ \dotplus A_\nu) = B^\circ \dotplus A^- = B^\circ.$$

Thus, putting $X = \sum_{\nu=1}^{\infty} G_\nu$, we have $A \prec X \prec B$. Therefore $\mathcal{T}$ is normal by definition.

<u>CHAPTER III</u>

<u>CONTINUOUS MAPPINGS</u>

§15 <u>Mappings</u>

Let R and S be two abstract spaces. A correspondence $\mathfrak{a}$, which assigns to every point $x \in R$ a point $\mathfrak{a}(x) \in S$, is called a <u>mapping</u> of R into S, and $\mathfrak{a}(x)$ the <u>image</u> of x.

Let $\mathfrak{a}$ be a mapping of R into S. For a point set $A \subset S$, the totality of points whose images belong to A, is called the <u>inverse image</u> of A and denoted by $\mathfrak{a}^{-1}(A)$, that is,

$$\mathfrak{a}^{-1}(A) = \{ x : \mathfrak{a}(x) \in A \}.$$

Concerning the inverse image, we have obviously the following:

(1) $\mathfrak{a}^{-1}(0) = 0, \qquad \mathfrak{a}^{-1}(S) = R,$

(2) $\mathfrak{a}^{-1}(A') = \mathfrak{a}^{-1}(A)',$

(3) $\mathfrak{a}^{-1}(\sum_{\lambda \in \Lambda} A_\lambda) = \sum_{\lambda \in \Lambda} \mathfrak{a}^{-1}(A_\lambda),$

(4) $\mathfrak{a}^{-1}(\prod_{\lambda \in \Lambda} A_\lambda) = \prod_{\lambda \in \Lambda} \mathfrak{a}^{-1}(A_\lambda),$

(5) $A \subset B$ implies $\mathfrak{a}^{-1}(A) \subset \mathfrak{a}^{-1}(B),$

(6) $AB = 0$ implies $\mathfrak{a}^{-1}(A)\, \mathfrak{a}^{-1}(B) = 0.$

Here the relations (1), (2), and (3) are essential for the inverse image $\mathfrak{a}^{-1}$, i.e., if a correspondence $\mathfrak{a}^{-1}$ which assigns to every point set $A \subset S$ a point set $\mathfrak{a}^{-1}(A) \subset R$, satisfies the relations (1), (2), and (3), then there exists uniquely a mapping $\mathfrak{a}$ of R into S such that $\mathfrak{a}^{-1}$ is a inverse image of $\mathfrak{a}$.

For a point set $X \subset R$, the totality of images $\mathfrak{a}(x)$ for $x \in X$ is called the <u>image</u> of X and denoted by $\mathfrak{a}(X)$, that is, $\mathfrak{a}(X) = \{ \mathfrak{a}(x) : x \in X \}$.

Concerning images of point sets, we have obviously the following :

(7) $X \subset Y$ implies $\mathfrak{a}(X) \subset \mathfrak{a}(Y),$

(8) $\mathfrak{a}(\sum_{\lambda \in \Lambda} X_\lambda) = \sum_{\lambda \in \Lambda} \mathfrak{a}(X_\lambda),$

(9) $\mathfrak{a}(\prod_{\lambda \in \Lambda} X_\lambda) \subset \prod_{\lambda \in \Lambda} \mathfrak{a}(X_\lambda),$

(10) $X \subset \mathfrak{a}^{-1}(A)$ implies $\mathfrak{a}(X) \subset A$.

As $\mathfrak{a}^{-1}(\mathfrak{a}(X)) = \{ x : \mathfrak{a}(x) \in \mathfrak{a}(X) \} \supset X$, we obtain

(11) $\mathfrak{a}^{-1}(\mathfrak{a}(X)) \supset X.$

Since $x \in \mathfrak{a}^{-1}(A)$ is by definition equivalent to $\mathfrak{a}(x) \in A$, we have

$$\mathfrak{a}(X\mathfrak{a}^{-1}(A)) = \{\mathfrak{a}(x) : x \in X\mathfrak{a}^{-1}(A)\}$$
$$= \{\mathfrak{a}(x) : x \in X, \mathfrak{a}(x) \in A\} = \mathfrak{a}(X)A,$$

that is, we have

(12) $\mathfrak{a}(X\mathfrak{a}^{-1}(A)) = \mathfrak{a}(X)A$.

In particular, putting $X = R$ in (12), we obtain

(13) $\mathfrak{a}(\mathfrak{a}^{-1}(A)) = \mathfrak{a}(R)A$.

Since we have by (2) and (11)

$$\mathfrak{a}^{-1}(\mathfrak{a}(X)') = \mathfrak{a}^{-1}(\mathfrak{a}(X))' \subset X',$$

we obtain by (7) and (13)

(14) $\mathfrak{a}(R)\mathfrak{a}(X)' \subset \mathfrak{a}(X')$.

If $\mathfrak{a}(R) = S$, then $\mathfrak{a}$ is called a mapping of R onto S. Every mapping $\mathfrak{a}$ of R into S may be considered as a mapping of R onto the subspace $\mathfrak{a}(R)$ of S.

For a mapping $\mathfrak{a}$ of R onto S, we see at once that

$$\mathfrak{a}^{-1}(\{a\}) \neq 0 \qquad \text{for every } a \in S.$$

If $\mathfrak{a}^{-1}(\{a\})$ is composed only of a single point for every $a \in S$, then $\mathfrak{a}$ is said to be a transformation from R to S, that is, a transformation from R to S is a one-to-one correspondence from R to S.

For a transformation $\mathfrak{a}$ from R to S, putting

$$\mathfrak{b}(a) = \mathfrak{a}^{-1}(\{a\}) \qquad \text{for every } a \in S,$$

we obtain obviously a transformation $\mathfrak{b}$ from S to R. This transformation $\mathfrak{b}$ is called the inverse transformation of $\mathfrak{a}$ and denoted by $\mathfrak{a}^{-1}$, that is, we have

$$\mathfrak{a}^{-1}(a) = \mathfrak{a}^{-1}(\{a\}) \qquad \text{for every } a \in S.$$

Concerning the inverse transformation $\mathfrak{a}^{-1}$ we have obviously

$$\mathfrak{a}(\mathfrak{a}^{-1}(a)) = a \qquad \text{for every } a \in S,$$
$$\mathfrak{a}^{-1}(\mathfrak{a}(x)) = x \qquad \text{for every } x \in R.$$

A mapping $\mathfrak{a}$ of R into S may be said to be a transformation, if $\mathfrak{a}$ is a transformation as a mapping of R onto the subspace $\mathfrak{a}(R)$ of S.

§16 Continuous mappings

Let R and S be topological spaces with topologies $\mathfrak{T}$ and $\mathfrak{T}$ respectively, and $\mathfrak{a}$ a mapping of R into S. A mapping $\mathfrak{a}$ is said to be continuous, if we have

$$\mathfrak{a}^{-1}(A) \in \mathfrak{T} \qquad \text{for every } A \in \mathfrak{T}.$$

With this definition we have obviously by §15(2)

Theorem 1. In order that a mapping $\mathfrak{a}$ be continuous, it is necessary and sufficient that we have

$$\mathfrak{a}^{-1}(A) \in \mathfrak{T}' \qquad \text{for every } A \in \mathfrak{T}'$$

for the totality of closed sets $\mathfrak{T}'$ and $\mathfrak{T}'$ respectively of R and S.

Theorem 2. In order that a mapping $\mathfrak{a}$ be continuous, it is necessary and sufficient that we have

$$\mathfrak{a}^{-1}(A^{\circ}) \subset \mathfrak{a}^{-1}(A)^{\circ} \qquad \text{for every } A \subset S.$$

Proof. If $\mathfrak{a}$ is continuous, then we have by §4(1) and §15(5)

$$\mathfrak{a}^{-1}(A^{\circ}) = \mathfrak{a}^{-1}(A^{\circ})^{\circ} \subset \mathfrak{a}^{-1}(A)^{\circ}.$$

Conversely, if $\mathfrak{a}^{-1}(A^{\circ}) \subset \mathfrak{a}^{-1}(A)^{\circ}$ for every $A \subset S$, then we have by the relations §4(1), §4(4), and §15(5)

$$\mathfrak{a}^{-1}(A^{\circ})^{\circ} \subset \mathfrak{a}^{-1}(A^{\circ}) = \mathfrak{a}^{-1}(A^{\circ\circ}) \subset \mathfrak{a}^{-1}(A^{\circ})^{\circ},$$

and hence $\mathfrak{a}^{-1}(A^{\circ}) \in \mathfrak{T}$ for every $A \subset S$, that is, $\mathfrak{a}$ is continuous by definition.

Recalling the formula §15(2), we conclude from Theorem 2 by duality

Theorem 3. In order that a mapping $\mathfrak{a}$ be continuous, it is necessary and sufficient that we have

$$\mathfrak{a}^{-1}(A^{-}) \supset \mathfrak{a}^{-1}(A)^{-} \qquad \text{for every } A \subset S.$$

Theorem 4. In order that a mapping $\mathfrak{a}$ be continuous, it is necessary and sufficient that we have

$$\mathfrak{a}(X)^{-} \supset \mathfrak{a}(X^{-}) \qquad \text{for every } X \subset R.$$

Proof. If $\mathfrak{a}$ is continuous, then we have by Theorem 3 and by the relation §15(11)

$$\mathfrak{a}^{-1}(\mathfrak{a}(X)^{-}) \supset \mathfrak{a}^{-1}(\mathfrak{a}(X))^{-} \supset X^{-},$$

and hence we obtain by the relations §15(7) and §15(13)

$$\mathfrak{a}(X)^{-} \supset \mathfrak{a}(\mathfrak{a}^{-1}(\mathfrak{a}(X)^{-})) \supset \mathfrak{a}(X^{-}).$$

Conversely, if $\mathfrak{a}(X)^- \supset \mathfrak{a}(X^-)$ for every $X \subset R$, then we have for every $A \subset S$ by the relation §15(13)

$$A^- \supset \mathfrak{a}(\mathfrak{a}^{-1}(A))^- \supset \mathfrak{a}(\mathfrak{a}^{-1}(A)^-),$$

and hence by the formula §15(11)

$$\mathfrak{a}^{-1}(A^-) \supset \mathfrak{a}^{-1}(\mathfrak{a}(\mathfrak{a}^{-1}(A)^-)) \supset \mathfrak{a}^{-1}(A)^-.$$

Therefore $\mathfrak{a}$ is continuous by Theorem 3.

Theorem 5. Let $\mathfrak{N}$ and $\mathfrak{M}$ be neighbourhood systems respectively of $\mathfrak{T}$ and $\mathfrak{S}$. In order that a mapping $\mathfrak{a}$ be continuous, it is necessary and sufficient that for any point $x \in R$ and for any open set $U \in \mathfrak{M}$ containing $\mathfrak{a}(x)$, we can find $V \in \mathfrak{N}$ such that $V \ni x$ and $\mathfrak{a}(V) \subset U$.

Proof. If $\mathfrak{a}$ is continuous and $\mathfrak{a}(x) \in U \in \mathfrak{M}$, then we have $x \in \mathfrak{a}^{-1}(U) \in \mathfrak{T}$, and hence we can find $V \in \mathfrak{N}$ such that $x \in V \subset \mathfrak{a}^{-1}(U)$. This relation yields by the relations §15(7) and §15(13)

$$\mathfrak{a}(V) \subset \mathfrak{a}(\mathfrak{a}^{-1}(U)) \subset U.$$

Conversely, if $\mathfrak{a}(x) \in U \in \mathfrak{M}$ implies $x \in V$, $\mathfrak{a}(V) \subset U$ for some $V \in \mathfrak{N}$, then for any $U \in \mathfrak{M}$ subject to $\mathfrak{a}^{-1}(U) \neq 0$, as $\mathfrak{a}(V) \subset U$ implies by the formula §15(11) $V \subset \mathfrak{a}^{-1}(U)$, we have

$$\mathfrak{a}^{-1}(U) = \sum_{\mathfrak{a}^{-1}(U) \supset V \in \mathfrak{N}} V,$$

and hence we conclude that $\mathfrak{a}^{-1}(U) \in \mathfrak{T}$ for every $U \in \mathfrak{M}$. Consequently we obtain by §15(3) that for any $A \in \mathfrak{S}$, if $A \neq 0$, then we have

$$\mathfrak{a}^{-1}(A) = \mathfrak{a}^{-1}\Big(\sum_{A \supset U \in \mathfrak{M}} U\Big) = \sum_{A \supset U \in \mathfrak{M}} \mathfrak{a}^{-1}(U) \in \mathfrak{T}.$$

Therefore $\mathfrak{a}$ is continuous by assumption.

Recalling §9 Theorem 1, we conclude immediately from Theorem 5

Theorem 6. If a mapping $\mathfrak{a}$ is continuous, then for every point set $X \subset R$ and for any point set $A \subset S$ including the image $\mathfrak{a}(X)$, $\mathfrak{a}$ also is continuous as a mapping of the subspace X into the subspace A by the relative topologies.

Theorem 7. If a mapping $\mathfrak{a}$ is continuous, then for every compact set $X \subset R$ its image $\mathfrak{a}(X)$ also is compact.

Proof. If $\mathfrak{a}(X) \subset \sum_{\lambda \in \Lambda} A_\lambda$ for a system of open sets $A_\lambda \in \mathfrak{S}$ ($\lambda \in \Lambda$), then we have by the formulas §15(11) and §15(3)

$$X \subset \mathfrak{a}^{-1}(\mathfrak{a}(X)) \subset \sum_{\lambda \in \Lambda} \mathfrak{a}^{-1}(A_\lambda).$$

As $\mathfrak{a}$ is continuous by assumption, we have $\mathfrak{a}^{-1}(A_\lambda) \in \mathfrak{T}$ for every $\lambda \in \Lambda$ and hence we can find a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \kappa$) such that $X \subset \sum_{\nu=1}^{\kappa} \mathfrak{a}^{-1}(A_{\lambda_\nu})$, because X is compact by assumption. Then we have by the relations §15(8) and §15(13)

$$\mathfrak{a}(X) \subset \sum_{\nu=1}^{\kappa} \mathfrak{a}(\mathfrak{a}^{-1}(A_{\lambda_\nu})) \subset \sum_{\nu=1}^{\kappa} A_{\lambda_\nu}.$$

Therefore $\mathfrak{a}(X)$ is compact by definition.

Theorem 8. If a mapping $\mathfrak{a}$ is continuous, then $A \prec B \subset S$ implies $\mathfrak{a}^{-1}(A) \prec \mathfrak{a}^{-1}(B)$.

Proof. If $A \prec B$, that is, if $A^- \subset B^\circ$, then we have by Theorems 2 and 3

$$\mathfrak{a}^{-1}(A)^- \subset \mathfrak{a}^{-1}(A^-) \subset \mathfrak{a}^{-1}(B^\circ) \subset \mathfrak{a}^{-1}(B)^\circ,$$

and hence $\mathfrak{a}^{-1}(A) \prec \mathfrak{a}^{-1}(B)$ by definition.

Theorem 9. If two continuous mappings $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of R into S coincide in a dense set $X \subset R$, that is, $\mathfrak{a}_1(x) = \mathfrak{a}_2(x)$ for every $x \in X$, and if S is a Hausdorff space by its topology $\mathfrak{S}$, then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ coincide over the whole space R.

Proof. If $\mathfrak{a}_1(x_0) \neq \mathfrak{a}_2(x_0)$ for a point $x_0 \in R$, then we can find two open sets $A_1, A_2 \in \mathfrak{S}$ such that $\mathfrak{a}_1(x_0) \in A_1$, $\mathfrak{a}_2(x_0) \in A_2$, and $A_1 A_2 = 0$, because S is a Hausdorff space by assumption. Then we can find by Theorem 5 two open sets $Y_1, Y_2 \in \mathfrak{T}$ such that $x_0 \in Y_1$, $x_0 \in Y_2$, $\mathfrak{a}_1(Y_1) \subset A_1$, $\mathfrak{a}_2(Y_2) \subset A_2$, and hence $\mathfrak{a}_1(X Y_1 Y_2)\mathfrak{a}_2(X Y_1 Y_2) = 0$. Since $\mathfrak{a}_1$ coincides with $\mathfrak{a}_2$ in X by assumption, we have $\mathfrak{a}_1(X Y_1 Y_2) = \mathfrak{a}_2(X Y_1 Y_2)$. Therefore we obtain $\mathfrak{a}_1(X Y_1 Y_2) = 0$, and hence $X Y_1 Y_2 = 0$, contradicting the assumption that X is dense in R and $Y_1 Y_2 \neq 0$.

Let R_1, R_2, and R_3 be abstract spaces. For two mappings $\mathfrak{a}_1$ of R_1 into R_2 and $\mathfrak{a}_2$ of R_2 into R_3, putting

$$\mathfrak{a}_3(x) = \mathfrak{a}_2(\mathfrak{a}_1(x)) \qquad \text{for every } x \in R_1,$$

we obtain a mapping $\mathfrak{a}_3$ of R_1 into R_3. This mapping $\mathfrak{a}_3$ is called the composition of $\mathfrak{a}_1$ and $\mathfrak{a}_2$, and denoted by $\mathfrak{a}_2\mathfrak{a}_1$. With this definition we see easily that we have

$$(\mathfrak{a}_2\mathfrak{a}_1)^{-1}(A) = \mathfrak{a}_1^{-1}(\mathfrak{a}_2^{-1}(A)) \qquad \text{for every } A \subset R_3.$$

Let R_1, R_2, and R_3 now be topological spaces. We see at once by definition that if both mappings $\mathfrak{a}_1$ of R_1 into R_2 and $\mathfrak{a}_2$ of R_2 into R_3 are continuous, then the composition $\mathfrak{a}_2\mathfrak{a}_1$ also is continuous.

§17 Open mappings, closed mappings

Let R and S be topological spaces respectively with topologies $\mathfrak{T}$ and $\mathfrak{S}$, and $\mathfrak{a}$ a mapping of R onto S, that is, $\mathfrak{a}(R) = S$. If

$$\mathfrak{a}(X) \in \mathfrak{S} \qquad \text{for every } X \in \mathfrak{T},$$

then $\mathfrak{a}$ is said to be open. With this definition we have

Theorem 1. *In order that a mapping $\mathfrak{a}$ be open, it is necessary and sufficient that we have*

$$\mathfrak{a}(X^{\circ}) \subset \mathfrak{a}(X)^{\circ} \qquad \text{for every } X \subset R.$$

Proof. If $\mathfrak{a}$ is open, then we have by definition

$$\mathfrak{a}(X^{\circ}) = \mathfrak{a}(X^{\circ})^{\circ} \qquad \text{for every } X \subset R,$$

and hence we obtain by the relations §4(1) and §15(7)

$$\mathfrak{a}(X^{\circ}) = \mathfrak{a}(X^{\circ})^{\circ} \subset \mathfrak{a}(X)^{\circ}.$$

Conversely, if $\mathfrak{a}(X^{\circ}) \subset \mathfrak{a}(X)^{\circ}$ for every $X \subset R$, then we have by the relations §4(4) and §4(1) for every $X \in \mathfrak{T}$

$$\mathfrak{a}(X) = \mathfrak{a}(X^{\circ}) \subset \mathfrak{a}(X)^{\circ},$$

and hence $\mathfrak{a}(X) = \mathfrak{a}(X)^{\circ} \in \mathfrak{S}$.

For the totality of closed sets $\mathfrak{T}'$ in R and $\mathfrak{S}'$ in S, if

$$\mathfrak{a}(X) \in \mathfrak{S}' \qquad \text{for every } X \in \mathfrak{T}',$$

then $\mathfrak{a}$ is said to be closed. With this definition, we have

Theorem 2. *In order that a mapping $\mathfrak{a}$ be closed, it is necessary and sufficient that we have*

$$\mathfrak{a}(X^{-}) \supset \mathfrak{a}(X)^{-} \qquad \text{for every } X \subset R.$$

Proof. If $\mathfrak{a}$ is closed, then we have by definition $\mathfrak{a}(X^{-}) = \mathfrak{a}(X^{-})^{-}$ for every $X \subset R$, and hence by the formulas §4(1) and §15(7)

$$\mathfrak{a}(X^{-}) = \mathfrak{a}(X^{-})^{-} \supset \mathfrak{a}(X)^{-}.$$

Conversely, if $\mathfrak{a}(X^{-}) \supset \mathfrak{a}(X)^{-}$ for every $X \subset R$, then we have by the relations §4(4) and §4(1) for every $X \in \mathfrak{T}'$

$$\mathfrak{a}(X) = \mathfrak{a}(X^{-}) \supset \mathfrak{a}(X)^{-},$$

and hence $\mathfrak{a}(X) = \mathfrak{a}(X)^{-} \in \mathfrak{S}'$.

Recalling §16 Theorem 4, we conclude from Theorem 2

Theorem 3. *In order that a mapping $\mathfrak{a}$ be continuous and closed, it is necessary and sufficient that we have*

$$\mathfrak{a}(X^{-}) = \mathfrak{a}(X)^{-} \qquad \text{for every } X \subset R.$$

Theorem 4. *If R is compact by $\mathfrak{T}$ and S is a Hausdorff space by $\mathfrak{S}$, then every continuous mapping of R onto S is closed.*

Proof. As R is compact by assumption, every closed set $X \in \mathfrak{T}'$ is compact by §7 Theorem 1. Thus, if a mapping $\mathfrak{a}$ of R onto S is continuous, then we see by §16 Theorem 7 that for every closed set $X \in \mathfrak{T}'$ the image $\mathfrak{a}(X)$ is compact, and hence $\mathfrak{a}(X)$ is closed by §13 Theorem 5, because S is a Hausdorff space by assumption.

Theorem 5. *If the topology $\mathfrak{T}$ of R is normal and there is a continuous and closed mapping $\mathfrak{a}$ of R onto S, then the topology $\mathfrak{S}$ of S also is normal.*

Proof. If $\mathfrak{S} \supset A > B$, then we have $\mathfrak{a}^{-1}(A^{\circ}) \supset \mathfrak{a}^{-1}(B^{-})$. As $\mathfrak{a}$ is continuous by assumption, $\mathfrak{a}^{-1}(A^{\circ})$ is open and $\mathfrak{a}^{-1}(B^{-})$ is closed. As $\mathfrak{T}$ is normal by assumption, there is then a point set $X \subset R$ such that

$$\mathfrak{a}^{-1}(A^{\circ}) > X > \mathfrak{a}^{-1}(B^{-}).$$

For such X, we have by the formulas §15(2)

$$\mathfrak{a}^{-1}(A^{\circ}) \supset X^{-},$$

$$\mathfrak{a}^{-1}(B^{-\prime}) = \mathfrak{a}^{-1}(B^{-})' \supset X^{\circ\prime} = X'^{-},$$

and hence by the relation §15(13)

$$A^{\circ} \supset \mathfrak{a}(X^{-}), \qquad B^{-\prime} \supset \mathfrak{a}(X'^{-}).$$

Since $\mathfrak{a}$ is closed by assumption, we obtain hence by Theorem 2 and §15(14)

$$A^{\circ} \supset \mathfrak{a}(X^{-}) \supset \mathfrak{a}(X)^{-},$$

$$B^{-\prime} \supset \mathfrak{a}(X')^{-} \supset \mathfrak{a}(X)'^{-} = \mathfrak{a}(X)^{\circ},$$

because $\mathfrak{a}(R) = S$. Consequently we have $A > \mathfrak{a}(X) > B$. Therefore $\mathfrak{S}$ is normal by definition.

Theorem 6. *For an open and closed mapping $\mathfrak{a}$ of R onto S, we have that $X > Y$ implies $\mathfrak{a}(X) > \mathfrak{a}(Y)$.*

Proof. If $X^{\circ} \supset Y^{-}$, then we have by Theorems 1 and 2

$$\mathfrak{a}(X)^\circ \supset \mathfrak{a}(X^\circ) \supset \mathfrak{a}(Y^-) \supset \mathfrak{a}(Y)^- .$$

We have obviously by definition

Theorem 7. *In order that a transformation $\mathfrak{a}$ from R to S be open or closed, it is necessary and sufficient that the inverse transformation $\mathfrak{a}^{-1}$ be continuous.*

A continuous transformation $\mathfrak{a}$ from a topological space R to a topological space S is called a *homeomorphism*, if the inverse transformation $\mathfrak{a}^{-1}$ is continuous. A topological space R is said to be *homeomorphic* to a topological space S, if there is a homeomorphism from R to S.

§18 Partition topologies

Let $\mathfrak{a}$ be a mapping of a topological space R with a topology $\mathfrak{T}$ onto an abstract space S. Putting

$$\mathfrak{P} = \{ P : \ \mathfrak{a}^{-1}(P) \in \mathfrak{T} \}$$

we see easily by the relations (1), (2), and (3) in §15 that $\mathfrak{P}$ satisfies the topology conditions, that is, $\mathfrak{P}$ is a topology on S. This topology $\mathfrak{P}$ is called the *partition topology* of S by a mapping $\mathfrak{a}$.

For the partition topology $\mathfrak{P}$ by $\mathfrak{a}$, we see at once by definition that $\mathfrak{a}$ is continuous for every topology $\mathfrak{V} \subset \mathfrak{P}$ on S. Conversely, if $\mathfrak{a}$ is continuous for a topology $\mathfrak{V}$ on S, then we have by definition $\mathfrak{a}^{-1}(A) \in \mathfrak{T}$ for every $A \in \mathfrak{V}$, and hence $\mathfrak{V} \subset \mathfrak{P}$. Therefore we can say that the partition topology $\mathfrak{P}$ of S is the strongest among the topologies for which $\mathfrak{a}$ becomes continuous.

Let R be now an abstract space. A system of point sets $P_\lambda \neq 0$ $(\lambda \in \Lambda)$ is called a *partition* of R, if

$$\sum_{\lambda \in \Lambda} P_\lambda = R \qquad \text{and} \qquad P_\lambda P_\rho = 0 \quad \text{for } \lambda \neq \rho .$$

A partition P_λ $(\lambda \in \Lambda)$ of R may be considered as a space, considering every P_λ as a point. This space is called a *partition space* of R. For a partition space P_λ $(\lambda \in \Lambda)$, putting

$$P_\lambda = \mathfrak{f}(x) \qquad \text{for } P_\lambda \ni x, \quad x \in R,$$

we obtain a mapping $\mathfrak{f}$ of R onto the partition space P_λ $(\lambda \in \Lambda)$. This mapping $\mathfrak{f}$ is called a *partition mapping*.

For a mapping $\mathfrak{a}$ of R onto S, putting

$$P_a = \mathfrak{a}^{-1}(\{a\}) \qquad \text{for every } a \in S,$$

we obtain obviously a partition P_a $(a \in S)$ of R. This partition of R is called the *partition* of R by a mapping $\mathfrak{a}$. For this partition $\mathfrak{a}^{-1}(\{a\})$ $(a \in S)$, putting

$$a = \mathfrak{b}(\mathfrak{a}^{-1}(\{a\})) \qquad \text{for every } a \in S,$$

we obtain a transformation from this partition space to S. If R and S are topological spaces and $\mathfrak{a}$ is continuous, then we see easily that this transformation also is continuous for the partition topology of the partition space.

Let R be a regular space by a topology $\mathfrak{T}$. By virtue of §10 Theorem 1, we see that for two points $x, y \in R$, if $x \in \{y\}^-$, then we have $\{x\}^- \subset \{y\}^-$, and if $x \bar{\in} \{y\}^-$, then $x \in \{y\}^{-\prime}$ and hence $\{x\}^- \subset \{y\}^{-\prime}$, namely $\{x\}^- \{y\}^- = 0$. Consequently we have for every $x, y \in R$

$$\{x\}^- = \{y\}^- \quad \text{or} \quad \{x\}^- \{y\}^- = 0 .$$

Therefore the totality of $\{x\}^-$ for all $x \in R$ constitutes a partition of R. This partition is called the *topological partition* of R by $\mathfrak{T}$. For the topological partition of R, every topological set of R is obviously by §10 Theorem 1 an inverse image of a point set of the topological partition space by the partition mapping. Furthermore, we see also by §10 Theorem 1 that for the partition topology of the topological partition space, the partition mapping is open and closed. Consequently, we conclude by Theorems §13,1; §16,8; and §17,6 that the partition topology of the topological partition space is separative and regular.

§19 Weak topologies

Let $\mathfrak{a}$ be a mapping of an abstract space R into a topological space S with a topology $\mathfrak{V}$. Putting

$$\mathfrak{T}_0 = \{ \mathfrak{a}^{-1}(A) ; \ \ A \in \mathfrak{V} \},$$

we see easily by the formulas (1), (2), and (3) in §15 that $\mathfrak{T}_0$ satisfies the topology conditions, that is, $\mathfrak{T}_0$ is a topology on R. This topology $\mathfrak{T}_0$ is called the weak topology of R by a mapping $\mathfrak{a}$. For the weak topology $\mathfrak{T}_0$ of R by $\mathfrak{a}$, it is evident by definition that $\mathfrak{a}$ becomes continuous, and hence naturally for every topology $\mathfrak{T} \supset \mathfrak{T}_0$ on R, $\mathfrak{a}$ also becomes continuous. Conversely, if $\mathfrak{a}$ is continuous for a topology $\mathfrak{T}$ on R, then we have obviously $\mathfrak{T} \supset \mathfrak{T}_0$ by definition. Therefore the weak topology of R by $\mathfrak{a}$ is the weakest topology on R for which $\mathfrak{a}$ becomes continuous. Furthermore we see easily by definition that for a neighbourhood system $\mathfrak{M}$ of $\mathfrak{T}'$,

$$\{\mathfrak{a}^{-1}(A) : \quad A \in \mathfrak{T}'\}$$

is a neighbourhood system of the weak topology of R by $\mathfrak{a}$.

Let $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) be a system of mappings of an abstract space R into topological spaces S_λ respectively with topologies $\mathfrak{T}'_\lambda$ ($\lambda \in \Lambda$). Corresponding to every $\lambda \in \Lambda$ we obtain the weak topology $\mathfrak{T}_\lambda$ of R by $\mathfrak{a}_\lambda$, as defined just above. For these weak topologies $\mathfrak{T}_\lambda$ ($\lambda \in \Lambda$), putting $\mathfrak{T}_0 = \bigcup_{\lambda\in\Lambda} \mathfrak{T}_\lambda$, we obtain a topology $\mathfrak{T}_0$ on R, as defined in §8. This topology $\mathfrak{T}_0$ is called the weak topology of R by $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$). For the weak topology $\mathfrak{T}_0$ of R by $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$), as $\mathfrak{T}_0 \supset \mathfrak{T}_\lambda$ for every $\lambda \in \Lambda$, each mapping $\mathfrak{a}_\lambda$ is continuous, as mentioned just above. On the other hand, if every $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) is continuous for a topology $\mathfrak{T}$ on R, then we have obviously $\mathfrak{T} \supset \mathfrak{T}_\lambda$ for every $\lambda \in \Lambda$, and hence $\mathfrak{T} \supset \mathfrak{T}_0$. Therefore the weak topology of R by a system of mappings $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) is the weakest among the topologies for which all $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) become continuous. For a neighbourhood system $\mathfrak{M}_\lambda$ of $\mathfrak{T}'_\lambda$ ($\lambda \in \Lambda$), $\{\mathfrak{a}_\lambda^{-1}(A) : \quad A \in \mathfrak{M}_\lambda\}$ is a neighbourhood system of $\mathfrak{T}_\lambda$ for every $\lambda \in \Lambda$, and hence by §8 Theorem 2

$$\{\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) : \quad U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu},\ \lambda_\nu \in \Lambda\ (\nu = 1, 2, \ldots, \varkappa),\ \varkappa = 1, 2, \ldots\}$$

is a neighbourhood system of $\mathfrak{T}_0$. Therefore we can state

Theorem 1. *Let $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) be a system of mappings of an abstract space R into topological spaces S_λ with topologies $\mathfrak{T}'_\lambda$ respectively. For a neighbourhood system $\mathfrak{M}_\lambda$ of $\mathfrak{T}'_\lambda$ ($\lambda \in \Lambda$), a system of point sets*

$$\{\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) : \quad U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu},\ \lambda_\nu \in \Lambda\ \ (\nu = 1, 2, \ldots, \varkappa),\ \varkappa = 1, 2, \ldots\}$$

is a neighbourhood system of the weak topology of R by $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$).

Recalling §9 Theorem 1, we conclude from the Theorem 1

Theorem 2. *Let $\mathfrak{T}$ be the weak topology of R by a system of mappings $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$). For a subspace S of R, the relative topology $\mathfrak{T}^S$ of $\mathfrak{T}$ in S coincides with the weak topology of S by $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$).*

Theorem 3. *The weak topology of R by a system of mappings $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) of R into regular spaces S_λ ($\lambda \in \Lambda$) is regular too.*

Proof. By virtue of §10 Theorem 4, we need only prove that the weak topology $\mathfrak{T}$ of R by a mapping $\mathfrak{a}$ of R into S with a regular topology $\mathfrak{T}'$ is regular. In this case, since we have by definition

$$\mathfrak{T} = \{\mathfrak{a}^{-1}(U) : \quad U \in \mathfrak{T}'\},$$

we see easily by the formulas (1), (2), and (3) in §15 that we have for every point set $A \subset S$

$$\mathfrak{a}^{-1}(A^\circ) = \mathfrak{a}^{-1}(A)^\circ, \qquad \mathfrak{a}^{-1}(A^-) = \mathfrak{a}^{-1}(A)^-.$$

Thus, if $\mathfrak{a}^{-1}(U) \ni x$ for an open set $U \in \mathfrak{T}'$, then $U \ni \mathfrak{a}(x)$, and hence we can find $V \in \mathfrak{T}'$ for which $U \supset V^- \supset V \ni \mathfrak{a}(x)$, because $\mathfrak{T}'$ is regular by assumption. For such $V \in \mathfrak{T}'$ we have

$$\mathfrak{a}^{-1}(U) \supset \mathfrak{a}^{-1}(V)^- \supset \mathfrak{a}^{-1}(V) \ni x.$$

Therefore $\mathfrak{T}$ is regular by definition.

Theorem 4. *Let $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) be a system of mappings of R into compact Hausdorff spaces S_λ with neighbourhood systems $\mathfrak{M}_\lambda$ ($\lambda \in \Lambda$). In order that the weak topology of R by $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) be compact, it is necessary and sufficient that for a system of points $a_\lambda \in S_\lambda$ ($\lambda \in \Lambda$), if*

$$\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) \neq 0$$

for every finite number of open sets $U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu}$ ($\nu = 1, 2, \ldots, \varkappa$) subject to $U_{\lambda_\nu} \ni a_{\lambda_\nu}$ ($\nu = 1, 2, \ldots, \varkappa$), then there is a point $x \in R$ for which

$$a_\lambda = \mathfrak{a}_\lambda(x) \qquad \text{for every } \lambda \in \Lambda.$$

Proof. If the weak topology $\mathfrak{T}$ of R by $\mathfrak{a}_\lambda$ ($\lambda \in \Lambda$) is compact, then for a system of points $a_\lambda \in S_\lambda$ ($\lambda \in \Lambda$) such that $a_{\lambda_\nu} \in U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu}$ ($\nu = 1, 2, \ldots, \varkappa$) implies $\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) \neq 0$, there is by §7 Theorem 3 a point $x \in R$ such that

$$x \in \prod_{a_\lambda \in U_\lambda \in \mathfrak{M}_\lambda} \Big(\prod_{\lambda\in\Lambda} \mathfrak{a}_\lambda^{-1}(U_\lambda)^- \Big).$$

For such a point x, we have by the formulas (9), (13) in §15 and by §16 Theorem 3

$$\mathfrak{a}_\lambda(x) \in \prod_{a_\lambda \in U_\lambda \in \mathfrak{M}_\lambda} \mathfrak{a}_\lambda(\mathfrak{a}_\lambda^{-1}(U_\lambda^-)) \subset \prod_{a_\lambda \in U_\lambda \in \mathfrak{M}_\lambda} U_\lambda^-$$

for every $\lambda \in \Lambda$, and hence $a_\lambda = \mathfrak{a}_\lambda(x)$ for every $\lambda \in \Lambda$, because S_λ is a Hausdorff space for every $\lambda \in \Lambda$ by assumption.

Conversely, we assume that the stated condition is satisfied. Let $X_\gamma \subset R$ $(\gamma \in \Gamma)$ be a system of closed sets by the weak topology $\mathfrak{T}$ of R such that $\prod_{\nu=1}^{\varkappa} X_{\gamma_\nu} \neq 0$ for every finite number of element $\gamma_\nu \in \Gamma$ $(\nu = 1, 2, \ldots, \varkappa)$. By virtue of Maximal Theorem, there is then a maximal system of point sets $\mathfrak{K}$ such that $\mathfrak{K} \ni X_\gamma$ for every $\gamma \in \Gamma$ and $\prod_{\nu=1}^{\varkappa} K_\nu \neq 0$ for every finite number of point sets $K_\nu \in \mathfrak{K}$ $(\nu = 1, 2, \ldots, \varkappa)$. For such a maximal system $\mathfrak{K}$, since

$$0 \neq \mathfrak{a}_\lambda\left(\prod_{\nu=1}^{\varkappa} K_\nu\right) \subset \prod_{\nu=1}^{\varkappa} \mathfrak{a}_\lambda(K_\nu)$$

for every finite number of point sets $K_\nu \in \mathfrak{K}$ $(\nu = 1, 2, \ldots, \varkappa)$, we have by §7 Theorem 3

$$\prod_{K \in \mathfrak{K}} \mathfrak{a}_\lambda(K)^- \neq 0 \quad \text{for every } \lambda \in \Lambda.$$

Thus, there is a system of points $a_\lambda \in S_\lambda$ such that

$$a_\lambda \in \prod_{K \in \mathfrak{K}} \mathfrak{a}_\lambda(K)^- \qquad \text{for every } \lambda \in \Lambda.$$

For such a_λ $(\lambda \in \Lambda)$, if $a_\lambda \in U_\lambda \in \mathfrak{M}_\lambda$ for an arbitrary $\lambda \in \Lambda$, then we have $U_\lambda \mathfrak{a}_\lambda(K) \neq 0$ for every $K \in \mathfrak{K}$. This relation yields by the formula §15(12)

$$\mathfrak{a}_\lambda(\mathfrak{a}_\lambda^{-1}(U_\lambda)K) = U_\lambda \mathfrak{a}_\lambda(K) \neq 0,$$

and hence $\mathfrak{a}_\lambda^{-1}(U_\lambda)K \neq 0$ for every $K \in \mathfrak{K}$. Consequently, $a_\lambda \in U_\lambda \in \mathfrak{M}_\lambda$ implies $\mathfrak{a}_\lambda^{-1}(U_\lambda) \in \mathfrak{K}$, because $\mathfrak{K}$ is a maximal system such that $\prod_{\nu=1}^{\varkappa} K_\nu \neq 0$ for every finite number of point sets $K_\nu \in \mathfrak{K}$ $(\nu = 1, 2, \ldots, \varkappa)$. Thus $a_{\lambda_\nu} \in U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu}$ $(\nu = 1, 2, \ldots, \varkappa)$ implies $\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) \neq 0$. Accordingly there is by assumption a point $x \in R$ for which $a_\lambda = \mathfrak{a}_\lambda(x)$ for every $\lambda \in \Lambda$. For such a point x we have that $\mathfrak{a}_\lambda(x) \in U_\lambda \in \mathfrak{M}_\lambda$ implies $\mathfrak{a}_\lambda^{-1}(U_\lambda) \in \mathfrak{K}$, as proved just above. Therefore $x \in \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu})$, $U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu}$ $(\nu = 1, 2, \ldots, \varkappa)$ implies

$$\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu})K \neq 0 \qquad \text{for every } K \in \mathfrak{K}$$

Since the totality of $\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu})$ for every finite number of open sets $U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu}$ $(\nu = 1, 2, \ldots, \varkappa)$ constitutes by Theorem 1 a neighbourhood system of the weak topology of R by $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$, we obtain hence $x \in K^-$ for every $K \in \mathfrak{K}$, and especially $x \in X_\gamma^- = X_\gamma$ for every $\gamma \in \Gamma$. Therefore R is compact by §7 Theorem 3.

In this Proof we see easily that the stated condition is necessary, when S_λ $(\lambda \in \Lambda)$ are Hausdorff spaces, and sufficient, when S_λ $(\lambda \in \Lambda)$ are compact. Furthermore we can prove likewise

<u>Theorem 5.</u> <u>Let $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$ be a system of mappings of R into compact spaces S_λ with neighbourhood systems $\mathfrak{M}_\lambda$. If for a system of closed sets $A_\lambda \subset S_\lambda$ $(\lambda \in \Lambda)$,</u>

$$\prod_{\nu=1}^{\varkappa} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) \neq 0$$

<u>for every finite number of open sets $U_{\lambda_\nu} \in \mathfrak{M}_{\lambda_\nu}$ $(\nu = 1, 2, \ldots, \varkappa)$ such that we have $U_{\lambda_\nu} \supset A_{\lambda_\nu}$ $(\nu = 1, 2, \ldots, \varkappa)$, implies $\prod_{\lambda \in \Lambda} \mathfrak{a}_\lambda^{-1}(A_\lambda) \neq 0$, then R is compact by the weak topology of R by $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$.</u>

§20 <u>Continuous functions</u>

The totality of real numbers may be considered as a space. This space will be called the <u>number space</u> in the sequel. In the number space, we define an <u>interval</u> (α, β) for two different real numbers $\alpha < \beta$ to mean

$$(\alpha, \beta) = \{\xi : \alpha < \xi < \beta\}$$

and a <u>closed interval</u> $[\alpha, \beta]$ to mean

$$[\alpha, \beta] = \{\xi : \alpha \leqq \xi \leqq \beta\}.$$

Furthermore we define for every real number α

$$(\alpha, +\infty) = \{\xi : \xi > \alpha\}, \qquad [\alpha, +\infty) = \{\xi : \xi \geqq \alpha\},$$

$$(-\infty, \alpha) = \{\xi : \xi < \alpha\}, \qquad (-\infty, \alpha] = \{\xi : \xi \leqq \alpha\}.$$

We see easily that the totality of intervals satisfies the condition in §2 Theorem 2. Thus there exists uniquely a topology in the number space such that the totality of intervals is a neighbourhood system. This topology is called the <u>number topology</u>. For the number topology we see at once by Theorems 1 and 2 in §3 that

$$(\alpha, \beta)^- = [\alpha, \beta], \qquad [\alpha, \beta]^\circ = (\alpha, \beta),$$
$$(\alpha, +\infty)^- = [\alpha, +\infty), \qquad [\alpha, +\infty)^\circ = (\alpha, +\infty),$$
$$(-\infty, \alpha)^- = (-\infty, \alpha], \qquad (-\infty, \alpha]^\circ = (-\infty, \alpha).$$

Therefore we can conclude easily by definition that the number space is a regular Hausdorff space by the number topology. The number topology is completely separable, because the totality of intervals (α, β) for rational numbers α, β constitutes obviously a neighbourhood system of the number topology. The number topology is further locally compact, that is, we have

Theorem 1. Every closed interval is compact by the number topology.

Proof. We suppose that $[\alpha, \beta] \subset \sum_{\lambda \in \Lambda} (\alpha_\lambda, \beta_\lambda)$ but $[\alpha, \beta]$ is covered by no finite number of intervals $(\alpha_\lambda, \beta_\lambda)$ for $\lambda \in \Lambda$. Let ξ_0 be the greatest lower bound of numbers ξ for which $[\alpha, \xi]$ is covered by no finite number of intervals $(\alpha_\lambda, \beta_\lambda)$ for $\lambda \in \Lambda$. Then, we have obviously $\xi_0 \leqq \beta$, and further $\xi_0 > \alpha$, because there is $\lambda \in \Lambda$ for which $\alpha_\lambda < \alpha < \beta_\lambda$. Thus there is $\lambda_0 \in \Lambda$ for which $\alpha_{\lambda_0} < \xi_0 < \beta_{\lambda_0}$, and we can find ξ_1 such that $\mathrm{Max}\{\alpha, \alpha_{\lambda_0}\} < \xi_1 < \xi_0$ and $[\alpha, \xi_1]$ is covered by a finite number of intervals $(\alpha_\lambda, \beta_\lambda)$. Then we have naturally that $[\alpha, \xi]$ is covered by a finite number of intervals $(\alpha_\lambda, \beta_\lambda)$ for every ξ subject to $\xi_0 < \xi < \beta_{\lambda_0}$, contradicting the definition of ξ_0. Therefore $[\alpha, \beta]$ is compact by the number topology.

Let R be an abstract space. A mapping of R into the number space is called a function on R. For a function φ on R, the image $\varphi(x)$ of a point $x \in R$ is called the value of φ at x. A function φ on R is said to be bounded, if

$$-\infty < \inf_{x \in R} \varphi(x) \leqq \sup_{x \in R} \varphi(x) < +\infty,$$

that is, if we can find a positive number γ such that

$$|\varphi(x)| \leqq \gamma \qquad \text{for every } x \in R.$$

We also say that a function φ on R is bounded in a point set A, if φ is bounded in the subspace A.

For two functions φ, ψ on R and for two real numbers α, β, putting

$$\omega(x) = \alpha\varphi(x) + \beta\psi(x) \qquad \text{for every } x \in R,$$

we obtain a function ω on R. This function ω will be denoted by $\alpha\varphi + \beta\psi$. We define likewise $\varphi \cup \psi$ and $\varphi \cap \psi$ to mean

$$\varphi \cup \psi(x) = \mathrm{Max}\{\varphi(x), \psi(x)\},$$
$$\varphi \cap \psi(x) = \mathrm{Min}\{\varphi(x), \psi(x)\}.$$

A sequence of functions φ_ν $(\nu = 1, 2, \dots)$ is said to be convergent to a function φ, if $\lim_{\nu \to \infty} \varphi_\nu(x) = \varphi(x)$ for every $x \in R$, and such a function φ is called a limit of φ_ν $(\nu = 1, 2, \dots)$. A sequence of functions φ_ν $(\nu = 1, 2, \dots)$ is said to be uniformly convergent to φ, if for any positive number ε we can find ν_0 such that

$$|\varphi_\nu(x) - \varphi(x)| \leqq \varepsilon \qquad \text{for every } x \in R \text{ and } \nu \geqq \nu_0.$$

It is evident by definition that the uniform convergence implies the convergence. We also say that φ_ν $(\nu = 1, 2, \dots)$ is convergent or uniformly convergent to φ in a point set A, if it is so in the subspace A.

We see at once by definition that if two sequences of functions φ_ν and ψ_ν $(\nu = 1, 2, \dots)$ are convergent or uniformly convergent to φ and ψ respectively, then all $\alpha\varphi_\nu + \beta\psi_\nu$, $\varphi_\nu \cup \psi_\nu$, $\varphi_\nu \cap \psi_\nu$ $(\nu = 1, 2, \dots)$ are convergent or uniformly convergent respectively to $\alpha\varphi + \beta\psi$, $\varphi \cup \psi$, $\varphi \cap \psi$. Furthermore we can prove easily : in order that a sequence of functions φ_ν $(\nu = 1, 2, \dots)$ be uniformly convergent in a point set A, it is necessary and sufficient that for any $\varepsilon > 0$ we can find ν_0 such that $|\varphi_\nu(x) - \varphi_\mu(x)| \leqq \varepsilon$ for every $x \in A$ and $\nu, \mu \geqq \nu_0$.

Let R be now a topological space with a topology $\mathcal{T}$. A function φ on R is said to be continuous, if φ is so as a mapping of R into the number space with the number topology. With this definition, we have

Theorem 2. In order that a function φ on R be continuous, it is necessary and sufficient that both point sets

$$\{x : \varphi(x) > \alpha\} \quad \text{and} \quad \{x : \varphi(x) < \alpha\}$$

are open, or both point sets

$$\{x : \varphi(x) \geqq \alpha\} \quad \text{and} \quad \{x : \varphi(x) \leqq \alpha\}$$

are closed for every real number α.

Proof. If φ is continuous, then, since both $(\alpha, +\infty)$ and $(-\infty, \alpha)$ are open by the number topology, the inverse image of $(\alpha, +\infty)$, namely $\{x: \varphi(x) > \alpha\}$, and the inverse image of $(-\infty, \alpha)$, namely $\{x: \varphi(x) < \alpha\}$ are open; and further the inverse images of closed sets $[\alpha, +\infty)$ and $(-\infty, \alpha]$: $\{x: \varphi(x) \geqq \alpha\}$ and $\{x: \varphi(x) \leqq \alpha\}$ are closed.

Conversely, if both $\{x: \varphi(x) > \alpha\}$ and $\{x: \varphi(x) < \alpha\}$ are open for every real number α, then the inverse image of every interval (α, β)

$$\{x: \alpha < \varphi(x) < \beta\} = \{x: \varphi(x) > \alpha\}\{x: \varphi(x) < \beta\}$$

is open, and hence φ is continuous by definition, because the totality of intervals (α, β) is a neighbourhood system of the number topology. If both $\{x: \varphi(x) \geqq \alpha\}$ and $\{x: \varphi(x) \leqq \alpha\}$ are closed for every real number α, then both

$$\{x: \varphi(x) < \alpha\} = \{x: \varphi(x) \geqq \alpha\}',$$
$$\{x: \varphi(x) > \alpha\} = \{x: \varphi(x) \leqq \alpha\}'$$

are open for every real number α, and hence φ is continuous, as proved just above.

Since both $[\alpha, +\infty)$ and $(-\infty, \alpha]$ are closed by the number topology, we obtain by §16 Theorem 4

Theorem 3. For a continuous function φ on R, if $\varphi(x) \geqq \alpha$ for $x \in A$, then we also have $\varphi(x) \geqq \alpha$ for $x \in A^-$; and if $\varphi(x) \leqq \alpha$ for $x \in A$, then we also have $\varphi(x) \leqq \alpha$ for $x \in A^-$.

Recalling §16 Theorem 5 we obtain immediately

Theorem 4. Let $\mathfrak{N}$ be a neighbourhood system of $\mathfrak{T}$. In order that a function φ on R be continuous, it is necessary and sufficient that for any $a \in R$ and for any $\varepsilon > 0$ we can find $U \in \mathfrak{N}$ such that $a \in U$ and $|\varphi(x) - \varphi(a)| \leqq \varepsilon$ for every $x \in U$.

By virtue of Theorem 4 we can prove easily

Theorem 5. For two continuous functions φ and ψ on R, all functions $\alpha\varphi + \beta\psi$, $\varphi \cup \psi$, $\varphi \frown \psi$ are continuous for every real numbers α and β.

Theorem 6. If a sequence of continuous functions φ_ν $(\nu = 1, 2, \dots)$

on R is uniformly convergent to a function φ on R, then φ also is continuous.

Proof. For any $\varepsilon > 0$ we can find by assumption ν_0 such that

$$|\varphi_{\nu_0}(x) - \varphi(x)| \leqq \tfrac{1}{3}\varepsilon \quad \text{for every } x \in R,$$

and for each point $a \in R$ we can find by Theorem 4 an open set $U \in \mathfrak{T}$ such that $a \in U$ and

$$|\varphi_{\nu_0}(x) - \varphi_{\nu_0}(a)| \leqq \tfrac{1}{3}\varepsilon \quad \text{for every } x \in U.$$

Then we have for every point $x \in U$

$$|\varphi(x) - \varphi(a)| \leqq |\varphi_{\nu_0}(x) - \varphi(x)| + |\varphi_{\nu_0}(x) - \varphi_{\nu_0}(a)| + |\varphi_{\nu_0}(a) - \varphi(a)| \leqq \varepsilon.$$

Therefore φ is continuous by Theorem 4.

Theorem 7. If a sequence of continuous functions φ_ν $(\nu = 1, 2, \dots)$ on R is uniformly convergent in a dense set A, then φ_ν $(\nu = 1, 2, \dots)$ is uniformly convergent.

Proof. For any $\varepsilon > 0$ we can find by assumption ν_0 such that

$$-\varepsilon \leqq \varphi_\nu(x) - \varphi_\mu(x) \leqq \varepsilon \qquad \text{for every } x \in A \text{ and } \nu, \mu \geqq \nu_0.$$

As $\varphi_\nu - \varphi_\mu$ is continuous, we obtain by Theorem 3

$$-\varepsilon \leqq \varphi_\nu(x) - \varphi_\mu(x) \leqq \varepsilon \qquad \text{for every } x \in R,\ \nu, \mu \geqq \nu_0,$$

because $A^- = R$ by assumption. Therefore φ_ν $(\nu = 1, 2, \dots)$ is uniformly convergent.

A function φ on R is said to be continuous in a point set A, if φ is continuous in the subspace A by the relative topology $\mathfrak{T}^A$. With this definition, it is evident by §16 Theorem 6 that if φ is continuous in a point set A, then φ also is continuous in every point set $B \subset A$.

Recalling §9 Theorem 1, we obtain by Theorem 4

Theorem 8. Let U_λ $(\lambda \in \Lambda)$ be a system of open sets. If a function φ on R is continuous in U_λ for every $\lambda \in \Lambda$, then φ also is continuous in $\sum_{\lambda \in \Lambda} U_\lambda$.

Theorem 9. For a compact set A, every continuous function φ is bounded in A, and we can find a maximum point a and a minimum point b in A, that is, $a, b \in A$ and $\varphi(a) \geqq \varphi(x) \geqq \varphi(b)$ for every $x \in A$.

Proof. For every real number $\xi < \sup_{x \in A} \varphi(x)$, we have obviously

$A\{x: \varphi(x) \geqq \xi\} \neq 0$ and $\{x: \varphi(x) \geqq \xi\}$ is closed by Theorem 2. Since A is compact by assumption, we can find by §7 Theorem 3 a point $a \in A \prod_{\xi < \alpha} \{x: \varphi(x) \geqq \xi\}$ for $\alpha = \sup_{x \in A} \varphi(x)$. For such a point a, we have obviously $\varphi(a) = \sup_{x \in A} \varphi(x)$. We also can prove likewise that there is a point $b \in A$ for which $\varphi(b) = \inf_{x \in A} \varphi(x)$.

Theorem 10. *Let φ_λ $(\lambda \in \Lambda)$ be a system of continuous functions on R. For a compact set A, if $\inf_{\lambda \in \Lambda} \varphi_\lambda(x) = 0$ for every $x \in A$, and for any $\lambda_1, \lambda_2 \in \Lambda$, we can find $\lambda \in \Lambda$ such that $\varphi_{\lambda_1} \frown \varphi_{\lambda_2}(x) \geqq \varphi_\lambda(x)$ for every $x \in A$, then for any $\varepsilon > 0$ we can find $\lambda \in \Lambda$ such that*

$$\varphi_\lambda(x) < \varepsilon \qquad \text{for every } x \in A.$$

Proof. If there is a positive number ε such that

$$A\{x: \varphi_\lambda(x) \geqq \varepsilon\} \neq 0 \qquad \text{for every } \lambda \in \Lambda,$$

then for each finite number of elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \varkappa)$ there is by assumption $\lambda \in \Lambda$ for which we have

$$\operatorname*{Min}_{\nu=1,2,\ldots,\varkappa} \{\varphi_{\lambda_\nu}(x)\} \geqq \varphi_\lambda(x) \qquad \text{for every } x \in A,$$

and hence $A \prod_{\nu=1}^{\varkappa} \{x: \varphi_{\lambda_\nu}(x) \geqq \varepsilon\} \supset A\{x: \varphi_\lambda(x) \geqq \varepsilon\} \neq 0$. Since A is compact by assumption and $\{x: \varphi_\lambda(x) \geqq \varepsilon\}$ is closed by Theorem 2, we can find by §7 Theorem 3 a point $a \in A \prod_{\lambda \in \Lambda} \{x: \varphi_\lambda(x) \geqq \varepsilon\}$. For such a point a, we have obviously $\varphi_\lambda(a) \geqq \varepsilon$ for every $\lambda \in \Lambda$, contradicting the assumption that we have $\inf_{\lambda \in \Lambda} \varphi_\lambda(a) = 0$.

As a special case of Theorem 10, we have: *if a sequence of continuous functions $\varphi_1(x) \geqq \varphi_2(x) \geqq \cdots$ is convergent to 0 in a compact set A, then φ_ν $(\nu = 1, 2, \ldots)$ is uniformly convergent in A to 0.*

§21 Fields of functions

Let R be an abstract space. A function φ on R is said to be a constant, if φ takes merely a single value for every point of R, and a constant will be denoted by its single value.

A collection $\mathfrak{f}$ of bounded functions on R is called a field, if

1) $\mathfrak{f}$ contains a constant 1;

2) $\mathfrak{f} \ni \varphi, \psi$ implies $\mathfrak{f} \ni \alpha\varphi + \beta\psi$ for every real numbers α, β;

3) $\mathfrak{f} \ni \varphi, \psi$ implies $\mathfrak{f} \ni \varphi \smile \psi, \varphi \frown \psi$.

With this definition, we see easily that for any system of fields $\mathfrak{f}_\lambda$ $(\lambda \in \Lambda)$, the intersection $\prod_{\lambda \in \Lambda} \mathfrak{f}_\lambda$ also is a field. The totality of bounded functions on R is obviously a field. Therefore, for any collection of bounded functions $\mathfrak{f}_0$ there exists the least fields containing $\mathfrak{f}_0$, which will be called the field generated by $\mathfrak{f}_0$.

A field $\mathfrak{f}$ is said to be closed, if $\mathfrak{f}$ contains every bounded function which is a limit of some uniformly convergent sequence of functions in $\mathfrak{f}$. We also see likewise that for any collection of bounded functions $\mathfrak{f}_0$ there exists the least closed field containing $\mathfrak{f}_0$, which will be called the closed field generated by $\mathfrak{f}_0$.

Theorem 1. *The closed field generated by a field $\mathfrak{f}$ is composed of bounded functions which are limits of some uniformly convergent sequences of functions in $\mathfrak{f}$.*

Proof. Let $\bar{\mathfrak{f}}$ be the totality of bounded functions which are limits of some uniformly convergent sequences of functions in $\mathfrak{f}$. If φ and ψ are limits of uniformly convergent sequences of functions φ_ν and $\psi_\nu \in \mathfrak{f}$ $(\nu = 1, 2, \ldots)$ respectively, then we have that $\alpha\varphi + \beta\psi$ is a limit of the uniformly convergent sequences $\alpha\varphi_\nu + \beta\psi_\nu \in \mathfrak{f}$ $(\nu = 1, 2, \ldots)$, that is, $\alpha\varphi + \beta\psi \in \bar{\mathfrak{f}}$ for every real numbers α and β, and we obtain likewise $\varphi \smile \psi$, $\varphi \frown \psi \in \bar{\mathfrak{f}}$. Therefore $\bar{\mathfrak{f}}$ is a field. If $\varphi_\nu \in \bar{\mathfrak{f}}$ is uniformly convergent to a function φ, then we can find $\psi_\nu \in \mathfrak{f}$ $(\nu = 1, 2, \ldots)$ such that

$$|\psi_\nu(x) - \varphi_\nu(x)| \leqq \frac{1}{2^\nu} \qquad \text{for every } x \in R,$$

and we see easily that ψ_ν $(\nu = 1, 2, \ldots)$ is uniformly convergent to φ, that is, $\varphi \in \bar{\mathfrak{f}}$. Therefore $\bar{\mathfrak{f}}$ is a closed field. Furthermore it is evident that $\bar{\mathfrak{f}}$ is the least closed field containing $\mathfrak{f}$, that is, $\bar{\mathfrak{f}}$ is the closed field generated by $\mathfrak{f}$.

Let R now be a topological space with a topology $\mathfrak{T}$. The totality of bounded continuous functions on R is obviously by Theorems 5 and 6 in §20 a closed field. Therefore we have

Theorem 2. *For any collection of bounded continuous functions $\mathfrak{f}$*

on R, the closed field generated by $\mathfrak{f}$ is composed only of bounded continuous functions.

Theorem 3. Let R be compact by $\mathfrak{T}$ and $\mathfrak{f}$ a closed field of continuous functions on R. If for any separated points $x, y \in R$ we can find $\varphi \in \mathfrak{f}$ such that $\varphi(x) \neq \varphi(y)$, then $\mathfrak{f}$ contains all continuous functions on R.

Proof. For each pair of separated points a, b we can find by assumption $\varphi \in \mathfrak{f}$ such that $\varphi(a) \neq \varphi(b)$. For such φ, putting

$$\psi = \frac{3}{\varphi(a) - \varphi(b)}\varphi - \frac{3\varphi(b)}{\varphi(a)-\varphi(b)} - 1,$$

we have obviously $\psi \in \mathfrak{f}$ and $\psi(a) = 2$, $\psi(b) = -1$. Therefore for each pair of separated points x, y there is a function $\varphi \in \mathfrak{f}$ such that

$$\varphi(x) = 2, \qquad \varphi(y) = -1.$$

Let A be a closed set different from 0. If $a \overline{\in} A$, then a is separated from every point of A, and hence corresponding to every point $y \in A$ we can find $\varphi_y \in \mathfrak{f}$ such that $\varphi_y(y) = 2$ and $\varphi_y(a) = -1$. For such φ_y ($y \in A$) we have obviously

$$A \subset \sum_{y \in A}\{x : \varphi_y(x) > 1\}.$$

Since A is compact by §7 Theorem 1 and $\{x : \varphi_y(x) > 1\}$ is open by §20 Theorem 2, there is a finite number of points $y_\nu \in A$ ($\nu = 1, 2, \dots, \kappa$) such that

$$A \subset \sum_{\nu=1}^{\kappa}\{x : \varphi_{y_\nu}(x) > 1\}.$$

For such points y_ν ($\nu = 1, 2, \dots, \kappa$), putting

$$\psi = (\varphi_{y_1} \cup \varphi_{y_2} \cup \dots \cup \varphi_{y_\kappa}) \cap 1,$$

we obtain a function $\psi \in \mathfrak{f}$ such that $\psi(a) = -1$, $\psi(x) = 1$ for every $x \in A$, and $\psi(x) \leq 1$ for every $x \in R$. Therefore, for each point $a \overline{\in} A$ there is a function $\varphi \in \mathfrak{f}$ such that $\varphi(a) = -1$, $\varphi(x) = 1$ for every $x \in A$, and $\varphi(x) \leq 1$ for every $x \in R$.

Let A and B be two closed sets different from 0 but subject to $AB = 0$. For each point $y \in B$, we have obviously $y \overline{\in} A$ and hence there is a function $\varphi_y \in \mathfrak{f}$ such that $\varphi_y(y) = -1$, $\varphi_y(x) = 1$ for every $x \in A$, and $\varphi_y(x) \leq 1$ for every $x \in R$, as proved just above. For such φ_y ($y \in B$), we can find likewise a finite number of points $y_\nu \in B$ ($\nu = 1,$

$2, \dots, \kappa$) such that

$$B \subset \sum_{\nu=1}^{\kappa}\{x : \varphi_{y_\nu}(x) < 0\}.$$

Then, putting $\psi = (\varphi_{y_1} \cap \varphi_{y_2} \cap \dots \cap \varphi_{y_\kappa}) \cup 0$, we obtain a function $\psi \in \mathfrak{f}$ such that

$$\psi(x) = \begin{cases} 1 & \text{for every } x \in A, \\ 0 & \text{for every } x \in B, \end{cases}$$

and $0 \leq \psi(x) \leq 1$ for every $x \in R$. Furthermore, we also see that for any closed set $A \neq 0$ there is a function $\varphi \in \mathfrak{f}$ such that we have $0 \leq \varphi(x) \leq 1$ for every $x \in R$ and $\varphi(x) = 1$ for every $x \in A$, or $\varphi(x) = 0$ for every $x \in A$.

Let ψ be an arbitrary continuous function on R. As ψ is bounded by §20 Theorem 9, putting

$$\psi_0 = \psi - \inf_{x \in R}\psi(x), \qquad \alpha = \sup_{x \in R}\psi(x) - \inf_{x \in R}\psi(x),$$

we obtain a continuous function ψ_0 on R and

$$0 \leq \psi_0(x) \leq \alpha \qquad \text{for every } x \in R.$$

If $\alpha = 0$, then ψ is obviously a constant and hence $\psi \in \mathfrak{f}$. Hence we assume that $\alpha > 0$. For every $\kappa = 1, 2, \dots$, since both

$$\{x : \psi_0(x) \geq \tfrac{\nu}{\kappa}\alpha\} \quad \text{and} \quad \{x : \psi_0(x) \leq \tfrac{\nu-1}{\kappa}\alpha\}$$

are closed by §20 Theorem 2, and we have obviously

$$\{x : \psi_0(x) \geq \tfrac{\nu}{\kappa}\alpha\}\{x : \psi_0(x) \leq \tfrac{\nu-1}{\kappa}\alpha\} = 0$$

for every $\nu = 1, 2, \dots, \kappa$, there is $\varphi_\nu \in \mathfrak{f}$ ($\nu = 1, 2, \dots, \kappa$) such that $0 \leq \varphi_\nu(x) \leq 1$ for every $x \in A$ and

$$\varphi_\nu(x) = \begin{cases} 1 & \text{for } \psi_0(x) \geq \tfrac{\nu}{\kappa}\alpha \\ 0 & \text{for } \psi_0(x) \leq \tfrac{\nu-1}{\kappa}\alpha \end{cases}$$

for every $\nu = 1, 2, \dots, \kappa$. Then, putting

$$\varphi = \frac{\alpha}{\kappa}\sum_{\nu=1}^{\kappa}\varphi_\nu,$$

we have obviously that $\varphi \in \mathfrak{f}$ and $\frac{\mu-1}{\kappa}\alpha \leq \psi_0(x) \leq \frac{\mu}{\kappa}\alpha$ implies

$$\varphi_\nu(x) = \begin{cases} 1 & \text{for } \nu \leq \mu - 1 \\ 0 & \text{for } \nu - 1 \geq \mu \end{cases}$$

and hence

$$\varphi(x) = \frac{\alpha}{\kappa}\sum_{\nu=1}^{\mu-1}\varphi_\nu(x) + \frac{\alpha}{\kappa}\varphi_\mu(x) + \frac{\alpha}{\kappa}\sum_{\nu=\mu+1}^{\kappa}\varphi_\nu(x) = \frac{\mu-1}{\kappa}\alpha + \frac{\alpha}{\kappa}\varphi_\mu(x).$$

Thus we see that $\frac{\mu-1}{\kappa}\alpha \leq \psi_0(x) \leq \frac{\mu}{\kappa}\alpha$ implies $\frac{\mu-1}{\kappa}\alpha \leq \varphi(x) \leq \frac{\mu}{\kappa}\alpha$, and consequently we obtain $|\psi_0(x) - \varphi(x)| \leq \frac{\alpha}{\kappa}$ for every $x \in R$. Since

$\mathfrak{F}$ is closed by assumption, we conclude therefore $\psi_0 \in \mathfrak{F}$, and hence we obtain $\psi \in \mathfrak{F}$.

§22 Weak topologies by functions

Let R be an abstract space. For a collection of functions $\mathfrak{F}$ on R, considering every $\varphi \in \mathfrak{F}$ as a mapping of R into the number space with the number topology, we obtain the weak topology of R by $\mathfrak{F}$. This weak topology will be called the weak topology of R by a collection of functions $\mathfrak{F}$ and denoted by $\mathfrak{T}^{\mathfrak{F}}$.

Recalling §19 Theorem 1 we have obviously

Theorem 1. For a collection of functions $\mathfrak{F}$ on R, the totality of point sets $\{x: \alpha_\nu < \varphi_\nu(x) < \beta_\nu \ (\nu = 1, 2, \dots, \varkappa)\}$ for every finite number of functions $\varphi_\nu \in \mathfrak{F}$ and of intervals (α_ν, β_ν) $(\nu = 1, 2, \dots, \varkappa)$ is a neighbourhood system of the weak topology of R by $\mathfrak{F}$.

From Theorem 1 we conclude immediately

Theorem 2. For a collection of functions $\mathfrak{F}$ on R, the totality of point sets $\{x: |\varphi_\nu(x) - \varphi_\nu(a)| < \varepsilon \ (\nu = 1, 2, \dots, \varkappa)\}$ for every finite number of functions $\varphi_\nu \in \mathfrak{F}$ $(\nu = 1, 2, \dots, \varkappa)$ and for every $\varepsilon > 0$ is a neighbourhood system of a point a for the weak topology of R by $\mathfrak{F}$.

Theorem 3. In order that a point $a \in R$ be separated from a point $b \in R$ by the weak topology of R by a collection of functions $\mathfrak{F}$, it is necessary and sufficient that there is a function $\varphi \in \mathfrak{F}$ such that

$$\varphi(a) \neq \varphi(b).$$

Proof. For the weak topology $\mathfrak{T}^{\mathfrak{F}}$ of R, if a point a is separated from a point b by $\mathfrak{T}^{\mathfrak{F}}$, then we can find by Theorem 2 a finite number of functions $\varphi_\nu \in \mathfrak{F}$ $(\nu = 1, 2, \dots, \varkappa)$ and a positive number ε such that $\{x: |\varphi_\nu(x) - \varphi_\nu(a)| < \varepsilon \ (\nu = 1, 2, \dots, \varkappa)\} \bar{\ni} b$, and hence there is ν_0 for which $|\varphi_{\nu_0}(b) - \varphi_{\nu_0}(a)| \geqq \varepsilon$.

Conversely, if $\varphi(a) \neq \varphi(b)$ for some $\varphi \in \mathfrak{F}$, then, putting $\varepsilon = |\varphi(a) - \varphi(b)|$, we have $b \bar{\in} \{x: |\varphi(x) - \varphi(a)| < \frac{1}{2}\varepsilon\}$, and hence a is separated from b by Theorem 2.

It is evident by the definition of weak topologies in §19 that for two collections of functions $\mathfrak{F}_1$ and $\mathfrak{F}_2$, if $\mathfrak{F}_1 \subset \mathfrak{F}_2$, then the weak topology of R by $\mathfrak{F}_1$ is weaker than that of R by $\mathfrak{F}_2$, that is

$$\mathfrak{F}_1 \subset \mathfrak{F}_2 \quad \text{implies} \quad \mathfrak{T}^{\mathfrak{F}_1} < \mathfrak{T}^{\mathfrak{F}_2}.$$

However we have

Theorem 4. For the closed field $\bar{\mathfrak{F}}$ generated by a collection of bounded function $\mathfrak{F}$ on R, the weak topology of R by $\mathfrak{F}$ coincides with that of R by $\bar{\mathfrak{F}}$, that is, $\mathfrak{T}^{\mathfrak{F}} = \mathfrak{T}^{\bar{\mathfrak{F}}}$.

Proof. It is evident by definition that every function $\varphi \in \mathfrak{F}$ is continuous by the weak topology $\mathfrak{T}^{\mathfrak{F}}$. Accordingly, every $\varphi \in \bar{\mathfrak{F}}$ also is continuous by $\mathfrak{T}^{\mathfrak{F}}$ on account of §21 Theorem 2. Therefore we obtain $\mathfrak{T}^{\bar{\mathfrak{F}}} < \mathfrak{T}^{\mathfrak{F}}$ by the definition of weak topologies. On the other hand we conclude $\mathfrak{T}^{\bar{\mathfrak{F}}} > \mathfrak{T}^{\mathfrak{F}}$ from $\bar{\mathfrak{F}} > \mathfrak{F}$. Consequently we have $\mathfrak{T}^{\bar{\mathfrak{F}}} = \mathfrak{T}^{\mathfrak{F}}$.

Let R be a topological space with a topology $\mathfrak{T}$ in the sequel. Recalling the definition of weak topologies in §19, we see that every weak topology of R by continuous functions is weaker than $\mathfrak{T}$. A collection of bounded continuous functions $\mathfrak{F}$ on R is called a trunk of $\mathfrak{T}$, if $\mathfrak{T}$ coincides with the weak topology of R by $\mathfrak{F}$, i.e., if $\mathfrak{T} = \mathfrak{T}^{\mathfrak{F}}$. With this definition we have

Theorem 5. In order that a field of bounded continuous functions $\mathfrak{F}$ on R be a trunk of $\mathfrak{T}$, it is necessary and sufficient that for any $a \in A \in \mathfrak{T}$ we can find $\varphi \in \mathfrak{F}$ such that $\varphi(a) = 1$ and $\varphi(x) = 0$ for every $x \bar{\in} A$.

Proof. If $\mathfrak{T} = \mathfrak{T}^{\mathfrak{F}}$, then for any $a \in A \in \mathfrak{T}$ we can find by Theorem 2 a finite number of functions $\varphi_\nu \in \mathfrak{F}$ $(\nu = 1, 2, \dots, \varkappa)$ and a positive number ε such that

$$\{x: |\varphi_\nu(x) - \varphi_\nu(a)| < \varepsilon \ (\nu = 1, 2, \dots, \varkappa)\} \subset A.$$

For such $\varphi_\nu \in \mathfrak{F}$ $(\nu = 1, 2, \dots, \varkappa)$ and ε, putting

$$\varphi(x) = 1 - \frac{1}{\varepsilon} \operatorname{Min}\{\varepsilon, \max_{\nu=1,2,\dots,\varkappa} |\varphi_\nu(x) - \varphi_\nu(a)|\},$$

we have obviously $\varphi(a) = 1$, $\varphi(x) = 0$ for every $x \bar{\in} A$, and $\varphi \in \mathfrak{F}$, because $\mathfrak{F}$ is a field by assumption.

Conversely, if for any $a \in A \in \mathfrak{T}$ there is a function $\varphi \in \mathfrak{F}$ such that

$\varphi(a) = 1$, $\varphi(x) = 0$ for every $x \bar{\in} A$, then we have obviously --

$$\{x : |\varphi(x) - \varphi(a)| < 1\} \subset A,$$

and hence we conclude by Theorem 2 and §8 Theorem 1 that $\mathfrak{T}^{\mathfrak{f}} \supset \mathfrak{T}$. On the other hand we have $\mathfrak{T}^{\mathfrak{f}} \subset \mathfrak{T}$, because $\mathfrak{f}$ is composed only of continuous functions on R. Therefore we obtain $\mathfrak{T}^{\mathfrak{f}} = \mathfrak{T}$.

Theorem 6. For a trunk $\mathfrak{f}$ of $\mathfrak{T}$, every collection of bounded continuous functions including $\mathfrak{f}$ also is a trunk of $\mathfrak{T}$.

Proof. For a collection of bounded continuous functions $\bar{\mathfrak{f}}$ containing a trunk $\mathfrak{f}$, we have obviously by definition

$$\mathfrak{T}^{\mathfrak{f}} \subset \mathfrak{T}^{\bar{\mathfrak{f}}} \subset \mathfrak{T},$$

and hence $\mathfrak{T}^{\bar{\mathfrak{f}}} = \mathfrak{T}$, because $\mathfrak{T}^{\mathfrak{f}} = \mathfrak{T}$ by assumption.

Theorem 7. If $\mathfrak{T}$ is a weak topology of R by a collection of functions $\mathfrak{f}$, then there is a trunk of $\mathfrak{T}$.

Proof. By virtue of Theorem 1, denoting by $\mathfrak{f}_0$ the totality of bounded functions $(\varphi \smile \alpha) \frown \beta$ for every $\varphi \in \mathfrak{f}$ and for every real numbers $\alpha < \beta$, we see that the weak topology of R by $\mathfrak{f}_0$ coincides with that of R by $\mathfrak{f}$. Therefore $\mathfrak{f}_0$ is a trunk of $\mathfrak{T}$.

Recalling §19 Theorem 2, we have obviously

Theorem 8. For a subspace $S \subset R$, every trunk of $\mathfrak{T}$ also is a trunk of the relative topology $\mathfrak{T}^S$ in S.

§23 Completely regular topologies

Let R be a topological space with a topology $\mathfrak{T}$. A topology $\mathfrak{T}$ is said to be completely regular, if for any $a \in A \in \mathfrak{T}$ we can find a continuous function φ on R such that $\varphi(a) = 1$ and $\varphi(x) = 0$ for every $x \bar{\in} A$.

Theorem 1. If $\mathfrak{T}$ is completely regular, then $\mathfrak{T}$ is regular.

Proof. For any $a \in A \in \mathfrak{T}$, there is by assumption a continuous function φ on R such that $\varphi(a) = 1$ and $\varphi(x) = 0$ for every $x \bar{\in} A$. For such φ, putting $X = \{x : \varphi(x) > \frac{1}{2}\}$, we obtain by Theorems 2 and 3 in §20 an open set X such that

$$a \in X \subset X^- \subset \{x : \varphi(x) \geq \tfrac{1}{2}\} \subset A.$$

Therefore $\mathfrak{T}$ is regular by definition.

Theorem 2. The condition that $\mathfrak{T}$ is completely regular, is equivalent to one of the conditions that $\mathfrak{T}$ coincides with a weak topology of R by functions; that there is a trunk of $\mathfrak{T}$; and that the totality of bounded continuous functions is a trunk of $\mathfrak{T}$.

Proof. By virtue of §22 Theorem 5, $\mathfrak{T}$ is completely regular, if and only if the totality of bounded continuous functions on R is a trunk of $\mathfrak{T}$. Therefore we obtain our assertion by Theorems 6 and 7 in §22.

Recalling §22 Theorem 8, we conclude from Theorem 2

Theorem 3. If $\mathfrak{T}$ is completely regular, then the relative topology $\mathfrak{T}^S$ also is completely regular for every subspace $S \subset R$.

For a function φ on a subspace S of R, if there is a continuous function ψ on R such that $\psi(x) = \varphi(x)$ for every $x \in S$, then ψ is said to be a continuous extension of φ over R. If a function φ on a subspace S has a continuous extension over R, then φ must be by §16 Theorem 6 a continuous function on S by the relative topology $\mathfrak{T}^S$. For a dense set S, if a function φ on S has a continuous extension over R, then we see by §16 Theorem 9 that the continuous extension is uniquely determined.

Theorem 4. (Urysohn) If $\mathfrak{T}$ is normal in an open set A and S is a closed set included in A, then every bounded continuous function φ on S by the relative topology $\mathfrak{T}^S$ has a continuous extension ψ over R, such that $\psi(x) = \sup_{y \in S} \varphi(y)$ for every $x \bar{\in} A$.

Proof. Since the totality of rational numbers is countable, we denote by α_ν ($\nu = 1, 2, \ldots$) the totality of rational numbers in the interval $(\inf_{x \in S} \varphi(x), \sup_{x \in S} \varphi(x))$. Then there is a sequence of open sets $X_\nu \prec A$ ($\nu = 1, 2, \ldots$) such that

$$\{x : \varphi(x) < \alpha_\nu\} \subset X_\nu, \qquad X_\nu^- \{x : \varphi(x) > \alpha_\nu\} = 0,$$

and $X_\nu \prec X_\mu$ for $\alpha_\nu < \alpha_\mu$. Indeed, we suppose that $X_1, \ldots, X_{\nu-1}$ are already determined in such a manner. Since S is closed by assumption, for every real number ξ both point sets

$$\{x : \varphi(x) \geqq \xi\} \quad \text{and} \quad \{x : \varphi(x) \leqq \xi\}$$

are closed sets in R by Theorems 2 in §20 and 8 in §9. Then, since

$$\{x : \varphi(x) < \alpha_\nu\} = \sum_{\rho=1}^{\infty} \{x : \varphi(x) \leqq \alpha_\nu - \tfrac{1}{\rho}\},$$

$$\{x : \varphi(x) > \alpha_\nu\} = \sum_{\rho=1}^{\infty} \{x : \varphi(x) \geqq \alpha_\nu + \tfrac{1}{\rho}\},$$

$$\{x : \varphi(x) < \alpha_\nu\}^- \subset \{x : \varphi(x) \leqq \alpha_\nu\},$$

$$\{x : \varphi(x) > \alpha_\nu\}^- \subset \{x : \varphi(x) \geqq \alpha_\nu\},$$

for a point set B such that $\sum_{\mu<\nu} X_\mu^- + S \prec B \prec A$, we can find by §11 Theorem 2 open sets X_ν and Y_ν such that

$$\sum_{\alpha_\mu<\alpha_\nu,\ \mu<\nu} X_\mu^- + \{x : \varphi(x) < \alpha_\nu\} \prec X_\nu \subset B,$$

$$\sum_{\alpha_\mu>\alpha_\nu,\ \mu<\nu} B^- X_\mu' + \{x : \varphi(x) > \alpha_\nu\} \subset Y_\nu \prec A,$$

and $X_\nu Y_\nu = 0$. For such an open set X_ν we have obviously by the first relation

$$\{x : \varphi(x) < \alpha_\nu\} \subset X_\nu \prec A, \qquad X_\mu \prec X_\nu \quad \text{for } \alpha_\mu < \alpha_\nu,\ \mu < \nu,$$

and by the second relation

$$X_\nu^- \{x : \varphi(x) > \alpha_\nu\} = 0, \qquad X_\mu \succ X_\nu \quad \text{for } \alpha_\mu > \alpha_\nu,\ \mu < \nu,$$

because $X_\nu Y_\nu = 0$ implies $X_\nu^- \subset Y_\nu'$, namely $X_\nu^- Y_\nu = 0$.

For such a sequence of open sets X_ν $(\nu = 1, 2, \ldots)$, putting

$$\psi(x) = \begin{cases} \inf_{X_\nu \ni x} \alpha_\nu & \text{for } x \in \sum_{\nu=1}^{\infty} X_\nu, \\ \sup_{x \in S} \varphi(x) & \text{for } x \,\overline{\in}\, \sum_{\nu=1}^{\infty} X_\nu, \end{cases}$$

we obtain a bounded function ψ on R. This function satisfies obviously for every $\nu = 1, 2, \ldots$

$$\{x : \psi(x) < \alpha_\nu\} \subset X_\nu \subset \{x : \psi(x) \leqq \alpha_\nu\}.$$

Accordingly we have for every real number ξ

$$\{x : \psi(x) < \xi\} = \sum_{\alpha_\nu < \xi} \{x : \psi(x) < \alpha_\nu\}$$

$$= \sum_{\alpha_\nu < \xi} \{x : \psi(x) \leqq \alpha_\nu\} = \sum_{\alpha_\nu < \xi} X_\nu,$$

and hence $\{x : \psi(x) < \xi\}$ is open for every real number ξ. Furthermore we have $\{x : \psi(x) \leqq \xi\} = \prod_{\alpha_\nu > \xi} \{x : \psi(x) < \alpha_\nu\}$

$$= \prod_{\alpha_\nu > \xi} \{x : \psi(x) \leqq \alpha_\nu\} = \prod_{\alpha_\nu > \xi} X_\nu = \prod_{\alpha_\nu > \xi} X_\nu^-,$$

because $\alpha_\nu > \alpha_\mu$ implies $X_\nu \supset X_\mu^- \supset X_\mu$. From this relation we conclude that $\{x : \psi(x) \leqq \xi\}$ is closed, and hence $\{x : \psi(x) > \xi\}$ is open. Therefore ψ is by §20 Theorem 2 a continuous function on R.

This continuous function ψ is a continuous extension of φ. Be-

cause, if $\varphi(x) < \alpha_\nu$ for a point $x \in S$, then we have $x \in X_\nu$, and hence $\psi(x) \leqq \alpha_\nu$, that is, $\varphi(x) < \alpha_\nu$ implies $\psi(x) \leqq \alpha_\nu$. Consequently we have $\varphi(x) \geqq \psi(x)$ for every $x \in S$. On the other hand, if $\psi(x) < \alpha_\nu$ for a point $x \in S$, then we have $x \in X_\nu$, and hence $\varphi(x) \leqq \alpha_\nu$, because $X_\nu \{x : \varphi(x) > \alpha_\nu\} = 0$. Consequently we obtain likewise that $\psi(x) \geqq \varphi(x)$ for every $x \in S$. Therefore we conclude $\varphi(x) = \psi(x)$ for every $x \in S$.

Theorem 5. *If a topology $\mathfrak{T}$ is regular and locally normal, then the topology $\mathfrak{T}$ is completely regular.*

Proof. For $a \in A \in \mathfrak{T}$ we can find by assumption an open set N such that $a \in N \subset A$ and $\mathfrak{T}$ is normal in N. Since $\mathfrak{T}$ is regular by assumption, we have $\{a\}^- \subset N$ by §10 Theorem 1. If $\{a\}^- \neq N$, then there is a point $b \in N$ such that $\{a\}^- \not\ni b$, and we have by §10 Theorem 1

$$\{a\}^- \{b\}^- = 0, \qquad \{a\}^- + \{b\}^- \subset N.$$

For such a and b, putting

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in \{a\}^- \\ 1 & \text{for } x \in \{b\}^- \end{cases}$$

we obtain a bounded continuous function φ on the subspace $\{a\}^- + \{b\}^-$. By virtue of Theorem 4, there is then a continuous function ψ on R such that $\psi(a) = 0$, $\psi(x) = 1$ for every $x \,\overline{\in}\, N$. For such ψ, putting $\omega = 1 - \psi$ we obtain a continuous function ω on R such that $\omega(a) = 1$ and $\omega(x) = 0$ for every $x \,\overline{\in}\, A$. If $\{a\}^- = N$, then putting

$$\omega(x) = \begin{cases} 1 & \text{for } x \in \{a\}^- \\ 0 & \text{for } x \,\overline{\in}\, \{a\}^- \end{cases}$$

we obtain a continuous function ω on R. Therefore $\mathfrak{T}$ is completely regular by definition.

Recalling §12 Theorem 3, we conclude from Theorem 5

Theorem 6. *If a topology $\mathfrak{T}$ is regular and locally compact, then $\mathfrak{T}$ is completely regular.*

Theorem 7. *If a topology $\mathfrak{T}$ is regular and compact, then in order that a collection of bounded continuous functions $\mathfrak{F}$ be a trunk of $\mathfrak{T}$ it is necessary and sufficient that for any pair of separated points x, y there is $\varphi \in \mathfrak{F}$ such that $\varphi(x) \neq \varphi(y)$.*

Proof. The necessity is evident by §22 Theorem 3. If $\mathcal{F}$ satisfies the stated condition, then the closed field generated by $\mathcal{F}$ is by §21 Theorem 3 composed of all continuous functions, and hence we see by Theorem 2 and §22 Theorem 4 that $\mathcal{F}$ is a trunk of $\mathfrak{T}$.

§24 Compactification

For a topological space R with a topology $\mathfrak{T}$, a topological space $\bar{R}$ with a regular compact topology $\bar{\mathfrak{T}}$ is called a compact extension of R, if R is included in $\bar{R}$ and $\mathfrak{T}$ coincides with the relative topology $\bar{\mathfrak{T}}^R$ of $\bar{\mathfrak{T}}$ in R. If a topological space R has a compact extension $\bar{R}$, then $\bar{R}$ is completely regular by §23 Theorem 6, and hence R also must be completely regular by §23 Theorem 3.

Let R be a topological space with a completely regular topology $\mathfrak{T}$, and $\mathcal{F}$ a trunk of $\mathfrak{T}$. A compact extension $\bar{R}$ of R is called a compactification of R by $\mathcal{F}$, if $\bar{R}$ satisfies the compactification conditions:

1) R is dense in $\bar{R}$,

2) every $\varphi \in \mathcal{F}$ has a continuous extension $\bar{\varphi}$ over $\bar{R}$,

3) every adding point $\bar{a} \in \bar{R} - R$ is separated from each other point $\bar{\xi} \in \bar{R}$ by $\mathcal{F}$, that is, we can find $\varphi \in \mathcal{F}$ such that for the continuous extension $\bar{\varphi}$ of φ over $\bar{R}$ we have $\bar{\varphi}(\bar{a}) \neq \bar{\varphi}(\bar{\xi})$.

We shall prove firstly that for every trunk $\mathcal{F}$ of $\mathfrak{T}$ there exists a compactification of R by $\mathcal{F}$. As every $\varphi \in \mathcal{F}$ is bounded, corresponding to each $\varphi \in \mathcal{F}$ we can find two numbers $\alpha_\varphi, \beta_\varphi$ such that

$$\alpha_\varphi \leqq \varphi(x) \leqq \beta_\varphi \qquad \text{for every } x \in R.$$

Considering $\mathcal{F}$ as a space, we denote by $\tilde{A}$ the totality of functions $\tilde{a}$ on $\mathcal{F}$ such that

$$\alpha_\varphi \leqq \tilde{a}(\varphi) \leqq \beta_\varphi \qquad \text{for every } \varphi \in \mathcal{F}$$

and there is no point $x \in R$ for which $\tilde{a}(\varphi) = \varphi(x)$ for every $\varphi \in \mathcal{F}$. If there is no such function $\tilde{a}$ on $\mathcal{F}$, then we assume naturally $\tilde{A} = 0$. Putting $\tilde{R} = R \dotplus \tilde{A}$, we obtain a space $\tilde{R}$ which includes R. Further-

more, corresponding to every $\varphi \in \mathcal{F}$, putting

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{for } x \in R \\ x(\varphi) & \text{for } x \in \tilde{A}, \end{cases}$$

we obtain a function $\tilde{\varphi}$ on $\tilde{R}$. Let $\tilde{\mathfrak{T}}$ be the weak topology of $\tilde{R}$ by $\{\tilde{\varphi} : \varphi \in \mathcal{F}\}$. Then for every system of real numbers ξ_φ ($\varphi \in \mathcal{F}$) such that $\alpha_\varphi \leqq \xi_\varphi \leqq \beta_\varphi$ there is a point $x \in \tilde{R}$ such that

$$\tilde{\varphi}(x) = \xi_\varphi \qquad \text{for every } \varphi \in \mathcal{F},$$

and the closed interval $[\alpha_\varphi, \beta_\varphi]$ is a compact Hausdorff space by the number topology. Accordingly we see by §19 Theorem 4 that $\tilde{\mathfrak{T}}$ is compact. Therefore, putting

$$\bar{R} = R^{\tilde{\mathfrak{T}}-},$$

we obtain by §12 Theorem 1 a compact space $\bar{R}$ by the relative topology $\tilde{\mathfrak{T}}^{\bar{R}}$ of $\tilde{\mathfrak{T}}$ in $\bar{R}$. Since the relative topology $\tilde{\mathfrak{T}}^{\bar{R}}$ is by §19 Theorem 3 the weak topology by $\{\tilde{\varphi} : \varphi \in \mathcal{F}\}$, $\tilde{\mathfrak{T}}^{\bar{R}}$ is regular by §23 Theorem 2, and hence $\bar{R}$ is a compact extension of R. Furthermore R is dense in $\bar{R}$ by §9 Theorem 7. $\tilde{\varphi}$ is obviously a continuous extension of φ over $\bar{R}$. If $\bar{a} \in \bar{R} - R$ and $\bar{a} \neq \bar{\xi} \in \bar{R}$, then there is $\varphi \in \mathcal{F}$ such that

$$\bar{a}(\varphi) \neq \begin{cases} \bar{\xi}(\varphi) & \text{for } \bar{\xi} \in \bar{R} - R \\ \varphi(\bar{\xi}) & \text{for } \bar{\xi} \in R, \end{cases}$$

that is, $\tilde{\varphi}(\bar{a}) \neq \tilde{\varphi}(\bar{\xi})$. Therefore $\bar{R}$ is a compactification of R by $\mathcal{F}$.

Now we can state

Theorem 1. For any trunk $\mathcal{F}$ of $\mathfrak{T}$ we can find a compactification of R by $\mathcal{F}$.

Theorem 2. Let $\bar{R}$ be a compactification of R by a trunk $\mathcal{F}$ of $\mathfrak{T}$. In order that a bounded continuous function φ on R have a continuous extension over $\bar{R}$, it necessary and sufficient that φ be contained in the closed field genrated by $\mathcal{F}$.

Proof. Let $\bar{\varphi}$ be the continuous extension of $\varphi \in \mathcal{F}$ over $\bar{R}$ and $\bar{\mathcal{F}}_0$ the closed field generated by $\{\bar{\varphi} : \varphi \in \mathcal{F}\}$. Recalling §21 Theorem 3, we see by the compactification condition 3) that $\bar{\mathcal{F}}_0$ is composed of all continuous functions on $\bar{R}$. Furthermore we see easily by §20 Theorem

7 that $\mathfrak{F}_0$ is the closed field generated by $\mathfrak{F}$ in R. Therefore we obtain our assertion.

Theorem 3. Let $\bar{R}$ and $\tilde{R}$ be two compactifications of R respectively by trunks $\mathfrak{F}$ and $\mathfrak{G}$ of $\mathfrak{T}$. If the closed field generated by $\mathfrak{F}$ includes $\mathfrak{G}$, then there exists uniquely a continuous mapping α of $\bar{R}$ onto $\tilde{R}$ such that $\alpha(x) = x$ for every $x \in R$, and such a mapping α is closed. Such a mapping α is a transformation from $\bar{R}$ to $\tilde{R}$, if and only if the closed field generated by $\mathfrak{F}$ coincides with that generated by $\mathfrak{G}$.

Proof. By virtue of Theorem 2, we can assume that both $\mathfrak{F}$ and $\mathfrak{G}$ are closed fields and $\mathfrak{F} \supset \mathfrak{G}$. Corresponding to every $\varphi \in \mathfrak{F}$ we obtain by the compactification condition 2) its continuous extension $\bar{\varphi}$ over $\bar{R}$, and corresponding to every $\varphi \in \mathfrak{G}$ its continuous extension $\tilde{\varphi}$ over $\tilde{R}$. For each adding point $\bar{a} \in \bar{R} - R$, we have by the compactification condition 1)

$$R \prod_{\nu=1}^{\kappa} \{\bar{x} : |\bar{\varphi}_\nu(\bar{x}) - \bar{\varphi}_\nu(\bar{a})| < \varepsilon, \ \bar{x} \in \bar{R}\} \neq 0$$

for every finite number of functions $\varphi_\nu \in \mathfrak{G}$ ($\nu = 1, 2, \ldots, \kappa$) and for every positive number ε, and hence also

$$R \prod_{\nu=1}^{\kappa} \{\tilde{x} : |\tilde{\varphi}_\nu(\tilde{x}) - \bar{\varphi}_\nu(\bar{a})| < \varepsilon, \ \tilde{x} \in \tilde{R}\} \neq 0.$$

Since $\tilde{R}$ is compact, we can find by §7 Theorem 3 a point

$$\tilde{a} \in \prod_{\varphi \in \mathfrak{G}, \, \varepsilon > 0} \{\tilde{x} : |\tilde{\varphi}(\tilde{x}) - \bar{\varphi}(\bar{a})| \leq \varepsilon, \ \tilde{x} \in \tilde{R}\}$$

and we have obviously

(*) $\tilde{\varphi}(\tilde{a}) = \bar{\varphi}(\bar{a})$ for every $\varphi \in \mathfrak{G}$,

that is, corresponding to each $\bar{a} \in \bar{R} - R$ we can find $\tilde{a} \in \tilde{R}$ satisfying (*). Such a point $\tilde{a}$ is by the compactification condition 3) uniquely determined and different from every point of R. Conversely we can prove likewise that corresponding to each adding point $\tilde{a} \in \tilde{R} - R$ there is at least a point $\bar{a} \in \bar{R} - R$ satisfying (*). Therefore, assigning to each adding point $\bar{a} \in \bar{R} - R$ such an adding point $\tilde{a} \in \tilde{R} - R$ satisfying (*), we obtain a mapping α of $\bar{R}$ onto $\tilde{R}$ such that $\alpha(x) = x$ for every $x \in R$. This mapping α is continuous. Because we have by (*)

$$\tilde{\varphi}(\alpha(\bar{x})) = \bar{\varphi}(\bar{x}) \quad \text{for every } \varphi \in \mathfrak{G}, \ \bar{x} \in \bar{R},$$

and hence for every finite number of functions $\varphi_\nu \in \mathfrak{G}$ ($\nu = 1, 2, \ldots, \kappa$) and for every positive number ε

$$\alpha^{-1}(\{\tilde{x} : |\tilde{\varphi}_\nu(\tilde{x}) - \tilde{\varphi}_\nu(\alpha(\bar{a}))| < \varepsilon \ (\nu = 1, 2, \ldots, \kappa)\})$$

$$= \{\bar{x} : |\bar{\varphi}_\nu(\bar{x}) - \bar{\varphi}_\nu(\bar{a})| < \varepsilon \ (\nu = 1, 2, \ldots, \kappa)\}.$$

Conversely, if α_1 is a continuous mapping of $\bar{R}$ onto $\tilde{R}$ such that $\alpha_1(x) = x$ for every $x \in R$, then for each function $\varphi \in \mathfrak{G}$, putting

$$\psi(\bar{x}) = \tilde{\varphi}(\alpha_1(\bar{x})) \quad \text{for every } \bar{x} \in \bar{R},$$

we obtain a continuous function ψ on $\bar{R}$ and we have

$$\psi(x) = \tilde{\varphi}(\alpha_1(x)) = \varphi(x) \quad \text{for every } x \in R.$$

Therefore we conclude by §16 Theorem 9

$$\psi(\bar{x}) = \bar{\varphi}(\bar{x}) = \tilde{\varphi}(\alpha(\bar{x})) \quad \text{for every } \bar{x} \in \bar{R},$$

and hence $\alpha_1(\bar{x}) = \alpha(\bar{x})$ by the compactification condition 3).

Such a mapping α is closed. Because for each closed set $\bar{A} \subset \bar{R}$, as $\bar{A}$ is compact by §7 Theorem 1, $\alpha(\bar{A})$ is compact by §16 Theorem 7. Furthermore, as $\alpha(\bar{A}) = \alpha(\sum_{x \in \bar{A}R} \{x\}^- \dotplus \bar{A}(\bar{R} - R)) = \sum_{x \in \bar{A}R} \{x\}^- \dotplus \alpha(\bar{A}(\bar{R} - R))$, $\alpha(\bar{A})$ is a topological set by the compactification condition 3). Thus $\alpha(\bar{A})$ is closed by §10 Theorem 2.

If $\mathfrak{F} = \mathfrak{G}$, then corresponding to each point $\tilde{a} \in \tilde{R}$ there exists uniquely a point $\bar{a} \in \bar{R}$ satisfying (*), as proved just above, and hence α is a transformation from $\bar{R}$ to $\tilde{R}$. Conversely, if such a mapping α is a transformation from $\bar{R}$ to $\tilde{R}$, then the inverse transformation α^{-1} is continuous, because α is closed, as proved just above. Thus for each $\varphi \in \mathfrak{F}$, putting $\psi(\tilde{x}) = \bar{\varphi}(\alpha^{-1}(\tilde{x}))$ for every $\tilde{x} \in \tilde{R}$, we obtain a continuous function ψ on $\tilde{R}$ and $\psi(x) = \bar{\varphi}(\alpha^{-1}(x)) = \varphi(x)$ for every $x \in R$. We conclude therefore $\varphi \in \mathfrak{G}$ by Theorem 2.

From Theorem 1 we conclude immediately

Theorem 4. (Tychonoff) In order that a topological space R has a compact extension, it is necessary and sufficient that R is completely regular.

We see at once by the compactification condition 3)

Theorem 5. Every compactification of R is a Hausdorff space, if R is separated by its topology $\mathfrak{T}$.

CHAPTER IV

UNIFORM SPACES

§25 Connectors

Let R be an abstract space. A correspondence U which assigns to every point $x \in R$ a point set $U(x) \subset R$, is called a connector, if $U(x) \ni x$ for every point $x \in R$. The correspondence which assignes to every point $x \in R$ the point set $\{x\}$, is obviously a connector, which will be called the identical connector and denoted by I.

For two connectors U, V, we shall write $U \geqq V$ or $V \leqq U$, if

$$U(x) \supset V(x) \qquad \text{for every } x \in R.$$

With this definition we have obviously that $U \geqq V$, $V \geqq W$ implies $U \geqq W$, and $I \leqq U$ for every connector U.

For two connectors U, V we define their intersection UV to mean

(1) $\quad UV(x) = U(x)V(x) \qquad$ for every $x \in R$.

The intersection UV is obviously a connector, and we have

(2) $\quad (UV)W = U(VW), \qquad UV = VU,$

(3) $\quad UI = I, \qquad UV \leqq U.$

For two connectors U, V we define their product $U \times V$ to mean

(4) $\quad U \times V(x) = \sum_{y \in U(x)} V(y) \qquad$ for every $x \in R$.

The product $U \times V$ is obviously a connector too, and we see easily that

(5) $\quad (U \times V) \times W = U \times (V \times W),$

(6) $\quad U \times I = I \times U = U,$

(7) $\quad U \leqq U \times V, \qquad V \leqq U \times V,$

(8) $\quad U \leqq V$ implies $U \times W \leqq V \times W, \qquad W \times U \leqq W \times V.$

Furthermore, for an arbitrary point set A and a connector U, we define $A \times U$ to mean

(9) $\quad A \times U = \sum_{x \in A} U(x).$

With this definition we have obviously

(10) $\quad U \times V(x) = U(x) \times V,$

(11) $\quad (A \times U) \times V = A \times (U \times V),$

(12) $\quad (\sum_{\lambda \in \Lambda} A_\lambda) \times U = \sum_{\lambda \in \Lambda} A_\lambda \times U,$

(13) $\quad A \times I = A, \qquad A \subset A \times U,$

(14) $\quad A \subset B$ implies $A \times U \subset B \times U,$

(15) $\quad U \leqq V$ implies $A \times U \subset A \times V.$

For a connector U, we define its inverse U^{-1} to mean

$$U^{-1}(x) = \{y : U(y) \ni x\} \qquad \text{for every } x \in R,$$

that is, we have

(16) $\quad y \in U^{-1}(x)$ is equivalent to $x \in U(y)$.

The inverse U^{-1} also is obviously a connector and we have

(17) $\quad (U^{-1})^{-1} = U,$

(18) $\quad (UV)^{-1} = U^{-1}V^{-1},$

(19) $\quad (U \times V)^{-1} = V^{-1} \times U^{-1},$

(20) $\quad U \leqq V$ implies $U^{-1} \leqq V^{-1}.$

The relation (17) is evident by definition. $y \in (UV)^{-1}(x)$ is equivalent by (16) to $x \in UV(y)$, namely $x \in U(y)V(y)$, which also is by (16) equivalent to $y \in U^{-1}(x)V^{-1}(x)$, namely $y \in U^{-1}V^{-1}(x)$. Hence we obtain the relation (18). $y \in (U \times V)^{-1}(x)$ is equivalent by (16) to $x \in U \times V(y)$, namely $x \in V(z)$, $z \in U(y)$ for some point $z \in R$. This relation is by definition equivalent to $z \in V^{-1}(x)$, $y \in U^{-1}(z)$ for some point $z \in R$, namely $y \in V^{-1} \times U^{-1}(x)$. Thus the relation (19) is proved. If $U \leqq V$, then $y \in U^{-1}(x)$ implies by (16) $x \in U(y) \subset V(y)$, and hence $y \in V^{-1}(x)$. Therefore we obtain the last relation (20).

As $U(x) \ni y$ implies by (16) $x \in U^{-1}(y)$, we obtain by (4)

(21) $\quad U(x) \ni y$ implies $U(x) \subset U^{-1} \times U(y)$.

If $AU(x) \neq 0$, then for a point $y \in AU(x)$ we have by the relation (21) $U(x) \subset U^{-1} \times U(y)$ and hence we obtain by (9)

(22) $\quad AU(x) \neq 0$ implies $U(x) \subset A \times U^{-1} \times U$.

For a connector U, we have

(23) $\quad (A \times U)B = 0$ implies $A(B \times U^{-1}) = 0$.

Becaase, if $x \in A(B \times U^{-1})$, then we can find by (9) a point $y \in R$ such that $x \in U^{-1}(y)$, $y \in B$, and hence we obtain by (16) $y \in U(x)$, $y \in B$, $x \in A$, namely $y \in (A \times U)B$.

A connector $\mathfrak{U}$ is said to be symmetric, if $\mathfrak{U}^{-1} = \mathfrak{U}$. With this definition we have obviously by (17), (18), (19)

Theorem 1. *For any connector* $\mathfrak{U}$, *all connectors* $\mathfrak{U}\mathfrak{U}^{-1}$, $\mathfrak{U} \times \mathfrak{U}^{-1}$, *and* $\mathfrak{U}^{-1} \times \mathfrak{U}$ *are symmetric.*

For a connector $\mathfrak{U}$ we define

$$\mathfrak{U}^0 = I, \quad \mathfrak{U}^\nu = \mathfrak{U}^{\nu-1} \times \mathfrak{U} \qquad \text{for every } \nu = 1, 2, \ldots .$$

With this definition we have

Theorem 2. *For an arbitrary point set* A *and a connector* $\mathfrak{U}$, *putting* $B = \sum_{\nu=1}^{\infty} A \times \mathfrak{U}^\nu$, *we have that* $\mathfrak{V} \leqq \mathfrak{U}$ *implies* $B \times \mathfrak{V} = B$, $B' \times \mathfrak{V}^{-1} = B'$.

Proof. We have by the relations (12), (15), and (11)

$$B \times \mathfrak{V} = \sum_{\nu=1}^{\infty} A \times \mathfrak{U}^\nu \times \mathfrak{V} \subset \sum_{\nu=1}^{\infty} A \times \mathfrak{U}^{\nu+1} \subset B,$$

and hence $B \times \mathfrak{V} = B$ by (13). From $B \times \mathfrak{V} = B$ we conclude $B'(B \times \mathfrak{V}) = 0$, and hence we obtain by (23) $B(B' \times \mathfrak{V}^{-1}) = 0$, namely $B' \times \mathfrak{V}^{-1} \subset B'$. Consequently we have $B' \times \mathfrak{V}^{-1} = B'$ by the relation (13).

Let R be now a topological space. For a connector $\mathfrak{U}$ we define its opener $\mathfrak{U}^\circ$ to mean

$$\mathfrak{U}^\circ(x) = \{\mathfrak{U}(x)\}^\circ \qquad \text{for every } x \in R,$$

and its closure $\mathfrak{U}^-$ to mean

$$\mathfrak{U}^-(x) = \{\mathfrak{U}(x)\}^- \qquad \text{for every } x \in R.$$

Similarly we can define further $\mathfrak{U}^{-\circ}, \mathfrak{U}^{\circ -}, \ldots$. For any connector $\mathfrak{U}$, its closure $\mathfrak{U}^-$ is obviously a connector too, but its opener $\mathfrak{U}^\circ$ is not necessarily so. The opener $\mathfrak{U}^\circ$ is a connector, if and only if every point $x \in R$ is an inner point of $\mathfrak{U}(x)$.

§26 Uniformities

A collection of connectors $\mathcal{U}$ is called a uniformity, if $\mathcal{U}$ satisfies the uniformity conditions:

1) $\mathcal{U} \ni \mathfrak{U} \leqq \mathfrak{V}$ implies $\mathcal{U} \ni \mathfrak{V}$,

2) $\mathcal{U} \ni \mathfrak{U}, \mathfrak{V}$ implies $\mathcal{U} \ni \mathfrak{U}\mathfrak{V}$,

3) For any $\mathfrak{U} \in \mathcal{U}$ we can find $\mathfrak{V} \in \mathcal{U}$ such that $\mathfrak{V}^{-1} \times \mathfrak{V} \leqq \mathfrak{U}$.

For a uniformity $\mathcal{U}$ we have

(4) $\mathcal{U} \ni \mathfrak{U}$ implies $\mathcal{U} \ni \mathfrak{U}^{-1}$.

Because, for any $\mathfrak{U} \in \mathcal{U}$ we can find by the condition 3) $\mathfrak{V} \in \mathcal{U}$ for which $\mathfrak{V}^{-1} \times \mathfrak{V} \leqq \mathfrak{U}$. Thus $\mathfrak{V}^{-1} \leqq \mathfrak{U}$ by the formula §25(7) and hence $\mathfrak{V} \leqq \mathfrak{U}^{-1}$ by the formulas (17) and (20) in §25. Therefore we have $\mathfrak{U}^{-1} \in \mathcal{U}$ by the condition 1). Furthermore, putting $\mathfrak{W} = \mathfrak{V}\mathfrak{V}^{-1}$, we obtain by the formula (7) and Theorem 1 in §25 that $\mathfrak{W} \times \mathfrak{W} \leqq \mathfrak{U}$ and $\mathfrak{W}^{-1} = \mathfrak{W}$. Therefore we have

(5) for any $\mathfrak{U} \in \mathcal{U}$ there is a symmetric connector $\mathfrak{V} \in \mathcal{U}$ such that

$$\mathfrak{V} \times \mathfrak{V} \leqq \mathfrak{U}.$$

From this relation we conclude immediately

(6) for any $\mathfrak{U} \in \mathcal{U}$ there is a symmetric connector $\mathfrak{V} \in \mathcal{U}$ such that

$$\mathfrak{V} \times \mathfrak{V} \times \mathfrak{V} \leqq \mathfrak{U}.$$

The totality of connectors is obviously a uniformity. This uniformity is called the discrete uniformity of a space R. We see at once by the condition 1) that a uniformity $\mathcal{U}$ is the discrete uniformity if and only if $\mathcal{U} \ni I$.

For a uniformity $\mathcal{U}$, a subset $\mathcal{B} \subset \mathcal{U}$ is said to be a basis of $\mathcal{U}$, if for any $\mathfrak{U} \in \mathcal{U}$ we can find $\mathfrak{V} \in \mathcal{B}$ such that $\mathfrak{V} \leqq \mathfrak{U}$. We see easily by the uniformity conditions 2) and 3) that every basis $\mathcal{B}$ of a uniformity $\mathcal{U}$ satisfies the basis conditions:

1) for any $\mathfrak{U}, \mathfrak{V} \in \mathcal{B}$ there is $\mathfrak{W} \in \mathcal{B}$ such that $\mathfrak{W} \leqq \mathfrak{U}\mathfrak{V}$;

2) for any $\mathfrak{U} \in \mathcal{B}$ there is $\mathfrak{V} \in \mathcal{B}$ such that $\mathfrak{V}^{-1} \times \mathfrak{V} \leqq \mathfrak{U}$.

Conversely we have

Theorem 1. *If a collection of connectors* $\mathcal{B}$ *satisfies the basis conditions* 1) *and* 2), *then there exists uniquely a uniformity* $\mathcal{U}$ *such that* $\mathcal{B}$ *is a basis of* $\mathcal{U}$.

Proof. If we denote by $\mathcal{U}$ the totality of connectors $\mathfrak{U}$ such that $\mathfrak{U} \geqq \mathfrak{V}$ for some $\mathfrak{V} \in \mathcal{B}$, then we see easily that $\mathcal{U}$ satisfies the uniformity conditions 1), 2), and 3), that is, $\mathcal{U}$ is a uniformity. Furthermore it is evident by definition that $\mathcal{B}$ is a basis of this uniformity $\mathcal{U}$. The uniqueness of such a uniformity also is evident by definition of basis.

A basis $\mathfrak{B}$ of a uniformity $\mathfrak{U}$ is said to be a *symmetric* basis of $\mathfrak{U}$, if $\mathfrak{B}$ is composed only of symmetric connectors. For any uniformity $\mathfrak{U}$, $\{UU^{-1}: U \in \mathfrak{U}\}$ is obviously a symmetric basis of $\mathfrak{U}$. Therefore we have

Theorem 2. *Every uniformity has a symmetric basis.*

A point $a \in R$ is said to be *separated* from a point $b \in R$ by a uniformity $\mathfrak{U}$, if there is $U \in \mathfrak{U}$ such that $U(a) \not\ni b$. If every point of R is separated from each other point of R by a uniformity $\mathfrak{U}$, then we shall say that R is *separated* by $\mathfrak{U}$, or that $\mathfrak{U}$ is *separative*. A space R associated with a uniformity is called a *uniform space*.

§27 Induced topologies

Let R be a uniform space with a uniformity $\mathfrak{U}$. Denoting by $\mathfrak{T}$ the totality of point sets X such that $x \in X$ implies $U(x) \subset X$ for some $U \in \mathfrak{U}$, we see easily that $\mathfrak{T}$ satisfies the topology conditions, that is, $\mathfrak{T}$ is a topology on R. This topology $\mathfrak{T}$ is called the *induced topology* of R by a uniformity $\mathfrak{U}$ and denoted by $\mathfrak{T}^{\mathfrak{U}}$, that is,

(1) $\mathfrak{T}^{\mathfrak{U}} = \{X: \ x \in X \text{ implies } U(x) \subset X \text{ for some } U \in \mathfrak{U}\}$.

For the induced topology $\mathfrak{T}^{\mathfrak{U}}$ of R by a uniformity $\mathfrak{U}$, we have for every point set A

(2) $A^\circ = \{x: \ A \supset U(x) \text{ for some } U \in \mathfrak{U}\}$.

Because, putting $B = \{x: \ A \supset U(x) \text{ for some } U \in \mathfrak{U}\}$, we have obviously by (1) $A^\circ \subset B \subset A$. For any point $a \in B$ there is $U \in \mathfrak{U}$ such that $U(a) \subset A$. For such U we can find by §26(5) $V \in \mathfrak{U}$ such that $V \times V \leqq U$. Then we have

$$A \supset U(a) \supset V(a) \times V = \sum_{x \in V(a)} V(x),$$

and hence $B \supset V(a)$. Therefore we obtain by (1) $B \in \mathfrak{T}^{\mathfrak{U}}$, and hence $B = A^\circ$.

For each $U \in \mathfrak{U}$ we can find by §26(5) $V \in \mathfrak{U}$ such that $V \times V \leqq U$, and then we have by (2) $V(x) \subset \{U(x)\}^\circ$ for every $x \in R$, that is, $V \leqq U^\circ$. Therefore we obtain

(3) $U^\circ \in \mathfrak{U}$ for every $U \in \mathfrak{U}$.

Thus for each $U \in \mathfrak{U}$ its opener U° is naturally a connector, that is, $U^\circ(x) \ni x$ for every $x \in R$, and hence we obtain by (2)

Theorem 1. *For a basis $\mathfrak{B}$ of a uniformity $\mathfrak{U}$,*

$$\{U^\circ(a): \ U \in \mathfrak{B}\}$$

is a neighbourhood system of a point a for the induced topology $\mathfrak{T}^{\mathfrak{U}}$.

Let $\mathfrak{B}$ be a basis of a uniformity $\mathfrak{U}$. Then we have

(4) $A^- = \prod_{U \in \mathfrak{B}} A \times U = \prod_{U \in \mathfrak{B}} (A \times U)^\circ$.

Because, for each point $a \in A^-$ we have by Theorem 1 $A V(a) \neq 0$ for every $V \in \mathfrak{U}$, and hence we obtain by the formula §25(22)

$$V(a) \subset A \times V^{-1} \times V \quad \text{for every } V \in \mathfrak{U}.$$

Therefore we conclude by the uniformity condition 3) and by (2)

$$A^- \subset (A \times U)^\circ \quad \text{for every } U \in \mathfrak{U}.$$

Conversely, for each point $a \in \prod_{U \in \mathfrak{B}} A \times U$, we have by the relations (9) and (16) in §25 $A U^{-1}(a) \neq 0$ for every $U \in \mathfrak{B}$. Since $\mathfrak{B}$ is a basis of $\mathfrak{U}$, we conclude hence by §26(4) $A U(a) \neq 0$ for every $U \in \mathfrak{U}$, and consequently $a \in A^-$ by (2). Therefore we obtain (4).

Since $A \times U^\circ$ is by the definition §25(9) open for every $U \in \mathfrak{U}$, we obtain by (4)

(5) $A^- \subset A \times U^\circ \subset (A \times U)^\circ$ for every $U \in \mathfrak{U}$.

Putting $A = W(x)$, we obtain hence for any connector W

(6) $W^- \leqq (W \times U)^\circ$ for every $U \in \mathfrak{U}$.

Theorem 2. *For every $U \in \mathfrak{U}$, $\sum_{\nu=1}^{\infty} A \times U^\nu$ is open and closed by the induced topology $\mathfrak{T}^{\mathfrak{U}}$.*

Proof. Putting $B = \sum_{\nu=1}^{\infty} A \times U^\nu$, we have by §25 Theorem 2 $B = B \times U$ and hence $B^- \subset B^\circ$ by the formula (5).

By virtue of formulas (3), (6), and §25(5), we have obviously

Theorem 3. *For any basis $\mathfrak{B}$ of $\mathfrak{U}$, both $\{U^\circ: \ U \in \mathfrak{B}\}$ and $\{U^-: \ U \in \mathfrak{B}\}$ also are basis of $\mathfrak{U}$.*

Such a connector U° is said to be *open*, and U^- *closed*.

Theorem 4. *R is a Hausdorff space by the induced topology $\mathfrak{T}^{\mathfrak{U}}$, if and only if R is separated by a uniformity $\mathfrak{U}$.*

Proof. If R is separated by the induced topology $\mathfrak{T}^{\mathfrak{U}}$, then R is separated by $\mathfrak{U}$ on account of Theorem 1. Conversely, if R is separated by $\mathfrak{U}$, then for each pair of different points x, y we can find by the formula §26(5) a symmetric connector $V \in \mathfrak{U}$ such that $V \times V(x) \not\ni y$, namely $(V(x) \times V)\{y\} = 0$. Then we have by §25(23)

$$V(x)\, V(y) = V(x)\, V^{-1}(y) = 0,$$

and hence naturally $V^{\circ}(x)\, V^{\circ}(y) = 0$. Therefore R is a Hausdorff space by $\mathfrak{T}^{\mathfrak{U}}$.

Theorem 5. Let $\mathfrak{B}$ be a basis of a uniformity $\mathfrak{U}$. For a topology $\mathfrak{T}$ on R, we have $\mathfrak{T} \supset \mathfrak{T}^{\mathfrak{U}}$ if and only if every point x is an inner point of $U(x)$ by $\mathfrak{T}$ for every $U \in \mathfrak{B}$, that is, $x \in \{U(x)\}^{\mathfrak{T}\circ}$ for every $x \in R$ and $U \in \mathfrak{B}$.

Proof. If $\mathfrak{T} \supset \mathfrak{T}^{\mathfrak{U}}$, then we have by §8(1) $\{U(x)\}^{\mathfrak{T}\circ} \supset U^{\circ}(x)$ for every $U \in \mathfrak{U}$, and hence we see by (3) that x is an inner point of $U(x)$ by $\mathfrak{T}$. Conversely, if $x \in \{U(x)\}^{\mathfrak{T}\circ}$ for every $U \in \mathfrak{B}$ and $x \in R$, then for each $U \in \mathfrak{B}$ we can find by (3) $V \in \mathfrak{B}$ such that $U^{\circ} \geqq V$, and hence

$$U^{\circ}(x) \supset \{V(x)\}^{\mathfrak{T}\circ} \ni x \qquad \text{for every } x \in R.$$

Thus we conslude $\mathfrak{T} \supset \mathfrak{T}^{\mathfrak{U}}$ by §8 Theorem 1.

Theorem 6. For a topology $\mathfrak{T}$ on R and a basis $\mathfrak{B}$ of $\mathfrak{U}$, we have $\mathfrak{T} \subset \mathfrak{T}^{\mathfrak{U}}$ if and only if $a \in A \in \mathfrak{T}$ implies $U(x) \subset A$ for some $U \in \mathfrak{B}$.

Proof. If $\mathfrak{T} \subset \mathfrak{T}^{\mathfrak{U}}$, then for $a \in A \in \mathfrak{T}$ we can find by Theorem 1 $U \in \mathfrak{B}$ such that $U^{\circ}(a) \subset A$. For such U we can find further by (3) $V \in \mathfrak{B}$ such that $V \leqq U^{\circ}$, and hence $V(a) \subset A$. Conversely, if $a \in A \in \mathfrak{T}$ implies $U(a) \subset A$ for some $U \in \mathfrak{B}$, then, as $U^{\circ} \leqq U$, we obtain $\mathfrak{T} \subset \mathfrak{T}^{\mathfrak{U}}$ by §8 Theorem 1.

§28 Comparation of uniformities

Let R be an abstract space. For two uniformities $\mathfrak{U}$ and $\mathfrak{V}$ on R, if $\mathfrak{U} \subset \mathfrak{V}$, then we shall say that $\mathfrak{U}$ is weaker than $\mathfrak{V}$ or $\mathfrak{V}$ is stronger than $\mathfrak{U}$. It is evident by the definition of basis that if $\mathfrak{V}$ includes a basis of $\mathfrak{U}$, then $\mathfrak{V}$ is stronger than $\mathfrak{U}$. Furthermore we

see easily by definition

Theorem 1. For a basis $\mathfrak{B}$ of a uniformity $\mathfrak{U}$ and for a basis $\mathfrak{A}$ of another uniformity $\mathfrak{V}$ on R, we have $\mathfrak{U} \subset \mathfrak{V}$, if and only if for each $U \in \mathfrak{B}$ we can find $V \in \mathfrak{A}$ such that $U \geqq V$.

The discrete uniformity is obviously stronger than each other uniformity. The trivial uniformity, which is composed only of a single connector U_0 such that $U_0(x) = R$ for every $x \in R$, is obviously weaker than every other uniformity.

Theorem 2. For a system of uniformities $\mathfrak{U}_\lambda$ $(\lambda \in \Lambda)$ on R there exist both the weakest stronger uniformity $\bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda$ and the strongest weaker uniformity $\bigcap_{\lambda \in \Lambda} \mathfrak{U}_\lambda$.

Proof. If we denote by $\mathfrak{B}_0$ the totality of connectors $U_{\lambda_1} U_{\lambda_2} \cdots U_{\lambda_\kappa}$ for every finite number of connectors $U_{\lambda_\nu} \in \mathfrak{U}_{\lambda_\nu}$, $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \kappa)$, then $\mathfrak{B}_0$ satisfies the basis conditions. Because, if $V_{\lambda_\nu}^{-1} \times V_{\lambda_\nu} \leqq U_{\lambda_\nu}$ $(\nu = 1, 2, \ldots, \kappa)$, then we have by the formulas (7) and (20) in §25

$$(V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_\kappa})^{-1} \times (V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_\kappa}) \leqq V_{\lambda_\nu}^{-1} \times V_{\lambda_\nu} \leqq U_{\lambda_\nu}$$

for every $\nu = 1, 2, \ldots, \kappa$, and hence

$$(V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_\kappa})^{-1} \times (V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_\kappa}) \leqq U_{\lambda_1} U_{\lambda_2} \cdots U_{\lambda_\kappa}.$$

Thus we see by §26 Theorem 1 that there exists uniquely a uniformity $\mathfrak{U}_0$ such that $\mathfrak{B}_0$ is a basis of $\mathfrak{U}_0$. For such a uniformity $\mathfrak{U}_0$, it is evident that $\mathfrak{U}_0 \supset \mathfrak{U}_\lambda$ for every $\lambda \in \Lambda$. For a uniformity $\mathfrak{U}$, if $\mathfrak{U} \supset \mathfrak{U}_\lambda$ for every $\lambda \in \Lambda$, then we have $\mathfrak{U} \supset \mathfrak{U}_0$ by the uniformity condition 2). Consequently $\mathfrak{U}_0$ is the weakest stronger uniformity of $\mathfrak{U}_\lambda$ $(\lambda \in \Lambda)$, namely $\mathfrak{U}_0 = \bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda$.

Let $\mathfrak{V}_\gamma$ $(\gamma \in \Gamma)$ be the system of all uniformities which are weaker than every $\mathfrak{U}_\lambda$ $(\lambda \in \Lambda)$. Putting $\mathfrak{V}_0 = \bigcup_{\gamma \in \Gamma} \mathfrak{V}_\gamma$, we see easily that $\mathfrak{V}_0$ is the strongest weaker uniformity of $\mathfrak{U}_\lambda$ $(\lambda \in \Lambda)$, namely $\mathfrak{V}_0 = \bigcap_{\lambda \in \Lambda} \mathfrak{U}_\lambda$.

In this Proof we find easily

Theorem 3. If $\mathfrak{B}_\lambda$ is a basis of $\mathfrak{U}_\lambda$ for every $\lambda \in \Lambda$, then

$$\{U_{\lambda_1} U_{\lambda_2} \cdots U_{\lambda_\kappa} : \ U_{\lambda_\nu} \in \mathfrak{B}_{\lambda_\nu},\ \lambda_\nu \in \Lambda\ (\nu = 1, 2, \ldots, \kappa),\ \kappa = 1, 2, \ldots\}$$

is a basis of the weakest stronger uniformity $\bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda$.

Theorem 4. If a uniformity $\mathfrak{U}$ is weaker than a uniformity $\mathfrak{V}$

on R, then the induced topology $\mathfrak{T}^{\mathfrak{U}}$ by $\mathfrak{U}$ is weaker than that $\mathfrak{T}^{\mathfrak{V}}$ by $\mathfrak{V}$, that is, $\mathfrak{U} \subset \mathfrak{V}$ implies $\mathfrak{T}^{\mathfrak{U}} \subset \mathfrak{T}^{\mathfrak{V}}$.

Proof. If $a \in A \in \mathfrak{T}^{\mathfrak{U}}$, then we can find by §27 Theorem 1 $U \in \mathfrak{U}$ such that $U^{\circ}(a) \subset A$. As $U^{\circ} \in \mathfrak{U}$ by §27(3), we obtain $\mathfrak{T}^{\mathfrak{U}} \subset \mathfrak{T}^{\mathfrak{V}}$ by §27 Theorem 6, if $\mathfrak{U} \subset \mathfrak{V}$.

Theorem 5. For a system of uniformities $\mathfrak{U}_\lambda$ ($\lambda \in \Lambda$), the induced topology by $\bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda$ coincides with the weakest stronger topology of the induced topologies $\mathfrak{T}^{\mathfrak{U}_\lambda}$ by $\mathfrak{U}_\lambda$ ($\lambda \in \Lambda$), that is, $\mathfrak{T}^{\bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda} = \bigcup_{\lambda \in \Lambda} \mathfrak{T}^{\mathfrak{U}_\lambda}$.

Proof. As $\bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda \supset \mathfrak{U}_\rho$ ($\rho \in \Lambda$), we have obviously by Theorem 4

$$\mathfrak{T}^{\bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda} \supset \bigcup_{\lambda \in \Lambda} \mathfrak{T}^{\mathfrak{U}_\lambda}.$$

For each finite number of connectors $U_{\lambda_\nu} \in \mathfrak{U}_{\lambda_\nu}$ ($\nu = 1, 2, \ldots, \varkappa$), we have by §27(3) $U_{\lambda_\nu}^{\mathfrak{T}^{\mathfrak{U}_{\lambda_\nu}} \circ}(x) \ni x$ ($\nu = 1, 2, \ldots, \varkappa$), and hence $\prod_{\nu=1}^{\varkappa} U_{\lambda_\nu}^{\mathfrak{T}^{\mathfrak{U}_{\lambda_\nu}} \circ}(x) \ni x$. Since $\prod_{\nu=1}^{\varkappa} U_{\lambda_\nu}^{\mathfrak{T}^{\mathfrak{U}_{\lambda_\nu}} \circ}(x) \in \bigcup_{\lambda \in \Lambda} \mathfrak{T}^{\mathfrak{U}_\lambda}$, x is an inner point of $U_{\lambda_1} U_{\lambda_2} \cdots U_{\lambda_\varkappa}(x)$ by $\bigcup_{\lambda \in \Lambda} \mathfrak{T}^{\mathfrak{U}_\lambda}$. Therefore we conclude by Theorem 3 and §27 Theorem 5 $\bigcup_{\lambda \in \Lambda} \mathfrak{T}^{\mathfrak{U}_\lambda} \supset \mathfrak{T}^{\bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda}$. Consequently we obtain our assertion.

§29 Relative uniformities

Let R be an abstract space and S a subspace of R. For a connector U in R we define the induced connector U^S of R in the subspace S to mean $U^S(x) = S\,U(x)$ for every $x \in S$. The induced connector U^S is obviously a connector in S.

Concerning induced connectors, we have obviously by definition

(1) $U \leqq V$ implies $U^S \leqq V^S$,

(2) $(U\,V)^S = U^S V^S$.

For every points $x, y \in S$, $y \in (U^{-1})^S(x)$ is by §25(16) equivalent to $x \in U(y)$, namely $x \in U^S(y)$, and hence further equivalent to $y \in U^{S-1}(x)$. Therefore we have

(3) $(U^{-1})^S = U^{S-1}$.

(4) $(U \times V)^S \geqq U^S \times V^S$,

(5) $(A \times U)^S \supset A^S \times U^S$.

Because, we have by the definition §25(9)

$$(A \times U)^S = S \sum_{x \in A} U(x) \supset \sum_{x \in SA} S\,U(x) = \sum_{x \in A^S} U^S(x)$$

and hence we obtain (5). Putting $A = V(x)$ in (5), we obtain

$$(V \times U)^S \geqq V^S \times U^S,$$

and hence we conclude (4).

For a uniformity $\mathfrak{U}$ on R, we see by the relations (1), (2), (3), and (4) that the system of connectors $\{U^S : U \in \mathfrak{U}\}$ satisfies the uniformity conditions 2) and 3) in S. If $U^S \leqq W$ for a connector W in S, then putting

$$V(x) = \begin{cases} U(x) \dotplus W(x) & \text{for } x \in S, \\ U(x) & \text{for } x \overline{\in} S, \end{cases}$$

we obtain a connector V in R such that $V \geqq U$. For this connector V, we have further for every $x \in S$

$$V^S(x) = U^S(x) \dotplus W(x) = W(x),$$

because $U^S \leqq W$ by assumption. Therefore $\{U^S : U \in \mathfrak{U}\}$ is a uniformity on S. This uniformity is called the relative uniformity of $\mathfrak{U}$ in a subspace S and denoted by $\mathfrak{U}^S$, that is,

(6) $\mathfrak{U}^S = \{U^S : U \in \mathfrak{U}\}$.

With this definition we have obviously by (1)

Theorem 1. For a basis $\mathfrak{B}$ of $\mathfrak{U}$, $\{U^S : U \in \mathfrak{B}\}$ is a basis of the relative uniformity $\mathfrak{U}^S$ of $\mathfrak{U}$ in a subspace S.

Furthermore we have obviously

(7) $\mathfrak{U} \subset \mathfrak{V}$ implies $\mathfrak{U}^S \subset \mathfrak{V}^S$.

Concerning induced topologies we have

Theorem 2. The induced topology by the relative uniformity $\mathfrak{U}^S$ of $\mathfrak{U}$ in S coincides with the relative topology of the induced topology $\mathfrak{T}^{\mathfrak{U}}$ by $\mathfrak{U}$, that is, $\mathfrak{T}^{(\mathfrak{U}^S)} = (\mathfrak{T}^{\mathfrak{U}})^S$.

Proof. For each $U \in \mathfrak{U}$, we have obviously $U^S(x) \supset U^{\circ S}(x) \ni x$ and $U^{\circ S}(x) \in (\mathfrak{T}^{\mathfrak{U}})^S$ by the definition of relative topologies. Thus we have $\mathfrak{T}^{(\mathfrak{U}^S)} \subset (\mathfrak{T}^{\mathfrak{U}})^S$ by §27 Theorem 5. If $a \in A^S \in (\mathfrak{T}^{\mathfrak{U}})^S$, then there is by §9 Theorem 2 and §27(1) $U \in \mathfrak{U}$ for which we have $U^S(a) \subset A^S$. Therefore we have $\mathfrak{T}^{(\mathfrak{U}^S)} \supset (\mathfrak{T}^{\mathfrak{U}})^S$ by §27 Theorem 6. Consequently we obtain our assertion.

§30 Uniformly continuous mappings

Let R and S be two abstract spaces and a a mapping of R into S. For a connector U in S we define the inverse image $a^{-1}U$ of U as

$$a^{-1}U(x) = a^{-1}(U(a(x))) \qquad \text{for every } x \in R.$$

The inverse image $a^{-1}U$ is obviously a connector in R, and we have

(1) $y \in a^{-1}U(x)$ is equivalent to $a(y) \in U(a(x))$.

Recalling the formulas (4) and (5) in §15, we obtain at once

(2) $U \leqq V$ implies $a^{-1}U \leqq a^{-1}V$,

(3) $a^{-1}(UV) = (a^{-1}U)(a^{-1}V)$.

We see by (1) that $y \in a^{-1}V^{-1}(x)$ is equivalent to $a(y) \in V^{-1}(a(x))$, namely $a(x) \in V(a(y))$ by §25(16), and hence to $x \in a^{-1}V(y)$, namely $y \in (a^{-1}V)^{-1}(x)$. Therefore we have

(4) $a^{-1}U^{-1} = (a^{-1}U)^{-1}$.

(5) $a^{-1}(U \times V) \geqq a^{-1}U \times a^{-1}V$,

(6) $a^{-1}(A \times U) \supset a^{-1}(A) \times a^{-1}U$.

Because, for any point $x \in a^{-1}(A) \times a^{-1}U$ there is by §25(9) $y \in R$ for which $x \in a^{-1}U(y)$, $y \in a^{-1}(A)$, namely $a(x) \in A \times U$. Therefore we obtain (6). Putting $A = V(x)$ in (6), we conclude (5). On account of (4) we see at once that $a^{-1}U$ is symmetric for every symmetric connector U.

Let R and S be now uniform spaces with uniformities $\mathfrak{U}$ and $\mathfrak{V}$ respectively. A mapping a of R into S is said to be uniformly continuous, if $a^{-1}U \in \mathfrak{U}$ for every $U \in \mathfrak{V}$.

Theorem 1. For a basis $\mathfrak{L}$ of $\mathfrak{U}$ and a basis $\mathfrak{C}$ of $\mathfrak{V}$, in order that a mapping a of R into S be uniformly continuous, it is necessary and sufficient that for each $U \in \mathfrak{C}$ we can find $V \in \mathfrak{L}$ such that $y \in V(x)$ implies $a(y) \in U(a(x))$.

Proof. If a is uniformly continuous, then for any $U \in \mathfrak{C}$, as $a^{-1}U \in \mathfrak{U}$, there is $V \in \mathfrak{L}$ such that $V \leqq a^{-1}U$. For such $V \in \mathfrak{L}$, $y \in V(x)$ implies $y \in a^{-1}U(x)$, namely $a(y) \in U(a(x))$. Conversely, if the stated condition is satisfied, then for any $V \in \mathfrak{V}$ there is $V_1 \in \mathfrak{C}$ such that $V_1 \leqq V$, and further we can find $U \in \mathfrak{L}$ by assumption

such that $y \in U(x)$ implies $a(y) \in V_1(a(x))$, namely $y \in a^{-1}V_1(x)$, and hence we obtain $U \leqq a^{-1}V_1 \leqq a^{-1}V$ by (2). This relation yields $a^{-1}V \in \mathfrak{U}$ by the uniformity condition 1). Therefore a is uniformly continuous by definition.

Theorem 2. If a mapping a of R into S is uniformly continuous, then a is continuous for the induced topologies $\mathfrak{T}^{\mathfrak{U}}$ and $\mathfrak{T}^{\mathfrak{V}}$.

Proof. By virtue of §27 Theorem 3, $\{U^\circ : U \in \mathfrak{U}\}$ is a basis of $\mathfrak{U}$, and $\{U^\circ : U \in \mathfrak{V}\}$ is a basis of $\mathfrak{V}$. Thus we see by §27 Theorem 1 that $\{U^\circ(x) : U \in \mathfrak{U}\}$ and $\{U^\circ(x) : U \in \mathfrak{V}\}$ are neighbourhood systems of a point x respectively by $\mathfrak{T}^{\mathfrak{U}}$ and $\mathfrak{T}^{\mathfrak{V}}$. Therefore we obtain our assertion by Theorems 1 and 5 in §16.

For a point set $A \subset R$, a mapping a of R into S is said to be uniformly continuous in A, if a is so as a mapping of the subspace A into S for the relative uniformity $\mathfrak{U}^A$ of $\mathfrak{U}$, that is, if

$$(a^{-1}U)^A \in \mathfrak{U}^A \qquad \text{for every } U \in \mathfrak{V}.$$

If a mapping a is uniformly continuous, then a is obviously uniformly continuous in every point set A of R.

Theorem 3. If a mapping a of R into S is continuous for the induced topologies $\mathfrak{T}^{\mathfrak{U}}$ and $\mathfrak{T}^{\mathfrak{V}}$, and uniformly continuous in a dense set A of R, then a is uniformly continuous.

Proof. For any $V \in \mathfrak{V}$ we can find by §27 Theorem 3 and the uniformity condition 3) $V_1 \in \mathfrak{V}$ such that $(V_1^-)^{-1} \times V_1^- \leqq V$. For such V_1, as $(a^{-1}V_1)^A \in \mathfrak{U}^A$ by assumption, there is by §29(6) $U_1 \in \mathfrak{U}$, for which we have $(a^{-1}V_1)^A = U_1^A$. For such $U_1 \in \mathfrak{U}$, we can find further by §22(3) a symmetric $U \in \mathfrak{U}$ such that $U^{-1} \times U \leqq U_1^\circ$. Then for any point $x \in R$, as A is dense by assumption, we can find a point $y \in A$ such that

$$y \in U^\circ(x) \subset U(x).$$

For such $y \in A$, we have by §25(21) and §4(8)

$$U(x) \subset U^{-1} \times U(y) \subset U_1^\circ(y) \subset (A\,U_1(y))^- = (A\,a^{-1}(V_1(a(y))))^-$$
$$\subset A^-\,a^{-1}(V_1(a(y)))^- \subset a^{-1}(V_1^-(a(y))),$$

because $A^- = R$ by assumption and $a^{-1}(X)^- \subset a^{-1}(X^-)$ for every $X \subset S$ by §16 Theorem 3. Thus we have naturally $x \in a^{-1}(V_1^-(a(y)))$, namely

$\alpha(x) \in V_1^-(\alpha(y))$, and hence we obtain by §25(21)

$$V_1^-(\alpha(y)) \subset (V_1^-)^{-1} \times V_1^-(\alpha(x)) \subset V(\alpha(x)).$$

Consequently we have for every $x \in R$

$$U(x) \subset \alpha^{-1}(V(\alpha(x))) = \alpha^{-1}V(x),$$

that is, $U \leqq \alpha^{-1}V$, and hence $\alpha^{-1}V \in \mathfrak{U}$, as $U \in \mathfrak{U}$. Therefore α is uniformly continuous by definition.

§31 Uniformly continuous functions

In the number space, for any positive number ε, putting

$$U_\varepsilon(\xi) = (\xi - \varepsilon, \xi + \varepsilon) \qquad \text{for every real number } \xi,$$

we obtain a symmetric connector U_ε in the number space. With this definition we see easily by §26 Theorem 1 that there exists uniquely a uniformity, of which $\{U_\varepsilon : \varepsilon > 0\}$ is a basis. This uniformity will be called the *number uniformity*.

Let R be a uniform space with a uniformity $\mathfrak{U}$. A function φ on R is said to be *uniformly continuous*, if it is so as a mapping of R into the number space with the number uniformity.

With this definition we have by §30 Theorem 1

Theorem 1. *Let $\mathfrak{B}$ be a basis of $\mathfrak{U}$. In order that a function φ on R be uniformly continuous, it is necessary and sufficient that for any $\varepsilon > 0$ we can find $U \in \mathfrak{B}$ such that $y \in U(x)$ implies*

$$|\varphi(x) - \varphi(y)| < \varepsilon.$$

On account of Theorem 1, every constant is obviously uniformly continuous, and we can prove easily following two Theorems.

Theorem 2. *For two uniformly continuous functions φ and ψ, we obtain uniformly continuous functions $\varphi \smile \psi$, $\varphi \frown \psi$, and $\alpha\varphi + \beta\psi$ for every real numbers α, β.*

Theorem 3. *For a sequence of uniformly continuous functions φ_ν $(\nu = 1, 2, \ldots)$, if it is uniformly convergent to a function φ, then φ is uniformly continuous too.*

Theorem 4. *Let U_ν $(\nu = 1, 2, \ldots)$ be a sequence of symmetric con-*

nectors such that for a symmetric connector U_0 we have

$$U_\nu \geqq U_{\nu+1} \times U_{\nu+1} \qquad (\nu = 0, 1, 2, \ldots)$$

For an arbitrary point set $A \neq 0$, we can find a function φ on the subspace $\sum_{\mu=1}^{\infty} A \times U_0^\mu$ such that for $x \in \sum_{\mu=1}^{\infty} A \times U_0^\mu$

$$y \in U_\nu(x) \quad \text{implies} \quad |\varphi(x) - \varphi(y)| \leqq \frac{1}{2^{\nu-1}},$$

$$0 \leqq \varphi(x) \begin{cases} = 0 & \text{for } x \in A, \\ \geqq \frac{1}{2^\nu} & \text{for } x \bar{\in} A \times U_\nu, \\ \geqq \mu & \text{for } x \bar{\in} A \times U_0^\mu, \\ \leqq \mu & \text{for } x \in A \times U_0^\mu. \end{cases}$$

Proof. Putting $V_\tau = U_0^\mu \times U_1^{\varepsilon_1} \times U_2^{\varepsilon_2} \times \cdots \times U_\varkappa^{\varepsilon_\varkappa}$

$$\varepsilon_\nu = 0, 1 \ (\nu = 1, 2, \ldots, \varkappa), \ \mu = 0, 1, 2, \ldots,$$

$$\tau = \mu + \sum_{\nu=1}^{\varkappa} \frac{\varepsilon_\nu}{2^\nu}.$$

we shall prove firstly by the induction for ν that

$$\tau = \frac{\alpha}{2^\nu}, \quad \tau' = \frac{\alpha+1}{2^\nu}, \quad \alpha \geqq 0 \quad \text{implies} \quad V_{\tau'} \geqq V_\tau \times V_\nu.$$

This relation is valid clearly for $\nu = 0$. We assume hence that this relation holds for $0, 1, \ldots, \nu - 1$. If α is even, then we have obviously $V_{\tau'} = V_\tau \times U_\nu$. If α is odd, then, putting $\alpha = 2\beta + 1$, $\sigma = \frac{\beta}{2^{\nu-1}}$, we have $\tau' = \frac{\beta+1}{2^{\nu-1}}$, and hence $V_{\tau'} \geqq V_\sigma \times V_{\nu-1}$ by assumption. This relation yields by the formulas (5) and (7) in §25

$$V_{\tau'} \geqq V_\sigma \times U_\nu \times U_\nu = V_\tau \times U_\nu,$$

because $\sigma + \frac{1}{2^\nu} = \tau$. Consequently we obtain our assertion. Thus we can conclude further that $\sigma > \tau \geqq 0$ implies $V_\sigma \geqq V_\tau$.

For every point $x \in \sum_{\mu=1}^{\infty} A \times U_0^\mu$, putting

$$\varphi(x) = \inf_{A \times V_\tau \ni x} \tau,$$

we obtain a function φ on the subspace $\sum_{\mu=1}^{\infty} A \times U_0^\mu$. This function φ satisfies obviously

$$0 \leqq \varphi(x) \begin{cases} = 0 & \text{for } x \in A, \\ \leqq \mu & \text{for } x \in A \times U_0^\mu, \\ \geqq \mu & \text{for } x \bar{\in} A \times U_0^\mu, \\ \geqq \frac{1}{2^\nu} & \text{for } x \bar{\in} A \times U_\nu. \end{cases}$$

For each point $x \in \sum_{\mu=1}^{\infty} A \times U_0^\mu$ and for any ν we can find α such that

$$\frac{\alpha}{2^\nu} - \frac{1}{2^\nu} \leqq \varphi(x) < \frac{\alpha}{2^\nu}.$$

Then, putting $\tau = \frac{\alpha}{2^\nu}$ we have obviously $x \in A \times V_\tau$, and hence, if $y \in U_\nu(x)$, then we have by the definition §25(9)

$$y \in A \times \mathfrak{V}_\tau \times \mathfrak{V}_\nu \leqq A \times \mathfrak{V}_{\tau'}$$

for $\tau' = \tau + \frac{1}{2^\nu}$, because $\mathfrak{V}_\tau \times \mathfrak{V}_\nu \leqq \mathfrak{V}_{\tau'}$, as proved just above. Accordingly we have by the definition of φ

$$\varphi(y) \leqq \tau' = \frac{\alpha}{2^\nu} + \frac{1}{2^\nu} \leqq \varphi(x) + \frac{1}{2^{\nu-1}}.$$

Since $\mathfrak{V}_\nu$ is symmetric by assumption, we obtain likewise

$$\varphi(x) \leqq \varphi(y) + \frac{1}{2^{\nu-1}}.$$

Therefore $y \in \mathfrak{V}_\nu(x)$ implies $|\varphi(x) - \varphi(y)| \leqq \frac{1}{2^{\nu-1}}$.

Theorem 5. *For a connector $\mathfrak{V} \in \mathfrak{U}$, if $(A \times \mathfrak{V})B = 0$, then we can find a uniformly continuous function φ on R such that*

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in A, \\ 1 & \text{for } x \in B, \end{cases}$$

and $0 \leqq \varphi(x) \leqq 1$ for every $x \in R$.

Proof. On account of the relation §27(5) we can find a sequence of symmetric connectors $\mathfrak{V}_\nu \in \mathfrak{U}$ $(\nu = 0, 1, 2, \dots)$ such that

$$\mathfrak{V} \geqq \mathfrak{V}_0, \quad \mathfrak{V}_\nu \geqq \mathfrak{V}_{\nu+1} \times \mathfrak{V}_{\nu+1} \quad (\nu = 0, 1, 2, \dots).$$

Then, putting $S = \sum_{\mu=1}^{\infty} A \times \mathfrak{V}^\mu$, we obtain by Theorem 4 a function φ on the subspace S such that

$$\varphi(x) \begin{cases} = 0 & \text{for } x \in A, \\ \geqq 1 & \text{for } x \bar{\in} A \times \mathfrak{V}_0, \end{cases}$$

$\varphi(x) \geqq 0$ for every $x \in S$, and $y \in \mathfrak{V}_\nu(x)$ implies $|\varphi(x) - \varphi(y)| \leqq \frac{1}{2^{\nu-1}}$. If we set further $\varphi(x) = 1$ for $x \bar{\in} S$, then, as $S \times \mathfrak{V}_\nu = S$, $S' \times \mathfrak{V}_\nu = S'$ by §25 Theorem 2, we obtain by Theorem 1 a uniformly continuous function φ on R. Putting $\psi = \varphi \frown 1$, we also obtain by Theorem 2 a uniformly continuous function ψ such that

$$\psi(x) = \begin{cases} 0 & \text{for } x \in A, \\ 1 & \text{for } x \bar{\in} A \times \mathfrak{V}_0, \end{cases}$$

and $0 \leqq \psi(x) \leqq 1$ for every $x \in R$. Since $(A \times \mathfrak{V})B = 0$ by assumption, this function ψ satisfies our requirement.

Theorem 6. *For every uniform space R with a uniformity $\mathfrak{U}$, the induced topology $\mathfrak{T}^{\mathfrak{U}}$ of R coincides with the weak topology of R by the totality of bounded uniformly continuous functions, and hence the induced topology $\mathfrak{T}^{\mathfrak{U}}$ of R is completely regular.*

Proof. For $a \in A \in \mathfrak{T}^{\mathfrak{U}}$, we can find by §27 Theorem 1 and §27(3) $\mathfrak{V} \in \mathfrak{U}$ such that $\mathfrak{V}(a) \subset A$, and hence $(\{a\} \times \mathfrak{V})A' = 0$. Therefore we can find by Theorem 5 a bounded uniformly continuous function φ such that $\varphi(a) = 1$ and $\varphi(x) = 0$ for any $x \bar{\in} A$. Thus $\mathfrak{T}^{\mathfrak{U}}$ coincides by §22 Theorem 5 with the weak topology of R by all bounded uniformly continuous functions, and hence $\mathfrak{T}^{\mathfrak{U}}$ is completely regular by §23 Theorem 2.

For a point set S of R, a function φ on R is said to be *uniformly continuous in* S, if φ is uniformly continuous as a function on the subspace S by the relative uniformity $\mathfrak{U}^S$.

Theorem 7. *If a function φ on R is continuous by the induced topology $\mathfrak{T}^{\mathfrak{U}}$, then φ is uniformly continuous in every compact set.*

Proof. By virtue of Theorems 2 in §29 and 2 in §12, we need only prove the case where R is compact by the induced topology $\mathfrak{T}^{\mathfrak{U}}$. If φ is continuous by $\mathfrak{T}^{\mathfrak{U}}$, then for any $\varepsilon > 0$ and for each point $x \in R$ we can find by Theorems 4 in §20 and 1 in §27 $\mathfrak{V}_x^\circ \in \mathfrak{U}$ such that

$$\mathfrak{V}_x^\circ(x) \ni y \quad \text{implies} \quad |\varphi(x) - \varphi(y)| < \tfrac{1}{2}\varepsilon.$$

For such $\mathfrak{V}_x^\circ$ there is by §26(5) a symmetric $\mathfrak{V}_x \in \mathfrak{U}$ for which we have $\mathfrak{V}_x \times \mathfrak{V}_x \leqq \mathfrak{V}_x^\circ$. Then we have obviously $R = \sum_{x \in R} \mathfrak{V}_x^\circ(x)$. Since R is compact by assumption, we can find a finite number of points $x_\nu \in R$ $(\nu = 1, 2, \dots, \kappa)$ such that

$$R = \sum_{\nu=1}^{\kappa} \mathfrak{V}_{x_\nu}^\circ(x_\nu) = \sum_{\nu=1}^{\kappa} \mathfrak{V}_{x_\nu}(x_\nu).$$

For such $\mathfrak{V}_{x_\nu}$ $(\nu = 1, 2, \dots, \kappa)$, putting $\mathfrak{V} = \mathfrak{V}_{x_1}\mathfrak{V}_{x_2}\cdots\mathfrak{V}_{x_\kappa}$, we obtain a symmetric $\mathfrak{V} \in \mathfrak{U}$. For this $\mathfrak{V} \in \mathfrak{U}$, if $\mathfrak{V}(x) \ni y$, then there is x_ν for which $\mathfrak{V}_{x_\nu}(x_\nu) \ni x$, and hence we obtain by the relation §25(21)

$$y \in \mathfrak{V}_{x_\nu} \times \mathfrak{V}_{x_\nu}(x_\nu) \subset \mathfrak{V}_{x_\nu}^\circ(x_\nu).$$

As $x \in \mathfrak{V}_{x_\nu}(x_\nu) \subset \mathfrak{V}_{x_\nu}^\circ(x_\nu)$ by the formula §25(7), we obtain hence

$$|\varphi(x_\nu) - \varphi(x)| < \tfrac{1}{2}\varepsilon, \quad |\varphi(x_\nu) - \varphi(y)| < \tfrac{1}{2}\varepsilon,$$

and consequently $|\varphi(x) - \varphi(y)| < \varepsilon$. Therefore φ is uniformly continuous by Theorem 1.

Let S be a subspace of R. For a function φ on S, a uniformly continuous function ψ on R is said to be a *uniformly continuous extension* of φ over R, if $\varphi(x) = \psi(x)$ for every $x \in S$.

Theorem 8. Every bounded uniformly continuous function φ on a subspace S of R with the relative uniformity $\mathfrak{U}^S$ has a uniformly continuous extension over R.

Proof. On account of Theorem 2, we can assume that

$$\sup_{x\in S}\varphi(x)=1, \qquad \inf_{x\in S}\varphi(x)=-1.$$

Since φ is uniformly continuous by the relative uniformity $\mathfrak{U}^S$, we can find by Theorem 1 $U\in\mathfrak{U}$ such that

$$U(x)\ni y,\quad x,y\in S, \text{ implies } |\varphi(x)-\varphi(y)|<\frac{1}{3}.$$

For such $U\in\mathfrak{U}$, putting

$$A=\{x:\ \varphi(x)<-\tfrac{1}{3}\},\quad B=\{x:\ \varphi(x)>\tfrac{1}{3}\},$$

we have $(A\times U)B=0$, because for each $y\in A\times U$ there is by §25(9) $x\in A$ such that $y\in U(x)$, $x\in A$, and hence

$$\varphi(x)<-\frac{1}{3},\qquad |\varphi(x)-\varphi(y)|<\frac{1}{3}$$

which yields $\varphi(y)<0$. Thus we can find by Theorem 5 a uniformly continuous function ψ on R such that

$$\psi(x)=\begin{cases}0 & \text{for } x\in A\\ 1 & \text{for } x\in B\end{cases}$$

and $0\leqq\psi(x)\leqq 1$ for every $x\in R$. For such ψ, putting

$$\psi_1=\frac{2}{3}\psi-\frac{1}{3},$$

we obtain by Theorem 2 a uniformly continuous function ψ_1 on R, and further, putting

$$\varphi_1(x)=\frac{3}{2}(\varphi(x)-\psi_1(x)) \qquad \text{for } x\in S,$$

we obtain a uniformly continuous function φ_1 on S by the relative uniformity $\mathfrak{U}^S$ such that

$$\sup_{x\in S}\varphi_1(x)=\sup_{x\in B}\varphi_1(x)=1,$$
$$\inf_{x\in S}\varphi_1(x)=\inf_{x\in A}\varphi_1(x)=-1.$$

Similarly, we obtain consecutively uniformly continuous functions ψ_ν on R and φ_ν on S ($\nu=2,3,\ldots$) such that for every $\nu=2,3,\ldots$ we have

$$\varphi_\nu(x)=\frac{3}{2}(\varphi_{\nu-1}(x)-\psi_\nu(x)) \qquad \text{for every } x\in S,$$
$$|\varphi_\nu(x)|\leqq 1 \qquad \text{for every } x\in S,$$
$$|\psi_\nu(x)|\leqq\frac{1}{3} \qquad \text{for every } x\in R.$$

Then we have for every $\nu=1,2,\ldots$ and for each $x\in S$

$$\varphi(x)=\sum_{\mu=1}^{\nu}(\tfrac{2}{3})^{\mu-1}\psi_\mu(x)+(\tfrac{2}{3})^{\nu-1}\varphi_\nu(x).$$

By virtue of Theorem 3, putting

$$\psi(x)=\sum_{\mu=1}^{\infty}(\tfrac{2}{3})^{\mu-1}\psi_\mu(x) \qquad \text{for every } x\in R,$$

we obtain a uniformly continuous function ψ on R. For this function ψ we have obviously $\varphi(x)=\psi(x)$ for every $x\in S$. Therefore ψ is a uniformly continuous extension of φ over R.

Remark. In Theorem 8 we can not take off the assumption that φ is bounded. For instance, putting

$$\varphi(\nu)=\nu^2 \qquad (\nu=1,2,\ldots)$$

we obtain a uniformly continuous function φ on the subspace $\{1,2,\ldots\}$ of the number space with the number uniformity. Such φ has no uniformly continuous extension over the number space. Because, for each uniformly continuous function φ on the number space we see easily that there is a positive number γ such that

$$|\varphi(x+\nu)-\varphi(x)|\leqq\gamma\nu \qquad \text{for every } \nu=1,2,\ldots \text{ and } x\in R.$$

§32 Bounded sets

Let R be a uniform space with a uniformity $\mathfrak{U}$. A point set A of R is said to be bounded, if for any $U\in\mathfrak{U}$ we can find a natural number σ and a finite number of points $a_\nu\in R$ ($\nu=1,2,\ldots,\kappa$) such that

$$A\subset\sum_{\nu=1}^{\kappa}U^\sigma(a_\nu).$$

Here $a_\nu\in R$ ($\nu=1,2,\ldots,\kappa$) may be found in A. Because, if we have $A\subset\sum_{\nu=1}^{\kappa}U^\sigma(a_\nu)$ for a symmetric $U\in\mathfrak{U}$, then we obtain by the formula §25(21) $A\subset\sum_{\nu=1}^{\kappa}U^{2\sigma}(x_\nu)$ for $x_\nu\in AU^\sigma(a_\nu)$ ($\nu=1,2,\ldots,\kappa$).

Theorem 1. If a point set A is bounded, then its closure A^- by the induced topology $\mathfrak{T}^{\mathfrak{U}}$ also is bounded.

Proof. For any $U\in\mathfrak{U}$ we can find by the relation §26(5) a symmetric $V\in\mathfrak{U}$ for which $V\times V\leqq U$. As A is bounded by assumption, we can find σ and a_ν ($\nu=1,2,\ldots,\kappa$) such that $A\subset\sum_{\nu=1}^{\kappa}V^\sigma(a_\nu)$. We have then by the formulas §27(5) and §25(12)

$$A^-\subset A\times V\subset\sum_{\nu=1}^{\kappa}V^{\sigma+1}(a_\nu).$$

Thus A^- is bounded by definition.

Theorem 2. In order that a point set A be bounded, it is necessary and sufficient that every uniformly continuous function on R be bounded in A.

Proof. Since the necessity is evident by definition, we shall prove the sufficiency. If a point set A is not bounded, then we can find by definition a symmetric $U_0 \in \mathfrak{U}$ such that

$$A \neq A \sum_{\nu=1}^{\kappa} U_0^{\sigma}(a_\nu)$$

for every $\sigma = 1, 2, \ldots$ and for every finite number of point $a_\nu \in R$ ($\nu = 1, 2, \ldots, \kappa$).

For such $U_0 \in \mathfrak{U}$, we can consider two cases. In the first case where we can find a point $a \in A$ such that

$$A \sum_{\nu=1}^{\infty} U_0^{\nu}(a) \neq A \sum_{\nu=1}^{\mu} U_0^{\nu}(a) \qquad \text{for every } \mu = 1, 2, \ldots,$$

considering by the relation §26(5) a sequence of symmetric vicinities $U_\nu \in \mathfrak{U}$ ($\nu = 1, 2, \ldots$) such that

$$U_\nu \geqq U_{\nu+1} \times U_{\nu+1} \qquad (\nu = 0, 1, 2, \ldots),$$

we obtain by §31 Theorem 4 a function φ on the subspace $\sum_{\nu=1}^{\infty} U_0^{\nu}(a)$ such that $U_\nu(x) \ni y$ implies $|\varphi(x) - \varphi(y)| < \frac{1}{2^{\nu-1}}$, and

$$\varphi(x) \geqq \mu \qquad \text{for } x \overline{\in} \sum_{\nu=1}^{\mu} U_0^{\nu}(a).$$

For such a function φ, putting

$$\varphi(x) = 0 \qquad \text{for } x \overline{\in} \sum_{\nu=1}^{\infty} U_0^{\nu}(a)$$

we obtain a uniformly continuous function φ on R, because we have by §25 Theorem 2 for every $\mu = 1, 2, \ldots$

$$\Big(\sum_{\nu=1}^{\infty} U_0^{\nu}(a) \Big) \times U_\mu = \sum_{\nu=1}^{\infty} U_0^{\nu}(a),$$

$$\Big(\sum_{\nu=1}^{\infty} U_0^{\nu}(a) \Big)' \times U_\mu = \Big(\sum_{\nu=1}^{\infty} U_0^{\nu}(a) \Big)'.$$

This uniformly continuous function φ is not bounded in A, because

$$\varphi(x) \geqq \mu \qquad \text{for } x \in A \sum_{\nu=1}^{\infty} U_0^{\nu}(a) - A \sum_{\nu=1}^{\mu} U_0^{\nu}(a).$$

In the second case where there is no such point $a \in A$, we can find a sequence of points $a_\nu \in A$ ($\nu = 1, 2, \ldots$) such that

$$\Big(\sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu) \Big) \Big(\sum_{\nu=1}^{\infty} U_0^{\nu}(a_\rho) \Big) = 0 \qquad \text{for } \mu \neq \rho,$$

because A is not bounded by assumption. For such $a_\nu \in A$ ($\nu = 1, 2, \ldots$), putting

$$\varphi(x) = \begin{cases} \mu & \text{for } x \in \sum_{\nu=1}^{\mu} U_0^{\nu}(a_\mu) \quad (\mu = 1, 2, \ldots), \\ 0 & \text{for } x \overline{\in} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu), \end{cases}$$

we obtain a function φ on R. This function φ is uniformly continuous, because we have by §25 Theorem 2

$$\Big(\sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu) \Big) \times U_0 = \sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu)$$

for every $\mu = 1, 2, \ldots$, and hence by the formula §25(12)

$$\Big(\sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu) \Big) \times U_0 = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu),$$

which yields by the formula §25(23)

$$\Big(\sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu) \Big)' \times U_0 = \Big(\sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} U_0^{\nu}(a_\mu) \Big)'.$$

Furthermore φ is not bounded in A, because $\varphi(a_\mu) \geqq \mu$ for every $\mu = 1, 2, \ldots$. Therefore our assertion is established.

We obtain by definition immediately

Theorem 3. For two uniformities $\mathfrak{U}$ and $\mathfrak{V}$ on an abstract space R, if $\mathfrak{U} > \mathfrak{V}$ and a point set A is bounded by $\mathfrak{U}$, then A also is bounded by $\mathfrak{V}$.

Theorem 4. Let α be a uniformly continuous mapping of a uniform space R with a uniformity $\mathfrak{U}$ into a uniform space S with a uniformity $\mathfrak{V}$. The image $\alpha(A)$ of every bounded set A is bounded too.

Proof. For each $V \in \mathfrak{V}$, as α is uniformly continuous by assumption, we have $\alpha^{-1}V \in \mathfrak{U}$, and hence we can find σ and a finite number of points $a_\nu \in R$ ($\nu = 1, 2, \ldots, \kappa$) such that $A \subset \sum_{\nu=1}^{\kappa} (\alpha^{-1}V)^{\sigma}(a_\nu)$. Then we have by the formula §30(5)

$$A \subset \sum_{\nu=1}^{\kappa} \alpha^{-1} V^{\sigma}(a_\nu) = \sum_{\nu=1}^{\kappa} \alpha^{-1}(V^{\sigma}(\alpha(a_\nu))),$$

and hence $\alpha(A) \subset \sum_{\nu=1}^{\kappa} V^{\sigma}(\alpha(a_\nu))$ by the formula §15(13). Therefore $\alpha(A)$ is bounded by definition.

§33 Totally bounded sets

Let R be a uniform space with a uniformity $\mathfrak{U}$. A point set A of R is said to be *totally bounded*, if for any $U \in \mathfrak{U}$ we can find a finite number of points $a_\nu \in R$ ($\nu = 1, 2, \ldots, \kappa$) such that

$$A \subset \sum_{\nu=1}^{\kappa} U(a_\nu).$$

Here $a_\nu \in R$ ($\nu = 1, 2, \dots, \varkappa$) may be found in A. Because, if we have $V^{-1} \times V \leqq U$, then $A \subset \sum_{\nu=1}^{\varkappa} V(a_\nu)$ implies by the formula §25(21)

$$A \subset \sum_{\nu=1}^{\varkappa} U(x_\nu) \qquad \text{for } x_\nu \in A\,V(a_\nu) \qquad (\nu = 1, 2, \dots, \varkappa).$$

Therefore we conclude by definition immediately

Theorem 1. A point set A is totally bounded, if and only if A is totally bounded by the relative uniformity $\mathfrak{U}^A$.

Every totally bounded set is obviously bounded by definition. For a totally bounded set A, every subset of A also is totally bounded, as we see at once by definition. Therefore every subset S of a totally bounded set A is bounded by the relative uniformity $\mathfrak{U}^S$.

Conversely we have

Theorem 2. For a point set A, if every subset S of A is bounded by the relative uniformity $\mathfrak{U}^S$, then A is totally bounded.

Proof. If A is not totally bounded, then there is by definition $U \in \mathfrak{U}$ such that $A \neq A \sum_{\nu=1}^{\varkappa} U(a_\nu)$ for every finite number of points $a_\nu \in A$ ($\nu = 1, 2, \dots, \varkappa$). For such $U \in \mathfrak{U}$ we can find a sequence of points $a_\nu \in A$ ($\nu = 1, 2, \dots$) such that $U(a_\nu) \,\overline{\ni}\, a_\mu$ for $\mu > \nu$. On account of the relation §26(5) there is a symmetric $V \in \mathfrak{U}$ for which $V \times V \leqq U$, and we have by the relation §25(22)

$$V(a_\nu)\,V(a_\mu) = 0 \qquad \text{for } \nu \neq \mu.$$

Putting $S = \{a_1, a_2, \dots\}$, we see then that S is not bounded by the relative uniformity $\mathfrak{U}^S$, because

$$V^S(a_\nu) = \{a_\nu\} \qquad (\nu = 1, 2, \dots).$$

Theorem 3. If a point set A is totally bounded, then its closure A^- by the induced topology $\mathfrak{T}^{\mathfrak{U}}$ also is totally bounded.

Proof. For any $U \in \mathfrak{U}$, there is by the relation §26(5) a symmetric $V \in \mathfrak{U}$ such that $V \times V \leqq U$, and $A \subset \sum_{\nu=1}^{\varkappa} V(a_\nu)$ implies by the formulas §27(5) and §25(12)

$$A^- \subset A \times V \subset \sum_{\nu=1}^{\varkappa} V(a_\nu) \times V \subset \sum_{\nu=1}^{\varkappa} U(a_\nu).$$

Therefore A^- is totally bounded by definition.

Theorem 4. If a point set A is compact by the induced topology $\mathfrak{T}^{\mathfrak{U}}$, then A is totally bounded.

Proof. For any $U \in \mathfrak{U}$, we have obviously $A \subset \sum_{x \in A} U^\circ(x)$. As A is compact by assumption, we can find a finite number of points $a_\nu \in A$ ($\nu = 1, 2, \dots, \varkappa$) for which

$$A \subset \sum_{\nu=1}^{\varkappa} U^\circ(a_\nu) \subset \sum_{\nu=1}^{\varkappa} U(a_\nu).$$

Therefore A is totally bounded by definition.

We have obviously by definition

Theorem 5. For two uniformities $\mathfrak{U}$ and $\mathfrak{V}$ on an abstract space R, if $\mathfrak{U} > \mathfrak{V}$ and a point set A is totally bounded by $\mathfrak{U}$, then A also is totally bounded by $\mathfrak{V}$.

Theorem 6. For two uniformities $\mathfrak{U}$ and $\mathfrak{V}$ on an abstract space R, if R is totally bounded by $\mathfrak{U}$ and every bounded uniformly continuous function by $\mathfrak{U}$ is uniformly continuous by $\mathfrak{V}$, then $\mathfrak{U}$ is weaker than $\mathfrak{V}$, that is, $\mathfrak{U} < \mathfrak{V}$.

Proof. If $\mathfrak{U}$ is not weaker than $\mathfrak{V}$, then we can find by definition $U_0 \in \mathfrak{U}$ such that $U_0 \,\overline{\in}\, \mathfrak{V}$. For such $U_0 \in \mathfrak{U}$, we have by the uniformity condition 1) $U_0 \,\overline{\geqq}\, V$ for every $V \in \mathfrak{V}$, and hence, corresponding to every $V \in \mathfrak{V}$ we obtain a pair of points x_V, y_V such that

$$U_0(x_V) \,\overline{\ni}\, y_V \in V(x_V) \qquad \text{for every } V \in \mathfrak{V}.$$

On account of the relation §26(6) we can find a symmetric $U_1 \in \mathfrak{V}$ such that $U_1 \times U_1 \times U_1 \leqq U_0$, and there is a finite number of points $a_\nu \in R$ ($\nu = 1, 2, \dots, \varkappa$) such that

$$R = \sum_{\nu=1}^{\varkappa} U_1(a_\nu),$$

because R is totally bounded by assumption. Then we can find ν_0 for which $\{V : x_V \in U_1(a_{\nu_0})\}$ is a basis of $\mathfrak{V}$. Because if $\{V : x_V \in U_1(a_\nu)\}$ is not a basis of $\mathfrak{V}$ for every $\nu = 1, 2, \dots, \varkappa$, then, corresponding to every $\nu = 1, 2, \dots, \varkappa$, we can find $V_\nu \in \mathfrak{V}$ such that we have not $V_\nu \geqq V$ for all $x_V \in U_1(a_\nu)$. As

$$\mathfrak{V} = \sum_{\nu=1}^{\varkappa} \{V : x_V \in U_1(a_\nu)\},$$

we conclude then $\mathfrak{V} \,\overline{\ni}\, V_1 V_2 \dots V_\varkappa$, contradicting the uniformity condition 2). For such ν_0, putting

$$\mathfrak{L} = \{V : x_V \in U_1(a_{\nu_0})\},$$

$$A = \{x_V : V \in \mathfrak{L}\}, \qquad B = \{y_V : V \in \mathfrak{L}\},$$

we obtain hence a basis $\mathfrak{B}$ of $\mathfrak{V}$, and we have by the relation §25(21)

$$A \times U_1 \subset U_1(a_{\nu_0}) \times U_1 \subset (U_1(x) \times U_1) \times U_1 \subset U_0(x)$$

for every $x \in A$. Thus we have for every $V \in \mathfrak{B}$

$$A \times U_1 \subset \prod_{V \in \mathfrak{B}} U_0(x_V) \not\ni y_V,$$

and consequently $(A \times U_1)B = 0$. Therefore we can find by §31 Theorem 5 a function φ on R such that φ is uniformly continuous by $\mathfrak{U}$,

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in A, \\ 1 & \text{for } x \in B, \end{cases}$$

and $0 \leqq \varphi(x) \leqq 1$ for every $x \in R$. But such a function φ is not uniformly continuous by $\mathfrak{V}$, because $V(x_V) \ni y_V$ for every $V \in \mathfrak{B}$ and $\mathfrak{B}$ is a basis of $\mathfrak{V}$.

Similarly as §32 Theorem 4, we also can prove

Theorem 7. Let a be a uniformly continuous mapping of R into a uniform space S with a uniformity $\mathfrak{V}$. The image $a(A)$ of a totally bounded set A of R is totally bounded by $\mathfrak{V}$ too.

§34 Weak uniformities

Let R be an abstract space. For a mapping a of R into a uniform space S with a uniformity $\mathfrak{V}$, putting

$$\mathfrak{B}_0 = \{ a^{-1} V : \quad V \in \mathfrak{V} \}$$

we see easily by the formulas (2), (3), (4), (5) in §30 that $\mathfrak{B}_0$ satisfies the basis conditions in §26, and hence there is by §26 Theorem 1 uniquely a uniformity $\mathfrak{U}_0$ on R such that $\mathfrak{B}_0$ is a basis of $\mathfrak{U}_0$. This uniformity $\mathfrak{U}_0$ is called the weak uniformity of R by a mapping a.

For the weak uniformity $\mathfrak{U}_0$ of R, it is evident by definition that a becomes uniformly continuous by $\mathfrak{U}_0$. Conversely, if a is uniformly continuous for a uniformity $\mathfrak{U}$ on R, then we have by definition $\mathfrak{U} \supset \mathfrak{B}_0$, and hence $\mathfrak{U} \supset \mathfrak{U}_0$. Therefore we can say that the the weak uniformity of R by a mapping a is the weakest uniformity of R for which a becomes uniformly continuous.

Let a_λ $(\lambda \in \Lambda)$ be a system of mappings of R into uniform spaces

S_λ with uniformities $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$. Corresponding to every mapping a_λ $(\lambda \in \Lambda)$, we can determine the weak uniformity $\mathfrak{U}_\lambda$ of R by a_λ, as defined just above. For these weak uniformities $\mathfrak{U}_\lambda$ of R $(\lambda \in \Lambda)$, putting $\mathfrak{W}_0 = \bigcup_{\lambda \in \Lambda} \mathfrak{U}_\lambda$, we obtain a uniformity $\mathfrak{U}_0$ on R, as defined already in §28. This uniformity $\mathfrak{U}_0$ is called the weak uniformity of R by a system of mappings a_λ $(\lambda \in \Lambda)$.

For the weak uniformity $\mathfrak{U}_0$ of R by a_λ $(\lambda \in \Lambda)$, as $\mathfrak{U}_0 \supset \mathfrak{U}_\lambda$ for every $\lambda \in \Lambda$, every mapping a_λ is uniformly continuous by $\mathfrak{U}_0$. On the other hand, if every a_λ $(\lambda \in \Lambda)$ is uniformly continuous for a uniformity $\mathfrak{U}$ on R, then we have obviously $\mathfrak{U} \supset \mathfrak{U}_\lambda$ for every $\lambda \in \Lambda$, and hence $\mathfrak{U} \supset \mathfrak{U}_0$. Therefore we can say that the weak uniformity of R by a system of mappings a_λ $(\lambda \in \Lambda)$ is the weakest uniformity of R for which every a_λ becomes uniformly continuous.

Since $\{ a_\lambda^{-1} V : \quad V \in \mathfrak{V}_\lambda \}$ is by definition a basis of the weak uniformity $\mathfrak{U}_\lambda$ of R by a mapping a_λ, we obtain by §28 Theorem 3 that the totality of $(a_{\lambda_1}^{-1} V_{\lambda_1})(a_{\lambda_2}^{-1} V_{\lambda_2}) \cdots (a_{\lambda_\kappa}^{-1} V_{\lambda_\kappa})$ for every $V_{\lambda_\nu} \in \mathfrak{V}_{\lambda_\nu}$, $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \kappa)$, $\kappa = 1, 2, \ldots$, is a basis of the weak uniformity $\mathfrak{U}_0$ of R by a system of mappings a_λ $(\lambda \in \Lambda)$. Therefore we have

Theorem 1. For a system of mappings a_λ $(\lambda \in \Lambda)$ of R into uniform spaces S_λ with uniformities $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$,

$$\left\{ \prod_{\nu=1}^{\kappa} a_{\lambda_\nu}^{-1} V_{\lambda_\nu} \quad : V_{\lambda_\nu} \in \mathfrak{V}_{\lambda_\nu},\ \lambda_\nu \in \Lambda\ (\nu = 1, 2, \ldots, \kappa),\ \kappa = 1, 2, \ldots \right\}$$

is a basis of the weak uniformity of R by a_λ $(\lambda \in \Lambda)$.

Recalling §29 Theorem 1, we conclude immediately from Theorem 1

Theorem 2. For a subspace S of R, the weak uniformity of S by a system of mappings a_λ of R into uniform spaces S_λ $(\lambda \in \Lambda)$ coincides with the relative uniformity $\mathfrak{U}^S$ of the weak uniformity $\mathfrak{U}$ of R by a_λ $(\lambda \in \Lambda)$.

Theorem 3. For the weak uniformity $\mathfrak{U}$ of R by a system of mappings a_λ of R into uniform spaces S_λ with uniformities $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$, the induced topology $\mathfrak{T}^{\mathfrak{U}}$ by $\mathfrak{U}$ coincides with the weak topology of R by a_λ $(\lambda \in \Lambda)$, considering every a_λ $(\lambda \in \Lambda)$ as a mapping of R into the topological space S_λ with the induced topology $\mathfrak{T}^{\mathfrak{V}_\lambda}$.

Proof. For each $\lambda \in \Lambda$, let $\mathfrak{U}_\lambda$ be the weak uniformity of R by a_λ, and $\mathfrak{T}_\lambda$ the weak topology of R by a_λ for the induced topology $\mathfrak{T}^{\mathfrak{V}_\lambda}$. As a_λ is continuous by §30 Theorem 2 for the induced topology $\mathfrak{T}^{\mathfrak{V}_\lambda}$, we have naturally $\mathfrak{T}^{\mathfrak{U}_\lambda} > \mathfrak{T}_\lambda$. For every $V \in \mathfrak{V}_\lambda$ and for every $x \in R$, as $V^\circ(a_\lambda(x)) \in \mathfrak{T}^{\mathfrak{V}_\lambda}$, we have by the definition of weak topologies

$$a_\lambda^{-1} V(x) \supset a_\lambda^{-1} V^\circ(x) = a_\lambda^{-1}(V^\circ(a_\lambda(x))) \in \mathfrak{T}.$$

Since $\{a_\lambda^{-1} V : V \in \mathfrak{V}_\lambda\}$ is a basis of $\mathfrak{U}_\lambda$, we conclude hence by §27 Theorem 5 that $\mathfrak{T}_\lambda > \mathfrak{T}^{\mathfrak{U}_\lambda}$. Therefore we have $\mathfrak{T}_\lambda = \mathfrak{T}^{\mathfrak{U}_\lambda}$ for every $\lambda \in \Lambda$, and hence by §28 Theorem 5 $\bigcup_{\lambda \in \Lambda} \mathfrak{T}_\lambda = \mathfrak{T}^{\mathfrak{U}}$, that is, $\mathfrak{T}^{\mathfrak{U}}$ is the weak topology of R by a_λ ($\lambda \in \Lambda$) for the induced topologies $\mathfrak{T}^{\mathfrak{V}_\lambda}$ ($\lambda \in \Lambda$).

Let $\mathfrak{F}$ be a collection of functions on R. Considering every φ of $\mathfrak{F}$ as a mapping of R into the number space with the number uniformity, we obtain the weak uniformity of R by φ. This weak uniformity of R is called the *weak uniformity* of R by $\mathfrak{F}$ and denoted by $\mathfrak{U}^{\mathfrak{F}}$.

For a finite number of functions φ_ν ($\nu = 1, 2, \dots, \kappa$) on R and for a positive number ε, we define a connector $U_{\varphi_1, \dots \varphi_\kappa, \varepsilon}$ to mean

$$U_{\varphi_1, \dots, \varphi_\kappa, \varepsilon}(a) = \{x : |\varphi_\nu(x) - \varphi_\nu(a)| < \varepsilon \quad (\nu = 1, 2, \dots, \kappa)\}$$

for every point $a \in R$. With this definition we obtain by Theorem 1

Theorem 4. *For a collection of functions* $\mathfrak{F}$ *on* R,

$$\{U_{\varphi_1, \dots, \varphi_\kappa, \varepsilon} : \varphi_\nu \in \mathfrak{F} \quad (\nu = 1, 2, \dots, \kappa), \kappa = 1, 2, \dots\}$$

is a basis of the weak uniformity of R *by* $\mathfrak{F}$.

By virtue of Theorem 3 we have

Theorem 5. *The weak topology of* R *by* $\mathfrak{F}$ *coincides with the induced topology by the weak uniformity of* R *by* $\mathfrak{F}$.

Theorem 6. *For any collection of bounded functions* $\mathfrak{F}$, R *is totally bounded by the weak uniformity* $\mathfrak{U}^{\mathfrak{F}}$ *of* R *by* $\mathfrak{F}$.

Proof. For every finite number of functions $\varphi_\nu \in \mathfrak{F}$ ($\nu = 1, 2, \dots, \kappa$) and for any $\varepsilon > 0$, we can find α_ν, β_ν ($\nu = 1, 2, \dots, \kappa$) such that

$$\alpha_\nu \leqq \varphi_\nu(x) \leqq \beta_\nu \qquad \text{for every } x \in R,$$

and further $\lambda_{\nu,\mu}$ ($\nu = 1, 2, \dots, \kappa$; $\mu = 0, 1, 2, \dots, \kappa_\nu$) such that

$$\alpha_\nu = \lambda_{\nu,0} < \lambda_{\nu,1} < \dots < \lambda_{\nu,\kappa_\nu} = \beta_\nu$$

$$\lambda_{\nu,\mu} - \lambda_{\nu,\mu-1} < \tfrac{1}{2}\varepsilon.$$

Then, determining points $x_{\mu_1, \dots, \mu_\kappa}$ such that

$$\lambda_{\nu,\mu_\nu - 1} \leqq \varphi_\nu(x_{\mu_1, \dots, \mu_\kappa}) \leqq \lambda_{\nu,\mu_\nu} \qquad (\nu = 1, 2, \dots, \kappa),$$

we have obviously

$$R < \sum_{\mu_1, \dots, \mu_\kappa} U_{\varphi_1, \dots, \varphi_\kappa, \varepsilon}(x_{\mu_1, \dots, \mu_\kappa}).$$

Therefore R is totally bounded by definition.

Theorem 7. *If a uniform space* R *is totally bounded by its uniformity* $\mathfrak{U}$, *then* $\mathfrak{U}$ *coincides with the weak uniformity of* R *by all bounded uniformly continuous functions on* R.

Proof. Let $\mathfrak{F}$ be the totality of bounded uniformly continuous functions on R, and $\mathfrak{V}$ the weak uniformity of R by $\mathfrak{F}$. Then we have obviously by definition $\mathfrak{V} < \mathfrak{U}$. Furthermore we see by §33 Theorem 6 that $\mathfrak{U} < \mathfrak{V}$. Consequently we have $\mathfrak{U} = \mathfrak{V}$.

Theorem 8. *For the weak uniformity* $\mathfrak{U}$ *of* R *by a collection of bounded functions* $\mathfrak{F}$ *on* R, *the totality of uniformly continuous functions coincides with the closed field generated by* $\mathfrak{F}$.

Proof. Let $\bar{R}$ be the compactification of R by $\mathfrak{F}$; $\bar{\mathfrak{F}}$ the continuous extension of $\mathfrak{F}$ over $\bar{R}$; and $\bar{\mathfrak{U}}$ the weak uniformity of $\bar{R}$ by $\bar{\mathfrak{F}}$. Then $\mathfrak{U}$ is by Theorem 2 the relative uniformity of $\bar{\mathfrak{U}}$ in R, that is, we have $\mathfrak{U} = \bar{\mathfrak{U}}^R$. Therefore every bounded uniformly continuous function φ on R has by §31 Theorem 8 a uniformly continuous extension over $\bar{R}$, and hence we have by §24 Theorem 2 that φ is contained in the closed field generated by $\mathfrak{F}$, because the induced topology $\mathfrak{T}^{\mathfrak{U}}$ of R by $\mathfrak{U}$ coincides by §29 Theorem 2 with the relative topology $(\mathfrak{T}^{\bar{\mathfrak{U}}})^R$ of the induced topology $\mathfrak{T}^{\bar{\mathfrak{U}}}$ by $\bar{\mathfrak{U}}$. Furthermore it is evident by Theorems 2 and 3 in §31 that the closed field generated by $\mathfrak{F}$ is composed only of uniformly continuous functions by $\mathfrak{U}$. Therefore we obtain our assertion.

Theorem 9. *For a topological space* R *with a completely regular topology* $\mathfrak{T}$, *there is a uniformity* $\mathfrak{U}$ *on* R *such that* $\mathfrak{T}$ *is the induced topology by* $\mathfrak{U}$.

Proof. For the totality of bounded continuous functions $\mathfrak{F}$, $\mathfrak{T}$ is by §23 Theorem 2 the weak topology of R by $\mathfrak{F}$. For the weak uniformity $\mathfrak{U}$ of R by $\mathfrak{F}$, we have hence by §24 Theorem 3 that $\mathfrak{T}$ is the induced topology by $\mathfrak{U}$.

Theorem 10. For a topological space R with a regular compact topology $\mathfrak{T}$, there exists uniquely a uniformity on R by which $\mathfrak{T}$ is the induced topology.

Proof. Since $\mathfrak{T}$ is completely regular by §23 Theorem 6, there is by Theorem 9 a uniformity on R by which $\mathfrak{T}$ is the induced topology. We conclude further by Theorems 4, 6 in §33 and 7 in §31 that such a uniformity is uniquely determined.

Recalling §32 Theorem 2, we obtain immediately by Theorem 6

Theorem 11. For a weak uniformity by a collection of functions on R, every bounded set is totally bounded.

§35 Completeness

Let R be a uniform space with a uniformity $\mathfrak{U}$. A system of point sets $A_\lambda \subset R$ ($\lambda \in \Lambda$) is said to be a Cauchy system by $\mathfrak{U}$, if we have $\prod_{\nu=1}^{\kappa} A_{\lambda_\nu} \neq 0$ for every finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \kappa$), and for any $U \in \mathfrak{U}$ we can find $\lambda \in \Lambda$ and $a \in R$ such that $A_\lambda \subset U(a)$.

For a Cauchy system A_λ ($\lambda \in \Lambda$), if there is a point $a \in R$ such that for any $U \in \mathfrak{U}$ we can find $\lambda \in \Lambda$ for which $A_\lambda \subset U(a)$, then such a point a is called a limit of A_λ ($\lambda \in \Lambda$).

Theorem 1. Let $\mathfrak{B}$ be a basis of $\mathfrak{U}$. For a Cauchy system A_λ ($\lambda \in \Lambda$), in order that a point a be a limit of A_λ ($\lambda \in \Lambda$), it is necessary and sufficient that $U(a) A_\lambda \neq 0$ for every $U \in \mathfrak{B}$ and $\lambda \in \Lambda$.

Proof. Since the necessity is evident by definition, we shall prove only the sufficiency. For any $U \in \mathfrak{U}$ we can find by the relation §26(6) a symmetric $V \in \mathfrak{U}$ such that $V \times V \times V \leqq U$. For such $V \in \mathfrak{U}$, as A_λ ($\lambda \in \Lambda$) is a Cauchy system by assumption, we can find $\lambda \in \Lambda$ and a point $x \in R$ such that $A_\lambda \subset V(x)$, and then $V(x)\, V(a) \neq 0$, because $V(a) A_\lambda \neq 0$ by assumption. Hence we obtain by the relation §25(22) $A_\lambda \subset V(x) \subset V(a) \times V \times V \subset U(a)$. Therefore a is a limit of A_λ ($\lambda \in \Lambda$) by definition.

A point set A of R is said to be complete by $\mathfrak{U}$, if every Cauchy system $A_\lambda \subset A$ ($\lambda \in \Lambda$) has a limit in A. If R is complete by $\mathfrak{U}$, then we shall say that $\mathfrak{U}$ is complete.

With this definition we have obviously

Theorem 2. A point set A is complete by a uniformity $\mathfrak{U}$, if and only if the relative uniformity $\mathfrak{U}^A$ is complete.

Theorem 3. If a point set A is complete by $\mathfrak{U}$ and A is a topological set by the induced topology $\mathfrak{T}^{\mathfrak{U}}$, then A is closed by $\mathfrak{T}^{\mathfrak{U}}$.

Proof. For each point $a \in A^-$, we obtain obviously a Cauchy system $A\, U(a)$ ($U \in \mathfrak{U}$). As A is complete by assumption, there is by Theorem 1 a limit $b \in A$ such that

$$A\, U(a)\, U(b) \neq 0 \qquad \text{for every } U \in \mathfrak{U}.$$

This relation yields $a \in \{b\}^-$. Because, if $a \bar{\in} \{b\}^-$, namely if $a \in \{b\}^{-\prime}$, then we can find by §27 Theorem 1 and §26(5) a symmetric $U \in \mathfrak{U}$ for which $U \times U(a) \subset \{b\}^{-\prime}$, that is, $(U(a) \times U)\{b\}^- = 0$, and then we have $U(a)\, U(b) = 0$ by the relation §25(23). Since the induced topology $\mathfrak{T}^{\mathfrak{U}}$ is regular by §31 Theorem 6, and A is a topological set by assumption, we obtain by §10 Theorem 1 $A \supset \{b\}^- \ni a$. Therefore A is closed by $\mathfrak{T}^{\mathfrak{U}}$.

Theorem 4. If a point set A is complete by a uniformity $\mathfrak{U}$, then for every closed set B by the induced topology $\mathfrak{T}^{\mathfrak{U}}$ the intersection AB also is complete by $\mathfrak{U}$.

Proof. For every Cauchy system $B_\lambda \subset AB$ ($\lambda \in \Lambda$) there is by Theorem 1 a limit $a \in A$ for which $B_\lambda U(a) \neq 0$ for every $U \in \mathfrak{U}$ and $\lambda \in \Lambda$, and hence $B\, U(a) \neq 0$ for every $U \in \mathfrak{U}$. As B is closed by assumption, we obtain $a \in B$ by §27 Theorem 1. Therefore AB is complete by definition.

Theorem 5. In order that a point set A be compact by the induced topology $\mathfrak{T}^{\mathfrak{U}}$, it is necessary and sufficient that A is complete and totally bounded by $\mathfrak{U}$.

Proof. By virtue of Theorems 1 in §12, 2 in §29, 1 in §33, and 2 in §35, we need only prove the case where $A = R$. Let R be compact by the induced topology $\mathfrak{T}^{\mathfrak{U}}$. Then R is by §33 Theorem 4 totally bounded by $\mathfrak{U}$. Furthermore, for a Cauchy system $A_\lambda \subset R$ ($\lambda \in \Lambda$), there is by §7 Theorem 3 a point $a \in \prod_{\lambda\in\Lambda} A_\lambda^-$. For such a point a we have obviously $A_\lambda U(a) \neq 0$ for every $U \in \mathfrak{U}$ and $\lambda \in \Lambda$, and hence such a point a is by Theorem 1 a limit of A_λ ($\lambda \in \Lambda$). Therefore R is complete by $\mathfrak{U}$.

Conversely, let R be totally bounded and complete by $\mathfrak{U}$. We denote by $\mathfrak{F}$ the totality of bounded uniformly continuous functions on R; by $\bar{R}$ a compactification of R by $\mathfrak{F}$; by $\bar{\mathfrak{F}}$ the continuous extension of $\mathfrak{F}$ over $\bar{R}$; and by $\bar{\mathfrak{U}}$ the weak uniformity of $\bar{R}$ by $\bar{\mathfrak{F}}$. Then we have $\mathfrak{U} = \bar{\mathfrak{U}}^R$ by §34 Theorem 2. For each point $\bar{a} \in \bar{R}$, $R\bar{U}(\bar{a})$ ($\bar{U} \in \bar{\mathfrak{U}}$) is obviously a Cauchy system. As R is by Theorem 2 complete by $\bar{\mathfrak{U}}$, there is by Theorem 1 a point $a \in R$ such that

$$R\bar{U}(\bar{a})\bar{U}(a) \neq 0 \qquad \text{for every } \bar{U} \in \bar{\mathfrak{U}}.$$

This relation yields $\bar{a} \in R$. Because, if $\bar{a} \,\overline{\in}\, R$, then $\bar{a}$ is separated from a by the compactification condition 3), and hence there is by §27 Theorem 1 and the relation §26(5) a symmetric $\bar{U} \in \bar{\mathfrak{U}}$ such that $\bar{U} \times \bar{U}(\bar{a}) \,\overline{\ni}\, a$, which yields $\bar{U}(\bar{a})\bar{U}(a) = 0$ by the relation §25(23). Therefore we obtain $R = \bar{R}$, and hence R is compact by the induced topology $\mathfrak{T}^{\mathfrak{U}}$.

Theorem 6. *Let R and S be uniform spaces respectively with uniformities $\mathfrak{U}$ and $\mathfrak{V}$. If a point set X is dense in R by the induced topology $\mathfrak{T}^{\mathfrak{U}}$ and S is complete by $\mathfrak{V}$, then for a uniformly continuous mapping a of the subspace X with the relative uniformity $\mathfrak{U}^X$ into S there is a uniformly continuous mapping $\bar{a}$ of R into S such that we have $\bar{a}(x) = a(x)$ for every $x \in X$.*

Proof. For each point $x \in R - X$, $a(XU(x))$ ($U \in \mathfrak{U}$) is a Cauchy system by $\mathfrak{V}$. Because, for every $V \in \mathfrak{V}$, as a is uniformly continuous by assumption for the relative uniformity $\mathfrak{U}^X$, there is $U \in \mathfrak{U}$ such that $U^X = a^{-1}V$. For such $U \in \mathfrak{U}$ there is by the uniformity condition 3) $U_1 \in \mathfrak{U}$ such that $U_1^{-1} \times U_1 \leqq U$, and we can find a point $y \in XU_1(x)$, since X is dense in R and x is an inner point of $U_1(x)$ for the induced topology $\mathfrak{T}^{\mathfrak{U}}$. Then we have by the relation §25(21)

$$XU_1(x) \subset X(U_1 \times U_1(y)) \subset XU(y) \subset a^{-1}V(y),$$

and hence we obtain by the formula §15(13)

$$a(XU_1(x)) \subset V(a(y)).$$

Therefore $a(XU(x))$ ($U \in \mathfrak{U}$) is a Cauchy system in S. As S is complete by assumption, there exists hence a limit $s \in S$, that is, for every $V \in \mathfrak{V}$ we can find $U \in \mathfrak{U}$ such that $a(XU(x)) \subset V(s)$.

Assigning to every point $x \in R - X$ such a limit $s \in S$, we obtain a mapping $\bar{a}$ of R into S such that

$$\bar{a}(x) = a(x) \qquad \text{for every } x \in X.$$

We shall prove now that such a mapping $\bar{a}$ is continuous for the induced topologies $\mathfrak{T}^{\mathfrak{U}}$ and $\mathfrak{T}^{\mathfrak{V}}$. As a is uniformly continuous by assumption, we see easily that for every point $x \in R$ and for every symmetric $V \in \mathfrak{V}$ we can find $U \in \mathfrak{U}$ such that

$$a(XU(x)) \subset V(\bar{a}(x)).$$

For an arbitrary point set $A \subset R$, if $x \in A^-$, then, since x is an inner point of $U(x)$, there is a point $a \in AU^\circ(x)$, and we can find then $U_1 \in \mathfrak{U}$ for which

$$a(XU_1(a)) \subset V(\bar{a}(a)).$$

As X is dense in R by assumption, we have then

$$XU_1^\circ(a)U^\circ(x) \neq 0,$$

and obviously

$$a(XU_1^\circ(a)U^\circ(x)) \subset V(\bar{a}(a))V(\bar{a}(x)).$$

Accordingly we obtain

$$\bar{a}(a) \in V \times V(\bar{a}(x)).$$

Recalling the relation §26(5), we conclude therefore

$$\bar{a}(A)V(\bar{a}(x)) \neq 0 \qquad \text{for every } V \in \mathfrak{V}.$$

This relation yields by the relation §26(4)

$$\bar{a}(x) \in \bar{a}(A) \times V \qquad \text{for every } V \in \mathfrak{V},$$

and hence $\bar{a}(x) \in \bar{a}(A)^-$ by the formula §27(4). Therefore we have $\bar{a}(A^-) \subset \bar{a}(A)^-$ for every point set $A \subset R$, and consequently $\bar{a}$ is continuous by §16 Theorem 4. Thus we conclude by §30 Theorem 3 that $\bar{a}$ is uniformly continuous.

Theorem 7. *Let $\mathcal{U}$ and $\mathcal{V}$ be two uniformities on a space R with the same induced topology, namely $\gamma^{\mathcal{U}} = \gamma^{\mathcal{V}}$. If $\mathcal{U} \subset \mathcal{V}$ and R is complete by $\mathcal{U}$, then R also is complete by $\mathcal{V}$.*

Proof. Let A_λ ($\lambda \in \Lambda$) be a Cauchy system by $\mathcal{V}$. As $\mathcal{U} \subset \mathcal{V}$ by assumption, A_λ ($\lambda \in \Lambda$) also is a Cauchy system by $\mathcal{U}$, and hence there is a limit $a \in R$, because R is complete by $\mathcal{U}$ by assumption. Then, for each $V \in \mathcal{V}$, as $\gamma^{\mathcal{U}} = \gamma^{\mathcal{V}}$ by assumption, we can find by the relations (2), (3) in §27 $U \in \mathcal{U}$ and $\lambda \in \Lambda$ such that $V(a) \supset U(a) \supset A_\lambda$. Accordingly a also is a limit by $\mathcal{V}$.

Theorem 8. *Let α_λ ($\lambda \in \Lambda$) be a system of mappings of an abstract space R into uniform spaces S_λ with complete separative uniformities $\mathcal{V}_\lambda$ ($\lambda \in \Lambda$). In order that the weak uniformity of R by α_λ ($\lambda \in \Lambda$) be complete, it is necessary and sufficient that for a system of points $x_\lambda \in S_\lambda$ ($\lambda \in \Lambda$), if*

$$\prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1}(V_{\lambda_\nu}(x_{\lambda_\nu})) \neq 0$$

for every finite number of connectors $V_{\lambda_\nu} \in \mathcal{V}_{\lambda_\nu}$, $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \varkappa$), then we can find a point $x \in R$ such that $x_\lambda = \alpha_\lambda(x)$ for every $\lambda \in \Lambda$.

Proof. For a system of points $x_\lambda \in S_\lambda$ ($\lambda \in \Lambda$) subject to the stated condition,

$$\prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1}(V_{\lambda_\nu}(x_{\lambda_\nu})) \quad (V_{\lambda_\nu} \in \mathcal{V}_{\lambda_\nu},\ \lambda_\nu \in \Lambda,\ \varkappa = 1, 2, \ldots)$$

is by §34 Theorem 1 a Cauchy system by the weak uniformity $\mathcal{U}$ of R by α_λ ($\lambda \in \Lambda$). Because, for any finite number of connectors $V_{\lambda_\nu} \in \mathcal{V}_{\lambda_\nu}$ ($\nu = 1, 2, \ldots, \varkappa$) we can find $U_{\lambda_\nu} \in \mathcal{V}_{\lambda_\nu}$ such that $U_{\lambda_\nu}^{-1} \times U_{\lambda_\nu} \leqq V_{\lambda_\nu}$, and for a point $x \in \prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1}(U_{\lambda_\nu}(x_{\lambda_\nu}))$, as $\alpha_{\lambda_\nu}(x) \in U_{\lambda_\nu}(x_{\lambda_\nu})$ ($\nu = 1, 2, \ldots, \varkappa$), we have by the relation §25(21)

$$U_{\lambda_\nu}(x_{\lambda_\nu}) \subset U_{\lambda_\nu}^{-1} \times U_{\lambda_\nu}(\alpha_{\lambda_\nu}(x)) \subset V_{\lambda_\nu}(\alpha_{\lambda_\nu}(x)),$$

and consequently

$$\prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1}(U_{\lambda_\nu}(x_{\lambda_\nu})) \subset \prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1} V_{\lambda_\nu}(x).$$

Thus, if $\mathcal{U}$ is complete, then there is a limit $x \in R$. For a limit $x \in R$, we have $\alpha_\lambda(x) = x_\lambda$ for every $\lambda \in \Lambda$. Because, if we have $\alpha_\lambda(x) \neq x_\lambda$ for some $\lambda \in \Lambda$, then, as $\mathcal{V}_\lambda$ is separative by assumption, we can find $V_\lambda \in \mathcal{V}_\lambda$ such that $V_\lambda(\alpha_\lambda(x)) V_\lambda(x_\lambda) = 0$, and hence

$(\alpha_\lambda^{-1} V_\lambda(x)) \alpha_\lambda^{-1}(V_\lambda(x_\lambda)) = 0$, contradicting Theorem 1.

Conversely, for a Cauchy system $A_\gamma \subset R$ ($\gamma \in \Gamma$) by the weak uniformity $\mathcal{U}$ of R, we see easily by §34 Theorem 1 that for each $\lambda \in \Lambda$, $\alpha_\lambda(A_\gamma)$ ($\gamma \in \Gamma$) is a Cauchy system by $\mathcal{V}_\lambda$. As $\mathcal{V}_\lambda$ is complete by assumption, there is hence a limit $x_\lambda \in S_\lambda$, and for every finite number of connectors $V_{\lambda_\nu} \in \mathcal{V}_{\lambda_\nu}$ ($\nu = 1, 2, \ldots, \varkappa$) we can find $\gamma_\nu \in \Gamma$ ($\nu = 1, 2, \ldots, \varkappa$) such that

$$\prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1}(V_{\lambda_\nu}(x_{\lambda_\nu})) \supset \prod_{\nu=1}^{\varkappa} A_{\gamma_\nu} \neq 0.$$

Therefore, if $\mathcal{U}$ satisfies the stated relation, then there is a point $x \in R$ such that $\alpha_\lambda(x) = x_\lambda$ for every $\lambda \in \Lambda$. For such a point x we have obviously

$$A_\gamma \prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1} V_{\lambda_\nu}(x) = A_\gamma \prod_{\nu=1}^{\varkappa} \alpha_{\lambda_\nu}^{-1}(V_{\lambda_\nu}(x_{\lambda_\nu})) \neq 0$$

for every $\gamma \in \Gamma$ and for every finite number of connectors $V_{\lambda_\nu} \in \mathcal{V}_{\lambda_\nu}$ ($\nu = 1, 2, \ldots, \varkappa$). Thus x is by Theorem 1 a limit of A_γ ($\gamma \in \Gamma$). Therefore $\mathcal{U}$ is complete by definition.

§36 Sequential uniformities

Let R be a uniform space with a uniformity $\mathcal{U}$. We shall say that R is *sequential*, or that $\mathcal{U}$ is *sequential*, if $\mathcal{U}$ has a basis composed of at most countable connectors. A sequence of symmetric connectors U_ν ($\nu = 1, 2, \ldots$) is said to be *decreasing*, if

$$U_\nu \geqq U_{\nu+1} \times U_{\nu+1} \qquad \text{for every } \nu = 1, 2, \ldots .$$

On account of the relation §26(5), we see easily that if $\mathcal{U}$ is sequential, then $\mathcal{U}$ has a basis $\{U_1, U_2, \ldots\}$ which is a decreasing sequence. Such a basis is called a *decreasing basis* of $\mathcal{U}$.

Recalling §29 Theorem 1 we have obviously

Theorem 1. *If $\mathcal{U}$ is sequential, then the relative uniformity $\mathcal{U}^S$ also is so for every subspace S of R.*

Theorem 2. *If a point set S is dense in R by the induced topology $\gamma^{\mathcal{U}}$ and the relative uniformity $\mathcal{U}^S$ is sequential, then $\mathcal{U}$ is sequential too.*

Proof. If the relative uniformity $\mathfrak{U}^S$ is sequential, then we see easily by the relation §26(5) that there is a decreasing sequence $U_\nu \in \mathfrak{U}$ ($\nu = 1, 2, \ldots$) for which U_ν^S ($\nu = 1, 2, \ldots$) is a decreasing basis of $\mathfrak{U}^S$. For each $V \in \mathfrak{U}$, there is then ν for which $U_\nu^S \leqq V^S$, that is,

$$S\, U_\nu(y) \subset S\, V(y) \qquad \text{for every } y \in S.$$

This relation yields by the formula §4(8) for the induced topology $\mathcal{T}^{\mathfrak{U}}$

$$U_\nu^\circ(y) \subset (S\, U_\nu(y))^- \subset V^-(y) \qquad \text{for every } y \in S,$$

because $S^- = R$ by assumption. For each $x \in R$, we can find a point $y \in S\, U_{\nu+2}(x)$, because S is dense in R and x is an inner point of $U_{\nu+2}(x)$. For such a point y we have by the relations §25(21) and §27(5) $U_{\nu+2}(x) \subset U_{\nu+2} \times U_{\nu+2}(y) \subset U_{\nu+1}(y) \subset U_\nu^\circ(y) \subset V^-(y) \subset V \times V(y)$. Then, as $x \in U_{\nu+2}(x) \subset V \times V(y)$, we obtain further by the relations §25(21) and §25(19) $V \times V(y) \subset V \times V \times V \times V(x)$, and hence we have $U_{\nu+2} \leqq V \times V \times V \times V$. Thus we conclude by the relation §26(5) that for every $V \in \mathfrak{U}$ we can find ν such that $U_\nu \leqq V$, that is, $\{U_1, U_2, \ldots\}$ is a basis of $\mathfrak{U}$.

Theorem 3. If $\mathfrak{U}$ is sequential, then the induced topology $\mathcal{T}^{\mathfrak{U}}$ is normal and sequential.

Proof. Let $U_\nu \in \mathfrak{U}$ ($\nu = 1, 2, \ldots$) be a decreasing basis of $\mathfrak{U}$. For two closed sets A, B by the induced topology $\mathcal{T}^{\mathfrak{U}}$, if $AB = 0$, then putting

$$A_1 = \{x : U_{\nu+1}(x)A \neq 0,\ U_\nu(x)B = 0 \ \text{ for some } \nu\},$$
$$B_1 = \{x : U_{\nu+1}(x)B \neq 0,\ U_\nu(x)A = 0 \ \text{ for some } \nu\},$$

we have obviously $A \subset A_1$ and $B \subset B_1$ by §27 Theorem 1. Furthermore we have $A_1 B_1 = 0$. Because, if $A_1 B_1 \neq 0$, then for any point $x \in A_1 B_1$ we can find ν and μ such that

$$U_{\nu+1}(x)A \neq 0, \qquad U_\nu(x)B = 0,$$
$$U_{\mu+1}(x)B \neq 0, \qquad U_\mu(x)A = 0,$$

and hence $\mu > \nu+1$, $\nu > \mu+1$, contradicting $\mu < \mu+2$.

For each point $x \in A$, as $x \bar{\in} B$, we can find by §27 Theorem 1 ν such that $U_\nu(x)B = 0$, and for every point $y \in U_{\nu+2}(x)$ we have naturally $U_{\nu+2}(y)A \ni x$ and further by the formula §25(21)

$$U_{\nu+1}(y)B \subset (U_{\nu+2} \times U_{\nu+1}(x))B \subset U_\nu(x)B = 0.$$

Thus we have $U_{\nu+2}(x) \subset A_1$ for such ν, and hence we conclude $A \subset A_1^\circ$ by the formula §27(2). We also obtain likewise $B \subset B_1^\circ$. Therefore $\mathcal{T}^{\mathfrak{U}}$ is normal by §11 Theorem 4. Furthermore it is evident by §27 Theorem 1 that $\mathcal{T}^{\mathfrak{U}}$ is sequential.

A sequence of points $a_\nu \in R$ ($\nu = 1, 2, \ldots$) is said to be a Cauchy sequence by $\mathfrak{U}$, if for any $U \in \mathfrak{U}$ we can find ν_0 and a point $x \in R$ such that $a_\nu \in U(x)$ for every $\nu \geqq \nu_0$, that is, if the system of point sets $\{a_\nu, a_{\nu+1}, \ldots\}$ ($\nu = 1, 2, \ldots$) is a Cauchy system by $\mathfrak{U}$.

A point sequence $a_\nu \in R$ ($\nu = 1, 2, \ldots$) is said to be convergent by $\mathfrak{U}$ to a limit $a \in R$, if $\lim_{\nu\to\infty} a_\nu = a$ by the induced topology $\mathcal{T}^{\mathfrak{U}}$. With this definition we see at once by §27 Theorem 1 that we have $\lim_{\nu\to\infty} a_\nu = a$ if and only if for each $U \in \mathfrak{U}$ we can find ν_0 such that

$$a_\nu \in U(a) \qquad \text{for every } \nu \geqq \nu_0.$$

Thus we conclude easily that if $\lim_{\nu\to\infty} a_\nu = a$, then a_ν ($\nu = 1, 2, \ldots$) is a Cauchy sequence by $\mathfrak{U}$ and a is a limit of a Cauchy system $\{a_\nu, a_{\nu+1}, \ldots\}$ ($\nu = 1, 2, \ldots$).

Theorem 4. If $\mathfrak{U}$ is sequential, then in order that a point set A be complete by $\mathfrak{U}$, it is necessary and sufficient that every Cauchy sequence $a_\nu \in A$ ($\nu = 1, 2, \ldots$) is convergent to a limit $a \in A$.

Proof. Since the necessity is evident by definition, we need only prove the sufficiency. Let $\{U_1, U_2, \ldots\}$ be a decreasing basis of $\mathfrak{U}$. For a Cauchy system $A_\lambda \subset A$ ($\lambda \in \Lambda$) we can find by §35 Theorem 2 $a_\nu \in A$ and $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots$) such that

$$A_{\lambda_\nu} \subset U_\nu(a_\nu) \qquad \text{for every } \nu = 1, 2, \ldots .$$

Such a_ν ($\nu = 1, 2, \ldots$) is a Cauchy sequence. Because, for $\mu \geqq \nu$ we have

$$0 \neq A_{\lambda_\nu} A_{\lambda_\mu} \subset U_\nu(a_\nu) U_\mu(a_\mu) \subset U_\nu(a_\nu) U_\nu(a_\mu),$$

and hence we obtain by the relation §25(22)

$$a_\mu \in U_\nu \times U_\nu(a_\nu) \subset U_{\nu-1}(a_\nu) \qquad \text{for } \mu \geqq \nu.$$

Therefore there is by assumption a limit $a \in A$. For every $\nu = 1, 2, \ldots$, we can find then $\mu > \nu$ such that $a_\mu \in U_{\nu+1}(a)$, and we have

$$A_{\lambda_\mu} \subset U_\mu(a_\mu) \subset U_{\nu+1} \times U_\mu(a) \subset U_\nu(a).$$

Consequently such a limit a is a limit of A_λ $(\lambda \in \Lambda)$. Thus A is complete by $\mathfrak{U}$.

Theorem 5. If $\mathfrak{U}$ is sequential, then in order that $\mathfrak{U}$ be complete, it is necessary and sufficient that every closed totally bounded set is compact for the induced topology $\mathfrak{T}^{\mathfrak{U}}$.

Proof. Since the necessity is evident by Theorems 4 and 5 in §35, we need only prove the sufficiency. For a Cauchy sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$, it is obvious by definition that $\{a_\nu : \nu = 1, 2, \ldots\}$ is totally bounded, and hence its closure $\{a_\nu : \nu = 1, 2, \ldots\}^-$ also is so by §33 Theorem 3. Accordingly $\{a_\nu : \nu = 1, 2, \ldots\}^-$ is compact by assumption, and hence there is by §7 Theorem 2 a point a such that

$$\{a_\nu, a_{\nu+1}, \ldots\} U(a) \neq 0 \quad \text{for every } U \in \mathfrak{U} \text{ and } \nu = 1, 2, \ldots .$$

From this relation we conclude by §35 Theorem 1 that $\lim_{\nu \to \infty} a_\nu = a$. Thus $\mathfrak{U}$ is complete by Theorem 4.

Theorem 6. If $\mathfrak{U}$ is sequential and complete, then the induced topology $\mathfrak{T}^{\mathfrak{U}}$ is of the second category.

Proof. Let $\{U_1, U_2, \ldots\}$ be a decreasing basis of $\mathfrak{U}$. For an open set $A \neq 0$, if $A = \sum_{\nu=1}^{\infty} A_\nu$, $A_\nu^{-\circ} = 0$ $(\nu = 1, 2, \ldots)$, then we can find by §27 Theorem 1 a sequence of points $a_\nu \in A$ $(\nu = 1, 2, \ldots)$ and an increasing sequence of natural numbers μ_ν $(\nu = 1, 2, \ldots)$ such that

$$A \supset U_{\mu_1 - 1}(a_1), \quad (A_1^- \dotplus \cdots \dotplus A_\nu^-)' A\, U_{\mu_\nu}(a_\nu) \supset U_{\mu_{\nu+1}-1}(a_{\nu+1})$$

for every $\nu = 1, 2, \ldots$, because $(A_1^- \dotplus \ldots \dotplus A_\nu^-)^\circ = 0$. Then we have obviously $a_\rho \in U_{\mu_\nu}(a_\nu)$ for $\rho \geqq \nu$, and hence a_ν $(\nu = 1, 2, \ldots)$ is a Cauchy sequence. As $\mathfrak{U}$ is complete by assumption, there is hence a limit a, and for each $\nu = 1, 2, \ldots$ we can find ρ such that $U_{\mu_\nu}(a) \ni a_\rho$, $\rho \geqq \nu$, and hence $a \in U_{\mu_\nu}(a_\rho) \subset U_{\mu_\nu} \times U_{\mu_\nu}(a_\nu) \subset A$. But we have $a \,\overline{\in}\, \sum_{\nu=1}^{\infty} A_\nu^-$, contradicting the assumption $\sum_{\nu=1}^{\infty} A_\nu = A$. Therefore $\mathfrak{U}$ is of the second category by definition.

Theorem 7. If $\mathfrak{U}$ is sequential and the induced topology $\mathfrak{T}^{\mathfrak{U}}$ is separable, then $\mathfrak{T}^{\mathfrak{U}}$ is completely separable.

Proof. Let $\{U_1, U_2, \ldots\}$ be a decreasing basis of $\mathfrak{U}$ and $\{a_1, a_2, \ldots\}$ a dense set of R by $\mathfrak{T}^{\mathfrak{U}}$. Then $U_\nu^\circ(a_\mu)$ $(\nu, \mu = 1, 2, \ldots)$

is a neighbourhood system of $\mathfrak{T}^{\mathfrak{U}}$. Because for each $x \in R$ and for any $\nu = 1, 2, \ldots$ we can find μ such that $a_\mu \in U_{\nu+2}(x)$, and hence

$$x \in U_{\nu+2}(a_\mu) \subset U_{\nu+1}^\circ(a_\mu) \subset U_{\nu+1}(a_\mu) \subset U_\nu(x).$$

Thus $\mathfrak{T}^{\mathfrak{U}}$ is completely separable by definition.

§37 Completion

Let R be a uniform space with a uniformity $\mathfrak{U}$. A uniform space $\overline{R}$ with a complete uniformity $\overline{\mathfrak{U}}$ is said to be a complete extension of R, if $\overline{R}$ contains R as a subspace and $\mathfrak{U}$ coincides with the relative uniformity $\overline{\mathfrak{U}}^R$ of $\overline{\mathfrak{U}}$ in R. A complete extension $\overline{R}$ with a complete uniformity $\overline{\mathfrak{U}}$ is called a completion of R, if we have the completion conditions:

1) R is dense in $\overline{R}$ by the induced topology $\mathfrak{T}^{\overline{\mathfrak{U}}}$,

2) every $x \in \overline{R} - R$ is separated from the other points by $\overline{\mathfrak{U}}$.

We shall prove firstly that there is a completion of R. We denote by $\mathfrak{f}$ the totality of bounded uniformly continuous functions on R. Then the induced topology $\mathfrak{T}^{\mathfrak{U}}$ of $\mathfrak{U}$ coincides by §31 Theorem 6 with the weak topology of R by $\mathfrak{f}$. We denote further by $\widetilde{R}$ the compactification of R by $\mathfrak{f}$; by $\widetilde{\mathfrak{f}}$ the continuous extension of $\mathfrak{f}$ over $\widetilde{R}$; and by $\widetilde{\mathfrak{T}}$ the topology of $\widetilde{R}$. Then we see at once that the induced topology $\mathfrak{T}^{\mathfrak{U}}$ is the relative topology $\widetilde{\mathfrak{T}}^R$ of $\widetilde{\mathfrak{T}}$ in R, and $\widetilde{\mathfrak{T}}$ is the weak topology of $\widetilde{R}$ by $\widetilde{\mathfrak{f}}$. In the sequel, we consider point sets in $\widetilde{R}$ and employ the topological notations for the topology $\widetilde{\mathfrak{T}}$ of $\widetilde{R}$. We set

$$\overline{R} = \prod_{U \in \mathfrak{U}} \Big(\sum_{x \in R} U(x)^{-\circ} \Big),$$

$$\overline{U}(\overline{x}) = \sum_{U(x)^{-\circ} \ni \overline{x}} \overline{R}\, U(x)^{-\circ} \qquad \text{for } \overline{x} \in \overline{R} \text{ and } x \in R,$$

corresponding to every $U \in \mathfrak{U}$. Then we see easily that $R \subset \overline{R} \subset \widetilde{R}$ and $\overline{U}$ is a connector in the subspace $\overline{R}$. Furthermore we have obviously

(*) $\quad U \geqq V$ implies $\overline{U} \geqq \overline{V}$.

Since $\overline{U}(\overline{x}) \ni \overline{y}$ is equivalent to that $U(x)^{-\circ} \ni \overline{x}, \overline{y}$ for some $x \in R$, we have naturally

(**) $\quad \overline{U}^{-1} = \overline{U}$.

For a closed $V \in \mathcal{U}$, if $\bar{V}(\bar{x}) \ni \bar{y}$, $\bar{V}(\bar{y}) \ni \bar{z}$, then we can find two points $x, y \in R$ such that

$$V(x)^{-\circ} \ni \bar{x}, \bar{y}, \qquad V(y)^{-\circ} \ni \bar{y}, \bar{z},$$

and hence $V(x)^{-\circ} V(y)^{-\circ} \neq 0$. As R is dense in $\bar{R}$, we obtain then $R V(x)^{-\circ} V(y)^{-\circ} \neq 0$, and hence naturally $R V(x)^{-} V(y)^{-} \neq 0$. As V is closed by assumption, we conclude further $V(x) V(y) \neq 0$ by the formula §9(6). This relation yields by the relation §25(22)

$$V(y) \subset V \times V^{-1} \times V(x).$$

Therefore, if $V \times V^{-1} \times V \leqq U$, then we have $V(y)^{-\circ} \subset U(x)^{-\circ}$, and hence $U(x)^{-\circ} \ni \bar{x}, \bar{z}$, that is, $\bar{U}(\bar{x}) \ni \bar{z}$. Thus we have

(***) $V \times V^{-1} \times V \leqq U$ implies $\bar{V} \times \bar{V} \leqq \bar{U}$ for a closed $V \in \mathcal{U}$.

Recalling §27 Theorem 3 and the relation §26(6), we can conclude now that the totality of $\bar{U}$ ($U \in \mathcal{U}$) satisfies the basis conditions in §26, and hence there is by §26 Theorem 1 uniquely a uniformity $\bar{\mathcal{U}}$ on $\bar{R}$ such that $\bar{U}$ ($U \in \mathcal{U}$) is a basis of $\bar{\mathcal{U}}$. For such a uniformity $\bar{\mathcal{U}}$, $\mathcal{U}$ is the relative uniformity of $\bar{\mathcal{U}}$, that is, $\mathcal{U} = \bar{\mathcal{U}}^{R}$. Because, for each open $U \in \mathcal{U}$, if $U(x) \ni y$, then, as both x and y are inner points of $U(x)$ by the relative topology $\tilde{\mathcal{T}}^{R}$ and R is dense in $\tilde{R}$ by $\tilde{\mathcal{T}}$, we have by §9 Theorem 4 $R U(x)^{-\circ} \supset U(x) \ni x, y$, and hence $R \bar{U}(x) \ni y$, namely we have $\bar{U}^{R}(x) \ni y$. Thus $\bar{U}^{R} \geqq U$ for every open $U \in \mathcal{U}$. Recalling §27 Theorem 3, we conclude hence $\bar{\mathcal{U}}^{R} \subset \mathcal{U}$. On the other hand, for each closed $U \in \mathcal{U}$, as $R U(y)^{-\circ} \subset U(y)$ by the formula (10) and Theorem 4 in §9, we have for every point $x \in R$

$$\bar{U}^{R}(x) = \sum_{U(y)^{-\circ} \ni x} R U(y)^{-\circ} \subset \sum_{U(y) \ni x} U(y) = U^{-1} \times U(x),$$

that is, $\bar{U}^{R} \leqq U^{-1} \times U$. Thus we conclude $\bar{\mathcal{U}}^{R} \supset \mathcal{U}$ by the relation §26(5) and Theorem 3 in §27. Consequently we have $\bar{\mathcal{U}}^{R} = \mathcal{U}$.

Next we shall prove that the induced topology $\mathcal{T}^{\bar{\mathcal{U}}}$ by $\bar{\mathcal{U}}$ coincides with the relative topology $\tilde{\mathcal{T}}^{\bar{R}}$ of $\tilde{\mathcal{T}}$ in $\bar{R}$. For every $U \in \mathcal{U}$ we have obviously by the relation §9(10)

$$\bar{U}(\bar{x})^{\bar{R}\circ} = \bar{U}(\bar{x}) \ni \bar{x},$$

and hence we obtain $\tilde{\mathcal{T}}^{\bar{R}} \supset \mathcal{T}^{\bar{\mathcal{U}}}$ by §27 Theorem 5. On the other hand, if $\bar{a} \in \bar{R} A$, $A \in \tilde{\mathcal{T}}$, then we can find by §22 Theorem 5 $\tilde{\varphi} \in \tilde{\mathcal{F}}$ such that

$$\tilde{\varphi}(\bar{a}) = 1 \quad \text{and} \quad \tilde{\varphi}(\bar{x}) = 0 \quad \text{for } \bar{x} \,\bar{\in}\, A.$$

Since such $\tilde{\varphi}$ is uniformly continuous in R by $\mathcal{U}$, we can find $U \in \mathcal{U}$ for which $U(x) \ni y$ implies $|\tilde{\varphi}(x) - \tilde{\varphi}(y)| < \frac{1}{3}$. For such $U \in \mathcal{U}$, if we have $U(x)^{-\circ} \ni \bar{a}, \bar{x}$, then we obtain by §20 Theorem 3

$$|\tilde{\varphi}(x) - \tilde{\varphi}(\bar{a})| \leqq \frac{1}{3}, \qquad |\tilde{\varphi}(x) - \tilde{\varphi}(\bar{x})| \leqq \frac{1}{3},$$

and hence $\tilde{\varphi}(\bar{x}) \geqq \frac{1}{3}$. Thus we conclude

$$\bar{U}(\bar{a}) = \sum_{U(x)^{-\circ} \ni \bar{a}} \bar{R}\, U(x)^{-\circ} \subset \bar{R} A.$$

This relation yields $\tilde{\mathcal{T}}^{\bar{R}} \subset \mathcal{T}^{\bar{\mathcal{U}}}$ by §27 Theorem 6. Therefore we obtain our assertion $\mathcal{T}^{\bar{\mathcal{U}}} = \tilde{\mathcal{T}}^{\bar{R}}$. From this fact we conclude at once that R is dense in $\bar{R}$ by the induced topology $\mathcal{T}^{\bar{\mathcal{U}}}$ and each point $\bar{x} \in \bar{R} - R$ is separated from the other points of $\bar{R}$ by $\bar{\mathcal{U}}$.

Finally we shall prove that $\bar{R}$ is complete by $\bar{\mathcal{U}}$. For a Cauchy system $A_{\lambda} \subset \bar{R}$ ($\lambda \in \Lambda$) by $\bar{\mathcal{U}}$, we have $\prod_{\lambda \in \Lambda} A_{\lambda}^{-} \neq 0$, because $\tilde{R}$ is compact by $\tilde{\mathcal{T}}$. Thus we can find a point $a \in \prod_{\lambda \in \Lambda} A_{\lambda}^{-}$. For each $U \in \bar{\mathcal{U}}$ there is by the relation §26(6) and §27 Theorem 3 an open $V \in \bar{\mathcal{U}}$ for which $V^{-1} \times V \times V \leqq U$. For such $V \in \bar{\mathcal{U}}$, we can find $\lambda_0 \in \Lambda$ and a point $b \in \bar{R}$ such that $A_{\lambda_0} \subset V(b)$. As R is dense in $\bar{R}$ by the induced topology $\mathcal{T}^{\bar{\mathcal{U}}}$, there is a point $c \in R V(b)$, and we have by the formula §25(21) $V(b) \subset V^{-1} \times V(c)$. This relation yields by the formula §29(5)

$$(R V(b)) \times V^{R} \subset (V^{-1} \times V)^{R} \times V^{R}(c) \subset U^{R}(c).$$

Therefore we can find by §31 Theorem 5 $\varphi \in \mathcal{F}$ such that

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in R V(b), \\ 1 & \text{for } x \in R - U^{R}(c). \end{cases}$$

For the continuous extension $\tilde{\varphi}$ of φ over $\tilde{R}$, we have by §20 Theorem 3

$$\tilde{\varphi}(x) = \begin{cases} 0 & \text{for } x \in (R V(b))^{-}, \\ 1 & \text{for } x \in (R - U^{R}(c))^{-}. \end{cases}$$

As V is an open connector, we can find $B \in \tilde{\mathcal{T}}$ such that $V(b) = \bar{R} B$, and we have by the formula §4(8)

$$(R V(b))^{-} = (R B)^{-} \supset (R^{-} B^{\circ})^{-} \supset (\bar{R} B)^{-} = V(b)^{-}.$$

Thus we have $a \in A_{\lambda_0}^{-} \subset V(b)^{-} \subset (R V(b))^{-}$, and hence $\tilde{\varphi}(a) = 0$. Consequently we obtain by the formula §4(8)

$$a \in (R - U^{R}(c))^{-\prime} = (R\, U^{R}(c)^{\prime})^{-\prime} \subset (R^{-} U^{R}(c)^{\prime\circ})^{-\prime} = U^{R}(c)^{-\circ}$$

We conclude hence $a \in \bar{R}$ by the construction of $\bar{R}$, because $v^R \in \mathfrak{V}$ as proved just above. Furthermore we have by the formula §9(6) $a \in \bar{R} A_\lambda^- = A_\lambda^{\gamma^{\bar{\mathfrak{R}}}-}$ for every $\lambda \in \Lambda$, and consequently by §27(3) $A_\lambda v(a) \neq 0$ for every $\lambda \in \Lambda$ and $v \in \bar{\mathfrak{V}}$. Thus a is by §35 Theorem 1 a limit of $A_\lambda (\lambda \in \Lambda)$. Therefore $\bar{R}$ is a completion of R.

Let $\bar{R}$ and $\bar{\bar{R}}$ be two completions of R. By virtue of §35 Theorem 6, we can find a uniformly continuous mapping α of $\bar{R}$ into $\bar{\bar{R}}$ such that $\alpha(x) = x$ for $x \in R$, and b of $\bar{\bar{R}}$ into $\bar{R}$ such that $b(x) = x$ for $x \in R$. Then we have $b\alpha(x) = x$ for $x \in \bar{R}$. Because, if $b\alpha(x) \neq x$ for some $x \in \bar{R} - R$, then, as x is separated from the other points, we can find open sets $A, B \subset \bar{R}$ such that $x \in A$, $b\alpha(x) \in B$, $AB = 0$. As both mappings α and b are continuous for the induced topologies by §30 Theorem 2, $\alpha^{-1} b^{-1}(B)$ is open and contains x. Consequently we have $R A \alpha^{-1} b^{-1}(B) \neq 0$, as R is dense in $\bar{R}$. From this relation we conclude by §15(12) $b\alpha(RA)B \neq 0$. As $b\alpha(RA) = RA$, we obtain hence $RAB \neq 0$, contradicting $AB = 0$. We also can prove likewise that $\alpha b(x) = x$ for every $x \in \bar{\bar{R}}$. Therefore we have $b = \alpha^{-1}$.

Now we can state

Theorem 1. Every uniform space R has a completion uniquely within a homeomorphism.

Recalling §36 Theorem 2, we obtain immediately by definition

Theorem 2. For a sequential uniform space R, its completion is sequential too.

Furthermore we have obviously by definition

Theorem 3. If a uniform space R is separated by its uniformity $\mathfrak{V}$, then its completion is separated too.

CHAPTER V

METRIC SPACES

§38 Quasi-metric

Let R be an abstract space. A function $m(a, b)$ of a pair of points in R is called a quasi-metric, if we have the metric conditions:

1) $0 \leq m(a, b) < +\infty$,

2) $m(a, a) = 0$,

3) $m(a, b) = m(b, a)$,

4) $m(a, b) + m(b, c) \geq m(a, c)$.

A space R associated with a quasi-metric m is called a quasi-metric space. A quasi-metric m will be called a metric, if we have further

5) $m(a, b) = 0$ implies $a = b$.

A space R associated with a metric m is called a metric space.

Let R be a quasi-metric space with a quasi-metric m in the sequel. For each point $a \in R$ and for every positive number ε, a point set

$$U_\varepsilon(a) = \{x : m(a, x) < \varepsilon\}$$

is called a sphere with a radius ε and a center a.

Concerning spheres we have obviously

(1) $U_\varepsilon(a) \subset U_\delta(a)$ for $\varepsilon \leq \delta$,

(2) $U_\varepsilon(a) = \sum_{\lambda \in \Lambda} U_\lambda(a)$ for $\varepsilon = \sup_{\lambda \in \Lambda} \lambda$,

(3) $U_\varepsilon(a) \supset U_\delta(b)$ for $\varepsilon - \delta \geq m(a, b)$.

Because, for every point $x \in U_\delta(b)$ we have $m(a,x) \leq m(a,b) + m(b,x) < \varepsilon$.

(4) $U_\varepsilon(a) U_\delta(b) = 0$ for $\varepsilon + \delta \leq m(a, b)$.

Because, for every point $x \in U_\varepsilon(a)$ we have by 3) and 4)

$$m(b, x) \geq m(a, b) - m(a, x) > \delta.$$

Corresponding to every positive number ε, we obtain a connector U_ε in R as $R \ni x \longrightarrow U_\varepsilon(x)$ for the sphere $U_\varepsilon(x)$. This connector U_ε will be called a sphere connector with a radius ε by the quasi-metric m. If we need indicate the quasi-metric m, we shall write U_ε^m.

Concerning sphere connectors, we have obviously by definition

(5) $U_\varepsilon \leqq U_\delta$ for $\varepsilon \leqq \delta$,

(6) $U_\varepsilon U_\delta \geqq U_{Min\{\varepsilon,\delta\}}$,

(7) $U_\varepsilon^{-1} = U_\varepsilon$.

On account of the metric condition 4), we conclude easily

(8) $U_\varepsilon \times U_\delta \leqq U_{\varepsilon+\delta}$.

For a point set A and a point a we define $m(A,a)$ to mean

(9) $m(A,a) = \inf_{x\in A} m(x,a)$,

but $m(0,a) = +\infty$. With this definition we have obviously

(10) $m(A,x) = 0$ for $x \in A$,

(11) $m(A,x) \leqq m(B,x)$ for $A \supset B$,

(12) $m(\sum_{\lambda\in\Lambda} A_\lambda, a) = \inf_{\lambda\in\Lambda} m(A_\lambda, a)$.

We also can prove easily by the metric condition 4)

(13) $m(U_\varepsilon(a), x) \geqq m(a,x) - \varepsilon$,

(14) $m(A,a) + m(a,b) \geqq m(A,b)$.

§39 Induced uniformities and topologies

Let R be a quasi-metric space with a quasi-metric m. We see easily by the formulas (6), (7), (8) in §38 that the totality of sphere connectors U_ε for all $\varepsilon > 0$ satisfies the basis conditions in §26, and hence there exists by §26 Theorem 1 uniquely a uniformity $\mathfrak{U}$ on R, of which U_ε ($\varepsilon > 0$) is a basis. This uniformity $\mathfrak{U}$ is called the induced uniformity of R by m and denoted by $\mathfrak{U}^m$.

With this definition we see at once that $U_{\frac{1}{\nu}}$ ($\nu = 1, 2, \ldots$) is a basis of the induced uniformity $\mathfrak{U}^m$. Therefore we have

Theorem 1. The induced uniformity $\mathfrak{U}^m$ by a quasi-metric m is sequential.

Theorem 2. $m(x,y)$ is uniformly continuous by $\mathfrak{U}^m$: for any $\varepsilon > 0$ we can find a sphere connector U_δ such that $U_\delta(x_1) \ni x$, $U_\delta(y_1) \ni y$ implies $|m(x_1,y_1) - m(x,y)| < \varepsilon$.

Proof. On account of the metric conditions 3), 4) we have

$$|m(x_1,y_1) - m(x,y)| \leqq m(x_1,x) + m(y_1,y).$$

Thus, putting $\delta = \frac{1}{2}\varepsilon$, we obtain our assertion.

We also can prove likewise

Theorem 3. $m(A,x)$ ($x \in R$) is a uniformly continuous function on R by the induced uniformity $\mathfrak{U}^m$.

Theorem 4. R is separated by the induced uniformity $\mathfrak{U}^m$, if and only if m is a metric.

Proof. It is evident by definition that we have

$$\prod_{\varepsilon>0} U_\varepsilon(a) = \{a\} \quad \text{for every } a \in R,$$

if and only if m is a metric.

Theorem 5. Let m_1 and m_2 be two quasi-metrics on a space R. For the induced uniformities $\mathfrak{U}^{m_1}$ and $\mathfrak{U}^{m_2}$, we have $\mathfrak{U}^{m_1} \supset \mathfrak{U}^{m_2}$, if and only if for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$m_1(x,y) < \delta \text{ implies } m_2(x,y) < \varepsilon.$$

Proof. If $\mathfrak{U}^{m_1} \supset \mathfrak{U}^{m_2}$, then we have $U_\varepsilon^{m_2} \in \mathfrak{U}^{m_1}$ for every $\varepsilon > 0$, and hence we can find $\delta > 0$ such that $U_\delta^{m_1} \leqq U_\varepsilon^{m_2}$, that is, $m_1(x,y) < \delta$ implies $m_2(x,y) < \varepsilon$. Conversely, if for any $\varepsilon > 0$ we can find $\delta > 0$ such that $U_\delta^{m_1} \leqq U_\varepsilon^{m_2}$, then we have obviously $\mathfrak{U}^{m_1} \supset \mathfrak{U}^{m_2}$.

For the induced uniformity $\mathfrak{U}^m$, the induced topology by $\mathfrak{U}^m$ is called the induced topology by m and denoted by $\mathfrak{T}^m$.

For the induced topology $\mathfrak{T}^m$ we have

(1) $A^- = \{x : m(A,x) = 0\}$,

(2) $A^\circ = \{x : m(A',x) > 0\}$

for every point set A. Because, if $m(A,a) = 0$, then we have obviously $A U_\varepsilon(a) \neq 0$ for every $\varepsilon > 0$, and hence $a \in A^-$ by §27 Theorem 1. Conversely, if $a \in A^-$, then we have $A U_\varepsilon(a) \neq 0$ for every $\varepsilon > 0$, and hence $m(A,a) = 0$ by the definition §38(9). Thus we obtain the formula (1). Recalling the formula §4(3), we obtain by (1)

$$A^\circ = A'^{-\prime} = \{x : m(A',x) = 0\}' = \{x : m(A',x) > 0\}.$$

Recalling Theorems 2 in §20 and 2 in §30, we obtain immediately by §38 Theorem 3

Theorem 6. Every sphere $U_\varepsilon(a)$ is open by the induced topology $\mathfrak{T}^m$.

Theorem 7. Let m_1 and m_2 be two quasi-metrics on a space R. For the induced topologies $\mathfrak{T}^{m_1}$ and $\mathfrak{T}^{m_2}$, we have $\mathfrak{T}^{m_1} > \mathfrak{T}^{m_2}$, if and only if for any $\varepsilon > 0$ and for any $a \in R$ we can find $\delta > 0$ such that $m_1(a, x) < \delta$ implies $m_2(a, x) < \varepsilon$.

Proof. By virtue of Theorem 1 in §27, if $\mathfrak{T}^{m_1} > \mathfrak{T}^{m_2}$, then for any $\varepsilon > 0$ and $a \in R$ we can find $\delta > 0$ such that $U_\delta^{m_1}(a) \subset U_\varepsilon^{m_2}(a)$. We also can prove likewise the inverse.

Every subspace S of R may be considered as a quasi-metric space by the same quasi-metric m. This same quasi-metric m in a subspace S is called the relative quasi-metric of m in S and denoted by m^S, that is, $m^S(x, y) = m(x, y)$ for $x, y \in S$.

With this definition we have obviously

(3) $\quad U_\varepsilon^{m^S} = (U_\varepsilon^m)^S$ for every $\varepsilon > 0$.

Therefore we obtain

Theorem 8. The induced uniformity and topology by the relative quasi-metric m^S coincide respectively with the relative uniformity and topology of the induced uniformity and topology by m, that is,

$$\mathfrak{U}^{m^S} = (\mathfrak{U}^m)^S, \qquad \mathfrak{T}^{m^S} = (\mathfrak{T}^m)^S.$$

§40 Completion

Let R be a quasi-metric space with a quasi-metric m. By virtue of §27 Theorem 1, the system of spheres $\{U_\varepsilon(a) : \varepsilon > 0\}$ is a neighbourhood system of a point a for the induced topology $\mathfrak{T}^m$. Thus we see easily that $\lim_{\nu\to\infty} a_\nu = a$ is equivalnet to $\lim_{\nu\to\infty} m(a_\nu, a) = 0$.

As the system of sphere connectors $\{U_\varepsilon : \varepsilon > 0\}$ is a basis of the induced uniformity $\mathfrak{U}^m$, we see further that a point sequence $a_\nu \in R$ ($\nu = 1, 2, \dots$) is a Cauchy sequence if and only if $\lim_{\nu,\mu\to\infty} m(a_\nu, a_\mu) = 0$.

A point set A is said to be complete by m, if A is complete by the induced uniformity $\mathfrak{U}^m$. Recalling Theorems 4 in §36 and 1 in §39, we have then obviously

Theorem 1. A is complete by m if and only if $\lim_{\nu,\mu\to\infty} m(a_\nu, a_\mu) = 0$,

$a_\nu \in A$ ($\nu = 1, 2, \dots$) implies $\lim_{\nu\to\infty} m(a_\nu, a) = 0$ for some point $a \in A$.

We shall say that R is complete by m or that m is complete, if R is complete by m as a point set. A complete quasi-metric space $\bar{R}$ with a quasi-metric $\bar{m}$ is called a completion of R, if $\bar{R}$ is a completion of R for the induced uniformities $\mathfrak{U}^{\bar{m}}$, $\mathfrak{U}^m$, and m is a relative quasi-metric of $\bar{m}$, that is, $\bar{m}(x, y) = m(x, y)$ for $x, y \in R$.

Theorem 2. Every quasi-metric space R has a completion uniquely within a homeomorphism.

Proof. By virtue of §37 Theorem 1, considering R as a uniform space with the induced uniformity $\mathfrak{U}^m$, we obtain a completion $\bar{R}$ of R uniquely within a homeomorphism. Since $\mathfrak{U}^m$ is sequential, the uniformity $\bar{\mathfrak{U}}$ of $\bar{R}$ also is sequential by §37 Theorem 2. Therefore we see by Theorems 2 in §14 and 3 in §36 that for any $a \in \bar{R}$ we can find $a_\nu \in R$ ($\nu = 1, 2, \dots$) such that $\lim_{\nu\to\infty} a_\nu = a$. For a pair of points $a, b \in \bar{R}$, if

(*) $\quad \lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} b_\nu = b, \quad a_\nu, b_\nu \in R \ (\nu = 1, 2, \dots),$

then both a_ν and b_ν ($\nu = 1, 2, \dots$) are Cauchy sequences, and hence

$$\lim_{\nu,\mu\to\infty} m(a_\nu, a_\mu) = 0, \qquad \lim_{\nu,\mu\to\infty} m(b_\nu, b_\mu) = 0.$$

Since we obtain by the metric conditions 3) and 4)

$$|m(a_\nu, b_\nu) - m(a_\mu, b_\mu)| \leq m(a_\nu, a_\mu) + m(b_\nu, b_\mu),$$

we conclude thus

$$\lim_{\nu,\mu\to\infty} |m(a_\nu, b_\nu) - m(a_\mu, b_\mu)| = 0.$$

Therefore $m(a_\nu, b_\nu)$ ($\nu = 1, 2, \dots$) is convergent for every a_ν, b_ν ($\nu = 1, 2, \dots$) subject to the condition (*), and hence tends to the same limit. Thus, putting

$$\bar{m}(a, b) = \lim_{\nu\to\infty} m(a_\nu, b_\nu),$$

we see easily that $\bar{m}$ satisfies the metric conditions 1), 2), 3), 4), and hence $\bar{m}$ is a quasi-metric on $\bar{R}$. Furthermore we have obviously

$$\bar{m}(x, y) = m(x, y) \qquad \text{for } x, y \in R.$$

Finally we shall prove that $\bar{\mathfrak{U}}$ coincides with the induced uniformity of $\bar{R}$ by $\bar{m}$, that is, $\bar{\mathfrak{U}} = \mathfrak{U}^{\bar{m}}$. For any $\varepsilon > 0$, we can find $U \in \bar{\mathfrak{U}}$ such that $U^R(x) \ni y$ implies $m(x, y) < \frac{1}{3}\varepsilon$, and hence $U^\circ(x) \ni a, b$ implies

$$\bar{m}(a, b) \leqq \bar{m}(x, a) + \bar{m}(x, b) \leqq \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon < \varepsilon,$$

because we conclude $U^\circ(x) \subset (R\,U(x))^-$ from $R^- = \bar{R}$ and $\bar{m}(a, b)$ is continuous by §39 Theorem 2. For such $U \in \bar{\mathfrak{U}}$, we can find by the relation §26(5) a symmetric $V \in \bar{\mathfrak{U}}$ such that $V \times V \leqq U^\circ$, and if $V(a) \ni b$, then for a point $x \in R\,V(a)$ we have

$$a \in V(x) \subset U^\circ(x), \qquad b \in V \times V(x) \subset U^\circ(x),$$

and hence $\bar{m}(a, b) < \varepsilon$. Therefore we conclude $\bar{\mathfrak{U}} > \mathfrak{U}^{\bar{m}}$ by definition.

On the other hand, for any $U \in \bar{\mathfrak{U}}$ we can find by the relation §26 (6) a symmetric $V \in \bar{\mathfrak{U}}$ such that $V \times V \times V \leqq U$, and then further $\varepsilon > 0$ such that $m(x, y) < \varepsilon$, $x, y \in R$ implies $V(x) \ni y$. For such V and ε, if $\bar{m}(a, b) < \frac{1}{3}\varepsilon$, $a, b \in \bar{R}$, then we can find two points $x \in R\,V(a)$ and $y \in R\,V(b)$ such that $\bar{m}(a, x) < \frac{1}{3}\varepsilon$, $\bar{m}(b, y) < \frac{1}{3}\varepsilon$ and hence $m(x, y) \leqq \bar{m}(a, x) + \bar{m}(a, b) + \bar{m}(b, y) < \varepsilon$. For such $x, y \in R$ we have thus $V(x) \ni y$, and consequently

$$b \in V(y) \subset V(x) \times V \subset V(a) \times V \times V \subset U(a).$$

Therefore we conclude $\mathfrak{U}^{\bar{m}} > \bar{\mathfrak{U}}$. Thus we obtain $\bar{\mathfrak{U}} = \mathfrak{U}^{\bar{m}}$.

Recalling §37 Theorem 3, we have obviously

Theorem 3. For a metric space, its completion is a metric space.

§41 Metrization

The induced uniformity by a quasi-metric is sequential by §39 Theorem 1. Conversely we have

Theorem 1. For a uniform space R with a sequential uniformity $\mathfrak{U}$, we can find a quasi-metric m on R such that $\mathfrak{U}$ coincides with the induced uniformity $\mathfrak{U}^m$ by m.

Proof. Let $\{U_0, U_1, U_2, \dots\}$ be a decreasing basis of $\mathfrak{U}$. By virtue of §31 Theorem 4, corresponding to each point $a \in R$ there is a function φ_a on the subspace $\sum_{\mu=1}^{\infty} U_0^{\mu}(a)$ such that

$$U_\nu(x) \ni y \quad \text{implies} \quad |\varphi_a(x) - \varphi_a(y)| \leqq \frac{1}{2^{\nu-1}},$$

$$\varphi_a(x) \begin{cases} \geqq \frac{1}{2^\nu} & \text{for } x \,\bar{\in}\, U_\nu(a), \\ \geqq 1 & \text{for } x \,\bar{\in}\, U_0(a), \end{cases}$$

$\varphi_a(a) = 0$ and $\varphi_a(x) \geqq 0$ for every $x \in \sum_{\mu=1}^{\infty} U_0^{\mu}(a)$. For such φ_a putting

$$\omega_a(x) = \begin{cases} \operatorname{Min}\{\varphi_a(x), 1\} & \text{for } x \in \sum_{\mu=1}^{\infty} U_0^{\mu}(a), \\ 1 & \text{for } x \,\bar{\in}\, \sum_{\mu=1}^{\infty} U_0^{\mu}(a), \end{cases}$$

we obtain a positive function ω_a on R such that

$$\omega_a(x) = \begin{cases} 0 & \text{for } x = a, \\ 1 & \text{for } x \,\bar{\in}\, U_0(a), \end{cases}$$

$$U_\nu(x) \ni y \quad \text{implies} \quad |\omega_a(x) - \omega_a(y)| \leqq \frac{1}{2^{\nu-1}},$$

$$U_\nu(a) \,\bar{\ni}\, x \quad \text{implies} \quad \omega_a(x) \geqq \frac{1}{2^\nu},$$

because we have by §25 Theorem 2

$$\Big(\sum_{\mu=1}^{\infty} U_0^{\mu}(a)\Big) \times U_\nu = \sum_{\mu=1}^{\infty} U_0^{\mu}(a), \quad \Big(\sum_{\mu=1}^{\infty} U_0^{\mu}(a)\Big)' \times U_\nu = \Big(\sum_{\mu=1}^{\infty} U_0^{\mu}(a)\Big)'.$$

For such functions ω_a $(a \in R)$, putting

$$\varphi(x, y) = \inf_{a \in R} \{\omega_a(x) + \omega_a(y)\},$$

we have obviously

$$0 \leqq \varphi(x, y) \leqq 1, \qquad \varphi(x, y) = \varphi(y, x), \qquad \varphi(x, x) = 0.$$

Furthermore, if we set

$$m(x, y) = \sup_{z \in R} |\varphi(x, z) - \varphi(y, z)|,$$

then we see easily that m satisfies the metric conditions 1), 2), 3), 4), that is, m is a quasi-metric on R. We shall prove now that $\mathfrak{U} = \mathfrak{U}^m$ for this quasi-metric m.

If $U_\nu(x) \ni y$, then we have

$$\omega_a(x) \leqq \omega_a(y) + \frac{1}{2^{\nu-1}},$$

and hence for every point $z \in R$

$$\varphi(x, z) \leqq \varphi(y, z) + \frac{1}{2^{\nu-1}}.$$

We also obtain likewise

$$\varphi(y, z) \leqq \varphi(x, z) + \frac{1}{2^{\nu-1}}.$$

Thus $U_\nu(x) \ni y$ implies $m(x, y) \leqq \frac{1}{2^{\nu-1}}$. Therefore we have $\mathfrak{U} > \mathfrak{U}^m$.

On the other hand, as $U_{\nu+1} \times U_{\nu+1} \leqq U_\nu$, we see easily that $U_\nu(x) \,\bar{\ni}\, y$ implies $U_{\nu+1}(x)\,U_{\nu+1}(y) = 0$, and hence we have

$$U_{\nu+1}(x) \,\bar{\ni}\, a \quad \text{or} \quad U_{\nu+1}(y) \,\bar{\ni}\, a \qquad \text{for every } a \in R.$$

This relation yields

$$\omega_a(x) \geqq \frac{1}{2^{\nu+1}} \quad \text{or} \quad \omega_a(y) \geqq \frac{1}{2^{\nu+1}} \qquad \text{for every } a \in R,$$

and consequently $\varphi(x, y) \geqq \frac{1}{2^{\nu+1}}$. Thus $U_\nu(x) \,\bar{\ni}\, y$ implies

$$m(x,y) \geqq |\varphi(x,y) - \varphi(y,y)| \geqq \frac{1}{2^{\nu+1}},$$

that is, $m(x,y) < \frac{1}{2^{\nu+1}}$ implies $U_\nu(x) \ni y$. Therefore we conclude $\mathfrak{U}^m \supset \mathfrak{U}$. Consequently we obtain $\mathfrak{U} = \mathfrak{U}^m$.

Theorem 2. (Urysohn) For a topological space R, if its topology $\mathfrak{T}$ is regular and completely separable, then we can find a quasi-metric m on R such that $\mathfrak{T}$ coincides with the induced topology $\mathfrak{T}^m$.

Proof. Let $\{U_1, U_2, \dots\}$ be a neighbourhood system of $\mathfrak{T}$. As $\mathfrak{T}$ is normal by §14 Theorem 3, for a pair U_ν, U_μ subject to $U_\nu \prec U_\mu$, we can find by §23 Theorem 4 a continuous function $\varphi_{\nu,\mu}$ such that we have $0 \leqq \varphi_{\nu,\mu}(x) \leqq 1$ for every $x \in R$ and

$$\varphi_{\nu,\mu}(x) = \begin{cases} 0 & \text{for } x \in U_\nu, \\ 1 & \text{for } x \,\bar{\in}\, U_\mu. \end{cases}$$

Then the totality of such functions $\varphi_{\nu,\mu}$ $(U_\nu \prec U_\mu)$ is by §22 Theorem 5 a trunk of $\mathfrak{T}$, because for any $a \in A \in \mathfrak{T}$ we can find ν, μ such that $a \in U_\nu \prec U_\mu \subset A$. Therefore we see that there is a sequence of functions φ_ν $(\nu = 1, 2, \dots)$ which is a trunk of $\mathfrak{T}$. Let $\mathfrak{U}$ be the weak uniformity of R by φ_ν $(\nu = 1, 2, \dots)$. Then $\mathfrak{U}$ is sequential by §34 Theorem 4, and we see by §34 Theorem 3 that $\mathfrak{T}$ coincides with the induced topology of R by $\mathfrak{U}$. Therefore we obtain our assertion by Theorem 1.

Let m_λ $(\lambda \in \Lambda)$ be a system of quasi-metrics on an abstract space R. For the induced uniformity $\mathfrak{U}^{m_\lambda}$ by m_λ $(\lambda \in \Lambda)$, the weakest stronger uniformity $\bigcup_{\lambda\in\Lambda} \mathfrak{U}^{m_\lambda}$ is called the induced uniformity by a system of quasi-metrics m_λ $(\lambda \in \Lambda)$.

Theorem 3. For a uniform space R with a uniformity $\mathfrak{U}$ there is a system of quasi-metrics m_λ $(\lambda \in \Lambda)$ on R such that $\mathfrak{U}$ coincides with the induced uniformity by m_λ $(\lambda \in \Lambda)$.

Proof. Corresponding to every $U \in \mathfrak{U}$, we obtain by §26(5) a decreasing sequence $V_{U,\nu} \in \mathfrak{U}$ $(\nu = 1, 2, \dots)$ such that $V_{U,1} \times V_{U,1} \leqq U$, and there is by §26 Theorem 1 a uniformity $\mathfrak{U}_U$ on R of which $V_{U,\nu}$ $(\nu = 1, 2, \dots)$ is a basis. Then we have obviously $\mathfrak{U} = \bigcup_{U\in\mathfrak{U}} \mathfrak{U}_U$, and $\mathfrak{U}_U$ is sequential. Therefore we obtain our assertion by Theorem 1.

CHAPTER VI

LINEAR SPACES

§42 Fundamental definitions

A space R is called a commutative group, if for every two elements $a, b \in R$ we have $a + b \in R$ such that

1) $a + b = b + a$,

2) $(a + b) + c = a + (b + c)$,

3) for any $a, b \in R$ we can find $c \in R$ such that $a = b + c$.

Such an element $c \in R$ is uniquely determined. Because, if

$$a = b + c = b + c_1,$$

then we can find by 3) $e, d \in R$ such that

$$c = c_1 + d, \quad c_1 = a + e,$$

and we have by 1) and 2)

$$c_1 = a + e = (b + c) + e = (b + (c_1 + d)) + e$$
$$= ((b + c_1) + d) + e = (a + e) + d = c_1 + d = c.$$

Therefore we denote such $c \in R$ by $a - b$. Then we have naturally

$$a = b + (a - b)$$

for every $a, b \in R$. Especially $b = b + (b - b)$, and hence by 1), 2)

$$a = (b + (b - b)) + (a - b) = (b + (a - b)) + (b - b) = a + (b - b).$$

Consequently we obtain $a - a = b - b$ for every $a, b \in R$, that is, for every $a \in R$ we obtain the same element $a - a$. This uniquely determined element $a - a$ $(a \in R)$ is called the zero element of R and denoted by 0.

For each $a \in R$ we define $-a$ to mean $0 - a$. Then we have

$$-(-a) = a,$$

because $0 = a + (-a) = (-a) + (-(-a))$. As

$$b + (a + (-b)) = a + (b + (-b)) = a + 0 = a,$$

we have for every $a, b \in R$

$$a + (-b) = a - b.$$

A commutative group R is said to be a linear space, if for every $a \in R$ and for every real number α we have $\alpha a \in R$ such that

4) $\alpha(\beta a) = (\alpha\beta) a$,

5) $\alpha a + \beta a = (\alpha + \beta) a$,

6) $\alpha a + \alpha b = \alpha(a + b)$,

7) $1 a = a$.

As $0a + 0a = (0+0)a = 0a$ by 5), we have $0a = 0$ for every $a \in R$. Therefore we have $a + (-1)a = 0a = 0$ by 5), 7), and hence

$$(-1)a = -a.$$

Furthermore we have for every $a, b \in R$

$$\alpha(a - b) = \alpha a - \alpha b,$$

because $\alpha(a - b) = \alpha(a + (-b)) = \alpha a + \alpha(-1) b = \alpha a - \alpha b$.

For a subset A of a linear space R, an element $a \in R$ is said to be a linear combination from A, if we can find a finite number of elements $x_\nu \in A$ and of real numbers α_ν $(\nu = 1, 2, \dots, \kappa)$ such that

$$a = \sum_{\nu=1}^{\kappa} \alpha_\nu x_\nu .$$

An element $a \in R$ is said to be linearly independent from A, if a is not a linear combination from A. A subset $A \subset R$ is said to be linearly independent, if every element of A is linearly independent from the other elements of A, that is, if $\sum_{\nu=1}^{\kappa} \alpha_\nu x_\nu = 0$ implies $\alpha_\nu = 0$ $(\nu = 1, 2, \dots, \kappa)$ for every finite number of different elements $x_\nu \in A$ $(\nu = 1, 2, \dots, \kappa)$.

§43 Manifolds

Let R be a linear space. A subset $A \neq 0$ of R is called a manifold of R. For two manifolds A, B of R we define $A \times B$ to mean

(1) $A \times B = \{x + y : x \in A, y \in B\}$.

With this definition we have obviously

(2) $A \times B = B \times A$, $(A \times B) \times C = A \times (B \times C)$.

(3) $\sum_{\lambda, \rho} (A_\lambda \times B_\rho) = (\sum_\lambda A_\lambda) \times (\sum_\rho B_\rho)$.

For a manifold A and an element $a \in R$ we define $A + a$ to mean

(4) $A + a = A \times \{a\} = \{x + a : x \in A\}$.

With this definition we see easily

(5) $(A + a) + b = (A + b) + a = A + (a + b)$,

(6) $\sum_{\lambda \in \Lambda} (A_\lambda + a) = \sum_{\lambda \in \Lambda} A_\lambda + a$,

(7) $\prod_{\lambda \in \Lambda} (A_\lambda + a) = \prod_{\lambda \in \Lambda} A_\lambda + a$ for $\prod_{\lambda \in \Lambda} A_\lambda \neq 0$,

(8) $(A \times B) + a = (A + a) \times B = A \times (B + a)$,

(9) $(A + a) \times (B + b) = (A \times B) + (a + b)$,

(10) $(A + a)' = A' + a$ for $A' \neq 0$,

(11) $(A + a) - (B + a) = (A - B) + a$ for $A \supset B$, $A - B \neq 0$.

For a manifold A and a real number α we define αA to mean

(12) $\alpha A = \{\alpha x : x \in A\}$.

With this definition we have obviously

(13) $\alpha \sum_{\lambda \in \Lambda} A_\lambda = \sum_{\lambda \in \Lambda} \alpha A_\lambda$,

(14) $\alpha \prod_{\lambda \in \Lambda} A_\lambda = \prod_{\lambda \in \Lambda} \alpha A_\lambda$ for $\prod_{\lambda \in \Lambda} A_\lambda \neq 0$,

(15) $\alpha A - \alpha B = \alpha(A - B)$ for $A \supset B$, $A \neq 0$,

(16) $(\alpha A)' = \alpha A'$ for $\alpha \neq 0$, $A' \neq 0$,

(17) $\alpha(\beta A) = (\alpha\beta) A$,

(18) $\alpha(A \times B) = (\alpha A) \times (\alpha B)$,

(19) $\alpha(A + a) = \alpha A + \alpha a$.

A manifold A is said to be linear, if $A \ni a, b$ implies $A \ni \alpha a + \beta b$ for every real numbers α, β, that is, if $A \times A = A$, $\alpha A = A$ for every real number $\alpha \neq 0$.

For a linear manifold S we have

(20) $SA \times SB \subset S(A \times B)$.

Because $SA \times SB \subset A \times B$, $SA \times SB \subset S \times S = S$.

For an arbitrary manifold A, the totality of linear combinations from A constitutes obviously a linear manifold. This linear manifold is called the linear manifold generated by A. We see easily that for a system of linear manifolds A_λ $(\lambda \in \Lambda)$ the intersection $\prod_{\lambda \in \Lambda} A_\lambda$ also is a linear manifold. Therefore we can say that the linear manifold generated by a manifold A is the least linear manifold containing A. Every linear manifold of a linear space R may be considered itself as a linear space. In this sense, a linear manifold will be called a *subspace* of R.

§44 Linear functionals

Functions on a linear space are called *functionals*. Let R be a linear space. A functional φ on R is said to be *linear*, if

$$\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$$

for every $x, y \in R$ and real numbers α, β.

Theorem 1. *For a linear functional φ_0 on a subspace S of R, we can find a linear functional φ on R such that $\varphi(x) = \varphi_0(x)$ for $x \in S$.*

Proof. By virtue of Maximal Theorem, we see easily that there is a maximal system of elements $x_\lambda \in R$ ($\lambda \in \Lambda$) such that for each $\rho \in \Lambda$, x_ρ is linearly independent from $S \dotplus \{x_\lambda : \lambda \neq \rho\}$. Then for every $x \in R$ we can find uniquely $x_0 \in S$ and a finite number of elements $\lambda_\nu \in \Lambda$ and real numbers $\alpha_\nu \neq 0$ ($\nu = 1, 2, \ldots, \kappa$) such that

$$x = x_0 + \sum_{\nu=1}^{\kappa} \alpha_\nu x_{\lambda_\nu}$$

Thus, putting $\varphi(x) = \varphi_0(x_0)$, we see easily that φ satisfies our requirement.

Theorem 2. *Corresponding to every element $x_0 \neq 0$ there exists a linear functional φ on R such that $\varphi(x_0) = 1$.*

Proof. Putting $S = \{\xi x_0 : -\infty < \xi < +\infty\}$ and $\varphi_0(\xi x_0) = \xi$ for every real number ξ, we obtain obviously a subspace S and a linear functional φ_0 on S. Therefore there is by Theorem 1 a linear functional φ on R such that $\varphi(x) = \varphi_0(x)$ for $x \in S$, and hence $\varphi(x_0) = 1$.

Theorem 3. *For a functional ψ defined on a manifold A of R, in order that there is a linear functional φ on R such that $\varphi(x) = \psi(x)$ for every $x \in A$, it is necessary and sufficient that $\sum_{\nu=1}^{\kappa} \xi_\nu x_\nu = 0$, $x_\nu \in A$ ($\nu = 1, 2, \ldots, \kappa$) implies $\sum_{\nu=1}^{\kappa} \xi_\nu \psi(x_\nu) = 0$.*

Proof. As the necessity is evident, we shall prove the suffi-

ciency. Let S be the linear manifold generated by A. Then for each $x \in S$ we can find a finite number of elements $x_\nu \in A$ and real numbers α_ν ($\nu = 1, 2, \ldots, \kappa$) such that $x = \sum_{\nu=1}^{\kappa} \alpha_\nu x_\nu$, and if

$$x = \sum_{\nu=1}^{\kappa} \alpha_\nu x_\nu = \sum_{\mu=1}^{\sigma} \beta_\mu y_\mu, \quad x_\nu, y_\mu \in A,$$

then we have by assumption

$$\sum_{\nu=1}^{\kappa} \alpha_\nu \psi(x_\nu) = \sum_{\mu=1}^{\sigma} \beta_\mu \psi(y_\mu).$$

Accordingly, putting $\varphi_0(x) = \sum_{\nu=1}^{\kappa} \alpha_\nu \psi(x_\nu)$ for $x = \sum_{\nu=1}^{\kappa} \alpha_\nu \psi(x_\nu)$, $x_\nu \in A$, we obtain a linear functional φ_0 on S such that $\varphi_0(x) = \psi(x)$ for $x \in A$. Furthermore there is by Theorem 1 a linear functional φ on R such that $\varphi(x) = \varphi_0(x)$ for $x \in S$, and hence $\varphi(x) = \psi(x)$ for $x \in A$.

Finally we shall prove the so-called Banach's extension theorem:

Theorem 4. *Let μ be a functional on R such that*

$$\mu(x + y) \leqq \mu(x) + \mu(y),$$
$$\mu(\alpha x) = \alpha\mu(x) \quad \text{for } \alpha \geqq 0.$$

For a linear functional φ_0 on a subspace S of R subject to

$$\varphi_0(x) \leqq \mu(x) \quad \text{for } x \in S,$$

we can find a linear functional φ on R such that

$$\varphi(x) = \varphi_0(x) \quad \text{for } x \in S,$$
$$\varphi(x) \leqq \mu(x) \quad \text{for every } x \in R.$$

Proof. We consider all linear functionals φ_λ on subspaces S_λ ($\lambda \in \Lambda$) such that $S \subset S_\lambda$, and

$$\varphi_\lambda(x) = \varphi_0(x) \quad \text{for } x \in S,$$
$$\varphi_\lambda(x) \leqq \mu(x) \quad \text{for } x \in S_\lambda.$$

For two elements $\lambda_1, \lambda_2 \in \Lambda$, we shall write $\varphi_{\lambda_1} < \varphi_{\lambda_2}$ if $S_{\lambda_1} \subset S_{\lambda_2}$ and

$$\varphi_{\lambda_1}(x) = \varphi_{\lambda_2}(x) \quad \text{for } x \in S_{\lambda_1}.$$

By virtue of Maximal Theorem we can find a maximal system $\Gamma \subset \Lambda$ such that for every $\lambda_1, \lambda_2 \in \Gamma$ we have $\varphi_{\lambda_1} < \varphi_{\lambda_2}$ or $\varphi_{\lambda_2} < \varphi_{\lambda_1}$. For such a maximal system Γ, putting $S_0 = \sum_{\lambda \in \Gamma} S_\lambda$ and

$$\psi_0(x) = \varphi_\lambda(x) \quad \text{for } x \in S_\lambda, \ \lambda \in \Gamma,$$

we obtain a linear functional ψ_0 on S_0 such that $\varphi_\lambda < \psi_0$ for every $\lambda \in \Gamma$, and hence there is $\lambda_0 \in \Gamma$ for which $\psi_0 = \varphi_{\lambda_0}$, as Γ is a maximal system subject to the indicated condition. For such $\lambda_0 \in \Gamma$, we need only prove $S_{\lambda_0} = R$. If there is an element $x_0 \notin S_{\lambda_0}$, then we have for every $x, y \in S_{\lambda_0}$

$$\varphi_{\lambda_0}(x) - \varphi_{\lambda_0}(y) = \varphi_{\lambda_0}(x-y) \leqq \mu(x-y)$$
$$\leqq \mu(x+x_0) + \mu(-y-x_0),$$

that is, $-\mu(-y-x_0) - \varphi_{\lambda_0}(y) \leqq \mu(x+x_0) - \varphi_{\lambda_0}(x)$ for every $x, y \in S_{\lambda_0}$. Thus we can find a real number γ such that

$$-\mu(-x-x_0) - \varphi_{\lambda_0}(x) \leqq \gamma \leqq \mu(x+x_0) - \varphi_{\lambda_0}(x)$$

for every $x \in S_{\lambda_0}$. For such γ, putting

$$\psi(x + \xi x_0) = \varphi_{\lambda_0}(x) + \xi\gamma$$

for every $x \in S_{\lambda_0}$ and for every real number ξ, we obtain a linear functional ψ on the linear manifold generated by $\{S_{\lambda_0}, x_0\}$. Furthermore we have obviously $\psi(x) = \varphi_{\lambda_0}(x)$ for $x \in S_{\lambda_0}$; for $\xi > 0$,

$$\psi(x+\xi x_0) = \varphi_{\lambda_0}(x) + \xi\gamma \leqq \varphi_{\lambda_0}(x) + \xi\left(\mu\left(\tfrac{1}{\xi}x + x_0\right) - \varphi_{\lambda_0}\left(\tfrac{1}{\xi}x\right)\right)$$
$$= \xi\mu\left(\tfrac{1}{\xi}x + x_0\right) = \mu(x + \xi x_0);$$

and for $\xi < 0$,

$$\psi(x+\xi x_0) \leqq \varphi_{\lambda_0}(x) + \xi\left(-\mu\left(-\tfrac{1}{\xi}x - x_0\right) - \varphi_{\lambda_0}\left(\tfrac{1}{\xi}x\right)\right)$$
$$= \mu(x + \xi x_0).$$

Thus we have $\varphi_{\lambda_0} \subset \psi$, contradicting that Γ is a maximal system subject to the indicated condition. Therefore we obtain $S_{\lambda_0} = R$, and hence φ_{λ_0} satisfies our requirement.

If both functionals φ and ψ on R are linear, then $\alpha\varphi + \beta\psi$ also is obviously a linear functional on R for every real numbers α, β. Thus the totality of linear functionals on R constitutes a linear space. This linear space, composed of all linear functionals on R, is called the <u>associated space</u> of R and denoted by $\tilde{R}$.

§45 Finite-dimensional linear spaces

Let R be a linear space. If we can find a natural number $\varkappa$ such that every $\varkappa + 1$ elements of R are not linearly independent, then R is said to be <u>finite-dimensional</u>, and the minimum of such $\varkappa$ is called the <u>dimension</u> of R. With this definition we see at once that if R is finite-dimensional with the dimension $\varkappa$, then we can find $\varkappa$ elements

$x_\nu \in R$ $(\nu = 1, 2, \ldots, \varkappa)$ such that x_ν $(\nu = 1, 2, \ldots, \varkappa)$ are linearly independent, and every $x \in R$ may be represented uniquely in a form

$$x = \sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu .$$

Such a system of elements x_ν $(\nu = 1, 2, \ldots, \varkappa)$ is called a <u>basis</u> of R. Conversely we see easily that if R has a basis of $\varkappa$ elements, then R is finite-dimensional with the dimension $\varkappa$.

<u>Theorem 1.</u> <u>If R is finite-dimensional with the dimension $\varkappa$, then the associated space $\tilde{R}$ of R also is finite-dimensional with the same dimension $\varkappa$.</u>

<u>Proof.</u> Let $x_\nu \in R$ $(\nu = 1, 2, \ldots, \varkappa)$ be a basis of R. Putting for every real numbers α_ν $(\nu = 1, 2, \ldots, \varkappa)$

$$\tilde{x}_\mu\left(\sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu\right) = \alpha_\mu \qquad (\mu = 1, 2, \ldots, \varkappa),$$

we obtain a basis $\tilde{x}_\nu \in \tilde{R}$ $(\nu = 1, 2, \ldots, \varkappa)$ of $\tilde{R}$. Because, if

$$\sum_{\nu=1}^{\varkappa} \alpha_\nu \tilde{x}_\nu = 0,$$

then we have for every $\mu = 1, 2, \ldots, \varkappa$

$$\alpha_\mu = \sum_{\nu=1}^{\varkappa} \alpha_\nu \tilde{x}_\nu(x_\mu) = 0;$$

and hence $\tilde{x}_\nu$ $(\nu = 1, 2, \ldots, \varkappa)$ are linearly independent. Furthermore for any $\tilde{x} \in \tilde{R}$ we have for every real numbers α_ν $(\nu = 1, 2, \ldots, \varkappa)$

$$\tilde{x}\left(\sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu\right) = \sum_{\mu=1}^{\varkappa} \tilde{x}(x_\mu)\, \tilde{x}_\mu\left(\sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu\right),$$

and hence $\tilde{x} = \sum_{\mu=1}^{\varkappa} \tilde{x}(x_\mu)\tilde{x}_\mu$.

<u>Theorem 2.</u> <u>If R is finite-dimensional, then for any linear functional φ on the associated space $\tilde{R}$ there exists uniquely $x \in R$ such that</u>

$$\varphi(\tilde{x}) = \tilde{x}(x) \quad \underline{\text{for every}}\ \tilde{x} \in \tilde{R}.$$

<u>Proof.</u> Let $x_\nu \in R$ $(\nu = 1, 2, \ldots, \varkappa)$ be a basis of R and $\tilde{x}_\nu \in \tilde{R}$ $(\nu = 1, 2, \ldots, \varkappa)$ a basis of $\tilde{R}$ such that

$$\tilde{x}_\mu\left(\sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu\right) = \alpha_\mu \qquad (\mu = 1, 2, \ldots, \varkappa),$$

as obtained in the previous Proof. Putting

$$x = \sum_{\nu=1}^{\varkappa} \varphi(\tilde{x}_\nu) x_\nu,$$

we have then for every real numbers α_μ $(\mu = 1, 2, \ldots, \varkappa)$

$$\sum_{\mu=1}^{\varkappa} \alpha_\mu \tilde{x}_\mu(x) = \sum_{\mu=1}^{\varkappa} \alpha_\mu \tilde{x}_\mu\left(\sum_{\nu=1}^{\varkappa} \varphi(\tilde{x}_\nu) x_\nu\right)$$
$$= \sum_{\mu=1}^{\varkappa} \alpha_\mu \varphi(\tilde{x}_\mu) = \varphi\left(\sum_{\mu=1}^{\varkappa} \alpha_\mu \tilde{x}_\mu\right),$$

and hence $\varphi(\tilde{x}) = \tilde{x}(x)$ for every $\tilde{x} \in \tilde{R}$. Furthermore, if we have

$\tilde{x}(x) = \tilde{x}(y)$ for every $\tilde{x} \in \tilde{R}$, then, putting

$$x = \sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu, \qquad y = \sum_{\nu=1}^{\varkappa} \beta_\nu x_\nu,$$

we obtain $\alpha_\nu = \tilde{x}_\nu(x) = \tilde{x}_\nu(y) = \beta_\nu$ $(\nu = 1, 2, \dots, \varkappa)$, and hence we conclude the uniqueness of such x.

On account of Theorem 2, we see that if R is finite-dimensional, then R coincides with the associated space $\tilde{\tilde{R}}$ of the associated space $\tilde{R}$, considering every $x \in R$ as a linear functional on $\tilde{R}$ by

$$x(\tilde{x}) = \tilde{x}(x) \qquad \text{for every } \tilde{x} \in \tilde{R}.$$

Thus we conclude by Theorem 1

Theorem 3. If the associated space $\tilde{R}$ of R is finite-dimensional with the dimension $\varkappa$, then R also is finite-dimensional with the same dimension $\varkappa$.

Recalling §44 Theorem 3, we obtain by Theorem 2

Theorem 4. If R is finite-dimensional, then for a functional φ on a manifold $\tilde{A}$ of the associated space $\tilde{R}$, in order that there is $x \in R$ such that $\varphi(\tilde{x}) = \tilde{x}(x)$ for every $\tilde{x} \in \tilde{A}$, it is necessary and sufficient that $\sum_{\nu=1}^{\varkappa} \xi_\nu \tilde{x}_\nu = 0$, $\tilde{x}_\nu \in \tilde{A}$ $(\nu = 1, 2, \dots, \varkappa)$ implies

$$\sum_{\nu=1}^{\varkappa} \xi_\nu \varphi(\tilde{x}_\nu) = 0.$$

For the associated space $\tilde{R}$ of a linear space R, a manifold $\tilde{A}$ of $\tilde{R}$ is said to be fundamental, if $\tilde{x}(x) = 0$ for every $\tilde{x} \in \tilde{A}$ implies $x = 0$. With this definition we have

Theorem 5. If a linear manifold $\tilde{A}$ of the associated space $\tilde{R}$ is finite-dimensional and fundamental, then R is finite-dimensional and $\tilde{A} = \tilde{R}$.

Proof. Let $\varkappa$ be the dimension of $\tilde{A}$. Then the associated space of $\tilde{A}$ is by Theorem 1 finite-dimensional with the dimension $\varkappa$ and every $x \in R$ may be considered as a linear functional on $\tilde{A}$ by

$$x(\tilde{x}) = \tilde{x}(x) \qquad \text{for every } \tilde{x} \in \tilde{A}.$$

Thus for every $x_\nu \in R$ $(\nu = 1, 2, \dots, \varkappa+1)$ we can find real numbers α_ν $(\nu = 1, 2, \dots, \varkappa+1)$ such that $\sum_{\nu=1}^{\varkappa+1} |\alpha_\nu| \neq 0$ and

$$\tilde{x}\Big(\sum_{\nu=1}^{\varkappa+1} \alpha_\nu x_\nu\Big) = \sum_{\nu=1}^{\varkappa+1} \alpha_\nu x_\nu(\tilde{x}) = 0 \qquad \text{for every } \tilde{x} \in \tilde{A},$$

and then we have $\sum_{\nu=1}^{\varkappa+1} \alpha_\nu x_\nu = 0$, because $\tilde{A}$ is fundamental by assumption.

Therefore R is by definition finite-dimensional and its dimension is not greater than $\varkappa$. Accordingly $\tilde{R}$ also is by Theorem 1 finite-dimensional and its dimension is not greater than $\varkappa$. If $\tilde{A} \neq \tilde{R}$, then every $\tilde{x} \in \tilde{A}$ is linearly independent from $\tilde{A}$, contradicting that the dimension of $\tilde{R}$ is not greater than $\varkappa$. Thus we have $\tilde{A} = \tilde{R}$.

§46 Quotient spaces

Let R be a linear space and A a linear manifold of R. For two elements $x, y \in R$ we define

$$x \equiv y \quad (A)$$

to mean $x - y \in A$. With this definition we have obviously

$$x \equiv x \quad (A),$$

$$x \equiv y \quad (A) \quad \text{implies} \quad y \equiv x \quad (A),$$

$$x \equiv y,\ y \equiv z \quad (A) \quad \text{implies} \quad x \equiv z \quad (A).$$

A manifold X of R is called a residue class by A, if we have

$$X = \{x : x \equiv x_0 \ (A)\} \qquad \text{for every } x_0 \in X.$$

Concerning residue classes, we see easily by definition that corresponding to each $x \in R$ there exists uniquely a residue class $X \ni x$ by A and for every residue classes X, Y by A we have $XY = 0$ or $X = Y$. A is obviously itself a residue class by A containing 0.

Since $x_1 \equiv x_2$, $y_1 \equiv y_2$ (A) implies

$$\alpha x_1 + \beta y_1 \equiv \alpha x_2 + \beta y_2 \quad (A)$$

for every real numbers α, β, we see that for every residue classes X, Y by A and for every real numbers α, β we can find uniquely a residue class Z by A such that $x \in X$, $y \in Y$ implies $\alpha x + \beta y \in Z$. Such a residue class Z will be denoted by $\alpha X + \beta Y$. Then we see easily that the totality of residue classes by A constitutes a linear space. This linear space is called a quotient space of R by a linear manifold A and denoted by R/A. The residue class A is obviously the zero element of R/A. Furthermore we have obviously

(1) $\quad A + x \in R/A \qquad$ for every $x \in R$,

(2) $A+(\alpha x+\beta y)=\alpha(A+x)+\beta(A+y)$,

(3) $(A+x)\times(A+y)=A+(x+y)$,

(4) $A+x=X$ for $x\in X\in R/A$.

Putting $\mathfrak{a}(x)=A+x$, we obtain a mapping of R onto the quotient space R/A. This mapping $\mathfrak{a}$ will be a quotient mapping of R by A. For the quotient mapping $\mathfrak{a}$ of R by A, the image $\mathfrak{a}(U)$ of a manifold U of R is denoted by U/A, and the inverse image $\mathfrak{a}^{-1}(V)$ of a manifold V of R/A by V^A. Then we have obviously for every manifold U of R

(5) $(U/A)^A=U\times A$, $U/A=(U\times A)/A$.

Furthermore we see easily by definition

(6) $(U\times V)/A=(U/A)\times(V/A)$,

(7) $\alpha U/A=\alpha(U/A)$,

(8) $(U\dotplus V)/A=(U/A)\dotplus(V/A)$,

(9) $UV/A\subset(U/A)(V/A)$,

(10) $U\supset V$ implies $U/A\supset V/A$.

Theorem 1. There exists a mapping $\mathfrak{a}$ of a quotient space R/A into R such that $\mathfrak{a}(X)\in X$ for every $X\in R/A$ and

$$\mathfrak{a}(\alpha X+\beta Y)=\alpha\,\mathfrak{a}(X)+\beta\,\mathfrak{a}(Y).$$

Proof. By virtue of Maximal Theorem, we see easily that there is a maximal system of residue classes $X_\lambda\in R/A$ ($\lambda\in\Lambda$) subject to the condition that the manifold X_λ ($\lambda\in\Lambda$) is linearly independent. Furthermore we obtain by Choice Axiom a system of elements $x_\lambda\in X_\lambda$ ($\lambda\in\Lambda$). Then, we conclude easily that for each $X\in R/A$ we can find uniquely a system of real numbers α_λ ($\lambda\in\Lambda$) such that $\alpha_\lambda=0$ except for a finite number of λ and

$$X=\sum_{\lambda\in\Lambda}\alpha_\lambda X_\lambda.$$

Thus, putting $\mathfrak{a}(X)=\sum_{\lambda\in\Lambda}\alpha_\lambda x_\lambda$ for $X=\sum_{\lambda\in\Lambda}\alpha_\lambda X_\lambda$, we have $\mathfrak{a}(X)\in X$ for every $X\in R/A$, and for $Y=\sum_{\lambda\in\Lambda}\beta_\lambda X_\lambda$

$$\mathfrak{a}(\alpha X+\beta Y)=\mathfrak{a}\Big(\sum_{\lambda\in\Lambda}(\alpha\alpha_\lambda+\beta\beta_\lambda)X_\lambda\Big)$$
$$=\sum_{\lambda\in\Lambda}(\alpha\alpha_\lambda+\beta\beta_\lambda)x_\lambda=\alpha\,\mathfrak{a}(X)+\beta\,\mathfrak{a}(Y).$$

Every functional φ on R/A may be considered as a functional on

R by the relation

$$\varphi(x)=\varphi(X) \quad \text{for } x\in X\in R/A.$$

Then we have obviously $\varphi(x)=\varphi(y)$ for $x-y\in A$. Conversely, if a functional φ on R satisfies

$$\varphi(x)=\varphi(y) \quad \text{for } x-y\in A,$$

then we see that φ may be considered as a functional on the quotient space R/A by the relation

$$\varphi(X)=\varphi(x) \quad \text{for } x\in X\in R/A.$$

If a functional φ on the quotient space R/A is linear, then φ also is obviously linear as a functional on R. Conversely, if a linear functional φ on R satisfies

$$\varphi(x)=0 \quad \text{for every } x\in A,$$

then we see easily that φ also is linear as a functional on the quotient space R/A. Therefore we have

Theorem 2. For a linear manifold A of R, the manifold of the associated space $\tilde{R}$

$$\{\tilde{x}:\ \tilde{x}(x)=0 \text{ for every } x\in A\}$$

coincides with the associated space $\widetilde{R/A}$ of the quotient space R/A as functionals on R/A.

Theorem 3. For a finite number of elements $\tilde{x}_\nu$ ($\nu=1,2,\ldots,\kappa$) of the associated space $\tilde{R}$, if we put

$$A=\{x:\ \tilde{x}_\nu(x)=0 \text{ for all } \nu=1,2,\ldots,\kappa\},$$

then the quotient space R/A is finite-dimensional and the associated space $\widetilde{R/A}$ of R/A coincides with the linear manifold generated by $\tilde{x}_\nu$ ($\nu=1,2,\ldots,\kappa$) as functionals on R/A.

Proof. Let B be the linear manifold generated by $\tilde{x}_\nu$ ($\nu=1,2,\ldots,\kappa$). Then B is obviously finite-dimensional. Furthermore B is fundamental as functionals on R/A. Because, for a residue class $X\in R/A$, if $\tilde{x}(X)=0$ for every $\tilde{x}\in B$, then we have

$$\tilde{x}_\nu(x)=0 \quad \text{for every } \nu=1,2,\ldots, \text{ and } x\in X,$$

and hence $A=X$. Therefore R/A is by §45 Theorem 5 finite-dimensional and its associated space coincides with B as functionals on R/A.

Theorem 4. *For a finite number of linear functionals* φ_ν *on* R *and real numbers* α_ν $(\nu = 1, 2, \dots, \varkappa)$, *in order that we can find* $x \in R$ *such that* $\varphi_\nu(x) = \alpha_\nu$ $(\nu = 1, 2, \dots, \varkappa)$, *it is necessary and sufficient that* $\sum_{\nu=1}^{\varkappa} \xi_\nu \varphi_\nu = 0$ *implies* $\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu = 0$.

Proof. As the necessity is evident, we need only prove the sufficiency. Putting $A = \{x : \varphi_\nu(x) = 0$ for every $\nu = 1, 2, \dots, \varkappa\}$, we obtain a linear manifold A of R and the quotient space R/A is finite-dimensional by Theorem 3. Thus there is by §45 Theorem 4 a residue class $X \in R/A$ such that $\varphi_\nu(X) = \alpha_\nu$ for every $\nu = 1, 2, \dots, \varkappa$, and then for an element $x \in X$ we have obviously $\varphi_\nu(x) = \alpha_\nu$ for every $\nu = 1, 2, \dots, \varkappa$.

§47 Product spaces

Let R and S be two linear spaces. The totality of pairs of elements (x, y) for $x \in R$, $y \in S$ is called the *product space* of R and S and denoted by (R, S). We define $\alpha(x_1, y_1) + \beta(x_2, y_2)$ to mean

$$\alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

for every real numbers α, β. Then we see easily that the product space (R, S) also is a linear space.

For manifolds A of R and B of S we define (A, B) to mean a manifold of the product space (R, S) composed of all (x, y) for $x \in A$, $y \in B$. With this definition we see at once that if both A and B are linear, then (A, B) also is linear.

For two linear spaces R and S, if there is a transformation $\mathfrak{a}$ from R to S such that

$$\mathfrak{a}(\alpha x + \beta y) = \alpha \mathfrak{a}(x) + \beta \mathfrak{a}(y)$$

for every $x, y \in R$ and real numbers α, β, then R is said to be *isomorphic* to S by a transformation $\mathfrak{a}$, and such a transformation $\mathfrak{a}$ is called a *linear transformation*.

With this definition, it is obvious that the linear manifold $(\{0\}, S)$ of the product space (R, S) is isomorphic to S by the transformation:

$$(0, y) \longrightarrow y \in S.$$

Similarly the linear manifold $(R, \{0\})$ is isomorphic to R by the transformation: $(x, 0) \rightarrow x \in R$.

Theorem 1. *The quotient space* $(R, S)/(\{0\}, S)$ *is isomorphic to* R *by the transformation:* $(\{x\}, S) \rightarrow x \in R$.

Proof. For each $x \in R$, $(\{x\}, S)$ is obviously by definition a residue class of (R, S) by the linear manifold $(\{0\}, S)$. Thus we see further that the indicated transformation is a linear transformation from $(R, S)/(\{0\}, S)$ to R.

A linear space R is said to be *isomorphic* to a linear space S, if there is a linear transformation by which R is isomorphic to S.

Theorem 2. *For a linear manifold* A *of a linear space* R, *there is a linear transformation* $\mathfrak{a}$ *from the product space* $(R/A, A)$ *to* R *such that* $\mathfrak{a}((B/A, A)) = B \times A$ *for every* $B \subset R$.

Proof. On account of §46 Theorem 1 there is a mapping $\mathfrak{a}_0$ of R/A into R such that $\mathfrak{a}_0(X) \in X$ for every $X \in R/A$ and

$$\mathfrak{a}_0(\alpha X + \beta Y) = \alpha \mathfrak{a}_0(X) + \beta \mathfrak{a}_0(Y).$$

Then, corresponding to every pair (X, y) for $X \in R/A$, $y \in A$ we obtain uniquely an element $\mathfrak{a}_0(X) + y \in R$. Conversely, for each $x \in R$ we have $A + x \in R/A$, $\mathfrak{a}_0(A + x) \in A + x$, and hence $\mathfrak{a}_0(A + x) - x \in A$. Thus, putting $y = \mathfrak{a}_0(A + x) - x$, we obtain

$$x = \mathfrak{a}_0(A + x) + y, \quad A + x \in R/A, \; y \in A.$$

Therefore, putting $\mathfrak{a}((X, y)) = \mathfrak{a}_0(X) + y$, we obtain a mapping $\mathfrak{a}$ of $(R/A, A)$ onto R. This mapping $\mathfrak{a}$ is a transformation. Because, if

$$\mathfrak{a}_0(X_1) + y_1 = \mathfrak{a}_0(X_2) + y_2, \quad X_1, X_2 \in R/A, \; y_1, y_2 \in A,$$

then we have $\mathfrak{a}_0(X_1) - \mathfrak{a}_0(X_2) = y_2 - y_1 \in A$, and hence $X_1 = X_2$, as $\mathfrak{a}_0(X_1) \in X_1$, $\mathfrak{a}_0(X_2) \in X_2$. Furthermore $\mathfrak{a}$ is linear, because we have

$$\begin{aligned}\mathfrak{a}(\alpha(X_1, y_1) + \beta(X_2, y_2)) &= \mathfrak{a}((\alpha X_1 + \beta X_2, \alpha y_1 + \beta y_2)) \\ &= \mathfrak{a}_0(\alpha X_1 + \beta X_2) + \alpha y_1 + \beta y_2 = \alpha(\mathfrak{a}_0(X_1) + y_1) + \beta(\mathfrak{a}_0(X_2) + y_2) \\ &= \alpha \mathfrak{a}((X_1, y_1)) + \beta \mathfrak{a}((X_2, y_2)).\end{aligned}$$

As $\mathfrak{a}_0(X) \in X$, we have for every $B \subset R$

$$\mathfrak{a}((B/A, A)) = \mathfrak{a}_0(B/A) \times A = B \times A.$$

CHAPTER VII

VICINITIES

§48 Fundamental definitions

Let R be a linear space. A manifold V of R is called a vicinity, if for any $x \in R$ we can find $\gamma > 0$ such that

$$\xi x \in V \qquad \text{for } 0 \leqq \xi \leqq \gamma .$$

With this definition we see that every vicinity V contains the zero element 0. For a vicinity V, every manifold $U > V$ also is obviously a vicinity by definition, and αV also is a vicinity for every real number $\alpha \neq 0$, because for any $x \in R$ we can find by definition $\gamma > 0$ such that $\xi(\frac{1}{\alpha}x) \in V$ for $0 \leqq \xi \leqq \gamma$, and hence $\xi x \in \alpha V$ for $0 \leqq \xi \leqq \gamma$. For two vicinities V_1 and V_2, the intersection $V_1 V_2$ also is a vicinity, because, if $\xi x \in V_1$ for $0 \leqq \xi \leqq \gamma_1$ and $\xi x \in V_2$ for $0 \leqq \xi \leqq \gamma_2$, then we have $\xi x \in V_1 V_2$ for $0 \leqq \xi \leqq \operatorname{Min}\{\gamma_1, \gamma_2\}$.

A manifold V is said to be symmetric, if $(-1)V = V$. A manifold V is called a star, if $\xi V \subset V$ for $0 \leqq \xi \leqq 1$. With this definition we see easily that for an arbitrary manifold V, $((-1)V)V$ is symmetric, $\sum_{0 \leqq \xi < \alpha} \xi V$ is a star, and $\sum_{|\xi| < \alpha} \xi V$ is a symmetric star for every positive number α.

A star V is said to be scalar-open, if

$$V = \sum_{0 \leqq \xi < 1} \xi V .$$

A star V is said to be scalar-closed, if

$$V = \prod_{\xi > 1} \xi V .$$

With this definition we see easily that a star V is scalar-open, if and only if for any $x \in V$ we can find a positive number ε such that we have $(1+\varepsilon)x \in V$; and a star V is scalar-closed, if and only if for any $x \bar{\in} V$ we can find a positive number $\varepsilon < 1$ such that $(1-\varepsilon)x \bar{\in} V$. Thus for an arbitrary star V, $\sum_{0 \leqq \xi < \alpha} \xi V$ is scalar-open, and $\prod_{\xi > \alpha} \xi V$ is scalar-closed for every positive number α.

A vicinity V is said to be of finite character, if we can find a positive number α such that

$$\lambda V \times \mu V \subset \alpha V \qquad \text{for } \lambda + \mu = 1, \ \lambda, \mu \geqq 0,$$

and the greatest lower bound of such α is called the character of V. If a vicinity V is not of finite character, then the character of V is defined as $+\infty$.

Theorem 1. If a vicinity V is of finite character and scalar-open or scalar-closed, then we have for its character χ

$$\lambda V \times \mu V \subset \chi V \qquad \text{for } \lambda + \mu = 1, \ \lambda, \mu \geqq 0.$$

Proof. We have by the definition of character χ that we can find a number sequence $\rho_\nu > 1$ $(\nu = 1, 2, \ldots)$ such that $\lim_{\nu \to \infty} \rho_\nu = 1$ and for every $\nu = 1, 2, \ldots$

$$\lambda V \times \mu V \subset \chi \rho_\nu V \qquad \text{for } \lambda + \mu \geqq 1, \ \lambda, \mu \geqq 0,$$

and hence by the formula §43(18)

$$\lambda \frac{1}{\rho_\nu} V \times \mu \frac{1}{\rho_\nu} V \subset \chi V \qquad \text{for } \lambda + \mu \geqq 1, \ \lambda, \mu \geqq 0.$$

If V is scalar-closed, then we have by the formula §43(14)

$$\lambda V \times \mu V \subset \prod_{\nu=1}^{\infty} \chi \rho_\nu V = \chi \prod_{\nu=1}^{\infty} \rho_\nu V = \chi V .$$

If V is scalar-open, then we obtain by the formulas (3), (13) in §43

$$\lambda V \times \mu V = \Big(\sum_{\nu=1}^{\infty} \lambda \frac{1}{\rho_\nu} V\Big) \times \Big(\sum_{\nu=1}^{\infty} \mu \frac{1}{\rho_\nu} V\Big) \subset \chi V .$$

We must remark that if the character χ of a vicinity V is less than 1, then $V = R$. Because, if $\chi < 1$, then we can find a positive number $\rho < 1$ such that $\rho V > \frac{1}{2} V \times \frac{1}{2} V > V$. Thus we have $V > \frac{1}{\rho^\nu} V$ for every $\nu = 1, 2, \ldots$, and hence we conclude $V = R$ by the definition of vicinities. A vicinity V with the character $\chi \leqq 1$ is said to be convex.

Theorem 2. Every convex vicinity is a star.

Proof. If a vicinity V is convex, then we can find a number sequence $\rho_\nu \geqq 1$ $(\nu = 1, 2, \ldots)$ such that $\lim_{\nu \to \infty} \rho_\nu = 1$ and

$$\lambda V \times \mu V \subset \rho_\nu V \qquad \text{for } \lambda + \mu = 1, \ \lambda, \mu \geqq 0.$$

Consequently we have $\lambda V \subset \rho_\nu V$ for $0 \leqq \lambda \leqq 1$ and $\nu = 1, 2, \ldots$. Thus we conclude $\lambda V \subset V$ for $0 \leqq \lambda < 1$, and hence V is a star by definition.

Theorem 3. If a convex vicinity V is scalar-open or scalar-closed, then we have for every finite number of positive numbers α_ν

$(\nu = 1, 2, \dots, \kappa)$

$$\alpha_1 V \times \alpha_2 V \times \cdots \times \alpha_\kappa V = (\alpha_1+\alpha_2+\cdots+\alpha_\kappa) V .$$

Proof. If a convex vicinity V is scalar-open or scalar-closed, then we have by Theorem 1

$$\lambda V \times \mu V = V \qquad \text{for } \lambda+\mu=1,\ \lambda, \mu \geqq 0.$$

Thus we have for positive numbers α, β by the formula §43(18)

$$\alpha V \times \beta V = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta} V \times \frac{\beta}{\alpha+\beta} V\right) = (\alpha+\beta) V .$$

Furthermore we obtain easily our assertion by the induction.

Theorem 4. If a symmetric vicinity V is of finite character, then $\prod_{\xi>0} \xi V$ is a linear manifold.

Proof. Recalling the formula §43(18), we see easily that if a vicinity V is of finite character, then there is a positive number α such that $V \times V \subset \alpha V$, and hence $x, y \in \prod_{\xi>0} \xi V$ implies

$$x + y \in \prod_{\xi>0} \xi \alpha V = \prod_{\xi>0} \xi V .$$

As V is symmetric by assumption, $x \in \prod_{\xi>0} \xi V$ implies by §48(14)

$$\alpha x \in \prod_{\xi>0} \xi \alpha V = \prod_{\xi>0} \xi V \qquad \text{for } \alpha \neq 0.$$

A vicinity V is said to be proper, if $\prod_{\xi>0} \xi V = \{0\}$.

§49 Pseudo-norms

Let R be a linear space. A functional $\|x\|$ on R is called a pseudo-norm on R, if $\|x\| \geqq 0$ for every $x \in R$, and

$$\|\xi x\| = |\xi| \|x\| \qquad \text{for every real number } \xi .$$

A pseudo-norm $\|x\|$ ($x \in R$) is said to be proper, if $\|x\| = 0$ implies $x = 0$. A pseudo-norm $\|x\|$ ($x \in R$) is said to be of finite character, if there is a positive number α such that we have

$$\|x + y\| \leqq \alpha (\|x\| + \|y\|)$$

for every $x, y \in R$, and then the greatest lower bound of such α is called the character of $\|x\|$. If a pseudo-norm $\|x\|$ is not of finite character, then its character is defined as $+\infty$. For the character χ of a pseudo-norm $\|x\|$ we have obviously by definition

$$\|x + y\| \leqq \chi (\|x\| + \|y\|)$$

for every $x, y \in R$. It is evident by definition that $\chi \geqq 1$. Especially, if $\chi = 1$, then a pseudo-norm $\|x\|$ is said to be convex. A pseudo-norm is said to be a norm, if it is proper and convex.

Theorem 1. If a pseudo-norm $\|x\|$ on R is of finite character with the character χ, then for every positive number α,

$$\{x : \|x\| \leqq \alpha\}$$

is a symmetric scalar-closed vicinity with the same character χ, and

$$\{x : \|x\| < \alpha\}$$

is a symmetric scalar-open vicinity with the same character χ.

Proof. Putting $V = \{x : \|x\| \leqq \alpha\}$, we have obviously that $x \in V$ implies $-x \in V$, that is, V is symmetric. If $x \notin V$, then $\|x\| > \alpha$, and hence there is a positive number ε such that

$$\|\xi x\| = \xi \|x\| > \alpha \qquad \text{for } \xi > 1 - \varepsilon ,$$

that is, $\xi x \notin V$ for $\xi > 1 - \varepsilon$. Thus V is a symmetric scalar-closed star. Furthermore we see easily that V is a vicinity.

For a positive number χ, if

$$\|x + y\| \leqq \chi (\|x\| + \|y\|) \qquad \text{for every } x, y \in R,$$

then $x, y \in V$ implies for $\lambda + \mu = 1$, $\lambda, \mu \geqq 0$

$$\|\lambda x + \mu y\| \leqq \chi (\lambda \|x\| + \mu \|y\|) \leqq \chi \alpha ,$$

and hence $\lambda V \times \mu V \subset \chi V$, because

$$\chi V = \{\chi x : \|x\| \leqq \alpha\} = \{x : \|x\| \leqq \chi \alpha\}.$$

Conversely, if $x, y \in V$ implies

$$\lambda x + \mu y \in \chi V \qquad \text{for } \lambda + \mu = 1,\ \lambda, \mu \geqq 0,$$

then for any $x, y \in R$, $\|x\| \neq 0$, $\|y\| \neq 0$, we have $\frac{\alpha}{\|x\|} x$, $\frac{\alpha}{\|y\|} y \in V$ and hence, putting

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|}, \qquad \mu = \frac{\|y\|}{\|x\| + \|y\|},$$

we obtain $\frac{\alpha}{\|x\| + \|y\|} (x + y) \in \chi V$. This relation yields

$$\|x + y\| \leqq \chi (\|x\| + \|y\|).$$

In the case, where $\|x\| \neq 0$, $\|y\| = 0$, we have obviously $\xi y \in V$ for every $\xi > 0$, and hence, putting

$$\lambda = \frac{\|x\|}{\|x\| + \xi}, \qquad \mu = \frac{\xi}{\|x\| + \xi},$$

we obtain likewise $\|x + y\| \leqq \chi (\|x\| + \xi)$ for every $\xi > 0$, which yields

$$\|x+y\| \leqq \chi \|x\|.$$

We also can dispose likewise the case where $\|x\| = \|y\| = 0$. Therefore the character of V coincides by definition with that of the pseudo-norm $\|x\|$. We also can prove likewise our assertion about $\{x : \|x\| < \alpha\}$.

Let V be a symmetric star vicinity. Putting

$$\|x\| = \inf_{x \in \xi V} |\xi|,$$

we obtain a pseudo-norm $\|x\|$ on R. Because, it is evident that we have $\|x\| \geqq 0$ for every $x \in R$, and for every real number α

$$\|\alpha x\| = \inf_{\alpha x \in \xi V} |\xi| = \inf_{x \in \xi V} |\alpha \xi| = |\alpha| \|x\|.$$

This pseudo-norm $\|x\|$ is called the *pseudo-norm of* V and denoted by $\|x\|_V$, that is,

(1) $$\|x\|_V = \inf_{x \in \xi V} |\xi|.$$

With this definition we have obviously

(2) $$\{x : \|x\|_V < \alpha\} \subset \alpha V \subset \{x : \|x\|_V \leqq \alpha\} \quad \text{for } \alpha > 0.$$

Furthermore we see easily that we have

(3) $\|x\|_{\alpha V} = \frac{1}{\alpha} \|x\|_V$,

(4) $V \subset U$ implies $\|x\|_V \geqq \|x\|_U$;

(5) $V \times V \subset U$ implies $\operatorname{Max}\{\|x\|_V, \|y\|_V\} \geqq \|x+y\|_U$.

Because, if $V \times V \subset U$, then we have

$$\|x+y\|_U = \inf_{x+y \in \xi U} |\xi| \leqq \inf_{x+y \in \xi V \times \xi V} |\xi| \leqq \inf_{x \in \xi V, y \in \xi V} |\xi| \leqq \operatorname{Max}\{\|x\|_V, \|y\|_V\}.$$

Theorem 2. *The character of a symmetric star vicinity* V *coincides with that of the pseudo-norm of* V. *Consequently, the pseudo-norm of* V *is convex, if and only if* V *is convex.*

Proof. Let χ_0 be the character of the pseudo-norm $\|x\|_V$ of V. If $\lambda V \times \mu V \subset \chi V$ for $\lambda + \mu = 1$, $\lambda, \mu \geqq 0$, then we obtain by the formula (2) that $\|x\|_V, \|y\|_V < 1$ implies

$$\|\lambda x + \mu y\|_V \leqq \chi \quad \text{for } \lambda + \mu = 1, \ \lambda, \mu \geqq 0,$$

and hence the character of $\{x : \|x\|_V < 1\}$ is not greater than χ. Thus we obtain by Theorem 1 $\chi \geqq \chi_0$, and consequently the character of V is not less than χ_0. On the other hand, if $\chi_0 < +\infty$, then the character of $\{x : \|x\|_V \leqq 1\}$ coincides by Theorem 1 with χ_0, and we see

by the formula (2) and §48 Theorem 1 that $x, y \in V$ implies

$$\|\lambda x + \mu y\| \leqq \chi_0 \quad \text{for } \lambda + \mu = 1, \ \lambda, \mu \geqq 0,$$

which yields $\lambda V \times \mu V \subset \chi_0(1+\varepsilon) V$ for every $\varepsilon > 0$, because we have by the formula (2)

$$\{x : \|x\|_V \leqq 1\} \subset \{x : \|x\|_V < 1+\varepsilon\} \subset (1+\varepsilon) V.$$

Thus we conclude that the character of V is not greater than χ_0, as we wish to prove.

We have obviously by definition the following two theorems:

Theorem 3. *If a symmetric vicinity* V *is scalar-closed, then we have for* $\alpha > 0$

$$\alpha V = \{x : \|x\|_V \leqq \alpha\},$$

and if V *is scalar-open, then we have for* $\alpha > 0$

$$\alpha V = \{x : \|x\|_V < \alpha\}.$$

Theorem 4. *The pseudo-norm* $\|x\|_V$ *of a symmetric star vicinity* V *is proper, if and only if* V *is proper.*

§50 Quasi-norm

A functional $|||x|||$ on R is called a *quasi-norm*, if

1) $|||x||| \geqq 0$ for every $x \in R$,

2) $|\alpha| \leqq |\beta|$ implies $|||\alpha x||| \leqq |||\beta x|||$,

3) $|||x+y||| \leqq |||x||| + |||y|||$,

4) $\lim_{\xi \to 0} |||\xi x||| = 0$.

We conclude immediately from 2) that $|||x||| = |||-x|||$ for every $x \in R$. On account of 3) we have hence for every real numbers α, β

$$\big|\, |||\alpha x||| - |||\beta x||| \,\big| \leqq |||(\alpha - \beta) x|||.$$

Therefore we see that $|||\xi x|||$ is a continuous function of ξ for every $x \in R$, and consequently $|||0||| = 0$ by 4). A quasi-norm $|||x|||$ is said to be *proper*, if $|||x||| = 0$ implies $x = 0$.

Theorem 1. *For a quasi-norm* $|||x|||$ *on* R *and for every* $\alpha > 0$

$$\{x : |||x||| \leqq \alpha\}$$

is a scalar-closed symmetric vicinity, and

$$\{x : |||x||| < \alpha\}$$

is a scalar-open symmetric vicinity.

Proof. Putting $V = \{x : |||x||| \le \alpha\}$, it is evident that V is symmetric. If $x \,\bar{\in}\, V$, then we have naturally $|||x||| > \alpha$. Since $|||\xi x|||$ is a non-decreasing continuous function of $\xi \ge 0$, as proved just above, there is a positive number ε such that

$$|||\xi x||| > \alpha \qquad \text{for } \xi > 1-\varepsilon.$$

Therefore V is scalar-closed by definition. On account of 4), for every $x \in R$ we can find $\gamma > 0$ such that $|||\xi x||| \le \alpha$ for $0 \le \xi \le \gamma$, and hence V is a vicinity. We also can prove likewise the other assertion about $\{x : |||x||| < \alpha\}$.

On account of 3) we have obviously

$$\tfrac{1}{2}|||x||| \le |||\tfrac{1}{2}x||| \qquad \text{for every } x \in R.$$

A quasi-norm $|||x|||$ is said to be of finite character, if we can find positive numbers α, γ such that

$$\tfrac{1}{2}|||x||| \ge |||\tfrac{\gamma}{2}x||| \qquad \text{for } |||x||| \le \alpha.$$

Theorem 2. If a quasi-norm $|||x|||$ is of finite character, then we can find a positive number α such that the characters of

$$\{x : |||x||| \le \xi\} \quad \text{and} \quad \{x : |||x||| < \xi\}$$

are bounded for $0 < \xi \le \alpha$.

Proof. If $|||x|||$ is of finite character, then we can find by definition positive numbers α, γ such that

$$\tfrac{1}{2}|||x||| \ge |||\tfrac{\gamma}{2}x||| \qquad \text{for } |||x||| \le \alpha.$$

Hence, if $|||x|||, |||y||| \le \xi \le \alpha$ and $\lambda + \mu = 1$, $\lambda, \mu \ge 0$, then we have by the postulates 2) and 3)

$$|||\tfrac{\gamma}{2}(\lambda x + \mu y)||| \le |||\tfrac{\gamma}{2}\lambda x||| + |||\tfrac{\gamma}{2}\mu y|||$$
$$\le |||\tfrac{\gamma}{2}x||| + |||\tfrac{\gamma}{2}y||| \le \tfrac{1}{2}|||x||| + \tfrac{1}{2}|||y||| \le \xi.$$

Consequently, putting $V_\xi = \{x : |||x||| \le \xi\}$, we have for $0 < \xi \le \alpha$

$$\lambda V_\xi \times \mu V_\xi \subset \tfrac{2}{\gamma} V_\xi \qquad \text{for } \lambda + \mu = 1, \ \lambda, \mu \ge 0.$$

Therefore the character of V_ξ is not greater than $\frac{2}{\gamma}$ for every positive number $\xi \le \alpha$. We also can prove likewise the other assertion about the character of $\{x : |||x||| < \xi\}$.

Theorem 3. For a sequence of symmetric star vicinities V_ν ($\nu = 1, 2, \dots$) such that

$$V_\nu \supset V_{\nu+1} \times V_{\nu+1} \qquad \text{for } \nu = 1, 2, \dots,$$

putting

$$V_\nu^0 = \{0\}, \quad V_\nu^1 = V_\nu \qquad (\nu = 1, 2, \dots),$$
$$U_\tau = V_1^{\varepsilon_1} \times V_2^{\varepsilon_2} \times \cdots \times V_\kappa^{\varepsilon_\kappa},$$
$$\tau = \sum_{\nu=1}^{\kappa} \frac{\varepsilon_\nu}{2^\nu}, \qquad \varepsilon_\nu = 0, 1,$$
$$U_\tau = R \qquad \text{for } \tau \ge 1,$$
$$|||x||| = \inf_{x \in U_\tau} \tau \qquad \text{for every } x \in R,$$

we obtain a quasi-norm $|||x|||$ on R such that $|||x||| \le 1$ for every $x \in R$ and we have for every $\nu = 1, 2, \dots$

$$\{x : |||x||| < \tfrac{1}{2^\nu}\} \subset V_\nu \subset \{x : |||x||| \le \tfrac{1}{2^\nu}\}.$$

Proof. From the construction of $|||x|||$ we conclude immediately

$$0 \le |||x||| \le 1 \qquad \text{for every } x \in R,$$
$$\{x : |||x||| < \tfrac{1}{2^\nu}\} \subset V_\nu \subset \{x : |||x||| \le \tfrac{1}{2^\nu}\} \qquad (\nu = 1, 2, \dots).$$

As every V_ν ($\nu = 1, 2, \dots$) is a symmetric star, U_τ also is a symmetric star for every $\tau > 0$, and hence, if $0 \ne |\alpha| \le |\beta|$, then $\beta x \in U_\tau$ implies $\alpha x \in U_\tau$. Thus we have that $|\alpha| \le |\beta|$ implies $|||\alpha x||| \le |||\beta x|||$.

For $\tau = \sum_{\nu=1}^{\kappa} \frac{\varepsilon_\nu}{2^\nu}$, $\tau' = \sum_{\nu=1}^{\kappa} \frac{\varepsilon'_\nu}{2^\nu}$ we have obviously

$$U_\tau \times U_{\tau'} = (V_1^{\varepsilon_1} \times V_1^{\varepsilon'_1}) \times \cdots \times (V_\kappa^{\varepsilon_\kappa} \times V_\kappa^{\varepsilon'_\kappa}) \subset U_{\tau+\tau'}.$$

Thus we obtain

$$|||x||| + |||y||| = \inf_{x \in U_\tau} \tau + \inf_{y \in U_{\tau'}} \tau' = \inf_{x \in U_\tau, y \in U_{\tau'}} (\tau + \tau')$$
$$\ge \inf_{x+y \in U_{\tau+\tau'}} (\tau + \tau') = |||x + y|||.$$

For every $x \in R$ and $\nu = 1, 2, \dots$, we can find by the definition of vicinities $\alpha > 0$ such that $\xi x \in V_\nu$ for $0 \le \xi \le \alpha$ and hence

$$|||\xi x||| \le \tfrac{1}{2^\nu} \qquad \text{for } 0 \le \xi \le \alpha.$$

Accordingly we have $\lim_{\xi \to 0} |||\xi x||| = 0$ for every $x \in R$. Therefore $|||x|||$ is a quasi-norm on R.

Theorem 4. If a symmetric star vicinity V is of finite character and $\lambda V \times \mu V \subset \chi V$ for $\lambda + \mu = 1$, $\lambda, \mu \ge 0$, then there is a quasi-norm $|||x|||$ on R such that

$$\{x : |||x||| < \tfrac{1}{2^\nu}\} \subset \frac{1}{(2\chi)^\nu} V \subset \{x : |||x||| \le \tfrac{1}{2^\nu}\},$$
$$|||\tfrac{1}{2\chi} x||| = \tfrac{1}{2}|||x||| \qquad \text{for } |||x||| < \tfrac{1}{2}.$$

Proof. Putting $\mathfrak{V}_\nu = \frac{1}{(2\varkappa)^\nu}\mathfrak{V}$ ($\nu = 1, 2, \ldots$), we have obviously

$$\mathfrak{V}_{\nu+1} \times \mathfrak{V}_{\nu+1} = \frac{1}{(2\varkappa)^{\nu+1}}(\mathfrak{V}\times\mathfrak{V}) < \frac{1}{(2\varkappa)^\nu}\mathfrak{V} = \mathfrak{V}_\nu .$$

Therefore we obtain a quasi-norm $|||x|||$, as described in Theorem 3, and we have for every $\nu = 1, 2, \ldots$

$$\{x : |||x||| < \frac{1}{2^\nu}\} < \frac{1}{(2\varkappa)^\nu}\mathfrak{V} < \{x : |||x||| \leqq \frac{1}{2^\nu}\}.$$

Furthermore, as we have for $\tau < \frac{1}{2}$

$$2\varkappa\,\mathfrak{U}_\tau = 2\varkappa\,\mathfrak{V}_2^{\varepsilon_2} \times \cdots \times 2\varkappa\,\mathfrak{V}_\varkappa^{\varepsilon_\varkappa} = \mathfrak{V}_1^{\varepsilon_2} \times \cdots \times \mathfrak{V}_{\varkappa-1}^{\varepsilon_\varkappa} = \mathfrak{U}_{2\tau},$$

we obtain for $|||x||| < \frac{1}{2}$

$$|||\frac{1}{2\varkappa}x||| = \inf_{x \in 2\varkappa\mathfrak{U}_\tau} \tau = \inf_{x \in \mathfrak{U}_{2\tau}} \tau = \frac{1}{2}|||x|||.$$

For a quasi-norm $|||x|||$ on R, putting

$$m(x, y) = |||x - y|||,$$

we obtain a quasi-metric m on R. This quasi-metric m is called the *induced quasi-metric* by a quasi-norm $|||x|||$ on R. A manifold A of R is said to be *complete* by a quasi-norm $|||x|||$ on R, if A is complete by the induced quasi-metric, that is, if $\lim_{\nu,\mu\to\infty} |||x_\nu - x_\mu||| = 0$ implies

$$\lim_{\nu\to\infty} |||x_\nu - x||| = 0 \quad \text{for some } x \in A.$$

A quasi-norm $|||x|||$ on R is said to be *complete*, if R is complete by it.

§51. Relative vicinities

Let $\mathfrak{S}$ be a linear manifold of a linear space R. For a vicinity $\mathfrak{V}$ in R, putting $\mathfrak{V}^{\mathfrak{S}} = \mathfrak{S}\mathfrak{V}$ we obtain a vicinity $\mathfrak{V}^{\mathfrak{S}}$ in the subspace $\mathfrak{S}$. This vicinity $\mathfrak{V}^{\mathfrak{S}}$ is called the *relative vicinity* of $\mathfrak{V}$ in the subspace $\mathfrak{S}$. Concerning relative vicinities we have obviously

Theorem 1. *If a vicinity $\mathfrak{V}$ is symmetric, scalar-closed, scalar-open, or a star, then its relative vicinity $\mathfrak{V}^{\mathfrak{S}}$ also is so respectively.*

Furthermore we conclude easily by definition

Theorem 2. *The character of the relative vicinity $\mathfrak{V}^{\mathfrak{S}}$ is not greater than that of $\mathfrak{V}$. Consequently, if a vicinity $\mathfrak{V}$ is convex, then the relative vicinity $\mathfrak{V}^{\mathfrak{S}}$ of $\mathfrak{V}$ also is so for every subspace $\mathfrak{S}$.*

Every pseudo-norm on R may be considered obviously as a pseudo-

norm on a subspace $\mathfrak{S}$. In this sense, we have obviously by definition

Theorem 3. *The pseudo-norm $\|x\|_{\mathfrak{V}}$ of a symmetric star vicinity $\mathfrak{V}$ coincides with the pseudo-norm $\|x\|_{\mathfrak{V}^{\mathfrak{S}}}$ of the relative vicinity $\mathfrak{V}^{\mathfrak{S}}$ in a subspace $\mathfrak{S}$, that is, $\|x\|_{\mathfrak{V}} = \|x\|_{\mathfrak{V}^{\mathfrak{S}}}$ for every $x \in \mathfrak{S}$.*

The image $\mathfrak{V}/\mathfrak{S}$ of a vicinity $\mathfrak{V}$ by the quotient mapping of R onto the quotient space $R/\mathfrak{S}$ is obviously by definition a vicinity in the quotient space $R/\mathfrak{S}$. This vicinity $\mathfrak{V}/\mathfrak{S}$ is called the *relative vicinity* of $\mathfrak{V}$ in the quotient space $R/\mathfrak{S}$.

Concerning relative vicinities in a quotient space, we obtain immediately by definition

Theorem 4. *If a vicinity $\mathfrak{V}$ is symmetric, scalar-open, scalar-closed, or a star, then its relative vicinity $\mathfrak{V}/\mathfrak{S}$ in a quotient space $R/\mathfrak{S}$ also is so respectively.*

Furthermore we conclude easily by definition

Theorem 5. *The character of the relative vicinity $\mathfrak{V}/\mathfrak{S}$ in a quotient space $R/\mathfrak{S}$ is not greater than that of $\mathfrak{V}$. Consequently, if a vicinity $\mathfrak{V}$ is convex, then the relative vicinity $\mathfrak{V}/\mathfrak{S}$ also is so.*

For a pseudo-norm $\|x\|$ on R, putting

$$\|X\| = \inf_{x \in X} \|x\| \qquad \text{for } X \in R/\mathfrak{S},$$

we obtain a pseudo-norm $\|X\|$ on the quotient space $R/\mathfrak{S}$. Because we have obviously $\|X\| \geqq 0$ for every $X \in R/\mathfrak{S}$, and

$$\|\alpha X\| = \inf_{x \in \alpha X} \|x\| = \inf_{x \in X} \|\alpha x\| = |\alpha| \inf_{x \in X} \|x\| = |\alpha|\|X\|$$

for every real number $\alpha \neq 0$. This pseudo-norm $\|X\|$ on the quotient space $R/\mathfrak{S}$ is called the *relative pseudo-norm* of $\|x\|$ in $R/\mathfrak{S}$.

Theorem 6. *The character of the relative pseudo-norm $\|X\|$ of a pseudo-norm $\|x\|$ in a quotient space $R/\mathfrak{S}$ is not greater than that of $\|x\|$. Consequently the relative pseudo-norm of a convex pseudo-norm is convex too.*

Proof. If $\|x + y\| \leqq \varkappa(\|x\| + \|y\|)$ for every $x, y \in R$, then we have by definition

$$\|X + Y\| = \inf_{x \in X+Y} \|x\| \leqq \inf_{x \in X, y \in Y} \|x + y\|$$
$$\leqq \inf_{x \in X, y \in Y} \varkappa(\|x\| + \|y\|) = \varkappa(\|X\| + \|Y\|).$$

Theorem 7. For a symmetric star vicinity $\mathfrak{V}$ in R, the relative pseudo-norm $\|x\|_{\mathfrak{V}}$ of $\mathfrak{V}$ in a quotient space R/S coincides with the pseudo-norm of the relative vicinity $\mathfrak{V}/S$ in R/S, that is, we have $\|X\|_{\mathfrak{V}} = \|X\|_{\mathfrak{V}/S}$ for $X \in R/S$.

Proof. Recalling the formulas (5) and (7) in §46, we obtain by definition

$$\|X\|_{\mathfrak{V}} = \inf_{x \in X} \|x\|_{\mathfrak{V}} = \inf_{x \in X} \left(\inf_{x \in \xi \mathfrak{V}} |\xi| \right)$$

$$= \inf_{X \subset (\xi \mathfrak{V}) \times S} |\xi| = \inf_{X \in \xi(\mathfrak{V}/S)} |\xi| = \|X\|_{\mathfrak{V}/S} .$$

§52 Adjoint norms

Let $\mathfrak{V}$ be a vicinity in a linear space R. For a linear functional φ on R, the adjoint norm $\|\varphi\|_{\mathfrak{V}}$ by a vicinity $\mathfrak{V}$ is defined as

(1) $$\|\varphi\|_{\mathfrak{V}} = \sup_{x \in \mathfrak{V}} |\varphi(x)| .$$

With this definition we have

(2) $$\|\varphi\|_{\alpha\mathfrak{V}} = |\alpha| \|\varphi\|_{\mathfrak{V}} \quad \text{for } \alpha \neq 0 .$$

Because we have for every $\alpha \neq 0$

$$\|\varphi\|_{\alpha\mathfrak{V}} = \sup_{x \in \alpha\mathfrak{V}} |\varphi(x)| = \sup_{x \in \mathfrak{V}} |\varphi(\alpha x)| = |\alpha| \|\varphi\|_{\mathfrak{V}} .$$

Furthermore we have obviously by definition

(3) $$\mathfrak{V} \subset \mathfrak{U} \quad \text{implies} \quad \|\varphi\|_{\mathfrak{V}} \leqq \|\varphi\|_{\mathfrak{U}} .$$

The totality of linear functional on R, which are bounded in $\mathfrak{V}$, is called the adjoint space of a vicinity $\mathfrak{V}$ and denoted by $\overline{R}_{\mathfrak{V}}$, that is,

$$\overline{R}_{\mathfrak{V}} = \{ \varphi : \|\varphi\|_{\mathfrak{V}} < +\infty \} .$$

Then we see easily by definition that $\overline{R}_{\mathfrak{V}}$ is a linear manifold of the associated space $\widetilde{R}$ of R.

Theorem 1. The adjoint norm $\|\bar{a}\|_{\mathfrak{V}}$ $(\bar{a} \in \overline{R}_{\mathfrak{V}})$ is a complete norm on the adjoint space $\overline{R}_{\mathfrak{V}}$ of a vicinity $\mathfrak{V}$.

Proof. From the definition (1) we conclude immediately that we have $\|\bar{a}\|_{\mathfrak{V}} \geqq 0$ and $\|\alpha \bar{a}\|_{\mathfrak{V}} = |\alpha| \|\bar{a}\|_{\mathfrak{V}}$, that is, $\|\bar{a}\|_{\mathfrak{V}}$ is a pseudo-norm on $\overline{R}_{\mathfrak{V}}$. For every $\bar{a}, \bar{b} \in \overline{R}_{\mathfrak{V}}$ we have by (1)

$$\|\bar{a} + \bar{b}\|_{\mathfrak{V}} = \sup_{x \in \mathfrak{V}} |\bar{a}(x) + \bar{b}(x)|$$

$$\leqq \sup_{x \in \mathfrak{V}} |\bar{a}(x)| + \sup_{x \in \mathfrak{V}} |\bar{b}(x)| = \|\bar{a}\|_{\mathfrak{V}} + \|\bar{b}\|_{\mathfrak{V}} .$$

If $\|\bar{a}\|_{\mathfrak{V}} = 0$, then we have by (1) $\bar{a}(x) = 0$ for every $x \in \mathfrak{V}$, and

hence $\bar{a} = 0$, because for every $x \in R$ we can find $\alpha > 0$ such that we have $\alpha x \in \mathfrak{V}$. Therefore $\|\bar{a}\|_{\mathfrak{V}}$ is a norm on $\overline{R}_{\mathfrak{V}}$.

If $\lim_{\nu,\mu\to\infty} \|\bar{a}_\nu - \bar{a}_\mu\|_{\mathfrak{V}} = 0$, then we have obviously by (1)

$$\lim_{\nu,\mu\to\infty} |\bar{a}_\nu(x) - \bar{a}_\mu(x)| = 0 \qquad \text{for every } x \in \mathfrak{V} .$$

Since for every $x \in R$ we can find $\alpha > 0$ such that $\alpha x \in \mathfrak{V}$, we obtain

$$\lim_{\nu,\mu\to\infty} |\bar{a}_\nu(x) - \bar{a}_\mu(x)| = 0 \qquad \text{for every } x \in R .$$

Therefore, putting $\varphi(x) = \lim_{\mu\to\infty} \bar{a}_\mu(x)$ for every $x \in R$, we obtain a linear functional φ on R. Then we have for every $\nu = 1, 2, \ldots$

$$|\bar{a}_\nu(x) - \varphi(x)| = \lim_{\mu\to\infty} |\bar{a}_\nu(x) - \bar{a}_\mu(x)|$$

$$\leqq \varliminf_{\mu\to\infty} \|\bar{a}_\nu - \bar{a}_\mu\|_{\mathfrak{V}}$$

for every $x \in \mathfrak{V}$, and hence by (1)

$$\|\bar{a}_\nu - \varphi\|_{\mathfrak{V}} \leqq \varliminf_{\mu\to\infty} \|\bar{a}_\nu - \bar{a}_\mu\|_{\mathfrak{V}} .$$

Consequently we obtain $\varphi \in \overline{R}_{\mathfrak{V}}$ and $\lim_{\nu\to\infty} \|\bar{a}_\nu - \varphi\|_{\mathfrak{V}} = 0$. Therefore the norm $\|\bar{a}\|_{\mathfrak{V}}$ is complete.

Theorem 2. If a vicinity $\mathfrak{V}$ is symmetric, scalar-closed, and convex, then for any $x_0 \bar{\in} \mathfrak{V}$ we can find a linear functional $\varphi_0 \in \overline{R}_{\mathfrak{V}}$ such that $\varphi_0(x_0) > \|\varphi_0\|_{\mathfrak{V}}$.

Proof. Putting $\psi(\xi x_0) = \xi$ for every real number ξ, we obtain obviously a linear functional ψ on the linear manifold generated by the single element x_0. Furthermore we have for every real number ξ

$$|\psi(\xi x_0)| = |\xi| = \frac{1}{\|x_0\|_{\mathfrak{V}}} \|\xi x_0\|_{\mathfrak{V}} .$$

Therefore we can find by §44 Theorem 4 a linear functional φ_0 on R such that

$$\varphi_0(\xi x_0) = \psi(\xi x_0) \qquad \text{for every real number } \xi ,$$

$$|\varphi_0(x)| \leqq \frac{1}{\|x_0\|_{\mathfrak{V}}} \|x\|_{\mathfrak{V}} \qquad \text{for every } x \in R ,$$

because the pseudo-norm $\|x\|_{\mathfrak{V}}$ $(x \in R)$ is convex by §49 Theorem 2. For such φ_0 we have by the definition (1)

$$\|\varphi_0\|_{\mathfrak{V}} = \sup_{x \in \mathfrak{V}} |\varphi_0(x)| \leqq \frac{1}{\|x_0\|_{\mathfrak{V}}}$$

because $\|x\|_{\mathfrak{V}} \leqq 1$ for $x \in \mathfrak{V}$ by the formula §49(2). Since $\mathfrak{V}$ is symmetric and scalar-closed by assumption, we have $\|x_0\|_{\mathfrak{V}} > 1$ by §49 Theorem 3, and hence $\|\varphi_0\|_{\mathfrak{V}} < 1 = \psi(x_0) = \varphi(x_0)$.

Theorem 3. For a vicinity $\mathfrak{V}$, the manifold

$$\mathfrak{V} = \{ x : |\bar{a}(x)| \leqq \|\bar{a}\|_{\mathfrak{V}} \quad \text{for all } \bar{a} \in \overline{R}_{\mathfrak{V}} \}$$

is the least symmetric scalar-closed convex vicinity including V, and we have $\bar{R}_V = \bar{R}_U$, $\|\bar{a}\|_V = \|\bar{a}\|_U$ $(\bar{a} \in \bar{R}_V)$.

Proof. We see at once by the construction of U that $U \supset V$ and U is symmetric and scalar-closed. For $\lambda + \mu = 1$, $\lambda, \mu \geqq 0$, if $x, y \in U$, then we have

$$|\bar{a}(\lambda x + \mu y)| \leqq \lambda |\bar{a}(x)| + \mu |\bar{a}(y)| \leqq \|\bar{a}\|_V,$$

and hence $\lambda x + \mu y \in U$. Therefore U is a symmetric scalar-closed convex vicinity. For every symmetric scalar-closed convex vicinity $W \supset V$, the intersection UW also is obviously such a one. If there is an element $x_0 \in U$ such that $x_0 \bar{\in} UW$, then we can find by Theorem 2 a linear functional $\varphi_0 \in \bar{R}_V$ such that $\varphi_0(x_0) > \|\varphi_0\|_V$, contradicting the construction of U. Consequently we have $U \subset UW \subset W$, and hence U is the least symmetric scalar-closed convex vicinity including V.

From $U \supset V$ we conclude by the formula (3) that we have $\bar{R}_U \subset \bar{R}_V$ and $\|\bar{a}\|_U \geqq \|\bar{a}\|_V$ for every $\bar{a} \in \bar{R}_U$. On the other hand, we have obviously by the construction of U that $\bar{R}_U \supset \bar{R}_V$ and $\|\bar{a}\|_U \leqq \|\bar{a}\|_V$ for every $\bar{a} \in \bar{R}_V$.

Theorem 4. For a symmetric star vicinity V we have

$$|\bar{a}(x)| \leqq \|\bar{a}\|_V \|x\|_V \quad \text{for every } \bar{a} \in \bar{R}_V \text{ and } x \in R,$$

$$\|x\|_V = \sup_{\|\bar{a}\|_V \leqq 1} |\bar{a}(x)| \quad \text{for every } x \in R.$$

Proof. For every $\xi > \|x\|_V$, as $\frac{1}{\xi}x \in V$ by the formula §49(2), we have by (1) $|\bar{a}(\frac{1}{\xi}x)| \leqq \|\bar{a}\|_V$, that is,

$$|\bar{a}(x)| \leqq \xi \|\bar{a}\|_V \quad \text{for every } \xi > \|x\|_V,$$

and hence we obtain $|\bar{a}(x)| \leqq \|\bar{a}\|_V \|x\|_V$. From this relation we conclude immediately $\|x\|_V \geqq \sup_{\|\bar{a}\|_V \leqq 1} |\bar{a}(x)|$.

For an element $x \in R$, if $\xi > \sup_{\|\bar{a}\|_V \leqq 1} |\bar{a}(x)|$, then $\sup_{\|\bar{a}\|_V \leqq 1} |\bar{a}(\frac{1}{\xi}x)| < 1$, and hence we obtain

$$|\bar{a}(\tfrac{1}{\xi}x)| \leqq \|\bar{a}\|_V \quad \text{for every } \bar{a} \in \bar{R}_V.$$

If V is convex, then $\{x : \|x\|_V \leqq 1\}$ is by §49 Theorem 1 a symmetric scalar-closed convex vicinity including V, and hence we conclude by Theorem 3 that $\|\frac{1}{\xi}x\|_V \leqq 1$, that is, $\|x\|_V \leqq \xi$. Accordingly, if V is convex, then we have $\|x\|_V \leqq \sup_{\|\bar{a}\|_V \leqq 1} |\bar{a}(x)|$ for every $x \in R$.

Theorem 5. Let V be a symmetric convex vicinity such that the pseudo-norm $\|x\|_V$ $(x \in R)$ is complete, and $\bar{R}_V$ the adjoint space of V. For a system of linear functionals $\bar{a}_\lambda \in \bar{R}_V$ $(\lambda \in \Lambda)$, if

$$\sup_{\lambda \in \Lambda} |\bar{a}_\lambda(x)| < +\infty \quad \text{for every } x \in R,$$

then we have $\sup_{\lambda \in \Lambda} \|\bar{a}_\lambda\|_V < +\infty$.

Proof. Since the pseudo-norm $\|x\|_V$ is complete by assumption, we see by Theorems 6 in §36 and 1 in §39 that R is of the second category for the induced topology by the induced quasi-metric. If we put

$$A_\nu = \{x : |\bar{a}_\lambda(x)| \leqq \nu \quad \text{for every } \lambda \in \Lambda\},$$

then we have obviously by assumption $R = \sum_{\nu=1}^{\infty} A_\nu$.

If $\lim_{\mu\to\infty} \|a_\mu - a\|_V = 0$, $a_\mu \in A_\nu$ $(\mu = 1, 2, \ldots)$, then, since we have by Theorem 4 $|\bar{a}_\lambda(a)| \leqq |\bar{a}_\lambda(a_\mu)| + |\bar{a}_\lambda(a_\mu) - \bar{a}_\lambda(a)| \leqq \nu + \|\bar{a}_\lambda\|_V \|a_\mu - a\|_V$, we obtain $|\bar{a}_\lambda(a)| \leqq \nu$ for every $\lambda \in \Lambda$, that is, $a \in A_\nu$. Consequently, A_ν is closed for the induced topology by Theorems 2 in §14, 3 in §36, and 1 §39. Therefore we can find ν_0 such that $A_{\nu_0}^{\circ} \neq 0$ for the induced topology, and then we can find further $a_0 \in A_{\nu_0}$ and $\varepsilon > 0$ such that $\|a_0 - x\|_V < \varepsilon$ implies $x \in A_{\nu_0}$. Then $\|x\|_V < \varepsilon$ implies $x + a_0 \in A_{\nu_0}$, and hence further $|\bar{a}_\lambda(x)| \leqq |\bar{a}_\lambda(a_0)| + |\bar{a}_\lambda(x + a_0)| \leqq |\bar{a}_\lambda(a_0)| + \nu$ for every $\lambda \in \Lambda$. As $\|x\|_V \leqq 1$ implies $\|\frac{\varepsilon}{2}x\|_V \leqq \frac{\varepsilon}{2} < \varepsilon$, we obtain hence by definition

$$\|\bar{a}_\lambda\|_V = \sup_{\|x\|_V \leqq 1} |\bar{a}_\lambda(x)| = \frac{2}{\varepsilon} \sup_{\|x\|_V \leqq 1} |\bar{a}_\lambda(\tfrac{\varepsilon}{2}x)| \leqq \frac{2}{\varepsilon}(|\bar{a}_\lambda(a_0)| + \nu),$$

and consequently $\sup_{\lambda \in \Lambda} \|\bar{a}_\lambda\|_V \leqq \frac{2}{\varepsilon}(\sup_{\lambda \in \Lambda} |\bar{a}_\lambda(a_0)| + \nu) < +\infty$.

Theorem 6. Let V be an arbitrary vicinity in R. For a linear manifold A of R, the manifold of the adjoint space $\bar{R}_V$ of V

$$\{\bar{a} : \bar{a}(x) = 0 \quad \text{for every } x \in A\}$$

coincides with the adjoint space of the relative vicinity V/A in the quotient space R/A, and we have $\|\bar{a}\|_{V/A} = \|\bar{a}\|_V$ for every $\bar{a} \in \bar{R}_V$ subject to the condition that $\bar{a}(x) = 0$ for every $x \in A$.

Proof. Every linear functional $\bar{a}$ on the quotient space R/A may be considered as a linear functional on R by the relation

$$\bar{a}(x) = \bar{a}(X) \quad \text{for } x \in X \in R/A,$$

and then we have obviously $\sup_{x \in V} |\bar{a}(x)| = \sup_{X \in V/A} |\bar{a}(X)|$. From This relation we conclude easily our assertion.

Theorem 7. Let V be a symmetric convex vicinity and γ a positive number. For a finite number of elements $x_\nu \in R$ and real numbers α_ν $(\nu = 1, 2, \ldots, \varkappa)$, in order that there is a linear functional φ on R such that

$$\|\varphi\|_V \leqq \gamma,$$
$$\varphi(x_\nu) = \alpha_\nu \quad (\nu = 1, 2, \ldots, \varkappa),$$

it is necessary and sufficient that we have

$$\left|\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu\right| \leqq \gamma \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right\|_V$$

for every real numbers ξ_ν $(\nu = 1, 2, \ldots, \varkappa)$.

Proof. If there is such a linear functional φ on R, then we have by Theorem 4 for every real numbers ξ_ν $(\nu = 1, 2, \ldots, \varkappa)$

$$\left|\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu\right| = \left|\varphi\left(\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right)\right| \leqq \gamma \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right\|_V.$$

Conversely, we suppose now that the indicated condition is satisfied. Then $\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu = 0$ implies obviously $\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu = 0$. Thus, if we denote by A the linear manifold generated by x_ν $(\nu = 1, 2, \ldots, \varkappa)$, then there is by §44 Theorem 3 a linear functional φ_0 on A such that

$$\varphi_0(x_\nu) = \alpha_\nu \quad (\nu = 1, 2, \ldots, \varkappa).$$

For such φ_0 we have by assumption

$$\left|\varphi_0\left(\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right)\right| = \left|\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu\right| \leqq \gamma \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right\|_V$$

for every real numbers ξ_ν $(\nu = 1, 2, \ldots, \varkappa)$, that is,

$$|\varphi_0(x)| \leqq \gamma \|x\|_V \quad \text{for every } x \in A.$$

As V is convex by assumption, the pseudo-norm $\|x\|_V$ is convex by §49 Theorem 2. Thus we can find by §44 Theorem 3 a linear functional φ on R such that

$$\varphi(x) = \varphi_0(x) \quad \text{for every } x \in A,$$
$$|\varphi(x)| \leqq \gamma \|x\|_V \quad \text{for every } x \in R,$$

and hence $\varphi(x_\nu) = \alpha_\nu$ $(\nu = 1, 2, \ldots, \varkappa)$ and $\|\varphi\|_V \leqq \gamma$.

Finally we shall prove the so-called Helly's theorem:

Theorem 8. Let V be a symmetric convex vicinity in R and γ a positive number. For a finite number of linear functionals $\bar{a}_\nu \in \bar{R}_V$ and real numbers α_ν $(\nu = 1, 2, \ldots, \varkappa)$, in order that for any positive number ε we can find an element $x_0 \in (\gamma + \varepsilon)V$ such that

$$\bar{a}_\nu(x_0) = \alpha_\nu \quad (\nu = 1, 2, \ldots, \varkappa),$$

it is necessary and sufficient that we have

$$\left|\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu\right| \leqq \gamma \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu\right\|_V$$

for every real numbers ξ_ν $(\nu = 1, 2, \ldots, \varkappa)$.

Proof. If there is an element $x_0 \in (\gamma + \varepsilon)V$ such that

$$\bar{a}_\nu(x_0) = \alpha_\nu \quad \text{for every } \nu = 1, 2, \ldots, \varkappa,$$

then we have $\|x_0\|_V \leqq \gamma + \varepsilon$ by the formula §49(2) and hence by Theorem 4

$$\left|\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu\right| = \left|\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu(x_0)\right| \leqq (\gamma + \varepsilon) \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu\right\|_V.$$

As $\varepsilon > 0$ may be arbitrary, we obtain thus for every ξ_ν $(\nu = 1, 2, \ldots, \varkappa)$

$$\left|\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu\right| \leqq \gamma \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu\right\|_V.$$

Conversely, we assume now that the indicated condition is satisfied. Putting $A = \{x : \bar{a}_\nu(x) = 0 \text{ for every } \nu = 1, 2, \ldots, \varkappa\}$, we obtain a linear manifold A of R, and we see by §46 Theorem 3 that the quotient space R/A is finite-dimensional and the associated space $\widehat{R/A}$ of R/A is composed of all linear combinations from $\bar{a}_\nu$ $(\nu = 1, 2, \ldots, \varkappa)$. As $\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu = 0$ implies by assumption $\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu = 0$, we can find by §45 Theorem 4 a residue class $X_0 \in R/A$ such that

$$\bar{a}_\nu(X_0) = \alpha_\nu \quad (\nu = 1, 2, \ldots, \varkappa).$$

For such X_0 we have by assumption

$$\left|\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu(X_0)\right| = \left|\sum_{\nu=1}^{\varkappa} \xi_\nu \alpha_\nu\right| \leqq \gamma \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu\right\|_V$$

for every real numbers ξ_ν $(\nu = 1, 2, \ldots, \varkappa)$, and hence

$$|\bar{a}(X_0)| \leqq \gamma \|\bar{a}\|_V \quad \text{for } \bar{a} \in \widehat{R/A}.$$

As $\|\bar{a}\|_V = \|\bar{a}\|_{V/A}$ for $\bar{a} \in \widehat{R/A}$ by Theorem 6, we obtain hence by Theorem 4 $\|X_0\|_{V/A} \leqq \gamma$, because V/A is a symmetric convex vicinity by Theorems 4 and 5 in §51. Since we have by §51 Theorem 7

$$\|X_0\|_{V/A} = \inf_{x \in X_0} \|x\|_V,$$

for any $\varepsilon > 0$ we can find therefore $x_0 \in X_0$ such that $\|x_0\|_V < \gamma + \varepsilon$. For such x_0 we have $x_0 \in (\gamma + \varepsilon)V$ by the formula §49(2) and obviously

$$\bar{a}_\nu(x_0) = \bar{a}_\nu(X_0) = \alpha_\nu \quad (\nu = 1, 2, \ldots, \varkappa).$$

CHAPTER VIII

LINEAR TOPOLOGY

§53 Definitions

Let R be a linear space. A system of vicinities $\mathfrak{V}$ in R is called a *linear topology* on R, if

1) $V \in \mathfrak{V}$, $V \subset U$ implies $U \in \mathfrak{V}$.
2) $U, V \in \mathfrak{V}$ implies $UV \in \mathfrak{V}$.
3) $V \in \mathfrak{V}$ implies $\xi V \in \mathfrak{V}$ for every $\xi \neq 0$;
4) for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that
$$\xi U \subset V \qquad \text{for } 0 \leqq \xi \leqq 1,$$
5) for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that
$$U \times U \subset V.$$

A linear space associated with a linear topology is called a *linear topological space*. If $\mathfrak{V}$ is composed only of a single vicinity R, then $\mathfrak{V}$ is obviously a linear topology on R by definition. This linear topology is called the *trivial linear topology* on R.

A subset $\mathfrak{U}$ of a linear topology $\mathfrak{V}$ is called a *basis* of $\mathfrak{V}$, if for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{U}$ and $\lambda > 0$ such that $\lambda U \subset V$.

Theorem 1. *For a linear topology $\mathfrak{V}$ on R there is a basis of $\mathfrak{V}$, which is composed only of symmetric star vicinities.*

Proof. For every $V \in \mathfrak{V}$ we can find by definition $U \in \mathfrak{V}$ such that $\xi U \subset V((-1)V)$ for $0 \leqq \xi \leqq 1$. For such U, we have obviously $\xi U \subset V$ for $|\xi| \leqq 1$, and hence $\sum_{|\xi| \leqq 1} \xi U \subset V$. Here $\sum_{|\xi|<1} \xi U$ is obviously a symmetric star vicinity and belongs to $\mathfrak{V}$ by the postulate 1). Therefore the totality of symmetric star vicinities contained in $\mathfrak{V}$ constitutes a basis of $\mathfrak{V}$.

Theorem 2. *For a basis $\mathfrak{U}$ of a linear topology $\mathfrak{V}$, if a manifold $A \ni 0$ is contained in every vicinity of $\mathfrak{U}$, then $V \times A$ $(V \in \mathfrak{U})$ also is a basis of $\mathfrak{V}$.*

Proof. For every $V \in \mathfrak{U}$ we have obviously by 1) $V \times A \in \mathfrak{V}$. Furthermore, for every $V \in \mathfrak{V}$, we can find by 5) $U \in \mathfrak{U}$ and $\lambda > 0$ such

$U \times U \subset \frac{1}{\lambda} V$, and hence $\lambda(U \times A) \subset \lambda(U \times U) \subset V$. Therefore we see by definition that $V \times A$ $(V \in \mathfrak{U})$ is a basis of $\mathfrak{V}$.

A linear topology $\mathfrak{V}$ is said to be of *finite character*, if there is a basis of $\mathfrak{V}$ which is composed only of symmetric star vicinities of finite character. A linear topology $\mathfrak{V}$ is said to be of *bounded character*, if there is a basis composed only of symmetric star vicinities whose chatacters are bounded. A linear topology $\mathfrak{V}$ is said to be *convex*, if there is a basis of $\mathfrak{V}$ composed only of symmetric convex vicinities.

Theorem 3 *If a collection of vicinites $\mathfrak{U}$ in R satisfies*

1) *for every* $V, U \in \mathfrak{U}$ *we can find* $W \in \mathfrak{U}$ *and* $\lambda > 0$ *such that*
$$\lambda W \subset VU,$$
2) *for any* $V \in \mathfrak{U}$ *we can find* $U \in \mathfrak{U}$ *and* $\lambda > 0$ *such that*
$$\xi U \subset V \qquad \text{for } |\xi| \leqq \lambda,$$
3) *for any* $V \in \mathfrak{U}$ *we can find* $U \in \mathfrak{U}$ *and* $\lambda > 0$ *such that*
$$\lambda U \times \lambda U \subset V$$

then there exists uniquely a linear topology $\mathfrak{V}$ of which $\mathfrak{U}$ is a basis.

Proof. Denoting by $\mathfrak{V}$ the totality of vicinities V such that we have $V \supset \lambda U$ for some $U \in \mathfrak{U}$ and $\lambda > 0$, we conclude easily from assumption that $\mathfrak{V}$ is a linear topology containing $\mathfrak{U}$ as a basis. Furthermore it is evident by definition that every linear topology containing $\mathfrak{U}$ as a basis coincides with $\mathfrak{V}$.

We shall say that a linear topology $\mathfrak{V}$ on R is *separative*, or that R is *separated* by $\mathfrak{V}$, if $\prod_{V \in \mathfrak{V}} V = \{0\}$.

As an immediate consequence of Theorem 3 we have

Theorem 4. *If a symmetric star vicinity V is of finite character, then there exists uniquely a linear topology of which V is a basis.*

§54 Induced topologies

Let R be a linear space. Every vicinity V in R may be considered as a connector in R by the correspondence: $R \ni x \longrightarrow V + x$.

For a linear topology $\mathfrak{V}$ on R, considering every vicinity $V \in \mathfrak{V}$ as a connector, we see easily that $\mathfrak{V}$ satisfies the basis conditions in §26, and hence there exists by §26 Theorem 1 uniquely a uniformity on R of which $\mathfrak{V}$ is a basis. This uniformity is called the <u>induced uniformity</u> by a linear topology $\mathfrak{V}$ and denoted by $\mathfrak{U}^{\mathfrak{V}}$. The indu[illegible] topology by this induced uniformity $\mathfrak{U}^{\mathfrak{V}}$ is called the <u>induced topology</u> by a linear topology $\mathfrak{V}$ and denoted by $\mathfrak{T}^{\mathfrak{V}}$.

Concerning the induced topology $\mathfrak{T}^{\mathfrak{V}}$ we have by the formulas (2), (4) in §27 for every manifold $A \subset R$

(1) $A^\circ = \{x : \ A \supset V + x \quad \text{for some } V \in \mathfrak{V}\},$

(2) $A^- = \prod_{V \in \mathfrak{V}} A \times V = \prod_{V \in \mathfrak{V}} (A \times V)^\circ .$

When we need indicate the induced topology $\mathfrak{T}^{\mathfrak{V}}$, we shall write $A^{\mathfrak{V}\circ}$ or $A^{\mathfrak{V}-}$ instead of A° or A^- respectively.

For every element $a \in R$ we obtain by (1)

$$\begin{aligned} A^\circ + a &= \{x + a : \ A \supset V + x \quad \text{for some } V \in \mathfrak{V}\} \\ &= \{x + a : \ A + a \supset V + x + a \quad \text{for some } V \in \mathfrak{V}\} \\ &= \{x : \ A + a \supset V + x \ \text{ for some } V \in \mathfrak{V}\} = (A + a)^\circ . \end{aligned}$$

That is, we have for every $a \in R$

(3) $A^\circ + a = (A + a)^\circ .$

From this relation we conclude by the formula §43(10)

(4) $A^- + a = (A + a)^- .$

For every real number $\alpha \neq 0$ we have by (1)

$$\begin{aligned} \alpha A^\circ &= \{\alpha x : \ A \supset V + x \quad \text{for some } V \in \mathfrak{V}\} \\ &= \{\alpha x : \alpha A \supset \alpha V + \alpha x \quad \text{for some } V \in \mathfrak{V}\} \\ &= \{x : \alpha A \supset V + x \ \text{ for some } V \in \mathfrak{V}\} = (\alpha A)^\circ \end{aligned}$$

and hence we have

(5) $\alpha A^\circ = (\alpha A)^\circ$ for $\alpha \neq 0$.

From this relation we conclude by the formula §43(16)

(6) $\alpha A^- = (\alpha A)^-$ for $\alpha \neq 0$.

By virtue of the formula §43(5), we obtain at once

(7) $A^- \subset A * V^\circ \subset (A \times V)^\circ$ for $V \in \mathfrak{V}$.

For every $V \in \mathfrak{V}$, as $V \subset V^-$, we have obviously $V^- \in \mathfrak{V}$ by definition.

For every $V \in \mathfrak{V}$ we can find by definition $U \in \mathfrak{V}$ such that $U \times U \subset V$ and hence by (7) $U \subset (U \times U)^\circ \subset V^\circ$. Thus we have

(8) $V^\circ \in \mathfrak{V}$ for $V \in \mathfrak{V}$.

For every manifolds A, B we have

(9) $A^\circ \times B^\circ \subset (A \times B)^\circ$,

(10) $A^- \times B^- \subset (A \times B)^-$.

Because, we have obviously $A^\circ \times B^\circ \subset A \times B$, and $A^\circ \times B^\circ$ is open by (3), as $A^\circ \times B^\circ = \sum_{x \in B^\circ} (A + x)^\circ$. Thus we obtain (9). For every $V \in \mathfrak{V}$ there is by definition $U \in \mathfrak{V}$ such that $U \times U \subset V$, and we have by (2)

$$A^- \times B^- \subset (A \times U) \times (B \times U) \subset (A \times B) \times V.$$

Therefore we obtain (10) by the formula (2)

On account of the formula (5) we have obviously

<u>Theorem 1.</u> <u>If a manifold A is symmetric, then both A^- and A° are symmetric too.</u>

<u>Theorem 2.</u> <u>If a manifold A is a star, then A^- is scalar-closed and A° is scalar-open.</u>

<u>Proof.</u> If A is a star, then A^- also is a star, because $\xi A \subset A$ implies $\xi A^- \subset A^-$ by the formula (6). Furthermore A^- is scalar-closed. Because, for any $x \,\bar{\in}\, A^-$ we can find by (1) $V \in \mathfrak{V}$ such that $V + x \subset A^{-\prime}$. For such $V \in \mathfrak{V}$ we can find further by definition $\varepsilon > 0$ such that $-\varepsilon x \in V$. Then we have $(1-\varepsilon)x \in V + x \subset A^{-\prime}$, that is, $(1-\varepsilon)x \,\bar{\in}\, A^-$. Therefore A^- is scalar-closed. We also can prove likewise that A° is scalar-open, if A is a star.

<u>Theorem 3.</u> <u>For a vicinity A, the characters of A^- and A° are not greater than that of A.</u>

<u>Proof.</u> Recalling (5), (6), (9), (10), we see that $\lambda A \times \mu A \subset \chi A$ implies $\lambda A^\circ \times \mu A^\circ \subset \chi A^\circ$, $\lambda A^- \times \mu A^- \subset \chi A^-$. Thus we obtain Theorem 3.

From Theorem 3 we conclude immediately

<u>Theorem 4.</u> <u>If a vicinity A is convex, then both A^- and A° are convex.</u>

<u>Theorem 5.</u> <u>For a linear manifold A, both A^- and A° are linear manifolds.</u>

Proof. If A is a linear manifold, then we have by definition $\alpha A \subset A$ and $A \times A \subset A$. Thus we obtain by (5) and (6)

$$\alpha A^{\circ} = (\alpha A)^{\circ} \subset A^{\circ}, \qquad \alpha A^{-} = (\alpha A)^{-} \subset A^{-},$$

and by (9), (10) further

$$A^{\circ} \times A^{\circ} \subset (A \times A)^{\circ} \subset A^{\circ}, \qquad A^{-} \times A^{-} \subset (A \times A)^{-} \subset A^{-}.$$

Therefore A^{-} and A° are linear manifolds.

For a basis $\mathfrak{B}$ of $\mathcal{V}$ we conclude immediately from (1) and (2)

(11) $$A^{\circ} = \{x : A \supset \lambda V + x \text{ for some } V \in \mathfrak{B} \text{ and } \lambda > 0\},$$

(12) $$A^{-} = \prod_{V \in \mathfrak{B}, \lambda > 0} A \times \lambda V = \prod_{V \in \mathfrak{B}, \lambda > 0} (A \times \lambda V)^{\circ}.$$

Theorem 6. For a basis $\mathfrak{B}$ of a linear topology $\mathcal{V}$, both

$$\{V^{-} : V \in \mathfrak{B}\} \quad \text{and} \quad \{V^{\circ} : V \in \mathfrak{B}\}$$

are basises of $\mathcal{V}$ too.

Proof. On account of (5) and (8), we see that $\{V^{\circ} : V \in \mathfrak{B}\}$ is a basis of $\mathcal{V}$. For any $U \in \mathcal{V}$, we can find by definition $V \in \mathfrak{B}$ and $\lambda > 0$ such that $\lambda V \times \lambda V \subset U$, and then we have by (6) and (7)

$$\lambda V^{-} = (\lambda V)^{-} \subset \lambda V \times \lambda V \subset U.$$

Therefore $\{V^{-} : V \in \mathfrak{B}\}$ is a basis of $\mathcal{V}$ by definition.

Theorem 7. Let $\mathfrak{B}$ be a basis of $\mathcal{V}$ which is composed only of symmetric star vicinities. We have $a \in A^{-}$ if and only if

$$\inf_{x \in A} \| a - x \|_V = 0 \qquad \text{for every } V \in \mathfrak{B}.$$

Proof. If $a \in A^{-}$, then we have by the formula (12)

$$a \in A \times \xi V \qquad \text{for every } V \in \mathfrak{B} \text{ and } \xi > 0.$$

Thus there is $x \in A$ such that $a \in \xi V + x$, and hence $a - x \in \xi V$. This relation yields by §49(2) $\| a - x \|_V \leq \xi$. As $\xi > 0$ may be arbitrary, we obtain hence $\inf_{x \in A} \| a - x \|_V = 0$ for every $V \in \mathfrak{B}$.

Conversely, if $\inf_{x \in A} \| a - x \|_V = 0$ for every $V \in \mathfrak{B}$, then for every $V \in \mathfrak{B}$ and $\lambda > 0$ we can find $x \in A$ such that $\| a - x \|_V < \lambda$. This relation yields by §49(2) $a - x \in \lambda V$, and hence $a \in A \times \lambda V$ for every $V \in \mathfrak{B}$ and $\lambda > 0$. Therefore we obtain by the formula (12) $a \in A^{-}$.

Theorem 8. If a topology $\mathcal{T}$ on a linear space R satisfies

1) $A \in \mathcal{T}$ implies $A + a \in \mathcal{T}$ for every $a \in R$,

2) $A \in \mathcal{T}$ implies $\lambda A \in \mathcal{T}$ for every $\lambda \neq 0$,

3) for $0 \in A \in \mathcal{T}$ and $x \in R$ we can find $\alpha > 0$ such that $\alpha x \in A$,

4) for $0 \in A \in \mathcal{T}$ we can find $B \in \mathcal{T}$ such that

$$0 \in \xi B \subset A \qquad \text{for } |\xi| \leq 1,$$

5) for $0 \in A \in \mathcal{T}$ we can find $C \in \mathcal{T}$ such that

$$C \times C \subset A, \quad 0 \in C,$$

then there exists uniquely a linear topology $\mathcal{V}$ on R such that $\mathcal{T}$ coincides with the induced topology by $\mathcal{V}$. Furthermore $\{A : 0 \in A \in \mathcal{T}\}$ is a basis of $\mathcal{V}$.

Proof. Every open set $A \ni 0$ is a vicinity by the assumption 3), 4). Furthermore we see easily that the totality of open sets containing 0 satisfies the conditions in Theorem 53.3, and hence there exists uniquely a linear topology $\mathcal{V}$ on R such that $\{A : 0 \in A \in \mathcal{T}\}$ is a basis of $\mathcal{V}$. For such $\mathcal{V}$, it is evident that $\mathcal{T} = \mathcal{T}^{\mathcal{V}}$.

For a linear topology $\mathcal{U}$ on R, if its induced topology $\mathcal{T}^{\mathcal{U}}$ coincides with $\mathcal{T}$, then we see by (8) that $\{A : 0 \in A \in \mathcal{T}\}$ is a basis of $\mathcal{U}$. Thus we obtain the uniqueness of such a linear topology $\mathcal{V}$.

We shall say that a manifold A is open or closed by a linear topology $\mathcal{V}$, if A is so by the induced topology $\mathcal{T}^{\mathcal{V}}$ by $\mathcal{V}$.

§55 Comparation of linear topologies

For two linear topologies $\mathcal{V}$ and $\mathcal{U}$ on a linear space R, if $\mathcal{V} \supset \mathcal{U}$, then we shall say that $\mathcal{V}$ is stronger than $\mathcal{U}$ or that $\mathcal{U}$ is weaker than $\mathcal{V}$.

Let $\mathcal{V}_\lambda$ $(\lambda \in \Lambda)$ be a system of linear topologies on a linear space R and $\mathfrak{B}_\lambda$ a basis of $\mathcal{V}_\lambda$ respectively. If we denote by $\mathfrak{B}$ the totality of

$$V_{\lambda_1} V_{\lambda_2} \dots V_{\lambda_\kappa}$$

for every finite number of vicinities $V_{\lambda_\nu} \in \mathfrak{B}_{\lambda_\nu}$ $(\nu = 1, 2, \dots, \kappa)$, then $\mathfrak{B}$ satisfies the conditions in Theorem 3 in §53. In fact, the condition 1) is satisfied obviously. For any $V_{\lambda_\nu} \in \mathfrak{B}_{\lambda_\nu}$, we can find $U_{\lambda_\nu} \in \mathfrak{B}_{\lambda_\nu}$ and $\rho_\nu > 0$ such that $\xi U_{\lambda_\nu} \subset V_{\lambda_\nu}$ for $0 \leq \xi \leq \rho_\nu$, and then, putting $\rho_0 = \operatorname{Min}_{\nu = 1, \dots, \kappa} \rho_\nu$, we have obviously for $0 \leq \xi \leq \rho_0$

$$\xi (U_{\lambda_1} U_{\lambda_2} \dots U_{\lambda_\kappa}) \subset V_{\lambda_1} V_{\lambda_2} \dots V_{\lambda_\kappa},$$

that is, the condition 2) is satisfied. For any $V_{\lambda_\nu} \in \mathfrak{L}_{\lambda_\nu}$ we can find $U_{\lambda_\nu} \in \mathfrak{L}_{\lambda_\nu}$ and $\rho_\nu > 0$ such that $\rho_\nu U_{\lambda_\nu} \times \rho_\nu U_{\lambda_\nu} \subset V_{\lambda_\nu}$, and further $W_{\lambda_\nu} \in \mathfrak{L}_{\lambda_\nu}$ and $\sigma_\nu > 0$ such that $\xi W_{\lambda_\nu} \subset U_{\lambda_\nu}$ for $0 \leqq \xi \leqq \sigma_\nu$. Then, putting $\rho = \operatorname{Min}_{\nu=1,2,\dots,\kappa} \sigma_\nu \rho_\nu$, we obtain $\rho(W_{\lambda_1} W_{\lambda_2} \cdots W_{\lambda_\kappa}) \times \rho(W_{\lambda_1} W_{\lambda_2} \cdots W_{\lambda_\kappa}) \subset V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_\kappa}$, that is, the condition 3) is satisfied too. Therefore there exists uniquely a linear topology $\mathfrak{V}_0$ such that $\mathfrak{L}$ is a basis of $\mathfrak{V}_0$. For this linear topology $\mathfrak{V}_0$ we have obviously $\mathfrak{V}_0 \supset \mathfrak{L}_\lambda$ and hence $\mathfrak{V}_0 \supset \mathfrak{V}_\lambda$ for every $\lambda \in \Lambda$. On the other hand, if a linear topology $\mathfrak{V}$ is stronger than $\mathfrak{V}_\lambda$ for every $\lambda \in \Lambda$, then we have obviously $\mathfrak{V} \supset \mathfrak{L}$ and hence we obtain $\mathfrak{V} \supset \mathfrak{V}_0$ by definition. Thus $\mathfrak{V}_0$ is the weakest stronger linear topology of $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$, and hence we may write $\mathfrak{V}_0 = \bigcup_{\lambda\in\Lambda} \mathfrak{V}_\lambda$.

The trivial linear topology on R is obviously weaker than every other linear topology, that is, it is the weakest linear topology on R.

For a system of linear topologies $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$ on R, putting

$$\mathfrak{V}_0 = \bigcup_{\mathfrak{V} \subset \mathfrak{V}_\lambda (\lambda \in \Lambda)} \mathfrak{V},$$

we obtain the strongest weaker linear topology of $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$. Because, we have obviously $\mathfrak{V}_0 \subset \mathfrak{V}_\lambda$ for every $\lambda \in \Lambda$, and if $\mathfrak{V} \subset \mathfrak{V}_\lambda$ for every $\lambda \in \Lambda$, then we have $\mathfrak{V} \subset \mathfrak{V}_0$. Therefore we obtain two following theorems:

Theorem 1. For a system of linear topologies $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$ on a linear space R, there exist the weakest stronger linear topology $\bigcup_{\lambda\in\Lambda} \mathfrak{V}_\lambda$ and the strongest weaker linear topology $\bigcap_{\lambda\in\Lambda} \mathfrak{V}_\lambda$.

Theorem 2. For a basis $\mathfrak{L}_\lambda$ of $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$, the totality of

$$V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_\kappa}$$

for every finite number of vicinities $V_{\lambda_\nu} \in \mathfrak{L}_{\lambda_\nu}$ $(\nu = 1, 2, \dots, \kappa)$ is a basis of the weakest stronger linear topology $\bigcup_{\lambda\in\Lambda} \mathfrak{V}_\lambda$.

If V_{λ_ν} is symmetric and convex for every $\nu = 1, 2, \dots, \kappa$, then $V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_\kappa}$ also is so obviously. Therefore we conclude immediately from Theorem 2

Theorem 3. If all linear topologies $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$ are convex, then $\bigcup_{\lambda\in\Lambda} \mathfrak{V}_\lambda$ also is convex.

Theorem 4. For two linear topologies $\mathfrak{V}$ and $\mathfrak{U}$ on a linear space R, the induced topology $\mathfrak{T}^{\mathfrak{V}}$ by $\mathfrak{V}$ is weaker than that $\mathfrak{T}^{\mathfrak{U}}$ by $\mathfrak{U}$, if and only if $\mathfrak{V} \subset \mathfrak{U}$.

Proof. If $\mathfrak{V} \subset \mathfrak{U}$, then the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$ by $\mathfrak{V}$ is weaker than that $\mathfrak{U}^{\mathfrak{U}}$ by $\mathfrak{U}$, and hence $\mathfrak{T}^{\mathfrak{V}} \subset \mathfrak{T}^{\mathfrak{U}}$ by §28 Theorem 4. Conversely, if $\mathfrak{T}^{\mathfrak{V}} \subset \mathfrak{T}^{\mathfrak{U}}$, then we have by the formulas (1) and (8) in §54

$$\{A : 0 \in A \in \mathfrak{T}^{\mathfrak{V}}\} \subset \mathfrak{U},$$

and hence $\mathfrak{V} \subset \mathfrak{U}$ by §54 Theorem 8.

Theorem 5. For a system of linear topologies $\mathfrak{V}_\lambda$ $(\lambda \in \Lambda)$, putting $\mathfrak{V} = \bigcup_{\lambda\in\Lambda} \mathfrak{V}_\lambda$, we have $\mathfrak{U}^{\mathfrak{V}} = \bigcup_{\lambda\in\Lambda} \mathfrak{U}^{\mathfrak{V}_\lambda}$, $\mathfrak{T}^{\mathfrak{V}} = \bigcup_{\lambda\in\Lambda} \mathfrak{T}^{\mathfrak{V}_\lambda}$.

Proof. Recalling Theorem 2, we obtain $\mathfrak{U}^{\mathfrak{V}} = \bigcup_{\lambda\in\Lambda} \mathfrak{U}^{\mathfrak{V}_\lambda}$ by §28 Theorem 3, and hence $\mathfrak{T}^{\mathfrak{V}} = \bigcup_{\lambda\in\Lambda} \mathfrak{T}^{\mathfrak{V}_\lambda}$ by §28 Theorem 5.

§56 Relative linear topologies

Let S be a linear manifold of a linear space R. For a linear topology $\mathfrak{V}$ on R, we see easily by definition that the totality of relative vicinities V^S $(V \in \mathfrak{V})$ constitutes a linear topology on the subspace S. This linear topology is called the relative linear topology of $\mathfrak{V}$ in the subspace S and denoted by $\mathfrak{V}^S$.

Concerning relative linear topologies we have obviously by definition

Theorem 1. For a basis $\mathfrak{L}$ of a linear topology $\mathfrak{V}$ on R, the collection of vicinities V^S $(V \in \mathfrak{L})$ is a basis of the relative linear topology $\mathfrak{V}^S$ in a subspace S.

The induced uniformity by the relative linear topology $\mathfrak{V}^S$ is obviously by definition the relative uniformity of the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$ by $\mathfrak{V}$ in a subspace S. Thus we have by §29 Theorem 2

Theorem 2. The induced topology by the relative linear topology $\mathfrak{V}^S$ coincides with the relative topology of the induced topology by $\mathfrak{V}$ in a subspace S, that is, $\mathfrak{T}^{\mathfrak{V}^S} = (\mathfrak{T}^{\mathfrak{V}})^S$.

We have obviously by definition

Theorem 3. The relative linear topology of a separative linear topology also is separative.

Recalling the formulas (5)-(10) in §46, we see easily that for a linear topology $\mathcal{V}$ on R the totality of relative vicinities V/S ($V \in \mathcal{V}$) constitutes a linear topology on the quotient space R/S. This linear topology on R/S is called the relative linear topology of $\mathcal{V}$ in the quotient space R/S and denoted by $\mathcal{V}^{R/S}$.

With this definition we have obviously

Theorem 4. For a basis $\mathfrak{U}$ of a linear topology $\mathcal{V}$ on R, V/S ($V \in \mathfrak{U}$) is a basis of the relative linear topology $\mathcal{V}^{R/S}$.

The quotient space R/S is obviously by definition a partition space of R. In this sense, we have

Theorem 5. The induced topology by the relative linear topology $\mathcal{V}^{R/S}$ is the partition topology of the induced topology $\mathcal{T}^{\mathcal{V}}$ by $\mathcal{V}$ and the quotient mapping is continuous and open.

Proof. If A is open by $\mathcal{V}$, then for any $x \in A$ we can find by the formula §54(1) $V \in \mathcal{V}$ such that $V + x \subset A$ and hence $V + x/S \subset A/S$. As $V + x/S = (V/S) \times (S + x/S)$ by the formulas (5) and (6) in §46, we see by the formula §54(1) that A/S is open by $\mathcal{V}^{R/S}$. Therefore the quotient mapping is open. We also can prove likewise that the quotient mapping is continuous. Consequently we obtain our assertion.

Theorem 6. If a linear manifold S is closed by a linear topology $\mathcal{V}$ on R, then the quotient space R/S is separated by the relative linear topology $\mathcal{V}^{R/S}$.

Proof. If $S + x \in V/S$ for all $V \in \mathcal{V}$, then we have by §46(5) $S + x \subset S \times V$ for all $V \in \mathcal{V}$, and hence $S + x \subset S^- = S$ by §54(2). Therefore R/S is separated by $\mathcal{V}^{R/S}$.

Let R and S be two linear topological spaces with linear topologies $\mathcal{V}$ and $\mathcal{U}$ respectively. We see easily that the totality of (V, U) for $V \in \mathcal{V}$, $U \in \mathcal{U}$ in the product space (R, S) satisfies the conditions in §53 Theorem 3, and hence there exists uniquely a linear topology on the product space (R, S) of which (V, U) for $V \in \mathcal{V}$, $U \in \mathcal{U}$ is a basis. This linear topology on (R, S) is called the product of $\mathcal{V}$ and $\mathcal{U}$, and denoted by $(\mathcal{V}, \mathcal{U})$. With this definition we have obviously

Theorem 7. For a basis $\mathfrak{U}$ of $\mathcal{V}$ and $\mathfrak{A}$ of $\mathcal{U}$, (V, U) for $V \in \mathfrak{U}$, $U \in \mathfrak{A}$ constitutes a basis of the product $(\mathcal{V}, \mathcal{U})$.

A linear topological space R is said to be isohomeomorphic to a linear topological space S, if we can find a linear transformation from R to S which is a homeomorphism for the induced topologies. With this definition we can prove easily by §16 Theorem 5

Theorem 8. Let R and S be two linear topological spaces with linear topologies $\mathcal{V}$ and $\mathcal{U}$ respectively. For a linear transformation $\mathfrak{a}$ from R to S, in order that R be isohomeomorphic to S by $\mathfrak{a}$, it is necessary and sufficient that for any $U \in \mathcal{U}$ we can find $V \in \mathcal{V}$ such that $\mathfrak{a}(V) \subset U$ and for any $V \in \mathcal{V}$ we can find $U \in \mathcal{U}$ such that $\mathfrak{a}(V) \supset U$.

Recalling §47 Theorem 1 we obtain by Theorem 8

Theorem 9. The quotient space $(R, S)/(\{0\}, S)$ is isohomeomorphic to R for the relative linear topology of the product $(\mathcal{V}, \mathcal{U})$.

Theorem 10. For a linear topological space R with a linear topology $\mathcal{V}$, if a linear manifold S is contained in every $V \in \mathcal{V}$, then the product space $(R/S, S)$ is isohomeomorphic to R for $(\mathcal{V}^{R/S}, \mathcal{V}^{S})$.

Proof. By virtue of §47 Theorem 2, $(R/S, S)$ is isomorphic to R by a linear transformation such that the image of $(V/S, S)$ is $V \times S$ for every $V \in \mathcal{V}$. As $S \subset V$ for every $V \in \mathcal{V}$ by assumption, the relative linear topology $\mathcal{V}^{S}$ is the trivial one, that is, $\mathcal{V}^{S}$ is composed only of S. Therefore we see by Theorem 8 that $(R/S, S)$ is isohomeomorphic to R.

§57 Bounded manifolds

Let R be a linear topological space with a linear topology $\mathcal{V}$. A manifold A of R is said to be bounded, if for any $V \in \mathcal{V}$ we can find $\lambda > 0$ such that $A \subset \lambda V$. With this definition we see at once that for a basis $\mathfrak{U}$ of $\mathcal{V}$, a manifold A is bounded if and only if for any $V \in \mathfrak{U}$ we can find $\lambda > 0$ such that $A \subset \lambda V$.

We see further by definition that if a manifold A is bounded, then αA also is bounded for every real number α. If both A and B are bounded by $\mathfrak{V}$, then $A \times B$ also is bounded. Because, for any $V \in \mathfrak{V}$ we can find by definition $U \in \mathfrak{V}$ such that $U \times U \subset V$, and $\alpha > 0$ such that $A \subset \alpha U$, $B \subset \alpha U$. Then we have $A \times B \subset \alpha U \times \alpha U \subset \alpha V$.

Recalling the formula §49(2) we have obviously

Theorem 1. For a basis $\mathfrak{B}$ of a linear topology $\mathfrak{V}$, a manifold A is bounded by $\mathfrak{V}$ if and only if $\sup_{x \in A} \|x\|_V < +\infty$ for every $V \in \mathfrak{B}$.

Theorem 2. If a manifold A is bounded by $\mathfrak{V}$, then the closure A^- by the induced topology $\mathfrak{T}^{\mathfrak{V}}$ is bounded too.

Proof. For any $V \in \mathfrak{V}$ we can find by definition $U \in \mathfrak{V}$ and $\lambda > 0$ such that $U \times U \subset V$, $A \subset \lambda U$. Then we have by §54(7)

$$A^- \subset \lambda U \times \lambda U \subset \lambda V.$$

Therefore A^- is bounded by definition.

Theorem 3. If there is no bounded open set except O, then every bounded set is nowhere dense by the induced topology $\mathfrak{T}^{\mathfrak{V}}$.

Proof. If a manifold A is bounded, then A^- also is bounded by Theorem 2, and hence $A^{-\circ} = O$ by assumption.

For a star vicinity V we have obviously

$$\alpha V \subset V^{\nu} \qquad \text{for } 0 \leqq \alpha < \nu.$$

Thus we obtain by §53 Theorem 1

Theorem 4. If a manifold A is bounded by a linear topology $\mathfrak{V}$, then A is bounded by the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$.

If a star vicinity V is of finite character and $V \times V \subset \alpha V$, $\alpha > 1$, then we have $V^{\nu} \subset \alpha^{\nu-1} V$. Therefore we have

Theorem 5. If a linear topology $\mathfrak{V}$ is of finite character, then every bounded manifold by the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$ is bounded by $\mathfrak{V}$.

Recalling the definition of relative linear topologies we have

Theorem 6. For a bounded manifold A in R by a linear topology $\mathfrak{V}$, A/S is bounded in the quotient space R/S by the relative linear topology $\mathfrak{V}^{R/S}$.

Let $\mathfrak{V}$ and $\mathfrak{U}$ be two linear topologies on a linear space R. $\mathfrak{V}$

is said to be equivalent to $\mathfrak{U}$, if bounded manifolds by $\mathfrak{V}$ coincide with those by $\mathfrak{U}$. A linear topology $\mathfrak{V}$ is said to be equivalently strongest, if $\mathfrak{V}$ is stronger than every other linear topology which is equivalent to $\mathfrak{V}$.

Recalling §55 Theorem 2, we can prove easily

Theorem 7. For every linear topology $\mathfrak{V}$ there exists an equivalently strongest linear topology which is equivalent to $\mathfrak{V}$.

A linear topology $\mathfrak{V}$ is said to be standard, if $\mathfrak{V}$ is convex and stronger than every other convex linear topology which is equivalent to $\mathfrak{V}$. With this definition we obtain by Theorems 2 and 3 in §55

Theorem 8. For every convex linear topology $\mathfrak{V}$ there exists a standard linear topology which is equivalent to $\mathfrak{V}$.

A manifold A of R is said to be totally bounded by a linear topology $\mathfrak{V}$, if A is totally bounded by the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$ by $\mathfrak{V}$.

§58 Sequential linear topologies

A linear topology $\mathfrak{V}$ is said to be sequential, if $\mathfrak{V}$ has a basis composed only of countable vicinities. With this definition we see easily by §52 Theorem 1 that if a linear topology $\mathfrak{V}$ is sequential, then $\mathfrak{V}$ has a basis $\{V_1, V_2, \dots\}$ such that V_ν ($\nu = 1, 2, \dots$) are symmetric stars,

$$V_\nu \supset V_{\nu+1} \times V_{\nu+1} \qquad (\nu = 1, 2, \dots),$$

and for each $V \in \mathfrak{V}$ we can find ν such that $V \supset V_\nu$. Such a basis $\{V_1, V_2, \dots\}$ is called a decreasing basis. If $\mathfrak{V}$ is sequential, then its induced uniformity is obviously sequential by definition. Furthermore we see easily that a decreasing basis $\{V_1, V_2, \dots\}$ of $\mathfrak{V}$ is a decreasing basis of the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$, considering every V_ν as a connector: $R \ni x \rightarrow V_\nu + x$.

Theorem 1. If a linear topology $\mathfrak{V}$ is sequential, then $\mathfrak{V}$ is equivalently strongest, and furthermore $\mathfrak{V}$ contains all vicinities V subject to the condition that for any bounded manifold A we can find $\lambda > 0$ for which $A \subset \lambda V$.

Proof. Let $\{V_1, V_2, \dots\}$ be a decreasing basis of a sequential linear topology $\mathfrak{P}$. For a vicinity U, if $U \bar{\in} \mathfrak{P}$, then corresponding to every $\nu = 1, 2, \dots$ we can find $x_\nu \in R$ such that

$$\nu U \bar{\ni} x_\nu \in V_\nu \qquad (\nu = 1, 2, \dots).$$

For such x_ν $(\nu = 1, 2, \dots)$, $\{x_1, x_2, \dots\}$ is obviously bounded by $\mathfrak{P}$ but not contained in νU for every $\nu = 1, 2, \dots$. Therefore $\mathfrak{P}$ is equivalently strongest by definition.

Let $\mathfrak{P}$ be an arbitrary linear topology on a linear space R. For any $V \in \mathfrak{P}$ we can find by definition a sequence of symmetric star vicinities $V_\nu \in \mathfrak{P}$ such that $V \supset V_1 \times V_1$, $V_\nu \supset V_{\nu+1} \times V_{\nu+1}$ $(\nu = 1, 2, \dots)$, and $\{V, V_1, V_2, \dots\}$ satisfies the conditions of §53 Theorem 3. Therefore we obtain

Theorem 2. *For a linear topology $\mathfrak{P}$ there is a system of sequential linear topologies $\mathfrak{P}_\lambda$ $(\lambda \in \Lambda)$ such that $\mathfrak{P} = \bigcup_{\lambda \in \Lambda} \mathfrak{P}_\lambda$.*

A linear topology $\mathfrak{P}$ is said to be of *single vicinity*, if $\mathfrak{P}$ has a basis composed only of a single vicinity. If a linear topology $\mathfrak{P}$ is of single vicinity, then there is obviously by §53 Theorem 3 a symmetric star vicinity $V \in \mathfrak{P}$ such that V is a basis of $\mathfrak{P}$. Every linear topology of single vicinity is naturally sequential.

If a linear topology $\mathfrak{P}$ is of single vicinity and a symmetric star vicinity V is a basis of $\mathfrak{P}$, then we can find by definition $\alpha > 0$ such that $V \times V \subset \alpha V$, and then $\lambda V \times \mu V \subset \alpha V$ for $\lambda + \mu = 1$, $\lambda, \mu \geqq 0$. Therefore we have

Theorem 3. *Every linear topology of single vicinity is of bounded character.*

For a symmetric star vicinity of finite character V there is by §53 Theorem 3 a linear topology of which V is a basis. Therefore we have

Theorem 4. *For a linear topology of finite character $\mathfrak{P}$ there is a system of linear topologies of single vicinity $\mathfrak{P}_\lambda$ $(\lambda \in \Lambda)$ such that we have $\mathfrak{P} = \bigcup_{\lambda \in \Lambda} \mathfrak{P}_\lambda$.*

Theorem 5. *In order that a linear topology $\mathfrak{P}$ be of single vicinity, it is necessary and sufficient that there is a bounded open manifold different from O for the induced topology $\mathfrak{T}^{\mathfrak{P}}$.*

Proof. If $\mathfrak{P}$ is of single vicinity and V is a basis of $\mathfrak{P}$, then V is obviously a bounded manifold and $V^\circ \ni O$ by the formula §54(8). Conversely, if there is a bounded open manifold $A \neq O$, then we can find by the formula §54(1) $V_0 \in \mathfrak{P}$ and $a_0 \in R$ such that $A \supset V_0 + a_0$. For each $V \in \mathfrak{P}$ there is a symmetric $U \in \mathfrak{P}$ such that $U \times U \subset V$, and we can find $\alpha > 0$ such that $\alpha U \supset A \supset V_0 + a_0$. Then we have

$$V_0 \subset \alpha U + (-a_0) \subset \alpha U \times \alpha U \subset \alpha V,$$

because $a_0 \in \alpha U$ implies $-a_0 \in \alpha U$. Therefore V_0 is a basis of $\mathfrak{P}$.

Recalling the formulas (2) and (3) in §49, we see easily

Theorem 6. *Let $\mathfrak{P}$ and $\mathfrak{U}$ be two linear topologies of single vicinity with symmetric star basises V and U respectively. In order that $\mathfrak{P} > \mathfrak{U}$, it is necessary and sufficient that we can find $\alpha > 0$ such that*

$$\alpha \|x\|_V \geqq \|x\|_U \quad \text{for every } x \in R.$$

A linear topology of single vicinity $\mathfrak{P}$ is said to be *normable*, if $\mathfrak{P}$ is convex. Thus, for a normable linear topology $\mathfrak{P}$ there is a symmetric convex vicinity $V \in \mathfrak{P}$ such that V is a basis of $\mathfrak{P}$. Furthermore we have obviously

Theorem 7. *For a convex linear topology $\mathfrak{P}$ there is a system of normable linear topologies $\mathfrak{P}_\lambda$ $(\lambda \in \Lambda)$ such that $\mathfrak{P} = \bigcup_{\lambda \in \Lambda} \mathfrak{P}_\lambda$*

Recalling §56 Theorem 7, we obtain obviously by definition

Theorem 8. *If both linear topological spaces R and S are sequential or of single vicinity, then the product space (R, S) also is so.*

§59 Completeness

Let R be a linear topological space with a linear topology $\mathfrak{P}$. A manifold A is said to be *complete*, if A is complete by the induced uniformity $\mathfrak{U}^{\mathfrak{P}}$. If every closed bounded manifold is complete, then we shall say that R is *conditionally complete*, or that $\mathfrak{P}$ is *conditionally complete*. If R is complete, then we shall say that $\mathfrak{P}$ is *complete*.

Let $\mathfrak{L}$ be a basis of φ composed only of symmetric star vicinities. Recalling the formulas (2) and (3) in §49, we see easily by definition that a system of manifolds A_λ $(\lambda \in \Lambda)$ is a Cauchy system for the induced uniformity $\mathcal{U}^{\varphi}$, if and only if for any $V \in \mathfrak{L}$ and $\varepsilon > 0$ we can find $\lambda \in \Lambda$ such that $\sup_{x, y \in A_\lambda} \|x - y\|_V < \varepsilon$ and $A_{\lambda_1} A_{\lambda_2} \cdots A_{\lambda_\kappa} \neq 0$ for every finite number of elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \kappa)$. We see further that a Cauchy system A_λ $(\lambda \in \Lambda)$ is convergent to a limit $a \in R$ if and only if for any $V \in \mathfrak{L}$ and $\varepsilon > 0$ we can find $\lambda \in \Lambda$ such that $\sup_{x \in A_\lambda} \|x - a\|_V < \varepsilon$.

We also see likewise that a sequence of elements $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is a Cauchy sequence if and only if $\lim_{\nu, \mu \to \infty} \|a_\nu - a_\mu\|_V = 0$ for every $V \in \mathfrak{L}$, and that $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is convergent to a limit $a \in R$ if and only if $\lim_{\nu \to \infty} \|a_\nu - a\|_V = 0$ for every $V \in \mathfrak{L}$.

It is evident by definition that every Cauchy sequence is bounded by φ. Therefore we obtain by §36 Theorem 4

Theorem 1. If a linear topology φ is sequential and conditionally complete, then φ is complete.

Furthermore we have by §36 Theorem 6

Theorem 2. If a linear topology φ is sequential and complete, then the induced topology γ^{φ} is of second category.

Theorem 3. If a linear topology φ is sequential and $\{V_1, V_2, \ldots\}$ is a decreasing basis of φ, then for any $x_\nu \in V_\nu$ the sequence of elements $x_1 + x_2 + \cdots + x_\nu$ $(\nu = 1, 2, \ldots)$ is a Cauchy sequence by φ and

$$x_{\nu+1} + x_{\nu+2} + \cdots + x_{\nu+\rho} \in V_\nu \qquad \text{for every } \rho = 1, 2, \ldots$$

Proof. As $V_{\nu+1} \times V_{\nu+1} \subset V_\nu$, we have by definition

$$x_{\nu+1} + x_{\nu+2} + \cdots + x_{\nu+\rho} \in V_{\nu+1} \times V_{\nu+2} \times \cdots \times V_{\nu+\rho} \subset V_\nu .$$

Thus we obtain for every $\rho = 1, 2, \ldots$

$$x_1 + x_2 + \cdots + x_{\nu+\rho} \in V_\nu + (x_1 + x_2 + \cdots + x_\nu),$$

and consequently $x_1 + \cdots + x_\nu$ $(\nu = 1, 2, \ldots)$ is a Cauchy sequence by φ.

Theorem 4. For two complete sequential linear topologies φ and $\mathcal{U}$, if $\varphi \subset \mathcal{U}$ and R is separated by φ, then we have $\varphi = \mathcal{U}$.

Proof. Let $\{V_1, V_2, \ldots\}$ be a decreasing basis of φ and $\{U_1, U_2, \ldots\}$ that of $\mathcal{U}$. For each $\nu = 1, 2, \ldots$, since $\sum_{\rho=1}^{\infty} \rho U_{\nu+1} = R$

and the induced topology γ^{φ} is by Theorem 2 of second category, we can find $\rho > 0$ such that $(\rho U_{\nu+1})^{\varphi-o} \neq 0$ for the induced topology γ^{φ}, and hence $U_{\nu+1}^{\varphi-o} \neq 0$ by the formula §54(5). Thus we can find an element $a \in R$ and μ such that $V_\mu + a \subset U_{\nu+1}^{\varphi-}$. This relation yields by the formulas (5), (10) in §54 that $-a \in U_{\nu+1}^{\varphi-}$ and

$$V_\mu = (V_\mu + a) + (-a) \subset U_{\nu+1}^{\varphi-} \times U_{\nu+1}^{\varphi-} \subset U_\nu^{\varphi-}.$$

Therefore we can find $\mu_1 < \mu_2 < \ldots$ such that

$$V_{\mu_\nu} \subset U_\nu^{\varphi-} \qquad \text{for every } \nu = 1, 2, \ldots$$

For every $a \in V_{\mu_{\nu+1}}$, as $a \in U_{\nu+1}^{\varphi-}$, we can find $x_1 \in U_{\nu+1}$ such that $a \in V_{\mu_{\nu+2}} + x_1$, and hence $x_1 \in U_{\nu+1}$, $a - x_1 \in V_{\mu_{\nu+2}}$. We can find further likewise by the induction x_ρ $(\rho = 2, 3, \ldots)$ such that

(*) $\quad x_\rho \in U_{\nu+\rho} \quad (\rho = 1, 2, \ldots),$

(**) $\quad a - (x_1 + x_2 + \cdots + x_\rho) \in V_{\mu_{\nu+\rho+1}}.$

On account of Theorem 3, we conclude from (*) that $x_1 + x_2 + \cdots + x_\rho$ $(\rho = 1, 2, \ldots)$ is a Cauchy sequence by $\mathcal{U}$ and

$$x_1 + x_2 + \cdots + x_\rho \in U_\nu \qquad \text{for every } \rho = 1, 2, \ldots$$

As $\mathcal{U}$ is complete by assumption, there is hence a limit x_0 of the sequence $x_1 + x_2 + \cdots + x_\rho$ $(\rho = 1, 2, \ldots)$ by $\mathcal{U}$, and we can find ρ_0 such that

$$x_0 \in U_\nu + (x_1 + x_2 + \cdots + x_{\rho_0}) \subset U_\nu \times U_\nu \subset U_{\nu-1}.$$

On the other hand, we conclude from (**) that a is a limit of $x_1 + x_2 + \cdots + x_\rho$ $(\rho = 1, 2, \ldots)$ for the induced topology γ^{φ}, and we have by assumption that $\gamma^{\varphi} \subset \gamma^{\mathcal{U}}$ and R is separated by γ^{φ}. Therefore we obtain $a = x_0$ and hence $a \in U_{\nu-1}$ for every $a \in V_{\mu_{\nu+1}}$, that is, $V_{\mu_{\nu+1}} \subset U_{\nu-1}$ for every $\nu = 2, 3, \ldots$ Consequently we have $\varphi \supset \mathcal{U}$.

Theorem 5. For a complete sequential linear topology φ on R, the relative linear topology $\varphi^{R/S}$ in a quotient space R/S is complete and sequential.

Proof. Let $\{V_1, V_2, \ldots\}$ be a decreasing basis of φ. On account of Theorem 4 in §56, $\{V_1/S, V_2/S, \ldots\}$ is a basis of the relative linear topology $\varphi^{R/S}$ in a quotient space R/S, and hence $\varphi^{R/S}$ is sequential. For a Cauchy sequence $X_\nu \in R/S$ $(\nu = 1, 2, \ldots)$, we can find by definition $\mu_1 < \mu_2 < \ldots$ such that

$$X_{\mu_{\nu+1}} - X_{\mu_\nu} \in \mathfrak{V}_\nu / S \qquad (\nu = 1, 2, \dots),$$

and then further $x_\nu \in \mathfrak{V}_\nu$ $(\nu = 1, 2, \dots)$ such that

$$S + x_\nu = X_{\mu_{\nu+1}} - X_{\mu_\nu} \qquad (\nu = 1, 2, \dots).$$

For such $x_\nu \in \mathfrak{V}_\nu$, there is by Theorem 3 a limit a of $x_1 + \dots + x_\nu$ $(\nu = 1, 2, \dots)$. As $S + (x_1 + \dots + x_\nu) = X_{\mu_{\nu+1}} - X_{\mu_1}$, we see easily that $(S + a) + X_{\mu_1}$ is a limit of X_{μ_ν} $(\nu = 1, 2, \dots)$ by the relative linear topology $\varphi^{R/S}$, and hence by §35 Theorem 1 a limit of X_ν $(\nu = 1, 2, \dots)$. Therefore R/S is complete by the relative linear topology.

Theorem 6. If both linear topologies φ on R and $\mathfrak{U}$ on S are complete or conditionally complete, then the product space (R, S) is complete or conditionally complete by the product $(\varphi, \mathfrak{U})$.

Proof. Let $A_\lambda \subset (R, S)$ $(\lambda \in \Lambda)$ be a Cauchy system by the product $(\varphi, \mathfrak{U})$. Then for any $\mathfrak{V} \in \varphi$ and $\mathfrak{U} \in \mathfrak{U}$ we can find by definition $\lambda_0 \in \Lambda$, $x_0 \in R$, and $y_0 \in S$ such that $A_{\lambda_0} \subset (\mathfrak{V} + x_0, \mathfrak{U} + y_0)$. From this relation we conclude easily $A_{\lambda_0} \times (\{0\}, S) \subset (\mathfrak{V} + x_0, S)$. Putting

$$(B_\lambda, S) = A_\lambda \times (\{0\}, S) \qquad (\lambda \in \Lambda),$$

we obtain hence $B_{\lambda_0} \subset \mathfrak{V} + x_0$. Therefore B_λ $(\lambda \in \Lambda)$ is a Cauchy system by φ. Furthermore, if $A_\lambda \subset A$ $(\lambda \in \Lambda)$ for a bounded manifold A by $(\varphi, \mathfrak{U})$, then, putting $(B, S) = A \times (\{0\}, S)$, we conclude easily that $B_\lambda \subset B$ $(\lambda \in \Lambda)$ and B is bounded by φ. Therefore there is by assumption a limit b of B_λ $(\lambda \in \Lambda)$ by φ. Putting

$$(R, C_\lambda) = A_\lambda \times (R, \{0\}) \qquad (\lambda \in \Lambda),$$

we obtain likewise a limit c of C_λ $(\lambda \in \Lambda)$ by $\mathfrak{U}$. Then for every $\mathfrak{V} \in \varphi$ and $\mathfrak{U} \in \mathfrak{U}$ we can find λ_0 and $\rho_0 \in \Lambda$ such that $B_{\lambda_0} \subset \mathfrak{V} + b$, $C_{\rho_0} \subset \mathfrak{U} + c$, and hence $(B_{\lambda_0}, C_{\rho_0}) \subset (\mathfrak{V} + b, \mathfrak{U} + c)$. As $B_\lambda B_{\lambda_0} \neq 0$, $C_\lambda C_{\rho_0} \neq 0$, we have $(B_\lambda, C_\lambda)(B_{\lambda_0}, C_{\rho_0}) \neq 0$, $A_\lambda = (B_\lambda, C_\lambda)$ for every $\lambda \in \Lambda$. Consequently we obtain $A_\lambda(\mathfrak{V} + b, \mathfrak{U} + c) \neq 0$ for every $\lambda \in \Lambda$, and hence (b, c) is a limit of A_λ $(\lambda \in \Lambda)$ by §35 Theorem 1. Therefore $(\varphi, \mathfrak{U})$ is complete or conditionally complete.

§60 Completion

Let R be a linear topological space with a linear topology φ. A linear topological space $\tilde{R}$ with a linear topology $\tilde{\varphi}$ is called a completion of R, if

1) $\tilde{R}$ is complete by $\tilde{\varphi}$,
2) $\tilde{R}$ contains R as a linear manifold,
3) φ is the relative linear topology of $\tilde{\varphi}$,
4) R is dense in $\tilde{R}$ for the induced topology $\mathfrak{T}^{\tilde{\varphi}}$ by $\tilde{\varphi}$,
5) $\{0\}^{\tilde{\varphi}-} \subset R$.

Theorem 1. For a linear topological space R there exists uniquely a completion of R within an isohomeomorphism.

Proof. We suppose firstly that R is separated by a linear topology φ. Considering R as a uniform space by the induced uniformity $\mathfrak{U}^{\varphi}$, we obtain by §37 Theorem 1 a completion $\tilde{R}$ of R with a uniformity $\mathfrak{U}$ uniquely within a homeomorphism. Then $\tilde{R}$ is separated by $\mathfrak{U}$ by §37 Theorem 3.

For every $\tilde{x}, \tilde{y} \in \tilde{R}$, both $R\mathfrak{U}(\tilde{x})$ and $R\mathfrak{U}(\tilde{y})$ $(\mathfrak{U} \in \mathfrak{U})$ are obviously Cauchy systems by $\mathfrak{U}$. We see easily further that

$$\alpha R\mathfrak{U}(\tilde{x}) \times \beta R\mathfrak{U}(\tilde{y}) \qquad (\mathfrak{U} \in \mathfrak{U})$$

is a Cauchy system for every real numbers α, β. Thus there exists uniquely its limit $\tilde{z} \in \tilde{R}$. For this limit $\tilde{z} \in \tilde{R}$ we define

$$\tilde{z} = \alpha \tilde{x} + \beta \tilde{y}.$$

Then we can prove easily that $\tilde{R}$ is a linear space and contains R as a linear manifold.

If $\tilde{z} = \alpha \tilde{x} + \beta \tilde{y}$, then for any $\mathfrak{U}_1 \in \mathfrak{U}$ we can find by definition $\mathfrak{U}_2 \in \mathfrak{U}$ such that $\alpha R\mathfrak{U}_2(\tilde{x}) \times \beta R\mathfrak{U}_2(\tilde{y}) \subset R\mathfrak{U}_1(\tilde{z})$. Furthermore, for $\mathfrak{U}_3 \times \mathfrak{U}_3 \leqq \mathfrak{U}_2$, $\mathfrak{U}_1 \times \mathfrak{U}_1 \leqq \mathfrak{U}_0$, we have

$$\alpha \mathfrak{U}_3(\tilde{x}) \times \beta \mathfrak{U}_3(\tilde{y}) \subset \mathfrak{U}_0(\tilde{z}).$$

Because, if $\tilde{x}_1 \in \mathfrak{U}_3(\tilde{x})$, $\tilde{y}_1 \in \mathfrak{U}_3(\tilde{y})$, but $\tilde{z}_1 = \alpha \tilde{x}_1 + \beta \tilde{y}_1 \bar{\in} \mathfrak{U}_0(\tilde{z})$, then we can find $\mathfrak{W}_0 \in \mathfrak{U}$ such that $\mathfrak{W}_0(\tilde{z}_1)\mathfrak{U}_1(\tilde{z}) = 0$, and $\mathfrak{W}_1 \in \mathfrak{U}$ such that

$$\mathfrak{W}_1 \leqq \mathfrak{U}_3, \qquad \alpha R\mathfrak{W}_1(\tilde{x}_1) \times \beta R\mathfrak{W}_1(\tilde{y}_1) \subset R\mathfrak{W}_0(\tilde{z}_1).$$

On the other hand, since we conclude

$$\mathfrak{W}_1(\tilde{x}_1) \subset \mathfrak{U}_3(\tilde{x}_1) \subset \mathfrak{U}_3 \times \mathfrak{U}_3(\tilde{x}) \subset \mathfrak{U}_2(\tilde{x}),$$

and similarly $W_1(\tilde{y}_1) \subset U_2(\tilde{x})$, we have

$$\alpha R W_1(\tilde{x}_1) \times \beta R W_1(\tilde{y}_1) \subset \alpha R U_2(\tilde{x}) \times \beta R U_2(\tilde{y}) \subset R U_1(\tilde{x}).$$

Consequently we obtain $\alpha R W_1(\tilde{x}_1) \times \beta R W_1(\tilde{y}_1) = 0$, contradicting that R is dense in $\tilde{R}$ by the induced topology $\mathcal{T}^{\mathfrak{U}}$. Therefore we have proved that if $\tilde{z} = \alpha\tilde{x} + \beta\tilde{y}$, then for any $U_0 \in \mathfrak{U}$ we can find $U \in \mathfrak{U}$ such that

$$(*) \qquad \alpha U(\tilde{x}) \times \beta U(\tilde{y}) \subset U_0(\tilde{z}).$$

We shall denote merely by $\mathcal{T}$ the induced topology $\mathcal{T}^{\mathfrak{U}}$ in $\tilde{R}$ by $\mathfrak{U}$ in the sequel. We obtain by (*) that for any $U \in \mathfrak{U}$ and $\tilde{x}, \tilde{a} \in \tilde{R}$ we can find $U_1 \in \mathfrak{U}$ such that $U(\tilde{x}) + \tilde{a} \supset U_1(\tilde{x} + \tilde{a})$. Thus for any $A \in \mathcal{T}$ and $\tilde{a} \in \tilde{R}$, if $\tilde{x} \in A$, then there is by §27(2) $U \in \mathfrak{U}$ such that $A \supset U(\tilde{x})$, and hence we can find $U_1 \in \mathfrak{U}$ such that

$$A + \tilde{a} \supset U(\tilde{x}) + \tilde{a} \supset U_1(\tilde{x} + \tilde{a}).$$

Therefore we conclude that $A \in \mathcal{T}$ implies $A + \tilde{a} \in \mathcal{T}$ for every $\tilde{a} \in \tilde{R}$. We also can prove likewise that $A \in \mathcal{T}$ implies $\alpha A \in \mathcal{T}$ for every $\alpha \neq 0$. Thus for an arbitrary manifold A of $\tilde{R}$ and $\alpha \neq 0$, as $\alpha A^\circ \in \mathcal{T}$, we obtain $\alpha A^\circ \subset (\alpha A)^\circ$. Accordingly we have further $(\alpha A)^\circ \subset \alpha(\frac{1}{\alpha}\alpha A)^\circ = \alpha A^\circ$. Therefore we have

$$(**) \qquad (\alpha A)^\circ = \alpha A^\circ \qquad \text{for } \alpha \neq 0.$$

If $0 \in A \in \mathcal{T}$, then we can find $U \in \mathfrak{U}$ such that $U^-(0) \subset A$. For such $U \in \mathfrak{U}$, as $U^R \in \mathfrak{U}^{\mathcal{P}}$, we can find a star $V \in \mathcal{P}$ such that

$$(V \times V) + x \subset R\, U(x) \qquad \text{for every } x \in R,$$

and further $U_1 \in \mathfrak{U}$ such that

$$V + x = R\, U_1(x) \qquad \text{for every } x \in R.$$

Then, as R is dense in $\tilde{R}$, for any $\tilde{x} \in \tilde{R}$ we can find $x \in R$ such that $\tilde{x} \in U_1^\circ(x)$, and hence we have

$$\tilde{x} \in U_1^\circ(x) \subset (V + x)^-.$$

From this relation we conclude by (**) for $0 < \alpha \leq 1$

$$\alpha\tilde{x} \in (\alpha V + \alpha x)^- \subset (V + \alpha x)^-.$$

Let α be a positive number such that $\alpha < 1$ and $\alpha x \in V$. Then we have

$$(V + \alpha x)^- \subset U^-(0) \subset A.$$

Thus, if $0 \in A \in \mathcal{T}$, then for any $\tilde{x} \in \tilde{R}$ we can find $\alpha > 0$ such that $\alpha\tilde{x} \in A$.

From (**) we conclude easily by §43(16)

$$(\alpha A)^- = \alpha A^- \qquad \text{for } \alpha \neq 0.$$

Thus, if A is a symmetric star, then A^- also is a symmetric star, because $\lambda A \subset A$ implies $\lambda A^- \subset A^-$. If $0 \in A \in \mathcal{T}$, then there is $U \in \mathfrak{U}$ such that $U^-(0) \subset A$ and for such $U \in \mathfrak{U}$ we can find a symmetric star $V \in \mathcal{P}$ such that $V \subset R\,U(0)$. Then we have for $|\xi| \leq 1$

$$A \supset U^-(0) \supset V^- \supset \xi V^- \supset \xi V^{-\circ} \ni 0.$$

Finally we see at once by (*) that if $0 \in A \in \mathcal{T}$, then there is $U \in \mathfrak{U}$ such that $U^\circ(0) \times U^\circ(0) \subset A$. Therefore there exists uniquely by §54 Theorem 8 a linear topology $\tilde{\mathcal{P}}$ on $\tilde{R}$ such $\mathcal{T}$ is the induced topology by $\tilde{\mathcal{P}}$. Then for the relative linear topology $\tilde{\mathcal{P}}^R$, as the induced topology in R by $\tilde{\mathcal{P}}^R$ coincides with that in R by $\mathcal{P}$, we conclude by §52 Theorem 4 that $\mathcal{P}$ coincides with $\tilde{\mathcal{P}}^R$. Furthermore, as the induced uniformity by $\tilde{\mathcal{P}}$ coincides with $\mathfrak{U}$ in a dense set R, we see by §30 Theorem 3 that $\mathfrak{U}$ is the induced uniformity in $\tilde{R}$ by $\tilde{\mathcal{P}}$. Therefore $\tilde{\mathcal{P}}$ is complete. We can conclude the uniqueness from the uniqueness of completion for uniformity.

If R is not separated by $\mathcal{P}$, then, putting $S = \{0\}^-$, we obtain by §54 Theorem 4 a closed linear manifold S, and hence the quotient space R/S is by §56 Theorem 6 separated by the relative linear topology $\mathcal{P}^{R/S}$. Therefore there exists uniquely a completion $\tilde{R}$ of R/S. Then we conclude easily that the product space $(\tilde{R}, S)$ is a completion of $(R/S, S)$. As $(R/S, S)$ is by §56 Theorem 10 isohomeomorphic to R, we see that there exists uniquely a completion of R within a homeomorphism.

<u>Theorem</u> 2. <u>For a linear topological space with a sequential linear topology, its completion also is of a sequential linear topology.</u>

<u>Proof</u>. The completion of a linear topological space R with a sequential linear topology is a completion of R for the induced uniformity, and hence it is by §37 Theorem 2 sequential too.

§61 <u>Finite-dimensional linear spaces</u>

Let a linear space R be finite-dimensional with the dimension $\varkappa$, and x_ν ($\nu = 1, 2, \ldots, \varkappa$) a basis of R. Every element of R may be represented uniquely as a linear combination of x_ν $(\nu = 1, 2, \ldots, \varkappa)$.

If we put

$$V_0 = \{ \sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu : \quad |\xi_\nu| \leqq 1 \quad (\nu = 1, 2, \ldots, \varkappa) \},$$

then we see easily that V_0 is a symmetric convex vicinity in R, and the pseudo-norm of V_0 is given by

$$\| \sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu \|_{V_0} = \max_{\nu=1,2,\ldots,\varkappa} |\xi_\nu| .$$

As V_0 is a symmetric convex vicinity, there exists by §53 Theorem 3 uniquely a linear topology $\mathfrak{V}_0$ on R of which V_0 is a basis. This linear topology $\mathfrak{V}_0$ is obviously of single vicinity and R is separated by $\mathfrak{V}_0$. Furthermore $\mathfrak{V}_0$ is complete. Because, if $a_\mu = \sum_{\nu=1}^{\varkappa} \alpha_{\nu,\mu} x_\nu$ $(\mu = 1, 2, \ldots)$, $\lim_{\mu,\rho\to\infty} \| a_\mu - a_\rho \|_{V_0} = 0$, then we have $\lim_{\mu,\rho\to\infty} | \alpha_{\nu,\mu} - \alpha_{\nu,\rho} | = 0$ for every $\nu = 1, 2, \ldots, \varkappa$, and hence, putting $\alpha_\nu = \lim_{\mu\to\infty} \alpha_{\nu,\mu}$, we obtain

$$\lim_{\mu\to\infty} \| a_\mu - \sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu \|_{V_0} = 0 .$$

If we set

$$\varphi_\mu (\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu) = \xi_\mu \qquad (\mu = 1, 2, \ldots, \varkappa),$$

then we see easily that the induced uniformity by $\mathfrak{V}_0$ coincides with the weak uniformity by these linear functionals φ_μ $(\mu = 1, 2, \ldots, \varkappa)$. By virtue of §57 Theorem 4, every bounded manifold by $\mathfrak{V}_0$ is bounded by the induced uniformity, and hence totally bounded by §34 Theorem 11.

If R is separated by a linear topology $\mathfrak{V}$, then we have $\mathfrak{V} = \mathfrak{V}_0$. Because, for any $V \in \mathfrak{V}$ we can find by definition a symmetric star $U \in \mathfrak{V}$ such that $U^{\varkappa} \subset V$, and $\alpha > 0$ such that $\xi x_\nu \in U$ for $0 \leqq \xi \leqq \alpha$, $\nu = 1, 2, \ldots, \varkappa$. Then we have for $|\xi_\nu| \leqq \alpha$ $(\nu = 1, 2, \ldots, \varkappa)$

$$\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu \in U^{\varkappa} \subset V,$$

that is, $\alpha V_0 \subset V$. Therefore we have $\mathfrak{V}_0 \supset \mathfrak{V}$ by definition. On the other hand, let $\mathfrak{L}$ be a closed star basis of $\mathfrak{V}$. Then we have

$$\prod_{V\in\mathfrak{L},\, \lambda>0} \lambda V = \{0\},$$

as R is separated by $\mathfrak{V}$ by assumption. As $\mathfrak{V}_0 \supset \mathfrak{V}$, we have by §55 Theorem 4 $\mathfrak{T}^{\mathfrak{V}_0} \supset \mathfrak{T}^{\mathfrak{V}}$, and hence λV is closed by $\mathfrak{T}^{\mathfrak{V}_0}$ for every $V \in \mathfrak{L}$ and $\lambda > 0$. If we put

$$A = \{ \sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu : \max_{\nu=1,2,\ldots,\varkappa} |\xi_\nu| = 1 \},$$

then A is obviously bounded by $\mathfrak{V}_0$ and hence totally bounded by the in-

duced uniformity $\mathfrak{U}^{\mathfrak{V}_0}$ by $\mathfrak{V}_0$, as proved just above. Furthermore we see easily that A is complete by $\mathfrak{V}_0$, and hence closed by the induced topology $\mathfrak{T}^{\mathfrak{V}_0}$. Thus A is by §35 Theorem 5 compact by $\mathfrak{T}^{\mathfrak{V}_0}$. As

$$\prod_{V\in\mathfrak{L},\, \lambda>0} (\lambda V) A = 0,$$

we can find by §7 Theorem 3 a finite number of vicinities $V_\mu \in \mathfrak{L}$ and $\lambda_\mu > 0$ $(\mu = 1, 2, \ldots, \sigma)$, such that $(\prod_{\mu=1}^{\sigma} \lambda_\mu V_\mu) A = 0$. Since $\prod_{\mu=1}^{\sigma} \lambda_\mu V_\mu$ is a star vicinity, we obtain hence $\prod_{\mu=1}^{\sigma} \lambda_\mu V_\mu \subset V_0$. Therefore we obtain $\mathfrak{V} \supset \mathfrak{V}_0$. Consequently we have $\mathfrak{V} = \mathfrak{V}_0$. Thus we can state

<u>Theorem 1.</u> <u>In a finite-dimensional linear space, there exists uniquely a separative linear topology, which is convex and of single vicinity, and</u>

$$\{ \sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu : \quad |\xi_\nu| \leqq 1 \quad (\nu = 1, 2, \ldots, \varkappa) \}$$

<u>is a basis of this linear topology for a basis x_ν $(\nu = 1, 2, \ldots, \varkappa)$ of R.</u>

For an arbitrary linear topology $\mathfrak{V}$ on R, putting $S = \{0\}^{\perp}$, we see by §56 Theorem 6 that the quotient space R/S is separated by the relative linear topology $\mathfrak{V}^{R/S}$ of $\mathfrak{V}$, and further by §56 Theorem 10 that R is isohomeomorphic to the product space $(R/S, S)$. Thus we have

<u>Theorem 2.</u> <u>In a finite-dimensional linear space, every linear topology is convex, complete, and of single vicinity; and every bounded manifold is totally bounded by the induced uniformity.</u>

As an immediate consequence of Theorem 2 we have

<u>Theorem 3.</u> <u>If a linear manifold A of a linear topological space is finite-dimensional, then A is closed by the induced topology.</u>

<u>Theorem 4.</u> <u>For a linear topological space R with a separative linear topology $\mathfrak{V}$ of finite character, if there is a vicinity $V_0 \in \mathfrak{V}$, which is totally bounded by the induced uniformity, then R is finite-dimensional.</u>

<u>Proof.</u> We can assume obviously that V_0 is a symmetric star vicinity of finite character. Then there is a positive number $\varkappa$ such that for the pseudo-norm $\|x\|_{V_0}$ of V_0 we have

(*) $$\| x + y \|_{V_0} \leqq \varkappa (\|x\|_{V_0} + \|y\|_{V_0}).$$

R is separated by V_0, that is, $\|x\|_{V_0} = 0$ implies $x = 0$. Because, if $\|a\|_{V_0} = 0$, $a \neq 0$, then we have $\|\xi a\|_{V_0} = 0$ for every real number ξ, and hence $\xi a \in V_0$ for every ξ. As $\mathfrak{V}$ is separative by assumption, there is $V \in \mathfrak{V}$ such that $\xi a \bar\in V$ for some ξ, and hence V_0 is not bounded by $\mathfrak{V}$, contradicting by §57 Theorem 5 the assumption that V_0 is totally bounded by the induced uniformity.

For a finite-dimensional linear manifold S, if $\inf_{x\in S}\|a-x\|_{V_0}=0$, then we can find a sequence $x_\mu \in S$ ($\mu = 1, 2, \dots$) such that $\lim_{\mu\to\infty}\|a-x_\mu\|_{V_0} = 0$, and hence by (*) $\lim_{\mu,\rho\to\infty}\|x_\mu - x_\rho\|_{V_0}=0$. Since S is by Theorem 2 complete by the linear topology on S of which V_0^S is a basis, there exists then $x_0\in S$ such that $\lim_{\mu\to\infty} \|x_\mu - x_0\|_{V_0}=0$. On account of (*), we obtain therefore $\|x_0 - a\|_{V_0}=0$, and consequently $a = x_0 \in S$. Thus if $a \bar\in S$, then, putting

$$\alpha = \inf_{x\in S}\|a-x\|_{V_0} > 0,$$

we can find $x_0\in S$ such that $\|a-x_0\|_{V_0} < 2\alpha$. For such $x_0 \in S$, putting

$$b = \frac{1}{2\|a-x_0\|_{V_0}}(a-x_0),$$

we have $\|b\|_{V_0} = \frac12$, and for every $x\in S$, as $x_0 + 2\|a-x_0\|_{V_0} x \in S$,

$$\|b - x\|_{V_0} = \frac{\|a - x_0 - 2\|a-x_0\|_{V_0} x\|}{2\|a-x_0\|_{V_0}} \geq \frac{\alpha}{4\alpha} = \frac14.$$

Therefore, if R is not finite-dimensional, then there is a sequence of elements $x_\nu \in R$ ($\nu = 1, 2, \dots$) such that

$$\|x_\nu\|_{V_0} = \frac12, \qquad \|x_\nu - x_\mu\|_{V_0} \geq \frac14 \qquad \text{for } \nu\neq\mu.$$

Then we have by the formula §49(2) $x_\nu \in V_0$ ($\nu = 1,2,\dots$), but $\{x_1, x_2, \dots\}$ is not totally bounded by the induced uniformity. Because, if

$$\{x_1,x_2,\dots\} \subset \sum_{\nu=1}^{\kappa}\left(\frac{1}{8\kappa}V_0 + a_\nu\right)$$

for some finite number of elements $a_\nu \in R$ ($\nu = 1, 2, \dots, \kappa$), then we can find ν and $\mu \neq \rho$ such that $x_\mu, x_\rho \in \left(\frac{1}{8\kappa}V_0 + a_\nu\right)$. This relation yields by §49(2) $\|x_\mu - a_\nu\|_{V_0} \leq \frac{1}{8\kappa}$, $\|x_\rho - a_\nu\|_{V_0}\leq \frac{1}{8\kappa}$, and hence by (*) $\|x_\mu - x_\rho\|_{V_0} \leq \frac18$, contradicting $\|x_\mu - x_\rho\| \geq \frac14$.

§62 Weak linear topologies

Let R be a linear topological space with a linear topology $\mathfrak{V}$. A functional φ on R is said to be *continuous* by $\mathfrak{V}$, if φ is continuous

by the induced topology $\mathfrak{T}^{\mathfrak{V}}$, and φ is said to be *uniformly continuous* by $\mathfrak{V}$, if φ is uniformly continuous by the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$.

With this definition we have

Theorem 1. *In order that a linear functional φ on R be continuous, it is necessary and sufficient that we can find a symmetric convex vicinity $V \in \mathfrak{V}$ such that φ is bounded in V, that is,*

$$\|\varphi\|_V = \sup_{x\in V}|\varphi(x)| < +\infty.$$

Proof. If φ is continuous, then, putting

$$V = \{x : |\varphi(x)| \leq 1\},$$

we have $V\in\mathfrak{V}$, because we can find $V_0\in\mathfrak{V}$ such that $V_0 \subset V$. Furthermore V is symmetric and convex, because $|\varphi(x)|, |\varphi(y)|\leq 1$ implies for $\lambda+\mu = 1$, $\lambda,\mu\geq 0$

$$|\varphi(\lambda x+\mu y)| = |\lambda\varphi(x)+\mu\varphi(y)| \leq \lambda|\varphi(x)|+\mu|\varphi(y)| \leq 1.$$

Conversely, if $\|\varphi\|_V < +\infty$ for a symmetric convex vicinity $V\in\mathfrak{V}$, then for any $x_0\in R$ and $\varepsilon>0$ we have by the formula §52(2)

$$\varepsilon V + x_0 \subset \{x + x_0 : |\varphi(x)| \leq \|\varphi\|_{\varepsilon V} = \varepsilon\|\varphi\|_V\}$$

$$= \{x : |\varphi(x) - \varphi(x_0)| \leq \varepsilon\|\varphi\|_V\}.$$

Therefore φ is uniformly continuous by the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$, and hence naturally continuous by the induced topology $\mathfrak{T}^{\mathfrak{V}}$.

In this Proof we obtain further

Theorem 2. *If a linear functional φ on R is continuous by $\mathfrak{V}$, then φ is uniformly continuous by $\mathfrak{V}$.*

Now let R be merely a linear space and φ a linear functional on R. Putting $V = \{x : |\varphi(x)|\leq 1\}$, we obtain a symmetric convex vicinity V in R, as proved just above. Thus there exists uniquely by §53 Theorem 4 a linear topology $\mathfrak{V}^\varphi$ on R of which V is a basis. By virtue of Theorem 1, we see at once that φ is continuous by this linear topology $\mathfrak{V}^\varphi$. For a linear topology $\mathfrak{V}$ on R, if φ is continuous by $\mathfrak{V}$, then $\mathfrak{V}$ must contain naturally V, and hence $\mathfrak{V} \supset \mathfrak{V}^\varphi$. Therefore $\mathfrak{V}^\varphi$ is the weakest linear topology by which φ be continuous. This linear topology $\mathfrak{V}^\varphi$ is called the *weak linear topology* of R by a linear functional φ. As $\varepsilon V + a = \{x : |\varphi(x)-\varphi(a)|\leq\varepsilon\}$, we see easily that the induced uniformity by $\mathcal{W}^{\varphi}$ coincides with the weak uniformity of R by φ, and hence the induced topology by $\mathcal{W}^{\varphi}$ coincides with the weak topology of R by φ.

For a collection of linear functionals $\mathcal{F}$ on R, there exists the weakest linear topology on R by which every linear functional of $\mathcal{F}$ be continuous. Because, for the weak linear topology $\mathcal{W}^{\varphi}$ by $\varphi \in \mathcal{F}$, putting $\mathcal{W} = \bigcup_{\varphi \in \mathcal{F}} \mathcal{W}^{\varphi}$, we see easily that every $\varphi \in \mathcal{F}$ is continuous by $\mathcal{W}$, and every linear topology on R, by which every $\varphi \in \mathcal{F}$ is continuous, is stronger than $\mathcal{W}$. This linear topology $\mathcal{W}$ on R is called the weak linear topology of R by $\mathcal{F}$. Every weak linear topology of R is obviously convex by §55 Theorem 3.

With this definition we obtain by §55 Theorem 2

Theorem 3. *For a collection of linear functionals $\mathcal{F}$ on R, the totality of manifolds $\{x : |\varphi_\nu(x)| \leqq 1 \quad (\nu = 1, 2, \ldots, \varkappa)\}$ for every finite number of linear functionals $\varphi_\nu \in \mathcal{F}$ $(\nu = 1, 2, \ldots, \varkappa)$ is a basis of the weak linear topology of R by $\mathcal{F}$.*

Recalling §55 Theorem 5, we obtain further

Theorem 4. *The induced topology and the induced uniformity by the weak linear topology by a collection of linear functionals $\mathcal{F}$ coincide respectively with the weak topology and the weak uniformity by $\mathcal{F}$.*

Theorem 5. *Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of linear functionals on R. For the weak linear topologies $\mathcal{W}^{\mathcal{F}}$, $\mathcal{W}^{\mathcal{G}}$ respectively by $\mathcal{F}$, $\mathcal{G}$, in order that $\mathcal{W}^{\mathcal{F}} > \mathcal{W}^{\mathcal{G}}$, it is necessary and sufficient that the linear manifold generated by $\mathcal{F}$ in the associated space $\tilde{R}$ of R contains $\mathcal{G}$.*

Proof. Let $\tilde{\mathcal{F}}$ be the linear manifold generated by $\mathcal{F}$ in the associated space $\tilde{R}$ of R. It is obvious by definition that every $\varphi \in \tilde{\mathcal{F}}$ is continuous by $\mathcal{W}^{\mathcal{F}}$. Thus the weak linear topology of R by $\tilde{\mathcal{F}}$ coincides by definition with that of R by $\mathcal{F}$. Therefore, if $\tilde{\mathcal{F}} > \mathcal{G}$, then $\mathcal{W}^{\mathcal{F}}$ is by definition stronger than $\mathcal{W}^{\mathcal{G}}$. Conversely, if $\mathcal{W}^{\mathcal{F}}$ is stronger than $\mathcal{W}^{\mathcal{G}}$, then for any $\psi \in \mathcal{G}$ we can find by Theorem 3 a finite number of linear functionals $\varphi_\nu \in \mathcal{F}$ $(\nu = 1, 2, \ldots, \varkappa)$ and $\varepsilon > 0$ such that

$$\{x : |\varphi_\nu(x)| \leqq \varepsilon \quad (\nu = 1, 2, \ldots, \varkappa)\} \subset \{x : |\psi(x)| \leqq 1\},$$

that is, $|\varphi_\nu(x)| \leqq \varepsilon$ $(\nu = 1, 2, \ldots, \varkappa)$ implies $|\psi(x)| \leqq 1$. Then, for any $\alpha > 0$, $|\varphi_\nu(x)| \leqq \frac{\varepsilon}{\alpha}$ $(\nu = 1, 2, \ldots, \varkappa)$ implies $|\psi(\alpha x)| \leqq 1$, and hence $|\psi(x)| \leqq \frac{1}{\alpha}$. Therefore we conclude that $\varphi_\nu(x) = 0$ $(\nu = 1, 2, \ldots, \varkappa)$ implies $\psi(x) = 0$. If we set

$$A = \{x : \varphi_\nu(x) = 0 \quad (\nu = 1, 2, \ldots, \varkappa)\},$$

then A is obviously a linear manifold of R, and considering φ_ν as a linear functional on the quotient space R/A, we see by §45 Theorem 6 that every linear functional on R/A is a linear combination of φ_ν $(\nu = 1, 2, \ldots, \varkappa)$. As $\psi(x) = 0$ for every $x \in A$, ψ may be considered as a linear functional on R/A, and hence ψ is a linear combination of φ_ν $(\nu = 1, 2, \ldots, \varkappa)$. Thus we have $\psi \in \tilde{\mathcal{F}}$ for every $\psi \in \mathcal{G}$.

Theorem 6. *In order that the weak linear topology of R by a collection of linear functionals $\mathcal{F}$ be of single vicinity, it is necessary and sufficient that the linear manifold generated by $\mathcal{F}$ in the associated space $\tilde{R}$ of R be finite-dimensional.*

Proof. If $\mathcal{F}$ is contained in a linear manifold generated by a finite number of linear functionals φ_ν $(\nu = 1, 2, \ldots, \varkappa)$, then the weak linear topology of R by $\mathcal{F}$ is by Theorem 3 of single vicinity with a basis

$$\{x : |\varphi_\nu(x)| \leqq 1 \quad (\nu = 1, 2, \ldots, \varkappa)\}.$$

Conversely, if the weak linear topology $\mathcal{W}^{\mathcal{F}}$ of R by $\mathcal{F}$ is of single vicinity, then we can find by Theorem 3 a finite number of linear functionals $\varphi_\nu \in \mathcal{F}$ such that $\{x : |\varphi_\nu(x)| \leqq 1 \quad (\nu = 1, 2, \ldots, \varkappa)\}$ is a basis of $\mathcal{W}^{\mathcal{F}}$, and hence we conclude by Theorem 5 that $\mathcal{F}$ is contained in the linear manifold generated by φ_ν $(\nu = 1, 2, \ldots, \varkappa)$.

Theorem 7. *In order that a manifold A of R be bounded by the weak linear topology $\mathcal{W}^{\mathcal{F}}$ by a collection of linear functionals $\mathcal{F}$, it is necessary and sufficient that we have*

$$\sup_{x \in A} |\varphi(x)| < +\infty \quad \text{for every } \varphi \in \mathcal{F}.$$

Proof. If A is bounded by $\mathcal{W}^{\mathcal{F}}$, then for each $\varphi \in \mathcal{F}$, as we have by Theorem 3 $\{x : |\varphi(x)| \leqq 1\} \in \mathcal{W}^{\mathcal{F}}$, we can find by definition $\alpha > 0$ such that

$$A \subset \alpha\{x : |\varphi(x)| \leqq 1\} = \{x : |\varphi(x)| \leqq \alpha\},$$

and hence $\sup_{x \in A} |\varphi(x)| \leqq \alpha$. Conversely, if $\sup_{x \in A} |\varphi(x)| < +\infty$ for every $\varphi \in \mathcal{F}$, then for every finite number of linear functionals $\varphi_\nu \in \mathcal{F}$ ($\nu = 1, 2, \ldots, \kappa$) we can find $\alpha > 0$ such that $\sup_{x \in A} |\varphi_\nu(x)| \leqq \alpha$ for every $\nu = 1, 2, \ldots, \kappa$, and then we have

$$A \subset \alpha \{x : |\varphi_\nu(x)| \leqq 1 \quad \text{for every } \nu = 1, 2, \ldots, \kappa\}.$$

Thus A is bounded by Theorem 3.

Recalling §34 Theorem 11, we conclude immediately from Theorem 7

Theorem 8. *If a manifold A of R is bounded by the weak linear topology $\mathcal{V}^{\mathcal{F}}$ of R for a collection of linear functionals $\mathcal{F}$, then A is totally bounded by $\mathcal{V}^{\mathcal{F}}$.*

Theorem 9. *If a linear manifold A of R is closed by the weak linear topology $\mathcal{V}^{\mathcal{F}}$ of R for a collection of linear functionals $\mathcal{F}$, then for any $x_0 \bar{\in} A$ we can find $\varphi \in \mathcal{F}$ such that $\varphi(x_0) \neq 0$ but $\varphi(x) = 0$ for every $x \in A$.*

Proof. For a linear manifold A of R, if $\varphi(x) = 0$ for every $x \in A$, $\varphi \in \mathcal{F}$, implies $\varphi(x_0) = 0$, then for every finite number of linear functionals $\varphi_\nu \in \mathcal{F}$ ($\nu = 1, 2, \ldots, \kappa$), considering φ_ν as linear functionals on A, $\sum_{\nu=1}^{\kappa} \xi_\nu \varphi_\nu = 0$ implies $\sum_{\nu=1}^{\kappa} \xi_\nu \varphi_\nu(x_0) = 0$, and hence there is by §46 Theorem 4 $a \in A$ such that $\varphi_\nu(a) = \varphi_\nu(x_0)$ ($\nu = 1, 2, \ldots, \kappa$). Thus, if A is closed by the weak linear topology $\mathcal{V}^{\mathcal{F}}$, then we must have $x_0 \in A$.

Theorem 10. *The weak linear topology of R by a collection of linear functionals $\mathcal{F}$ is separative, if and only if $\mathcal{F}$ is fundamental.*

Proof. If the weak linear topology $\mathcal{V}^{\mathcal{F}}$ is separative, then for any $x \neq 0$ we can find by Theorem 3 $\varphi \in \mathcal{F}$ such that $\varphi(x) \neq 0$, and hence $\mathcal{F}$ is fundamental. Conversely, if $\mathcal{F}$ is fundamental, then for any $x \neq 0$ we can find $\varphi \in \mathcal{F}$ such that $\varphi(x) \neq 0$, and hence we conclude by Theorem 3 that the weak linear topology $\mathcal{V}^{\mathcal{F}}$ is separative.

§63 Quasi-normed linear spaces

A linear space R associated with a quasi-norm $|||x|||$ ($x \in R$) is

called a *quasi-normed linear space*. With this definition we see easily that, putting $V_\alpha = \{x : |||x||| \leqq \alpha\}$ for $\alpha > 0$, we obtain a symmetric star vicinity V_α for every $\alpha > 0$, and we have

$$V_\alpha \subset V_\beta \quad \text{for} \quad 0 < \alpha < \beta,$$

$$V_\alpha \times V_\alpha \subset V_{2\alpha},$$

because $|||x + y||| \leqq |||x||| + |||y|||$ for every $x, y \in R$. Therefore there exists uniquely by §53 Theorem 3 a linear topology $\mathcal{V}_0$ on R such that V_α ($\alpha > 0$) is a basis of $\mathcal{V}_0$. This linear topology $\mathcal{V}_0$ is called the *induced linear topology* by a quasi-norm $|||x|||$ ($x \in R$).

Since $V_{\frac{1}{\nu}}$ ($\nu = 1, 2, \ldots$) is obviously a basis of $\mathcal{V}_0$, we obtain

Theorem 1. *The induced linear topology by a quasi-norm is sequential.*

Furthermore we have obviously by definition

Theorem 2. *A quasi-norm is proper if and only if the induced linear topology is separative.*

Theorem 3. *In order that the induced linear topology by a quasi-norm $|||x|||$ ($x \in R$) be of single vicinity, it is necessary and sufficient that we can find $\alpha > 0$ such that*

$$\lim_{\xi \to 0} \sup_{|||x||| \leqq \alpha} |||\xi x||| = 0.$$

Proof. If the induced linear topology $\mathcal{V}_0$ is of single vicinity, then we can find $\alpha > 0$ such that V_α is a basis of $\mathcal{V}_0$ for the notation indicated just above, and for any $\varepsilon > 0$ we can find $\lambda > 0$ such that $\lambda V_\alpha \subset V_\varepsilon$, that is,

$$|||\lambda x||| \leqq \varepsilon \quad \text{for} \quad |||x||| \leqq \alpha.$$

This relation yields obviously

$$\lim_{\xi \to 0} \sup_{|||x||| \leqq \alpha} |||\xi x||| = 0.$$

Conversely, if this relation holds for some $\alpha > 0$, then we see easily that V_α is a basis of $\mathcal{V}_0$ for such $\alpha > 0$.

Theorem 4. *If a quasi-norm is of finite character, then the induced linear topology is of single vicinity.*

Proof. If a quasi-norm $|||x|||$ ($x \in R$) is of finite character, then we can find by definition two positive numbers α, γ such that

$$\frac{1}{2}|||x||| \geqq |||\frac{\gamma}{2}x||| \quad \text{for } |||x||| \leqq \alpha,$$

and hence we have for every $\nu = 1, 2, \ldots$.

$$\frac{1}{2^\nu}|||x||| \geqq |||\frac{\gamma^\nu}{2^\nu}x||| \quad \text{for } |||x||| \leqq \alpha.$$

Thus we have

$$\lim_{\xi \to 0} \sup_{|||x||| \leqq \alpha} |||\xi x||| = 0.$$

Therefore V_α is a basis of the induced linear topology by Theorem 3.

Theorem 5. A quasi-norm $|||x|||$ $(x \in R)$ is uniformly continuous by the induced linear topology.

Proof. As we have by definition for every $x, y \in R$

$$|\, |||x||| - |||y|||\,| \leqq |||x - y|||,$$

we conclude easily by definition our assertion.

We have defined already in §50 the completeness of a quasi-norm. As the induced linear topology by a quasi-norm is sequential by Theorem 1, we have obviously by the definition of completeness in §59

Theorem 6. A quasi-norm is complete if and only if the induced linear topology is complete.

Recalling §50 Theorem 3, we obtain immediately

Theorem 7. For a sequential linear topology $\mathcal{T}$ on a linear space R there is a quasi-norm on R by which $\mathcal{T}$ is the induced linear topology.

Furthermore we have by §50 Theorem 4

Theorem 8. For a linear topology $\mathcal{T}$ of single vicinity there is a quasi-norm such that $\mathcal{T}$ is the induced linear topology and

$$|||\frac{1}{2\chi}x||| = \frac{1}{2}|||x||| \quad \text{for } |||x||| < \frac{1}{2}$$

for some positive number χ.

CHAPTER IX

ADJOINT SPACES

§64 Adjoint topology

Let R be a linear topological space with a linear topology $\mathcal{T}$. A linear functional φ on R is said to be bounded, if we have

$$\sup_{x \in A} |\varphi(x)| < +\infty$$

for every bounded manifold A of R by $\mathcal{T}$. With this definition we shall prove firstly

Theorem 1. Every continuous linear functional is bounded.

Proof. If a linear functional φ is continuous, then we can find by §62 Theorem 1 a symmetric convex vicinity $V \in \mathcal{T}$ such that

$$\sup_{x \in V} |\varphi(x)| < +\infty.$$

For any bounded manifold A we can find by definition $\alpha > 0$ such that $A \subset \alpha V$. Then we have by the formula §52(2)

$$\sup_{x \in A} |\varphi(x)| \leqq \sup_{x \in \alpha V} |\varphi(x)| = \|\varphi\|_{\alpha V} = \alpha \|\varphi\|_V < +\infty.$$

Therefore φ is bounded by definition.

Theorem 2. If a linear topology $\mathcal{T}$ is equivalently strongest or standard, then every bounded linear functional is continuous.

Proof. Let φ be a bounded linear functional on R. Putting

$$V_0 = \{x : |\varphi(x)| \leqq 1\},$$

we see easily that V_0 is a symmetric convex vicinity. Thus there is uniquely by §53 Theorem 4 a linear topology $\mathcal{T}_0$, of which V_0 is a basis. Then, for each bounded manifold A by $\mathcal{T}$, putting $\alpha = \sup_{x \in A} |\varphi(x)|$, we have $A \subset \alpha V_0$, because $\alpha V_0 = \{\alpha x : |\varphi(x)| \leqq 1\} = \{x : |\varphi(x)| \leqq \alpha\}$. Thus every bounded manifold by $\mathcal{T}$ also is bounded by $\mathcal{T}_0$. Therefore we see easily by §55 Theorem 2 that $\mathcal{T} \smile \mathcal{T}_0$ is equivalent to $\mathcal{T}$. As $\mathcal{T}$ is equivalently strongest or standard by assumption, we obtain hence $\mathcal{T} = \mathcal{T} \smile \mathcal{T}_0 \supset \mathcal{T}_0 \ni V_0$. Therefore φ is continuous by §62 Theorem 1.

The totality of bounded linear functionals on R is called the adjoint space of R by a linear topology $\mathcal{T}$ and denoted by $\overline{R}^{\mathcal{T}}$. With this definition we see easily that the adjoint space $\overline{R}^{\mathcal{T}}$ is a linear manifold of the associated space $\bar{R}$ of R.

Let $\mathfrak{A}$ be the totality of bounded manifolds of R by $\mathfrak{V}$. Corresponding to every $A \in \mathfrak{A}$, putting for $\bar{x} \in \bar{R}^{\mathfrak{V}}$

$$\bar{V}_A = \{\bar{x} : |\bar{x}(a)| \leq 1 \quad \text{for every } a \in A\},$$

we obtain a symmetric convex vicinity $\bar{V}_A$ in the adjoint space $\bar{R}^{\mathfrak{V}}$, because $|\bar{x}(a)| \leq 1$, $|\bar{y}(a)| \leq 1$ for every $a \in A$ implies

$$|(\lambda \bar{x} + \mu \bar{y})(a)| \leq \lambda |\bar{x}(a)| + \mu |\bar{y}(a)| \leq 1$$

for $\lambda + \mu = 1$, $\lambda, \mu \geq 0$. For such $\bar{V}_A$ we have obviously

$$\alpha \bar{V}_A = \bar{V}_{\frac{1}{\alpha} A} \qquad \text{for every } \alpha > 0.$$

As $A \in \mathfrak{A}$ implies $\alpha A \in \mathfrak{A}$ and $A, B \in \mathfrak{A}$ implies $A \dotplus B \in \mathfrak{A}$, we see easily that the system of vicinities $\bar{V}_A$ ($A \in \mathfrak{A}$) in $\bar{R}^{\mathfrak{V}}$ satisfies the conditions in §53 Theorem 3. Therefore there exists uniquely a linear topology $\bar{\mathfrak{V}}$ on $\bar{R}^{\mathfrak{V}}$ of which the system $\bar{V}_A$ ($A \in \mathfrak{A}$) is a basis. This linear topology $\bar{\mathfrak{V}}$ is called the adjoint topology of $\mathfrak{V}$. With this definition we see at once that a linear topology equivalent to $\mathfrak{V}$ has the same adjoint topology with $\mathfrak{V}$.

A collection of bounded manifolds $\mathfrak{R}$ in R by $\mathfrak{V}$ is called a root of $\mathfrak{V}$, if for any bounded manifold A by $\mathfrak{V}$ we can find $B \in \mathfrak{R}$ such that $A \subset B$. For a root $\mathfrak{R}$ of $\mathfrak{V}$, we see easily by definition that

$$\{\bar{x} : \sup_{x \in A} |\bar{x}(x)| < +\infty \quad \text{for every } A \in \mathfrak{R}\}$$

is the adjoint space of R by $\mathfrak{V}$, and the system

$$\{\bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in A\} \qquad (A \in \mathfrak{R})$$

is a basis of the adjoint topology $\bar{\mathfrak{V}}$ of $\mathfrak{V}$. Because for any bounded manifold B we can find $A \in \mathfrak{R}$ such that $B \subset A$, and then

$$\{\bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in B\} \supset \{\bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in A\}.$$

Theorem 3. The adjoint topology is convex and separative.

Proof. The adjoint topology $\bar{\mathfrak{V}}$ of $\mathfrak{V}$ is obviously convex by definition. For $\bar{a} \in \bar{R}^{\mathfrak{V}}$, if $\bar{a} \neq 0$, then we can find $a \in R$ such that $\bar{a}(a) > 1$. For such $a \in R$, putting $\bar{V} = \{\bar{x} : |\bar{x}(a)| \leq 1\}$, we have $\bar{a} \,\bar{\in}\, \bar{V}$ but $\bar{V} \in \bar{\mathfrak{V}}$, because a is itself a bounded manifold of R by $\mathfrak{V}$. Thus $\bar{\mathfrak{V}}$ is separative.

For each $a \in R$, putting

$$a(\bar{x}) = \bar{x}(a) \qquad \text{for every } \bar{x} \in \bar{R}^{\mathfrak{V}},$$

we can consider a as a linear functional on $\bar{R}^{\mathfrak{V}}$. Then we have

Theorem 4. Every element $a \in R$ is continuous by the adjoint topology $\bar{\mathfrak{V}}$ of $\mathfrak{V}$ as a linear functional on $\bar{R}^{\mathfrak{V}}$ by the relation

$$a(\bar{x}) = \bar{x}(a) \qquad \text{for } \bar{x} \in \bar{R}^{\mathfrak{V}}.$$

Proof. Every $a \in R$ is bounded by $\mathfrak{V}$ as a manifold. Thus, putting $\bar{V} = \{\bar{x} : |\bar{x}(a)| \leq 1\}$, we have $\bar{V} \in \bar{\mathfrak{V}}$, and hence we conclude by §6: Theorem 1 that a is continuous by $\bar{\mathfrak{V}}$ as a linear functional on $\bar{R}^{\mathfrak{V}}$.

Theorem 5. In order that a manifold $\bar{A}$ of the adjoint space $\bar{R}^{\mathfrak{V}}$ be bounded by the adjoint topology $\bar{\mathfrak{V}}$, it is necessary and sufficient that for every bounded manifold A of R by $\mathfrak{V}$ we have

$$\sup_{\bar{x} \in \bar{A},\, x \in A} |\bar{x}(x)| < +\infty.$$

Proof. If $\bar{A}$ is bounded by the adjoint topology $\bar{\mathfrak{V}}$, then for each bounded manifold A of R by $\mathfrak{V}$, putting

$$\bar{V}_A = \{\bar{x} : |\bar{x}(a)| \leq 1 \quad \text{for every } a \in A\},$$

we have $\bar{V}_A \in \bar{\mathfrak{V}}$, and hence we can find by definition $\alpha > 0$ such that $\bar{A} \leq \alpha \bar{V}_A$. For such $\alpha > 0$ we have

$$\sup_{\bar{x} \in \bar{A},\, x \in A} |\bar{x}(x)| \leq \sup_{\bar{x} \in \alpha \bar{V}_A,\, x \in A} |\bar{x}(x)| \leq \alpha \sup_{\bar{x} \in \bar{V}_A,\, x \in A} |\bar{x}(x)| \leq \alpha.$$

Conversely, if $\sup_{\bar{x} \in \bar{A},\, x \in A} |\bar{x}(x)| < +\infty$ for every bounded manifold A of R, then, putting $\alpha = \sup_{\bar{x} \in \bar{A},\, x \in A} |\bar{x}(x)|$, we have

$$\bar{A} \subset \{\bar{x} : |\bar{x}(a)| \leq \alpha \quad \text{for every } a \in A\}$$
$$= \{\alpha \bar{x} : |\bar{x}(a)| \leq 1 \quad \text{for every } a \in A\} = \alpha \bar{V}_A.$$

As the system of $\bar{V}_A$ for all bounded manifold A of R is by definition a basis of $\bar{\mathfrak{V}}$, $\bar{A}$ is therefore bounded by $\bar{\mathfrak{V}}$.

Theorem 6. For every $V \in \mathfrak{V}$, the manifold

$$\{\bar{x} : |\bar{x}(x)| \leq 1 \qquad \text{for every } x \in V\}$$

is closed and bounded by the adjoint topology $\bar{\mathfrak{V}}$ of $\mathfrak{V}$.

Proof. For each bounded manifold A of R we can find by definition $\alpha > 0$ such that $A \subset \alpha V$. Thus, putting

$$\bar{A} = \{\bar{x} : |\bar{x}(x)| \leq 1 \qquad \text{for every } x \in V\},$$

we have $\sup_{\bar{x} \in \bar{A},\, x \in A} |\bar{x}(x)| \leq \sup_{\bar{x} \in \bar{A},\, x \in \alpha V} |\bar{x}(x)| \leq \alpha \sup_{\bar{x} \in \bar{A},\, x \in V} |\bar{x}(x)| \leq \alpha$. Thus $\bar{A}$ is by Theorem 5 bounded by $\bar{\mathfrak{V}}$. Furthermore $\bar{A}$ is closed by $\bar{\mathfrak{V}}$, because $\bar{A} = \prod_{x \in V} \{\bar{x} : |\bar{x}(x)| \leqq 1\}$ and every $x \in \bar{V}$ may be considered by Theorem 4 as a continuous linear functional on the adjoint space $\bar{R}^{\mathcal{V}}$ by $\mathcal{V}$.

Theorem 7. If $\mathcal{V}$ is convex, then for any $\bar{V} \in \bar{\mathcal{V}}$,

$$\{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{V}\}$$

is a bounded manifold of R by $\mathcal{V}$.

Proof. For every closed convex vicinity $V \in \mathcal{V}$, as V is by §54 Theorem 2 scalar-closed, putting

$$\bar{A} = \{\bar{x} : |\bar{x}(x)| \leqq 1 \quad \text{for every } x \in V\},$$

we have by §52 Theorem 3

$$V = \{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{A}\}.$$

As $\bar{A}$ is bounded by Theorem 6, for any $\bar{V} \in \bar{\mathcal{V}}$ we can find $\alpha > 0$ such that $\alpha \bar{A} \subset \bar{V}$, and hence

$$\{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{V}\}$$
$$\subset \{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \alpha \bar{A}\}$$
$$= \{\tfrac{1}{\alpha}x : |\alpha \bar{x}(\tfrac{1}{\alpha}x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{A}\} = \tfrac{1}{\alpha} V.$$

Theorem 8. If $\mathcal{V}$ is equivalently strongest or standard, then for every bounded manifold $\bar{A}$ of $\bar{R}^{\mathcal{V}}$ we have

$$\{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{A}\} \in \mathcal{V}.$$

Proof. Putting $V_0 = \{x : |\bar{x}(x)| \leqq 1 \text{ for every } \bar{x} \in \bar{A}\}$, we see easily that V_0 is a symmetric convex vicinity in R. Thus there is by §53 Theorem 4 a linear topology $\mathcal{V}_0$ on R such that V_0 is a basis of $\mathcal{V}_0$. For each bounded manifold A of R by $\mathcal{V}$, as we have by definition $\{\bar{x} : |\bar{x}(x)| \leqq 1 \text{ for every } x \in A\} \in \bar{\mathcal{V}}$, there is $\alpha > 0$ such that

$$\bar{A} < \alpha \{\bar{x} : |\bar{x}(x)| \leqq 1 \quad \text{for every } x \in A\}$$
$$= \{\bar{x} : |\bar{x}(x)| \leqq 1 \quad \text{for every } x \in \tfrac{1}{\alpha} A\},$$

and hence $|\bar{x}(x)| \leqq 1$ for $\bar{x} \in \bar{A}$, $x \in \frac{1}{\alpha} A$. Therefore we have $\frac{1}{\alpha} A \subset V_0$. Consequently every bounded manifold A by $\mathcal{V}$ is bounded by $\mathcal{V}_0$ too. Thus $\mathcal{V} \smile \mathcal{V}_0$ is equivalent to $\mathcal{V}$. As $\mathcal{V}$ is equivalently strongest or standard by assumption, we obtain hence $\mathcal{V} = \mathcal{V} \smile \mathcal{V}_0 \supset \mathcal{V}_0$, and consequently $V_0 \in \mathcal{V}$.

A manifold $\bar{A}$ of the adjoint space $\bar{R}^{\mathcal{V}}$ is said to be uniformly bounded, if we can find $V \in \mathcal{V}$ such that

$$\sup_{\bar{x} \in \bar{A},\, x \in V} |\bar{x}(x)| < +\infty.$$

With this definition we see easily

Theorem 9. If a manifold $\bar{A}$ of the adjoint space $\bar{R}^{\mathcal{V}}$ is uniformly bounded by $\bar{\mathcal{V}}$, then $\bar{A}$ is bounded by $\bar{\mathcal{V}}$.

Theorem 10. If $\mathcal{V}$ is convex and every bounded manifold of $\bar{R}^{\mathcal{V}}$ is uniformly bounded by the adjoint topology $\bar{\mathcal{V}}$, then $\mathcal{V}$ is standard.

Proof. Let $\mathcal{V}_0$ be a standard linear topology equivalent to $\mathcal{V}$. The existence of such $\mathcal{V}_0$ is obvious by §57 Theorem 8. For every symmetric closed convex vicinity $V \in \mathcal{V}_0$, putting

$$\bar{A} = \{\bar{x} : |\bar{x}(x)| \leqq 1 \quad \text{for every } x \in V\},$$

we have by §52 Theorem 3

$$V = \{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{A}\}.$$

Recalling Theorem 6, we see that $\bar{A}$ is bounded by $\bar{\mathcal{V}}$, because $\bar{\mathcal{V}}$ also is the adjoint topology of $\mathcal{V}_0$. Thus $\bar{A}$ is uniformly bounded by assumption, and hence there is by definition $U \in \mathcal{V}$ such that $|\bar{x}(x)| \leqq 1$ for $\bar{x} \in \bar{A}$, $x \in U$. Hence we have $U < V$, and consequently $V \in \mathcal{V}$. Therefore we conclude $\mathcal{V} \supset \mathcal{V}_0$, and hence naturally $\mathcal{V} = \mathcal{V}_0$.

§65 Weak topology

Let R be a linear topological space with a linear topology $\mathcal{V}$, $\bar{R}^{\mathcal{V}}$ the adjoint space of R by $\mathcal{V}$, and $\bar{\mathcal{V}}$ the adjoint topology of $\mathcal{V}$. Every element $a \in R$ may be considered as a linear functional on $\bar{R}^{\mathcal{V}}$ by the relation: $a(\bar{x}) = \bar{x}(a)$ for every $\bar{x} \in \bar{R}^{\mathcal{V}}$. The weak linear topology of $\bar{R}^{\mathcal{V}}$ by the system of linear functionals a $(a \in R)$ in this sense, is called the weak topology of the adjoint space $\bar{R}^{\mathcal{V}}$. Since every $a \in R$ is continuous by §64 Theorem 4 as a linear functional on $\bar{R}^{\mathcal{V}}$ by $\bar{\mathcal{V}}$, the weak topology of $\bar{R}^{\mathcal{V}}$ is by definition weaker than the adjoint topology $\bar{\mathcal{V}}$. Recalling §62 Theorem 10, we see further that the weak topology of $\bar{R}^{\mathcal{V}}$ is convex and separative.

A manifold $\bar{A}$ of $\bar{R}^{\mathcal{V}}$ will be said to be weakly bounded, weakly totally bounded, weakly closed, or weakly compact, if $\bar{A}$ is so respectively by the weak topology of $\overline{R}^{\mathcal{V}}$. On the other hand we shall say merely that a manifold $\overline{A}$ of $\overline{R}^{\mathcal{V}}$ is bounded, totally bounded, closed, or open, if $\overline{A}$ is so respectively by the adjoint topology $\overline{\mathcal{V}}$ of $\mathcal{V}$.

Recalling §62 Theorem 7 we have obviously

Theorem 1. A manifold $\overline{A}$ of the adjoint space $\overline{R}^{\mathcal{V}}$ is weakly bounded, if and only if $\sup_{\overline{x} \in \overline{A}} |\overline{x}(x)| < +\infty$ for every $x \in R$.

We obtain further by §62 Theorem 8

Theorem 2. Every weakly bounded manifold $\overline{A}$ of $\overline{R}^{\mathcal{V}}$ is weakly totally bounded.

Theorem 3. For every vicinity V in R,

$$\{\overline{x} : |\overline{x}(x)| \leq 1 \quad \text{for every } x \in V\}$$

is weakly bounded.

Proof. For each $x \in R$ we can find $\alpha > 0$ such that $\alpha x \in V$. Putting $\overline{A} = \{\overline{x} : |\overline{x}(x)| \leq 1$ for every $x \in V\}$, we have for such $\alpha > 0$

$$|\overline{x}(\alpha x)| \leq 1 \qquad \text{for every } \overline{x} \in \overline{A},$$

and hence $\sup_{\overline{x} \in \overline{A}} |\overline{x}(x)| \leq \frac{1}{\alpha}$. Therefore $\overline{A}$ is weakly bounded by Theorem 1.

Theorem 4. For every vicinity $V \in \mathcal{V}$,

$$\{\overline{x} : |\overline{x}(x)| \leq 1 \quad \text{for every } x \in V\}$$

is weakly compact.

Proof. Putting $\overline{A} = \{\overline{x} : |\overline{x}(x)| \leq 1$ for every $x \in V\}$, we see by Theorem 3 that $\overline{A}$ is weakly bounded. Thus, corresponding to every $x \in R$ we can find by Theorem 1 $\gamma_x > 0$ such that

$$-\gamma_x \leq \overline{x}(x) \leq \gamma_x \qquad \text{for every } \overline{x} \in \overline{A}.$$

For a functional φ on R subject to the condition:

$$-\gamma_x \leq \varphi(x) \leq \gamma_x \qquad \text{for every } x \in R,$$

putting $\overline{A}_{x,\varepsilon} = \{\overline{x} : |\overline{x}(x) - \varphi(x)| < \varepsilon\}$, if $\overline{A} \prod_{\nu=1}^{\kappa} \overline{A}_{x_\nu,\varepsilon} \neq 0$ for every finite number of elements $x_\nu \in R$ ($\nu = 1, 2, \ldots, \kappa$) and $\varepsilon > 0$, then φ is a linear functional on R. Because, for two elements $x, y \in R$ and real numbers α, β, for any $\varepsilon > 0$ we can find $\overline{x} \in \overline{A}_{x,\varepsilon} \overline{A}_{y,\varepsilon} \overline{A}_{\alpha x+\beta y,\varepsilon}$. For such $\overline{x}$ we have naturally $|\overline{x}(x) - \varphi(x)| < \varepsilon$, $|\overline{x}(y) - \varphi(y)| < \varepsilon$,

$$|\overline{x}(\alpha x + \beta y) - \varphi(\alpha x + \beta y)| < \varepsilon,$$

and hence $|\alpha\varphi(x) + \beta\varphi(y) - \varphi(\alpha x + \beta y)| \leq (|\alpha| + |\beta|)\varepsilon$. As $\varepsilon > 0$ may be arbitrary, we otain therefore $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$.

Furthermore, for any $x \in V$ and $\varepsilon > 0$ we can find $\overline{x} \in \overline{A}\,\overline{A}_{x,\varepsilon}$, and we have $|\varphi(x)| \leq |\overline{x}(x)| + \varepsilon \leq 1 + \varepsilon$. Thus we conclude $|\varphi(x)| \leq 1$ for every $x \in V$. Accordingly φ is continuous by $\mathcal{V}$ and $\varphi \in \overline{A}$. Therefore $\overline{A}$ is weakly compact by §19 Theorem 4.

Theorem 5. If a bounded manifold $\overline{A}$ of $\overline{R}^{\mathcal{V}}$ is weakly closed, then $\overline{A}$ is weakly compact.

Proof. We can assume obviously that $\mathcal{V}$ is equivalently strongest. Then for a bounded manifold $\overline{A}$ of $\overline{R}^{\mathcal{V}}$, putting

$$V = \{x : |\overline{x}(x)| \leq 1 \quad \text{for every } \overline{x} \in \overline{A}\},$$

we have $V \in \mathcal{V}$ by §64 Theorem 8, and obviously

$$\overline{A} \subset \{\overline{x} : |\overline{x}(x)| \leq 1 \quad \text{for every } x \in V\}.$$

Therefore, if $\overline{A}$ is weakly closed, then we conclude by Theorem 4 that $\overline{A}$ is weakly compact.

Recalling §62 Theorem 9, we conclude easily

Theorem 6. In order that a linear manifold $\overline{A}$ of $\overline{R}^{\mathcal{V}}$ be weakly closed, it is necessary and sufficient that for any $\overline{x}_0 \bar{\in} \overline{A}$ we can find $x \in R$ such that $\overline{x}_0(x) \neq 0$ but $\overline{x}(x) = 0$ for every $\overline{x} \in \overline{A}$.

A functional φ on the adjoint space $\overline{R}^{\mathcal{V}}$ is said to be *weakly continuous*, if φ is continuous by the weak topology of $\overline{R}^{\mathcal{V}}$. As an immediate consequence of §62 Theorem 5 we have then

Theorem 7. If a linear functional φ on $\overline{R}^{\mathcal{V}}$ is weakly continuous, then we can find $x \in R$ such that $\varphi(\overline{x}) = \overline{x}(x)$ for every $\overline{x} \in \overline{R}^{\mathcal{V}}$.

Theorem 8. Let a linear topology $\mathcal{V}$ on R be equivalently strongest or standard. If a linear manifold $\overline{A}$ of $\overline{R}^{\mathcal{V}}$ is weakly closed, then putting $A = \{x : \overline{x}(x) = 0$ for every $\overline{x} \in \overline{A}\}$, $\overline{A}$ coincides with the adjoint space $\overline{R/A}^{\mathcal{V}}$ of R/A by the relative linear topology of $\mathcal{V}$, considering every $\overline{x} \in \overline{A}$ as a linear functional on the quotient space R/A.

Proof. We conclude by §62 Theorem 9

$$\overline{A} = \{\overline{x} : \overline{x}(x) = 0 \quad \text{for every } x \in A\}.$$

As every $\overline{x} \in \overline{A}$ is by §64 Theorem 2 continuous in R by $\mathcal{V}$, recalling the definition of the relative linear topology, we see that every $\bar{x} \in \bar{A}$ is continuous in the quotient space R/A by the relative linear topology of $\mathcal{V}$, and hence $\bar{A} \subset \overline{R/A}^{\mathcal{V}}$. Onthe other hand, we see easily that for every bounded manifold B of R, B/A also is a bounded manifold of the quotient space R/A, and hence we conclude $\overline{R/A}^{\mathcal{V}} \subset \bar{A}$. Thus we obtain our assertion $\bar{A} = \overline{R/A}^{\mathcal{V}}$.

§66 Normality

Let R be a linear topological space with a linear topology $\mathcal{V}$. $\mathcal{V}$ is said to be normal, if $\mathcal{V}$ contains all closed convex vicinities.

Theorem 1. If the induced topology $\gamma^{\mathcal{V}}$ by $\mathcal{V}$ is of the second category, then $\mathcal{V}$ is normal.

Proof. For an arbitrary vicinity V_0 we have by definition

$$R = \sum_{\nu=1}^{\infty} \nu V_0 .$$

As R is of the second category by $\gamma^{\mathcal{V}}$ by assumption, if V_0 is convex and closed by $\mathcal{V}$, then we can find hence $V \in \mathcal{V}$, $x \in R$, and ν such that $V + x \subset \nu V_0$, and then we have

$$V \subset \nu V_0 \times \nu V_0 \subset 2\nu V_0 ,$$

that is, $\frac{1}{2\nu} V \subset V_0$. Thus we conclude $V_0 \in \mathcal{V}$, if V_0 is a closed convex vicinity. Therefore $\mathcal{V}$ is normal by definition.

Recalling §59 Theorem 2, we obtain by Theorem 1

Theorem 2. If $\mathcal{V}$ is sequential and complete, then $\mathcal{V}$ is normal.

As the weak topology of the adjoint space $\bar{R}^{\mathcal{V}}$ is weaker than the adjoint topology $\overline{\mathcal{V}}$ of $\mathcal{V}$, every bounded manifold of $\bar{R}^{\mathcal{V}}$ is weakly bounded by definition. Conversely we have

Theorem 3. If $\mathcal{V}$ is normal and equivalently strongest or standard, then every weakly bounded manifold $\bar{A}$ of the adjoint space $\bar{R}^{\mathcal{V}}$ is bounded by the adjoint topology $\overline{\mathcal{V}}$, and hence the weak topology of $\bar{R}^{\mathcal{V}}$ is equivalent to the adjoint topology $\overline{\mathcal{V}}$.

Proof. Let $\bar{A}$ be a weakly bounded manifold of $\bar{R}^{\mathcal{V}}$. Putting

$$V = \{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{A}\},$$

we see easily that V is convex. By virtue of §65 Theorem 1, for every $x \in R$ we have $\sup_{\bar{x} \in \bar{A}} |\bar{x}(x)| < +\infty$, and hence we can find $\lambda > 0$ such that $\sup_{\bar{x} \in \bar{A}} |\bar{x}(\xi x)| \leqq 1$ for $|\xi| \leqq \lambda$. Thus V is a vicinity in R by definition. As $\mathcal{V}$ is equivalently strongest or standard by assumption, every $\bar{x} \in \bar{R}^{\mathcal{V}}$ is continuous by §64 Theorem 2, and we have obviously

$$V = \prod_{\bar{x} \in \bar{A}} \{x : |\bar{x}(x)| \leqq 1\}.$$

Hence V is closed by $\mathcal{V}$. As $\mathcal{V}$ is normal by assumption, we obtain therefore $V \in \mathcal{V}$ by definition, and we have obviously

$$\bar{A} \subset \{\bar{x} : |\bar{x}(x)| \leqq 1 \quad \text{for every } x \in V\}.$$

Recalling §64 Theorem 6, we conclude thus that $\bar{A}$ is bounded by $\overline{\mathcal{V}}$.

Theorem 4. Let $\mathcal{V}$ be normal and equivalently strongest or standard. For a sequence $\bar{a}_\nu \in \bar{R}^{\mathcal{V}}$ ($\nu = 1, 2, \ldots$), if $\bar{a}_\nu(x)$ ($\nu = 1, 2, \ldots$) is convergent for every $x \in R$, then there exists $\bar{a} \in \bar{R}^{\mathcal{V}}$ such that

$$\lim_{\nu \to \infty} \bar{a}_\nu(x) = \bar{a}(x) \quad \text{for every } x \in R.$$

Proof. By virtue of §65 Theorem 1, $\{\bar{a}_1, \bar{a}_2, \ldots\}$ is weakly bounded in $\bar{R}^{\mathcal{V}}$, and hence it is bounded by the adjoint topology $\overline{\mathcal{V}}$ of $\mathcal{V}$, according to Theorem 3. Therefore, putting

$$V = \{x : |\bar{a}_\nu(x)| \leqq 1 \quad \text{for every } \nu = 1, 2, \ldots\},$$

we have $V \in \mathcal{V}$ by §64 Theorem 8. If we set

$$\varphi(x) = \lim_{\nu \to \infty} \bar{a}_\nu(x) \quad \text{for every } x \in R,$$

then we see easily that φ is a linear functional on R. Furthermore, as $|a_\nu(x)| \leqq 1$ for every $x \in V$ and $\nu = 1, 2, \ldots$, we obtain $|\varphi(x)| \leqq 1$ for every $x \in V$, and hence $\varphi \in \bar{R}^{\mathcal{V}}$ by Theorems 1 in §62 and 1 in §64.

§67 Completeness

Let R be now merely a linear space and $\tilde{R}$ the associated space of R. Considering every $x \in R$ as a linear functional on $\tilde{R}$ by the relation: $x(\tilde{x}) = \tilde{x}(x)$ for every $\tilde{x} \in \tilde{R}$, we obtain a weak linear topology of $\tilde{R}$ by x ($x \in R$). This weak linear topology is called the weak topology of the associated space $\tilde{R}$. With this definition we have

Theorem 1. The weak topology of the associated space $\tilde{R}$ of a linear space R is complete.

Proof. For a functional φ on R, if for $\tilde{x} \in \tilde{R}$
$$\prod_{\nu=1}^{\kappa} \{ \tilde{x} : |\tilde{x}(x_\nu) - \varphi(x_\nu)| \leqq \xi \quad (\nu = 1, 2, \ldots, \kappa) \} \neq 0$$
for every finite number of elements $x_\nu \in R$ ($\nu = 1, 2, \ldots, \kappa$) and $\varepsilon > 0$, then we have for every real numbers α, β and $x, y \in R$ that for any $\varepsilon > 0$ we can find $\tilde{x} \in \tilde{R}$ such that
$$|\tilde{x}(x) - \varphi(x)| \leqq \varepsilon, \qquad |\tilde{x}(y) - \varphi(y)| \leqq \varepsilon,$$
$$|\tilde{x}(\alpha x + \beta y) - \varphi(\alpha x + \beta y)| \leqq \varepsilon,$$
and hence $|\alpha\varphi(x) + \beta\varphi(y) - \varphi(\alpha x + \beta y)| \leqq \varepsilon$. As $\varepsilon > 0$ may be arbitrary, we obtain $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$ for every $x, y \in R$ and real numbers α, β, that is, $\varphi \in \tilde{R}$. As the number uniformity is complete, we conclude by §35 Theorem 8 that $\tilde{R}$ is complete by the weak uniformity, which is by §55 Theorem 5 the induced uniformity by the weak topology of $\tilde{R}$.

Theorem 2. If a linear manifold $\tilde{A}$ of the associated space $\tilde{R}$ is fundamental in R, then $\tilde{A}$ is dense in $\tilde{R}$ by the weak topology.

Proof. Let $\tilde{a} \in \tilde{R}$ be arbitrary. For every finite number of elements $x_\nu \in R$ ($\nu = 1, 2, \ldots, \kappa$), $\sum_{\nu=1}^{\kappa} \xi_\nu x_\nu = 0$ implies obviously $\sum_{\nu=1}^{\kappa} \xi_\nu \tilde{a}(x_\nu) = 0$. Thus, considering every x_ν as a linear functional on $\tilde{A}$ by the relation: $x_\nu(\tilde{x}) = \tilde{x}(x_\nu)$ for $\tilde{x} \in \tilde{A}$, we can find by §46 Theorem 4 $\tilde{x} \in \tilde{A}$ such that $\tilde{x}(x_\nu) = \tilde{a}(x_\nu)$ for every $\nu = 1, 2, \ldots, \kappa$. Thus $\tilde{a}$ is a contact point of $\tilde{A}$ by the weak topology. Therefore $\tilde{A}$ is dense in $\tilde{R}$ by the weak topology.

Let R be next a linear topological space with a linear topology $\mathcal{V}$. We shall prove firstly:

Theorem 3. The adjoint topology $\overline{\mathcal{V}}$ of $\mathcal{V}$ is complete.

Proof. We can assume by definition that $\mathcal{V}$ is equivalently strongest or standard. Let $\bar{A}_\lambda$ ($\lambda \in \Lambda$) be a Cauchy system by the adjoint topology $\overline{\mathcal{V}}$. Since the weak topology of the adjoint space $\bar{R}^{\mathcal{V}}$ is weaker than $\overline{\mathcal{V}}$, $\bar{A}_\lambda$ ($\lambda \in \Lambda$) also is a Cauchy system by the weak topology. As the associated space $\tilde{R}$ of R is by Theorem 1 complete by the weak topology, there exists a limit $\tilde{a}_0 \in \tilde{R}$ of $\bar{A}_\lambda$ ($\lambda \in \Lambda$) for the weak topology.

As $\bar{A}_\lambda$ ($\lambda \in \Lambda$) is a Cauchy system by $\overline{\mathcal{V}}$, for each bounded manifold B of R we can find $\lambda_0 \in \Lambda$ such that $\bar{x}, \bar{y} \in \bar{A}_{\lambda_0}$ implies $|\bar{x}(x) - \bar{y}(x)| \leqq 1$ for every $x \in B$, because $\{\bar{x} : |\bar{x}(x)| \leqq 1$ for every $x \in B\} \in \overline{\mathcal{V}}$ by definition. For any $x \in B$ and $\varepsilon > 0$ we can find $\bar{y} \in \bar{A}_{\lambda_0}$ such that we have $|\bar{y}(x) - \tilde{a}_0(x)| < \varepsilon$, because $\tilde{a}_0$ is a limit of $\bar{A}_\lambda$ ($\lambda \in \Lambda$) by the weak topology of $\tilde{R}$. Then we have $|\bar{x}(x) - \tilde{a}_0(x)| < 1 + \varepsilon$ for every $\bar{x} \in \bar{A}_{\lambda_0}$. As $x \in B$ and $\varepsilon > 0$ may be arbitrary, we conclude hence
$$|\bar{x}(x) - \tilde{a}_0(x)| \leqq 1 \qquad \text{for every } \bar{x} \in \bar{A}_{\lambda_0} \text{ and } x \in B.$$
From this relation we conclude further that $\tilde{a}_0$ is a bounded linear functional on R by $\mathcal{V}$, that is, $\tilde{a}_0 \in \bar{R}^{\mathcal{V}}$, and for such λ_0
$$\bar{A}_{\lambda_0} \subset \{ \bar{x} : |\bar{x}(x) - \tilde{a}_0(x)| \leqq 1 \quad \text{for every } x \in B \}.$$
Thus, putting $\bar{V}_B = \{\bar{x} : |\bar{x}(x)| \leqq 1$ for every $x \in B\}$, we have $\bar{A}_{\lambda_0} \subset \bar{V}_B + \tilde{a}_0$. This relation yields by the definition of the adjoint topology $\overline{\mathcal{V}}$ that $\tilde{a}_0$ is a limit of $\bar{A}_\lambda$ ($\lambda \in \Lambda$) by $\overline{\mathcal{V}}$. Therefore $\bar{R}^{\mathcal{V}}$ is complete by $\overline{\mathcal{V}}$.

Theorem 4. If $\mathcal{V}$ is normal and equivalently strongest or standard, then the weak topology of the adjoint space $\bar{R}^{\mathcal{V}}$ is conditionally complete.

Proof. If a manifold $\bar{A}$ of $\bar{R}^{\mathcal{V}}$ is weakly bounded, then $\bar{A}$ is bounded by §66 Theorem 3. Thus, if $\bar{A}$ is furthermore weakly closed, then $\bar{A}$ is by §65 Theorem 5 weakly compact, and hence naturally complete by the weak topology. Therefore the weak topology of $\bar{R}^{\mathcal{V}}$ is conditionally complete by definition.

Recalling Theorems 1 in §58 and 2 in §66, we conclude immediately from Theorem 2

Theorem 5. If $\mathcal{V}$ is sequential and complete, then the weak topology of the adjoint space $\bar{R}^{\mathcal{V}}$ is conditionally complete.

§68 Reflexibility

Let R be a linear topological space with a linear topology $\mathcal{V}$, and $\bar{R}^{\mathcal{V}}$ the adjoint space of R with the adjoint topology $\overline{\mathcal{V}}$ of $\mathcal{V}$. The weak linear topology of R by $\bar{R}^{\mathcal{V}}$ will be called the weak topology of R

by $\mathfrak{V}$. If a manifold A of R is bounded, closed, or compact by the weak topology of R, then we shall say that A is weakly bounded, weakly closed, or weakly compact.

Theorem 1. If $\mathfrak{V}$ is convex, then every weakly bounded manifold of R is bounded.

Proof. Let a manifold A of R be weakly bounded. For each symmetric convex vicinity $V \in \mathfrak{V}$, the adjoint norm $\|\bar{x}\|_V$ is by §52 Theorem 1 complete in the adjoint space $\bar{R}_V$ of V. As $\bar{R}_V \subset \bar{R}^{\mathfrak{V}}$ by Theorems 1 in §62 and 1 in §64, we have by §62 Theorem 7

$$\sup_{x \in A} |\bar{x}(x)| < +\infty \qquad \text{for every } \bar{x} \in \bar{R}_V .$$

Considering every $x \in R$ as a linear functional on $\bar{R}_V$ by the relation: $x(\bar{x}) = \bar{x}(x)$ for every $\bar{x} \in \bar{R}_V$, we obtain then by §52 Theorem 5

$$\sup_{x \in A} \{ \sup_{\|\bar{x}\|_V \leqq 1} |\bar{x}(x)| \} < +\infty,$$

and hence $\sup_{x \in A} \|x\|_V < +\infty$ by §52 Theorem 4. Therefore A is bounded by §57 Theorem 1.

If $\mathfrak{V}$ is equivalently strongest or standard, then every $\bar{x} \in \bar{R}^{\mathfrak{V}}$ is by §64 Theorem 2 a continuous linear functional on R by $\mathfrak{V}$, and hence the weak topology of R is by definition weaker than $\mathfrak{V}$. However we have conversely

Theorem 2. If $\mathfrak{V}$ is convex, then every closed linear manifold of R is weakly closed.

Proof. Let A be a closed linear manifold of R. By virtue of §56 Theorem 6, the quotient space R/A is separated by the relative linear topology $\mathfrak{V}^{R/A}$ which also is convex by Theorems 5 in §51 and 4 in §56. For each $x_0 \bar{\in} A$ there exists by definition $X_0 \in R/A$ such that $x_0 \in X_0$, and then we can find by Theorems 2 in §52 and 1 in §62 a continuous linear functional φ on R/A such that $\varphi(X_0) \neq 0$. If we consider φ as a linear functional on R, then φ is obviously continuous, $\varphi(x) = 0$ for every $x \in A$, and $\varphi(x_0) = \varphi(X_0) \neq 0$. Therefore we can conclude that A is weakly closed.

By virtue of §64 Theorem 4, every $x \in R$ may be considered as a continuous linear functional on the adjoint space $\bar{R}^{\mathfrak{V}}$ by the adjoint topology $\bar{\mathfrak{V}}$.

Thus, if $\bar{R}^{\mathfrak{V}}$ is fundamental in R, then we can consider R as a linear manifold of the adjoint space of $\bar{R}^{\mathfrak{V}}$ by the adjoint topology $\bar{\mathfrak{V}}$. On account of §67 Theorem 5, we obtain immediately

Theorem 3. If the adjoint space $\bar{R}^{\mathfrak{V}}$ of R by $\mathfrak{V}$ is fundamental in R, then, denoting by $\bar{\bar{R}}^{\bar{\mathfrak{V}}}$ the adjoint space of $\bar{R}^{\mathfrak{V}}$ by the adjoint topology $\bar{\mathfrak{V}}$, R is dense in $\bar{\bar{R}}^{\bar{\mathfrak{V}}}$ by the weak topology of $\bar{\bar{R}}^{\bar{\mathfrak{V}}}$.

Even if $\bar{R}^{\mathfrak{V}}$ is fundamental in R, the linear topology $\mathfrak{V}$ of R is not necessarily the relative linear topology of the adjoint topology of the adjoint topology $\bar{\mathfrak{V}}$. If $\bar{R}^{\mathfrak{V}}$ is fundamental in R and $\mathfrak{V}$ coincides with the relative linear topology of the adjoint topology of the adjoint topology $\bar{\mathfrak{V}}$, considering R as a linear manifold of the adjoint space $\bar{\bar{R}}^{\bar{\mathfrak{V}}}$ of the adjoint space $\bar{R}^{\mathfrak{V}}$, then $\mathfrak{V}$ is said to be reflexive.

Theorem 4. A linear topology $\mathfrak{V}$ is reflexive if and only if $\mathfrak{V}$ is separative and standard.

Proof. If $\mathfrak{V}$ is reflexive, then $\mathfrak{V}$ is separative and convex by §64 Theorem 3, and for any bounded manifold $\bar{A}$ of the adjoint space $\bar{R}^{\mathfrak{V}}$ we have by the definition of the adjoint topology

$$\{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{A}\} \in \mathfrak{V}.$$

Thus we conclude by §64 Theorem 10 that $\mathfrak{V}$ is standard.

Conversely, if $\mathfrak{V}$ is separative and standard, then the adjoint space $\bar{R}^{\mathfrak{V}}$ is fundamental in R by Theorems 2 in §52 and 1 in §62. For each closed symmetric convex vicinity $V \in \mathfrak{V}$, putting

$$\bar{A} = \{\bar{x} : |\bar{x}(x)| \leqq 1 \quad \text{for every } x \in V\},$$

we obtain by §64 Theorem 6 a bounded manifold $\bar{A}$ of $\bar{R}^{\mathfrak{V}}$ and we have by §52 Theorem 3 $V = \{x : |\bar{x}(x)| \leqq 1$ for every $\bar{x} \in \bar{A}\}$. Furthermore, for every bounded manifold $\bar{A}$ of $\bar{R}^{\mathfrak{V}}$ we have by §64 Theorem 8

$$\{x : |\bar{x}(x)| \leqq 1 \quad \text{for every } \bar{x} \in \bar{A}\} \in \mathfrak{V}.$$

Therefore $\mathfrak{V}$ is reflexive.

If $\mathfrak{V}$ is reflexive, and further R coincides with the adjoint space $\bar{\bar{R}}^{\bar{\mathfrak{V}}}$ of the adjoint space $\bar{R}^{\mathfrak{V}}$, then we shall say that $\mathfrak{V}$ is regular, or that R is regular by $\mathfrak{V}$.

Theorem 5. In order that a linear topological space R be regular by its linear topology φ, it is necessary and sufficient that both φ and its adjoint topology $\overline{\varphi}$ be reflexive and the weak topology of R by φ be conditionally complete.

Proof. If φ is regular, then the adjoint topology $\overline{\varphi}$ is by definition naturally reflexive. Every weakly bounded manifold A of R is by Theorem 1 bounded, and hence if A is furthermore weakly closed, then A is weakly compact by §65 Theorem 5. Therefore the weak topology of R by φ is conditionally complete.

Conversely, we assume that both φ and $\overline{\varphi}$ are reflexive and the weak topology of R by φ is conditionally complete. Let $\overline{\overline{R}}^{\varphi}$ be the adjoint space of the adjoint space $\overline{R}^{\varphi}$. As $\overline{\varphi}$ is standard by Theorem 4, for each $\overline{\overline{x}} \in \overline{\overline{R}}^{\varphi}$ we can find by §64 Theorem 2 and §62 Theorem 1 a symmetric convex vicinity $\overline{V} \in \overline{\varphi}$ such that $|\overline{\overline{x}}(\overline{x})| \leq 1$ for every $\overline{x} \in \overline{V}$. For such $\overline{V} \in \overline{\varphi}$, we can find by definition a bounded manifold B of R such that, putting $\overline{V}_1 = \{\overline{x} : |\overline{x}(x)| \leq 1 \text{ for every } x \in B\}$, we have $\overline{\varphi} \ni \overline{V}_1 \subset \overline{V}$.

If we set
$$A = \{x : \sup_{\overline{x} \in \overline{V}_1} |\overline{x}(x)| < +\infty\},$$
$$U = \{x : |\overline{x}(x)| \leq 1 \text{ for every } \overline{x} \in \overline{V}_1\},$$
then A is a linear manifold of R and U is a symmetric convex vicinity in the subspace A. As $U \supset B$, we have
$$\{\overline{x} : |\overline{x}(x)| \leq 1 \text{ for every } x \in U\} \subset \overline{V}_1 \subset \overline{V},$$
and hence for every finite number of elements $\overline{x}_\nu \in \overline{R}^{\varphi}$ ($\nu = 1, 2, \ldots, \kappa$), considering the adjoint norm by U, we have
$$\Big|\sum_{\nu=1}^{\kappa} \xi_\nu \overline{\overline{x}}(\overline{x}_\nu)\Big| = \Big|\overline{\overline{x}}\Big(\sum_{\nu=1}^{\kappa} \xi_\nu \overline{x}_\nu\Big)\Big| \leq \Big\|\sum_{\nu=1}^{\kappa} \xi_\nu \overline{x}_\nu\Big\|_U .$$
Accordingly we can find by §52 Theorem 8 $x \in 2U$ such that
$$\overline{x}_\nu(x) = \overline{\overline{x}}(\overline{x}_\nu) \qquad (\nu = 1, 2, \ldots, \kappa).$$
Therefore $\overline{\overline{x}}$ is a contact point of $2U$ by the weak topology. On the other hand, as $2U = \{x : |\overline{x}(x)| \leq 2 \text{ for every } \overline{x} \in \overline{V}_1\}$, $\overline{V}_1 \in \overline{\varphi}$, we see by §65 Theorem 1 that $2U$ is weakly bounded and furthermore by definition that $2U$ is weakly closed in R. Since the weak topology of R is conditionally complete by assumption, $2U$ is weakly compact, and hence $2U$ is weakly closed in $\overline{\overline{R}}^{\varphi}$. Thus we obtain $\overline{\overline{x}} \in 2U$. Therefore we conclude $R = \overline{\overline{R}}^{\varphi}$.

Theorem 6. Let R be regular by its linear topology φ. For a closed linear manifold A of R, if the relative linear topology φ^A of φ and its adjoint topology $\overline{\varphi^A}$ are standard, then φ^A is regular.

Proof. As φ is separative, both φ^A and $\overline{\varphi^A}$ are reflexive by assumption. As A is a closed linear manifold of R by assumption, A is weakly closed by Theorem 2, and hence the weak topology of the subspace A is conditionally complete, because the weak topology of R is so by Theorem 5. Thus φ^A is regular by Theorem 5.

Theorem 7. If φ is reflexive, complete, and its adjoint topology $\overline{\varphi}$ is regular, then φ also is regular.

Proof. For the adjoint space $\overline{\overline{R}}^{\varphi}$ of the adjoint space $\overline{R}^{\varphi}$, considering R as a linear manifold of $\overline{\overline{R}}^{\varphi}$, φ coincides with the relative linear topology of the linear topology of $\overline{\overline{R}}^{\varphi}$, because φ is reflexive by assumption. Furthermore R is by §35 Theorem 3 a closed linear manifold of $\overline{\overline{R}}^{\varphi}$. As the adjoint topology $\overline{\varphi}$ is regular by assumption, the linear topology of $\overline{\overline{R}}^{\varphi}$ also is regular. Therefore φ is regular by Theorem 6.

§69 Sequential roots

Let R be a linear topological space with a linear topology φ. If there is a sequence of bounded manifolds A_ν ($\nu = 1, 2, \ldots$) in R such that A_ν ($\nu = 1, 2, \ldots$) is a root of φ, that is, for any bounded manifold A of R we can find ν for which $A \subset A_\nu$, then such a root A_ν ($\nu = 1, 2, \ldots$) is called a sequential root of φ.

Theorem 1. If φ has a sequential root A_ν ($\nu = 1, 2, \ldots$), then the adjoint topology $\overline{\varphi}$ of φ is sequential.

Proof. Putting $\overline{V}_\nu = \{\overline{x} : |\overline{x}(x)| \leq 1 \text{ for every } x \in A_\nu\}$, we obtain by definition a basis $\overline{V}_\nu$ ($\nu = 1, 2, \ldots$) of $\overline{\varphi}$. Thus $\overline{\varphi}$ is sequential by definition.

Theorem 2. If φ is convex and its adjoint topology $\overline{\varphi}$ is sequential, then φ has a sequential root.

Proof. For a decreasing basis $\overline{V}_\nu$ $(\nu = 1, 2, \dots)$ of $\overline{\mathfrak{V}}$, putting

$$A_\nu = \{ x : |\bar{x}(x)| \leq 1 \quad \text{for every } \bar{x} \in \overline{V}_\nu \},$$

we obtain by §64 Theorem 7 bounded manifolds A_ν $(\nu = 1, 2, \dots)$ in R by $\mathfrak{V}$. For each bounded manifold A of R, putting

$$\overline{V} = \{ \bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in A \},$$

we have $\overline{V} \in \overline{\mathfrak{V}}$ by definition, and hence we can find ν such that $\overline{V} \supset \overline{V}_\nu$. Then we have obviously

$$A \subset \{ x : |\bar{x}(x)| \leq 1 \quad \text{for every } \bar{x} \in \overline{V} \} \subset A_\nu .$$

Therefore A_ν $(\nu = 1, 2, \dots)$ is a sequential root of $\mathfrak{V}$.

Theorem 3. If $\mathfrak{V}$ is sequential, then its adjoint topology $\overline{\mathfrak{V}}$ has a sequential root.

Proof. For a decreasing basis V_ν $(\nu = 1, 2, \dots)$ of $\mathfrak{V}$, putting

$$\overline{A}_\nu = \{ \bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in V_\nu \},$$

we obtain by §64 Theorem 6 a sequence of bounded manifolds $\overline{A}_\nu$ $(\nu = 1, 2, \dots)$ in the adjoint space $\overline{R}^{\mathfrak{V}}$. As $\mathfrak{V}$ is equivalently strongest by §58 Theorem 1, for each bounded manifold $\overline{A}$ of $\overline{R}^{\mathfrak{V}}$, putting

$$V = \{ x : |\bar{x}(x)| \leq 1 \quad \text{for every } \bar{x} \in \overline{A} \},$$

we have $V \in \mathfrak{V}$ by §64 Theorem 8. For such $V \in \mathfrak{V}$, we can find ν such that $V \supset V_\nu$, and then

$$\overline{A} \subset \{ \bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in V \} \subset \overline{A}_\nu .$$

Thus $\overline{A}_\nu$ $(\nu = 1, 2, \dots)$ is a sequential root of $\overline{\mathfrak{V}}$.

Theorem 4. If $\mathfrak{V}$ is standard and its adjoint topology $\overline{\mathfrak{V}}$ has a sequential root, then $\mathfrak{V}$ is sequential.

Proof. For a sequential root $\overline{A}_\nu$ $(\nu = 1, 2, \dots)$ of $\overline{\mathfrak{V}}$, putting

$$V_\nu = \{ x : |\bar{x}(x)| \leq 1 \quad \text{for every } \bar{x} \in \overline{A}_\nu \},$$

we have by §64 Theorem 8 $V_\nu \in \mathfrak{V}$ $(\nu = 1, 2, \dots)$. For each closed convex $V \in \mathfrak{V}$, putting

$$\overline{A} = \{ \bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in V \},$$

we obtain by §64 Theorem 6 a bounded manifold $\overline{A}$ of $\overline{R}^{\mathfrak{V}}$, and hence we can find ν such that $\overline{A} \subset \overline{A}_\nu$. Since we have by §52 Theorem 3

$$V = \{ x : |\bar{x}(x)| \leq 1 \quad \text{for every } \bar{x} \in \overline{A} \},$$

we obtain thus $V \supset V_\nu$ for such ν. Therefore V_ν $(\nu = 1, 2, \dots)$ is

a basis of $\mathfrak{V}$.

Theorem 5. If $\mathfrak{V}$ is sequential, complete, and has a sequential root, then $\mathfrak{V}$ is of single vicinity.

Proof. If $\mathfrak{V}$ is sequential and complete, then the induced topology $\gamma^{\mathfrak{V}}$ by $\mathfrak{V}$ is of the second category by §59 Theorem 2. Furthermore if $\mathfrak{V}$ has a sequential root A_ν $(\nu = 1, 2, \dots)$, then we have

$$R = \sum_{\nu=1}^{\infty} A_\nu ,$$

because every $x \in R$ is itself a bounded manifold of R. Thus we can find ν such that $A_\nu^{-\circ} \neq 0$ by the induced topology $\gamma^{\mathfrak{V}}$. As $A_\nu^{-\circ}$ also is bounded by §57 Theorem 2, we can conclude by §58 Theorem 5 that $\mathfrak{V}$ is of single vicinity.

Theorem 6. If $\mathfrak{V}$ is of single vicinity, then its adjoint topology $\overline{\mathfrak{V}}$ also is of single vicinity.

Proof. Let V be a basis of $\mathfrak{V}$. By virtue of §64 Theorem 6, putting $\overline{V} = \{ \bar{x} : |\bar{x}(x)| \leq 1$ for every $x \in V \}$ we obtain a bounded manifold $\overline{V}$ of the adjoint space $\overline{R}^{\mathfrak{V}}$. As V is a bounded manifold of R, we have further $\overline{V} \in \overline{\mathfrak{V}}$ by definition. Therefore, recalling the formula §54(8), we conclude by §58 Theorem 5 that $\overline{\mathfrak{V}}$ is of single vicinity.

Theorem 7. If $\mathfrak{V}$ is standard and its adjoint topology $\overline{\mathfrak{V}}$ is of single vicinity, then $\mathfrak{V}$ also is of single vicinity.

Proof. Let $\overline{V}$ be a basis of $\overline{\mathfrak{V}}$. By virtue of §64 Theorem 7, putting $V = \{ x : |\bar{x}(x)| \leq 1$ for every $\bar{x} \in \overline{V} \}$, we obtain a bounded manifold V of R. As $\overline{V}$ is a bounded manifold of the adjoint space $\overline{R}^{\mathfrak{V}}$ we have further $V \in \mathfrak{V}$ by §64 Theorem 8. Therefore $\mathfrak{V}$ is of single vicinity by §58 Theorem 5.

Theorem 8. If both $\mathfrak{V}$ and its adjoint topology $\overline{\mathfrak{V}}$ are sequential, then $\overline{\mathfrak{V}}$ is of single vicinity.

Proof. Recalling Theorem 3, we conclude from assumption that $\overline{\mathfrak{V}}$ has a sequential root. On the other hand, $\overline{\mathfrak{V}}$ is obviously complete by §67 Theorem 3. Furthermore, $\overline{\mathfrak{V}}$ is sequential by assumption. Therefore, we can conclude by Theorem 5 that the adjoint topology $\overline{\mathfrak{V}}$ is of sigle vicinity.

Theorem 9. *If $\mathfrak{V}$ is convex, sequential, and its adjoint topology $\overline{\mathfrak{V}}$ is sequential, then $\mathfrak{V}$ is of single vicinity.*

Proof. By virtue of Theorem 8 we see at once by assumption that $\overline{\mathfrak{V}}$ is of single vicinity. As $\mathfrak{V}$ is standard by §58 Theorem 1, we conclude by Theorem 7 that $\mathfrak{V}$ is of single vicinity.

On account of §58 Theorem 1 and §68 Theorem 4, we have obviously

Theorem 10. *A sequential linear topology $\mathfrak{V}$ is reflexive if and only if $\mathfrak{V}$ is convex and separative.*

§70 Strongest convex linear topologies

Let R be a linear space. By virtue of §53 Theorem 3, we see easily that there exists uniquely a linear topology $\mathfrak{V}_0$ on R such that the totality of convex vicinities in R is a basis of $\mathfrak{V}_0$. This linear topology $\mathfrak{V}_0$ is called the *strongest convex linear topology* on R. Indeed, $\mathfrak{V}_0$ is obviously stronger than every other convex linear topology on R. With this definition we have

Theorem 1. *The strongest convex linear topology $\mathfrak{V}_0$ on R is standard, normal, and separative.*

Proof. It is obvious by definition that $\mathfrak{V}_0$ is standard and normal. By virtue of §44 Theorem 2, for any element $x_0 \neq 0$ we can find a linear functional φ on R such that $\varphi(x_0) = 1$. Then, putting

$$V = \{x : |\varphi(x)| \leqq \tfrac{1}{2}\}$$

we obtain a convex vicinity V, and hence $V \in \mathfrak{V}_0$ by definition, but $V \not\ni x_0$. Therefore $\mathfrak{V}_0$ is separative by definition.

Theorem 2. *If a manifold A of R is bounded by the strongest convex linear topology $\mathfrak{V}_0$, then A is contained in a finite-dimensional linear manifold of R.*

Proof. If there is a sequence $a_\nu \in A$ ($\nu = 1, 2, \dots$) such that each a_ν is linearly independent from the others, then we can find by §44 Theorem 3 a linear functional φ on R such that $\varphi(a_\nu) = \nu$ for every $\nu = 1, 2, \dots$, and putting $V = \{x : |\varphi(x)| \leqq 1\}$, we obtain a convex vicinity V

in R. For such V, we have by definition $V \in \mathfrak{V}_0$, but $a_{\nu+1} \overline{\in} \nu V$ for every $\nu = 1, 2, \dots$ Thus a_ν ($\nu = 1, 2, \dots$) is not bounded by $\mathfrak{V}_0$.

Theorem 3. *The adjoint space of R by the strongest convex linear topology $\mathfrak{V}_0$ coincides with the associated space $\overline{R}$ of R and the adjoint topology of $\mathfrak{V}_0$ coincides with the weak topology of $\overline{R}$.*

Proof. For every linear functional φ on R, we have obviously by definition $\{x : |\varphi(x)| \leqq 1\} \in \mathfrak{V}_0$. We conclude hence by Theorems 1 in §62 and 1 §64 that $\overline{R}$ is the adjoint space of R by $\mathfrak{V}_0$. For each bounded manifold A of R by $\mathfrak{V}_0$, as A is contained by Theorem 2 in a finite-dimensional linear manifold, we can find by §61 Theorem 1 a finite number of linearly independent elements $a_\nu \in R$ ($\nu = 1, 2, \dots, \kappa$) such that

$$A \subset \{\sum_{\nu=1}^{\kappa} \alpha_\nu a_\nu : |\alpha_\nu| \leqq 1 \quad \text{for every } \nu = 1, 2, \dots, \kappa\}.$$

Then, putting $\overline{V} = \{\bar{x} : |\bar{x}(a_\nu)| \leqq \frac{1}{\kappa}$ for every $\nu = 1, 2, \dots, \kappa\}$, we have obviously $\overline{V} \subset \{\bar{x} : |\bar{x}(x)| \leqq 1$ for every $x \in A\}$. Therefore we conclude by definition that the adjoint topology of $\mathfrak{V}_0$ coincides with the weak topology of $\overline{R}$.

Theorem 4. *If R has a basis of linearly independent countable elements $a_\nu \in R$ ($\nu = 1, 2, \dots$), then, putting for $\nu = 1, 2, \dots$*

$$A_\nu = \{\sum_{\mu=1}^{\nu} \alpha_\mu a_\mu : |\alpha_\mu| \leqq \nu \quad (\mu = 1, 2, \dots, \nu)\},$$

we obtain a sequential root A_ν ($\nu = 1, 2, \dots$) of the strongest convex linear topology $\mathfrak{V}_0$ on R, the weak topology of the associated space $\overline{R}$ of R is sequential, and $\mathfrak{V}_0$ is regular.

Proof. If a manifold A of R is bounded by $\mathfrak{V}_0$, then we can find by Theorem 2 ν_0 such that A is contained in the linear manifold S generated by a_ν ($\nu = 1, 2, \dots, \nu_0$). Then A is naturally bounded in S by the relative linear topology of $\mathfrak{V}_0$, and hence we can find by §61 Theorem 1 $\alpha > 0$ such that $A \subset \{\sum_{\mu=1}^{\nu_0} \alpha_\mu a_\mu : |\alpha_\mu| \leqq \alpha \ (\mu = 1, 2, \dots, \nu_0)\}$, and for $\nu \geqq \operatorname{Max}\{\alpha, \nu_0\}$ we have obviously $A \subset A_\nu$. Therefore A_ν ($\nu = 1, 2, \dots$) is a sequential root of $\mathfrak{V}_0$. Recalling Theorem 4, we see by §69 Theorem 1 that the weak topology of $\overline{R}$ is sequential, and consequently standard by §58 Theorem 1. Accordingly, we conclude easily by Theorems 2 in §64 and 7 in §65 that $\mathfrak{V}_0$ is regular

CHAPTER X

NORMED SPACES

§71 Induced linear topologies.

A linear space R associated with a norm $\|x\|$ ($x \in R$) is called a *normed space*. A normed space R is naturally a quasi-normed linear space, and hence we defined already in §63 the induced linear topology. This induced linear topology is called the *norm topology* of R. With this definition, we see easily that the norm topology is separative, of single vicinity, and further $\{x : \|x\| \leqq 1\}$ is a basis of it, because

$$\|\xi x\| = |\xi| \|x\| \qquad \text{for every real number } \xi.$$

This vicinity $\{x : \|x\| \leqq 1\}$ is called the *unit sphere* of R. The unit sphere is convex, because, $\|x\| \leqq 1$, $\|y\| \leqq 1$, $\lambda + \mu = 1$, $\lambda, \mu \geqq 0$ implies

$$\|\lambda x + \mu y\| \leqq \lambda \|x\| + \mu \|y\| \leqq \lambda + \mu = 1.$$

Therefore we have

Theorem 1. *The norm topology is separative, convex, of single vicinity, and the unit sphere is a convex basis of it.*

Conversely we have obviously by the formula §49(2)

Theorem 2. *If a linear topology $\mathfrak{V}$ on a linear space R is separative, convex, of single vicinity, and a symmetric convex vicinity V is a basis of $\mathfrak{V}$, then the pseudo-norm $\|x\|_V$ of V is a norm on R and the norm topology coincides with $\mathfrak{V}$.*

By virtue of §61 Theorem 4 we obtain immediately

Theorem 3. *If the unit sphere of a normed space R is totally bounded by the norm topology, then R is finite-dimensional.*

We have defined already in §50 the completeness of quasi-norms. As a norm is naturally a quasi-norm, we have by §63 Theorem 6

Theorem 4. *A normed space R is complete by the norm, if and only if R is complete by the norm topology.*

For two norms $\|x\|_1$ and $\|x\|_2$ on a linear space R, we shall say that $\|x\|_1$ is *weaker* than $\|x\|_2$, or that $\|x\|_2$ is *stronger* than $\|x\|_1$, if the norm topology by $\|x\|_1$ is weaker than that by $\|x\|_2$.

Theorem 6. *In order that a norm $\|x\|_1$ be weaker than a norm $\|x\|_2$ on R, it is necessary and sufficient that we can find $\alpha > 0$ such that*

$$\|x\|_1 \leqq \alpha \|x\|_2 \qquad \text{for every } x \in R.$$

Proof. For a normed space R, denoting by V the unit sphere of R, we see by Theorem 1 that the pseudo-norm $\|x\|_V$ of V coincides with the norm of R and V is a basis of the norm topology. Therefore we conclude our assertion by §58 Theorem 6.

For two norms $\|x\|_1$ and $\|x\|_2$ on a linear space R, we shall say that $\|x\|_1$ is *equivalent* to $\|x\|_2$, and write $\|x\|_1 \sim \|x\|_2$ ($x \in R$), if $\|x\|_1$ is weaker and stronger than $\|x\|_2$ at the same time, that is, the norm topology of $\|x\|_1$ coincides with that of $\|x\|_2$ on R.

With this definition we have obviously by Theorem 6

Theorem 7. *We have $\|x\|_1 \sim \|x\|_2$ ($x \in R$) if and only if we can find two positive numbers α, β such that*

$$\alpha \|x\|_2 \leqq \|x\|_1 \leqq \beta \|x\|_2 \qquad \text{for every } x \in R.$$

Recalling §59 Theorem 4, we obtain further

Theorem 8. *If a norm $\|x\|_1$ is weaker than a norm $\|x\|_2$ on R, and both $\|x\|_1$ and $\|x\|_2$ are complete, then $\|x\|_1$ is equivalent to $\|x\|_2$ on R.*

A normed space R is said to be *separable* or *completely separable*, if it is so by the norm topology, as defined in §14.

Recalling §36 Theorem 7 we have then

Theorem 9. *If a normed space R is separable, then R is completely separable.*

A linear manifold A of a normed space R may be considered itself as a normed space with the norm of R. In this sense, A will be called a *subspace* of R.

Theorem 10. *For a sequence of elements a_ν ($\nu = 1, 2, \ldots$) in a normed space R, the least closed linear manifold containing all a_ν ($\nu = 1, 2, \ldots$) is separable as a subspace of R.*

Proof. Let A be the linear manifold generated by a_ν ($\nu = 1, 2, \ldots$). The closure A^- by the norm topology also is a linear manifold by §54 Theorem 5, and hence A^- is the least closed linear manifold containing a_ν ($\nu = 1, 2, \ldots$). Denoting by B the totality of linear combinations $\sum_{\nu=1}^{\varkappa} \alpha_\nu a_\nu$ from a_ν ($\nu = 1, 2, \ldots$) for all rational numbers α_ν ($\nu = 1, 2, \ldots, \varkappa$), we see easily that B is countable and dense in A too, and hence A^- is separable by definition.

§72 Adjoint spaces

Let R be a normed space and U the unit sphere of R. The adjoint space of U associated with the adjoint norm of U, as defined in §52, is called the adjoint space of R. Thus the adjoint space $\overline{R}$ of R also is a normed space. Since the norm of R coincides obviously by definition with the pseudo-norm of U, we have by §52(1)

(1) $\qquad \|\overline{x}\| = \sup_{x \in U} |\overline{x}(x)| \qquad (\overline{x} \in \overline{R})$

and by §52 Theorem 4 further

(2) $\qquad |\overline{x}(x)| \leqq \|\overline{x}\|\|x\| \qquad (x \in R, \overline{x} \in \overline{R})$

(3) $\qquad \|x\| = \sup_{\|\overline{x}\| \leqq 1} |\overline{x}(x)| \qquad (x \in R, \overline{x} \in \overline{R}).$

Furthermore we have by §52 Theorem 1

Theorem 1. The norm of the adjoint space is complete.

Theorem 2. The adjoint space $\overline{R}$ of R coincides with the adjoint space of R by the norm topology, and the norm topology of $\overline{R}$ coincides with the adjoint topology of the norm topology of R.

Proof. As the unit sphere U of R is obviously by definition bounded by the norm topology of R, we see at once by definition that the adjoint space $\overline{R}$ of R is composed of all bounded linear functionals by the norm topology of R. Furthermore, for the unit sphere $\overline{U}$ of $\overline{R}$, as we have by (1)

$$\overline{U} = \{\overline{x} : |\overline{x}(x)| \leqq 1 \quad \text{for every } x \in U\},$$

we conclude by §64 Theorem 6 that $\overline{U}$ is bounded by the adjoint topology, and by definition further that $\overline{U}$ is a vicinity of the adjoint topology. Thus we obtain by definition that $\overline{U}$ is a basis of the adjoint topology. Therefore the norm topology of $\overline{R}$ coincides by §71 Theorem 1 with the adjoint topology of the norm topology of R.

Let $\overline{R}$ be the adjoint space of a normed space R and $\overline{U}$ the unit sphere of $\overline{R}$. For a linear manifold $\overline{A}$ of $\overline{R}$, if $\overline{A}$ is fundamental in R, then, putting

(4) $\qquad \|x\|_{\overline{A}} = \sup_{\overline{x} \in \overline{U}\overline{A}} |\overline{x}(x)| \qquad (x \in R),$

we obtain a norm $\|x\|_{\overline{A}}$ on R. Indeed, we see easily that $\|x\|_{\overline{A}}$ is a pseudo-norm on R. If $\|x\|_{\overline{A}} = 0$, then $\overline{x}(x) = 0$ for every $\overline{x} \in \overline{A}$, and hence $x = 0$, because $\overline{A}$ is fundamental in R by assumption. Furthermore, as $\overline{U}\overline{A} \subset \overline{U}$, we have by (3)

(5) $\qquad \|x\|_{\overline{A}} \leqq \|x\| \qquad (x \in R),$

that is, this norm $\|x\|_{\overline{A}}$ is weaker than that of R. If this norm $\|x\|_{\overline{A}}$ is equivalent to the norm of R, that is, if we can find $\lambda > 0$ such that $\lambda \|x\|_{\overline{A}} \geqq \|x\|$ for every $x \in R$, then $\overline{A}$ is said to be of finite character, and

(6) $\qquad \chi = \sup_{0 \neq x \in R} \dfrac{\|x\|}{\|x\|_{\overline{A}}}$

is called the character of a fundamental linear manifold $\overline{A}$ of $\overline{R}$. If $\overline{A}$ is not of finite character, then the character χ of $\overline{A}$ is defined as $+\infty$.

For a manifold $\overline{A}$ of the adjoint space $\overline{R}$, we denote by $\overline{A}^{w-}$ the closure of $\overline{A}$ by the weak topology of $\overline{R}$. Then we have

Theorem 3. For a fundamental linear manifold $\overline{A}$ of the adjoint space $\overline{R}$, we have

$$(\overline{U}\overline{A})^{w-} \supset \frac{1}{\gamma}\overline{U},$$

if and only if $\gamma \geqq \chi$ for the character χ of $\overline{A}$.

Proof. If $\chi < +\infty$, then for any $\overline{a} \in \frac{1}{\chi}\overline{U}$ and for every finite number of elements $x_\nu \in R$ ($\nu = 1, 2, \ldots, \varkappa$), we have by (6)

$$\left|\sum_{\nu=1}^{\varkappa} \xi_\nu \overline{a}(x_\nu)\right| = \left|\overline{a}\left(\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right)\right| \leqq \frac{1}{\chi}\left\|\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right\| \leqq \left\|\sum_{\nu=1}^{\varkappa} \xi_\nu x_\nu\right\|_{\overline{A}}.$$

Considering x_ν ($\nu = 1, 2, \ldots, \varkappa$) as linear functionals on $\overline{A}$ by the relation: $x_\nu(\overline{x}) = \overline{x}(x_\nu)$ for every $\overline{x} \in \overline{A}$, we conclude hence by §52 Theorem 8 that for any $\varepsilon > 0$ we can find $\overline{x}_0 \in (1+\varepsilon)(\overline{U}\overline{A})$ such that $\overline{x}_0(x_\nu) = \overline{a}(x_\nu)$ for every $\nu = 1, 2, \ldots, \varkappa$, and consequently we have $\frac{1}{1+\varepsilon}\overline{x}_0 \in \overline{U}\overline{A}$ and

$$\left|\frac{1}{1+\varepsilon}\overline{x}_0(x_\nu) - \overline{a}(x_\nu)\right| \leqq \frac{\varepsilon}{1+\varepsilon}|\overline{a}(x_\nu)| \qquad (\nu = 1, 2, \ldots, \varkappa).$$

Thus we obtain by definition that $\overline{a} \in (\overline{U}\overline{A})^{w-}$. Therefore we have

$$\frac{1}{\chi}\,\overline{U} \subset (\overline{U}\,\overline{A})^{w-}.$$

Conversely, if $\frac{1}{\gamma}\overline{U} \subset (\overline{U}\,\overline{A})^{w-}$ for a positive number γ, then for any $x \in R$ and $\varepsilon > 0$, corresponding to every $\bar{x} \in \overline{U}$ we can find $\bar{x}_0 \in \overline{U}\,\overline{A}$ such that $|\frac{1}{\gamma}\bar{x}(x) - \bar{x}_0(x)| < \varepsilon$, and hence for every $\bar{x} \in \overline{U}$

$$|\tfrac{1}{\gamma}\bar{x}(x)| \leqq \varepsilon + |\bar{x}_0(x)| \leqq \varepsilon + \|x\|_{\overline{A}}.$$

Thus we obtain $\frac{1}{\gamma}\|x\| \leqq \varepsilon + \|x\|_{\overline{A}}$ by the formula (3). Here $\varepsilon > 0$ may be arbitrary, and hence we conclude $\|x\| \leqq \gamma\|x\|_{\overline{A}}$ for every $x \in R$. Accordingly we have $\gamma \geqq \chi$ by the definition (6).

A fundamental linear manifold $\overline{A}$ of the adjoint space $\overline{R}$ is said to be *norm fundamental* in R, if the character of $\overline{A}$ is 1, that is, if

$$\|x\| = \sup_{\bar{x} \in \overline{U}\,\overline{A}} |\bar{x}(x)| \qquad \text{for every } x \in R.$$

With this definition we have obviously by Theorem 3

Theorem 4. *A fundamental linear manifold $\overline{A}$ of the adjoint space $\overline{R}$ is norm fundamental, if and only if $\overline{U}\,\overline{A}$ is dense in $\overline{U}$ by the weak topology of $\overline{R}$.*

For the unit sphere $\overline{U}$ of the adjoint space $\overline{R}$ we have obviously

$$\overline{U} = \{\bar{x} : |\bar{x}(x)| \leqq 1 \quad \text{for every } x \in U\}.$$

Thus we obtain immediately by §65 Theorem 4

Theorem 5. *The unit sphere $\overline{U}$ of the adjoint space $\overline{R}$ is weakly compact.*

Recalling Theorems 2 in §68 and 9 in §62, we obtain immediately

Theorem 6. *For a closed linear manifold A of a normed space R and an lement $x_0 \overline{\in} A$, we can find $\bar{a} \in \overline{R}$ such that $\bar{a}(x) = 0$ for every $x \in A$ and $\bar{a}(x_0) = 1$.*

Theorem 7. *If the adjoint space $\overline{R}$ is separable, then R is separable too.*

Proof. Let $\bar{a}_\nu \in \overline{R}$ $(\nu = 1, 2, \dots)$ be dense in $\overline{R}$ by the norm topology of $\overline{R}$. Recalling the definition of the adjoint norm, we can find a sequence $a_\nu \in R$ $(\nu = 1, 2, \dots)$ such that $\bar{a}_\nu(a_\nu) \geqq \frac{1}{2}\|\bar{a}_\nu\|$, $\|a_\nu\| = 1$. Let A be the least closed linear manifold containing a_ν $(\nu = 1, 2, \dots)$. Then A is by §71 Theorem 10 separable as a subspace of R. If A does not coincides with R, then, as A is weakly closed by §68 Theorem 2, for an element $a \in R - A$, we can find by §62 Theorem 9 $\bar{a} \in \overline{R}$ such that $\bar{a}(x) = 0$ for every $x \in A$, $\bar{a}(a) \neq 0$, and $\|\bar{a}\| = 1$. For such $\bar{a} \in \overline{R}$ we have for every $\nu = 1, 2, \dots$

$$\|\bar{a}_\nu - \bar{a}\| \geqq |\bar{a}_\nu(a_\nu) - \bar{a}(a_\nu)| = |\bar{a}_\nu(a_\nu)| \geqq \tfrac{1}{2}\|\bar{a}_\nu\|,$$
$$\|\bar{a}_\nu - \bar{a}\| \geqq \|\bar{a}\| - \|\bar{a}_\nu\| = 1 - \|\bar{a}_\nu\|,$$

and hence $\|\bar{a}_\nu - \bar{a}\| \geqq \operatorname{Max}\{\frac{1}{2}\|\bar{a}_\nu\|, 1 - \|\bar{a}_\nu\|\} \geqq \frac{1}{3}$, contradicting that $\bar{a}_\nu$ $(\nu = 1, 2, \dots)$ are dense in $\overline{R}$. Thus we obtain $A = R$, and hence R is separable by definition.

§73 Quotient spaces

Let R be a normed space. For a linear manifold A of R, we have defined the quotient space R/A in §46 and the relative pseudo-norm on R/A in §51 as $\|X\| = \inf_{x \in X}\|x\|$ for $X \in R/A$. This pseudo-norm $\|X\|$ is convex by §51 Theorem 6. If A is closed by the norm topology of R, then we see easily by §51 Theorem 7, §56 Theorems 4 and 5 that the relative pseudo-norm $\|X\|$ is proper and hence a norm on R/A. Thus for a closed linear manifold A of R we obtain a *quotient space* R/A as a normed space with the relative pseudo-norm, and we have

Theorem 1. *The norm topology of a quotient space R/A by a closed linear manifold A coincides with the relative linear topology of the norm topology of R on the quotient space R/A.*

Furthermore we obtain by §59 Theorem 5

Theorem 2. *If the norm of R is complete, then the norm of the quotient space R/A by a closed linear manifold A is complete too.*

Theorem 3. *If a linear manifold $\overline{A}$ of the adjoint space $\overline{R}$ is weakly closed, then for the linear manifold*

$$A = \{x : \bar{x}(x) = 0 \quad \text{for every } \bar{x} \in \overline{A}\},$$

considering every $\bar{x} \in \overline{A}$ as a linear functional on the quotient space R/A, $\overline{A}$ coincides with the adjoint space $\overline{R/A}$ of R/A as a normed space.

Proof. By virtue of §65 Theorem 8, $\overline{A}$ coincides with $\overline{R/A}$ as a linear space. Thus we need only prove that the norm of $\bar{A}$ coincides with that of $\overline{R/A}$. As $\|X\| \leq \|x\|$ for every $x \in X$, we have

$$\sup_{\|X\|\leq 1} |\bar{x}(X)| \geq \sup_{\|x\|\leq 1} |\bar{x}(x)| \quad \text{for every } \bar{x}\in\bar{A}.$$

On the other hand, for any $X \in R/A$ subject to $\|X\| < 1$, we can find $x \in X$ for which $\|x\|<1$, and hence

$$\sup_{\|X\|\leq 1}|\bar{x}(X)| = \sup_{\|X\|<1}|\bar{x}(X)| \leq \sup_{\|x\|\leq 1}|\bar{x}(x)|.$$

Therefore we have $\|\bar{x}\| = \sup_{\|X\|\leq 1}|\bar{x}(X)|$ for every $\bar{x}\in\bar{A}$.

For a linear manifold $\bar{A}$ of the adjoint space $\bar{R}$, putting

$$A=\{x:\ \bar{x}(x)=0 \quad \text{for every } \bar{x}\in\bar{A}\},$$

we obtain a closed linear manifold A of R. Then we can consider by Theorem 3 $\bar{A}$ as a linear manifold of the adjoint space $\overline{R/A}$ of the quotient space R/A. The <u>character</u> of a linear manifold $\bar{A}$ of the adjoint space $\bar{R}$ is defined by the character of $\bar{A}$ as a linear manifold of the adjoint space $\overline{R/A}$. Denoting by $\bar{A}^{w-}$ the closure of a manifold $\bar{A}$ of the adjoint space $\bar{R}$ by the weak topology, we conclude easily from §72 Theorem 3

<u>Theorem</u> 4. <u>For a linear manifold $\bar{A}$ of the adjoint space $\bar{R}$, we have $(\bar{U}\bar{A})^{w-} \supset \frac{1}{\gamma}\bar{U}\bar{A}^{w-}$ if and only if $\gamma \geq \chi$ for the character χ of $\bar{A}$ and the unit sphere $\bar{U}$ of $\bar{R}$.</u>

A linear manifold $\bar{A}$ of the adjoint space $\bar{R}$ is said to be <u>norm fundamental</u>, if $\bar{A}$ is so as a linear manifold of the adjoint space $\overline{R/A}$ of the quotient space R/A for $A=\{x: \bar{x}(x)=0$ for every $\bar{x}\in\bar{A}\}$.

With this definition we have obviously by Theorem 4

<u>Theorem</u> 5. <u>A linear manifold $\bar{A}$ of the adjoint space $\bar{R}$ is norm fundamental, if and only if $\bar{U}\bar{A}$ is dense in $\bar{U}\bar{A}^{w-}$ by the weak topology.</u>

<u>Theorem</u> 6. <u>For a linear manifold $\bar{A}$ of the adjoint space $\bar{R}$, putting</u> $A=\{x: \bar{x}(x)=0$ for every $\bar{x}\in\bar{A}\}$, <u>we have</u>

$$\bar{A}^{w-} = \{\bar{x}:\ \bar{x}(x)=0 \quad \text{for every } x\in A\}.$$

<u>Proof</u>. Putting $\bar{B} = \{\bar{x}: \bar{x}(x)=0$ for every $x\in A\}$, we obtain obviously by definition a weakly closed linear manifold $\bar{B}$ of $\bar{R}$, and on account of Theorem 3, $\bar{B}$ coincides with the adjoint space $\overline{R/A}$ of the quotient space R/A. As $\bar{A}$ is obviously fundamental in the

quotient space R/A, $\bar{A}$ is weakly dense in $\bar{B}$ by Theorem 5.

Finally we will prove

<u>Theorem</u> 7. (Banach) <u>When the norm of R is complete, for a linear manifold $\bar{A}$ of then adjoint space $\bar{R}$, if $\bar{U}\bar{A}$ is weakly closed for the unit sphere $\bar{U}$ of $\bar{R}$, then $\bar{A}$ also is weakly closed.</u>

<u>Proof</u>. As $\alpha(\bar{U}\bar{A}) = (\alpha\bar{U})(\alpha\bar{A}) = (\alpha\bar{U})\bar{A}$ for every $\alpha\neq 0$, we see by assumption that $(\alpha\bar{U})\bar{A}$ is weakly closed for every $\alpha>0$. For an element $\bar{a} \,\bar{\in}\, \bar{A}$, we can find $\varepsilon>0$ such that $\bar{A}(\varepsilon\bar{U}+\bar{a})=0$. Because, as $\bar{A}(2\|\bar{a}\|\bar{U})$ is closed naturally by the norm topology, we can find a positive number $\varepsilon<\|\bar{a}\|$ such that $\bar{A}(2\|\bar{a}\|\bar{U})(\varepsilon\bar{U}+\bar{a})=0$, and we have obviously $\varepsilon\bar{U}+\bar{a} \subset 2\|\bar{a}\|\bar{U}$. For such ε, we have obviously

$$\varepsilon\bar{U}+\bar{a} = \prod_{\|x\|\leq 1}\{\bar{x}:\ |\bar{x}(x)-\bar{a}(x)|\leq\varepsilon\},$$

$$\bar{A}(2\varepsilon\bar{U}+\bar{a})(\varepsilon\bar{U}+\bar{a})=0.$$

Since $2\varepsilon\bar{U}+\bar{a} \subset 4\|\bar{a}\|\bar{U}$, and $\bar{A}(4\|\bar{a}\|\bar{U})$ is weakly compact by §72 Theorem 5, we can find a finite number of elements $x_{1,\mu}$ $(\mu=1,2,\ldots,\mu_1)$ such that $\|x_{1,\mu}\|\leq 1$ and

$$\bar{A}\prod_{\mu=1}^{\mu_1}\{\bar{x}:\ |\bar{x}(x_{1,\mu})-\bar{a}(x_{1,\mu})|\leq\varepsilon\}(2\varepsilon\bar{U}+\bar{a})=0.$$

Similarly we can find consecutively $x_{\nu,\mu}$ $(\mu=1,2,\ldots,\mu_\nu;\ \nu=1,2,\ldots)$ such that $\|x_{\nu,\mu}\|\leq 1$ and

$$\bar{A}\prod_{\nu=1}^{\kappa}\prod_{\mu=1}^{\mu_\nu}\{\bar{x}:\ |\bar{x}(x_{\nu,\mu})-\bar{a}(x_{\nu,\mu})|\leq 2^\nu\varepsilon\}(2^\kappa\varepsilon\bar{U}+\bar{a})=0.$$

As $\sum_{\kappa=1}^{\infty} \bar{A}(2^\kappa\varepsilon\bar{U}+\bar{a})=\bar{A}$, we obtain hence

$$\bar{A}\prod_{\nu,\mu}\{\bar{x}:\ |\bar{x}(\tfrac{1}{2^\nu\varepsilon}x_{\nu,\mu})-\bar{a}(\tfrac{1}{2^\nu\varepsilon}x_{\nu,\mu})|\leq 1\}=0.$$

Thus we can find a sequence of elements x_ν $(\nu=1,2,\ldots)$ such that

$$\lim_{\nu\to\infty}\|x_\nu\|=0,$$

$$\bar{A}\prod_{\nu=1}^{\infty}\{\bar{x}:\ |\bar{x}(x_\nu)-\bar{a}(x_\nu)|\leq 1\}=0.$$

Let C be a space of number sequences $y=(\xi_\nu:\ \nu=1,2,\ldots)$ subject to the condition: $\lim_{\nu\to\infty}\xi_\nu=0$, with the norm

$$\|y\| = \sup_{\nu=1,2,\ldots}|\xi_\nu|.$$

Then we see easily that, putting

$$\bar{A}_0=\{(\bar{x}(x_\nu):\ \nu=1,2,\ldots):\ \bar{x}\in\bar{A}\},$$

we obtain a linear manifold $\bar{A}_0$ of C, and, putting

$$y_0=(\bar{a}(x_\nu):\ \nu=1,2,\ldots),$$

we have $\|y - y_0\| \geqq 1$ for every $y \in \bar{A}_0$. Thus we can find by §72 Theorem 6 a continuous linear functional φ on $\mathfrak{C}$ such that $\varphi(y) = 0$ for every $y \in \bar{A}_0$ and $\varphi(y_0) = 1$. For such φ, putting

$$\alpha_\mu = \varphi(y_\mu), \quad y_\mu = (\xi_\nu : \xi_\nu = 0 \text{ for } \nu \neq \mu, \ \xi_\mu = 1),$$

we see easily that $\varphi(\xi_1, \xi_2, \dots) = \sum_{\nu=1}^{\infty} \alpha_\nu \xi_\nu$, and hence $\sum_{\nu=1}^{\infty} |\alpha_\nu| < +\infty$. As R is complete by assumption, we obtain hence an element $x_0 \in R$ as

$$x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu,$$

and we have for every $\bar{x} \in \bar{A}$

$$\bar{x}(x_0) = \sum_{\nu=1}^{\infty} \alpha_\nu \bar{x}(x_\nu) = 0,$$

$$\bar{a}(x_0) = \sum_{\nu=1}^{\infty} \alpha_\nu \bar{a}(x_\nu) = \varphi(y_0) = 1$$

Therefore every $\bar{a} \bar{\in} \bar{A}$ is not a contact point of $\bar{A}$ by the weak topology, that is, $\bar{A}$ is weakly closed.

§74 Weak convergence

Let R be a normed space. A sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be norm convergent to a limit $a \in R$, if

$$\lim_{\nu \to \infty} \|a_\nu - a\| = 0,$$

that is, if $\lim_{\nu \to \infty} a_\nu = a$ for the norm topology. As the norm topology is separative, we see easily that such a limit a is determined uniquely, if it exists.

Concerning norm convergence, we can prove easily

Theorem 1. If $\lim_{\nu \to \infty} a_\nu = a$, $\lim_{\nu \to \infty} b_\nu = b$, then we have

$$\lim_{\nu \to \infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b.$$

Theorem 2. $\lim_{\nu \to \infty} a_\nu = a$, $\lim_{\nu \to \infty} \alpha_\nu = \alpha$ implies $\lim_{\nu \to \infty} \alpha_\nu a_\nu = \alpha a$.

Theorem 3. $\lim_{\nu \to \infty} a_\nu = a$ implies $\lim_{\nu \to \infty} \|a_\nu\| = \|a\|$.

Let $\bar{R}$ be the adjoint space of R. A sequence of elements $\bar{a}_\nu \in \bar{R}$ ($\nu = 1, 2, \dots$) is said to be weakly convergent to a limit $\bar{a} \in \bar{R}$, if

$$\lim_{\nu \to \infty} \bar{a}_\nu(x) = \bar{a}(x) \qquad \text{for every } x \in R,$$

and then we shall write

$$\text{w-}\lim_{\nu \to \infty} \bar{a}_\nu = \bar{a}.$$

We recognize easily by definition the uniqueness of such a limit $\bar{a} \in \bar{R}$, if it exists.

Theorem 4. $\text{w-}\lim_{\nu \to \infty} \bar{a}_\nu = \bar{a}$ implies $\underline{\lim}_{\nu \to \infty} \|\bar{a}_\nu\| \geqq \|\bar{a}\|$.

Proof. For any $\varepsilon > 0$ we can find by the definition of the adjoint norm $x \in R$ such that $\bar{a}(x) \geqq \|\bar{a}\| - \varepsilon$, $\|x\| = 1$. For such $x \in R$ we have by assumption $\bar{a}(x) = \lim_{\nu \to \infty} \bar{a}_\nu(x) \leqq \underline{\lim}_{\nu \to \infty} \|\bar{a}_\nu\|$, and hence

$$\underline{\lim}_{\nu \to \infty} \|\bar{a}_\nu\| \geqq \|\bar{a}\| - \varepsilon \qquad \text{for every } \varepsilon > 0.$$

Thus we obtain our assertion.

Theorem 5. Let R be complete. If $\bar{a}_\nu(x)$ ($\nu = 1, 2, \dots$) is convergent for every $x \in R$, then $\bar{a}_\nu$ ($\nu = 1, 2, \dots$) is weakly convergent and we have $\sup_{\nu = 1, 2, \dots} \|\bar{a}_\nu\| < +\infty$.

Proof. As R is complete by assumption, we conclude by Theorems 1 in §58 and 2 in §66 that the norm topology is standard and normal. Thus we obtain by §66 Theorem 4 that $\bar{a}_\nu$ ($\nu = 1, 2, \dots$) is weakly convergent, and further by §66 Theorem 3 that $\sup_{\nu = 1, 2, \dots} \|\bar{a}_\nu\| < +\infty$.

Theorem 6. If R is complete, then $\text{w-}\lim_{\nu \to \infty} \bar{a}_\nu = \bar{a}$, $\lim_{\nu \to \infty} a_\nu = a$ implies $\lim_{\nu \to \infty} \bar{a}_\nu(a_\nu) = \bar{a}(a)$.

Proof. As we have by the formula §72(2)

$$|\bar{a}_\nu(a_\nu) - \bar{a}(a)| \leqq |\bar{a}_\nu(a_\nu) - \bar{a}_\nu(a)| + |\bar{a}_\nu(a) - \bar{a}(a)|$$

$$\leqq \|\bar{a}_\nu\| \|a_\nu - a\| + |\bar{a}_\nu(a) - \bar{a}(a)|,$$

and $\sup_{\nu = 1, 2, \dots} \|\bar{a}_\nu\| < +\infty$ by Theorem 5, we conclude by assumption

$$\lim_{\nu \to \infty} |\bar{a}_\nu(a_\nu) - \bar{a}(a)| = 0.$$

Theorem 7. If R is separable, then every bounded sequence $\bar{a}_\nu \in \bar{R}$ ($\nu = 1, 2, \dots$) contains a weakly convergent subsequence.

Proof. Let $a_\rho \in R$ ($\rho = 1, 2, \dots$) be dense in R by the norm topology. From $\bar{a}_\nu \in \bar{R}$ ($\nu = 1, 2, \dots$) we can find by the diagonal method a subsequence $\bar{a}_{\nu_\mu}$ ($\mu = 1, 2, \dots$) such that $\bar{a}_{\nu_\mu}(a_\rho)$ ($\mu = 1, 2, \dots$) is convergent for every $\rho = 1, 2, \dots$. As $\bar{a}_\nu$ ($\nu = 1, 2, \dots$) is bounded by assumption, we can find $\gamma > 0$ such that $\|\bar{a}_{\nu_\mu}\| \leqq \gamma$ for every $\mu = 1, 2, \dots$ Let x be an arbitrary element of R. For any $\varepsilon > 0$ we can find by assumption ρ such that $\|a_\rho - x\| < \varepsilon$, and then for every $\mu, \mu' = 1, 2, \dots$

$$|\bar{a}_{\nu_\mu}(x) - \bar{a}_{\nu_{\mu'}}(x)| \leqq |\bar{a}_{\nu_\mu}(a_\rho) - \bar{a}_{\nu_\mu}(x)|$$

$$+ |\bar{a}_{\nu_{\mu'}}(a_\rho) - \bar{a}_{\nu_{\mu'}}(x)| + |\bar{a}_{\nu_\mu}(a_\rho) - \bar{a}_{\nu_{\mu'}}(a_\rho)|$$

$$\leq \|\bar{a}_{\nu_\mu}\| \|a_\rho - x\| + \|\bar{a}_{\nu_{\mu'}}\| \|a_\rho - x\| + |\bar{a}_{\nu_\mu}(a_\rho) - \bar{a}_{\nu_{\mu'}}(a_\rho)|$$

$$\leq 2\gamma\varepsilon + |\bar{a}_{\nu_\mu}(a_\rho) - \bar{a}_{\nu_{\mu'}}(a_\rho)|.$$

Thus we conclude that $\bar{a}_{\nu_\mu}(x)$ $(\mu = 1, 2, \ldots)$ is convergent for every $x \in R$, and hence, putting $\varphi(x) = \lim_{\mu\to\infty} \bar{a}_{\nu_\mu}(x)$ for every $x \in R$, we obtain a linear functional φ on R. Furthermore we have for every $x \in R$

$$|\varphi(x)| = \lim_{\mu\to\infty} |\bar{a}_{\nu_\mu}(x)| \leq \gamma \|x\|.$$

Therefore we conclude $\varphi \in \bar{R}$, and consequently $\bar{a}_{\nu_\mu}$ $(\mu = 1, 2, \ldots)$ is weakly convergent by definition.

A sequence of elements $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is said to be *weakly convergent* to a *limit* $a \in R$, if

$$\lim_{\nu\to\infty} \bar{x}(a_\nu) = \bar{x}(a) \qquad \text{for every } \bar{x} \in \bar{R},$$

and then we shall write $\text{w-}\lim_{\nu\to\infty} a_\nu = a$.

Every $a \in R$ may be considered as a linear functional on the adjoint space $\bar{R}$ of R by the relation: $a(\bar{x}) = \bar{x}(a)$ for every $\bar{x} \in \bar{R}$, and we have by the formula §72 (3)

$$\|a\| = \sup_{\|\bar{x}\| \leq 1} |\bar{x}(a)|.$$

Thus we have by Theorem 4

Theorem 8. $\text{w-}\lim_{\nu\to\infty} a_\nu = a$ *implies* $\underline{\lim}_{\nu\to\infty} \|a_\nu\| \geq \|a\|$.

As the adjoint space $\bar{R}$ is complete by §72 Theorem 1, we obtain by Theorems 5 and 6

Theorem 9. $\text{w-}\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a}$ *implies*

$$\lim_{\nu\to\infty} \bar{a}_\nu(a_\nu) = \bar{a}(a), \qquad \sup_{\nu=1,2,\ldots} \|a_\nu\| < +\infty.$$

§75 Regularity

Let R be a normed space. The norm topology of R is by §71 Theorem 1 and §58 Theorem 1 separative, standard, and hence reflexive by §68 Theorem 4. Furthermore we see by the formula §72(3) that R may be considered as a subspace of the adjoint space $\bar{\bar{R}}$ of the adjoint space $\bar{R}$ by the relation: $x(\bar{x}) = \bar{x}(x)$ for every $\bar{x} \in \bar{R}$ and $x \in R$, and then R is norm fundamental in $\bar{\bar{R}}$. Thus we obtain by §72 Theorem 4

Theorem 1. *The unit sphere* U *of* R *is weakly dense in the unit sphere* $\bar{\bar{U}}$ *of the adjoint space* $\bar{\bar{R}}$ *of the adjoint space* $\bar{R}$.

We shall consider the regularity of R by the norm topology in the sequel.

Theorem 2. R *is regular by the norm topology if and only if the unit sphere* U *of* R *is weakly compact.*

Proof. If R is regular, then the unit sphere U of R coincides with the unit sphere $\bar{\bar{U}}$ of $\bar{\bar{R}}$, and hence weakly compact by §72 Theorem 5. Conversely, if the unit sphere U of R is weakly compact, then U coincides by Theorem 1 with the unit sphere $\bar{\bar{U}}$ of $\bar{\bar{R}}$, and hence R coincides with $\bar{\bar{R}}$.

Theorem 3. *If* R *is regular, then every closed linear manifold of* R *is regular as a subspace of* R.

Proof. If a linear manifold A of R is closed. then A is weakly closed by §68 Theorem 2. As R is regular by assumption, the unit sphere U of R is weakly compact by Theorem 2. Accordingly UA is weakly compact. Since every bounded linear functional on A may be extended by §44 Theorem 4 to a bounded linear functional on R, the unit sphere UA of the subspace A is weakly compact. Thus A is regular by Theorem 2.

Theorem 4. *If* R *is complete and the adjoint space* $\bar{R}$ *of* R *is regular, then* R *also is regular.*

Proof. If the adjoint space $\bar{R}$ is regular, then the adjoint space $\bar{\bar{\bar{R}}}$ of $\bar{\bar{R}}$ is regular by definition. As R is complete by assumption, R may be considered as a closed linear manifold of $\bar{\bar{R}}$. Therefore R also is regular by Theorem 3.

Theorem 5. *If* R *is regular and separable by the norm topology, then the adjoint space* $\bar{R}$ *of* R *is separable by the norm topology.*

Proof. If R is regular, then R may be considered by definition as the adjoint space of the adjoint space $\bar{R}$. Thus, if R is separable, then $\bar{R}$ is separable by §72 Theorem 6.

Theorem 6. *If* R *is regular, then every bounded sequence of elements* $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ *contains a weakly convergent subsequence.*

Proof. Let A be the least closed linear manifold containing all $a_\nu \in R$ $(\nu = 1, 2, \ldots)$. As R is regular by assumption, A also is regular as a subspace of R by Theorem 3. Furthermore A is separable by §71 Theorem 10. Consequently we see by Theorem 5 that the adjoint space $\bar{A}$ of A is separable too. Thus we can conclude by §74 Theorem 7 that $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ contains a weakly convergent subsequence.

Theorem 7. If R is separable and every bounded sequence of elements $x_\nu \in R$ $(\nu = 1, 2, \ldots)$ contains a weakly convergent subsequence, then R is regular.

Proof. Let $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ be dense in R by the norm topology. Then we can find by the formula §72(3) a sequence of elements $\bar{a}_\nu \in \bar{R}$ $(\nu = 1, 2, \ldots)$ such that $\bar{a}_\nu(a_\nu) \geqq \frac{1}{2}\|a_\nu\|$, $\|\bar{a}_\nu\| = 1$ $(\nu = 1, 2, \ldots)$. Let φ be an arbitrary bounded linear functional on the adjoint space $\bar{R}$. As we have for every finite number of real numbers ξ_ν $(\nu = 1, 2, \ldots, \varkappa)$

$$\left| \sum_{\nu=1}^{\varkappa} \xi_\nu \varphi(\bar{a}_\nu) \right| = \left| \varphi\left(\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu \right) \right| \leqq \|\varphi\| \left\| \sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu \right\|,$$

we can find by §52 Theorem 8 $x_\rho \in R$ $(\rho = 1, 2, \ldots)$ such that

$$\bar{a}_\nu(x_\rho) = \varphi(\bar{a}_\nu) \qquad \text{for every } \nu = 1, 2, \ldots, \rho,$$

$$\|x_\rho\| \leqq \|\varphi\| + 1 \qquad \text{for every } \rho = 1, 2, \ldots .$$

From this sequence x_ρ $(\rho = 1, 2, \ldots)$ we can select by assumption a weakly convergent subsequence x_{ρ_μ} $(\mu = 1, 2, \ldots)$, and, putting $x = \text{w-}\lim_{\mu\to\infty} x_{\rho_\mu}$ we see easily that we have for every $\nu = 1, 2, \ldots$

$$\bar{a}_\nu(x) = \lim_{\mu\to\infty} \bar{a}_\nu(x_{\rho_\mu}) = \varphi(\bar{a}_\nu).$$

Therefore there exists an element $x \in R$ such that

$$\bar{a}_\nu(x) = \varphi(\bar{a}_\nu) \qquad \text{for every } \nu = 1, 2, \ldots .$$

For an arbitrary $\bar{x} \in \bar{R}$, considering $\bar{x}, \bar{a}_1, \bar{a}_2, \ldots$ instead of $\bar{a}_1, \bar{a}_2, \ldots$, we can find likewise an element $y \in R$ such that

$$\bar{x}(y) = \varphi(\bar{x}), \qquad \bar{a}_\nu(y) = \varphi(\bar{a}_\nu) \qquad \text{for every } \nu = 1, 2, \ldots .$$

Then we have naturally $\bar{a}_\nu(x) = \bar{a}_\nu(y)$ for every $\nu = 1, 2, \ldots$. Putting $\bar{A} = \{\bar{z} : \bar{z}(x) = \bar{z}(y)\}$, we obtain obviously a weakly closed linear manifold $\bar{A}$ of the adjoint space $\bar{R}$. If $\bar{A}$ does not coincide with the whole $\bar{R}$, then we can find by §62 Theorem 9 $x_0 \in R$ such that $x_0 \neq 0$ but $\bar{x}(x_0) = 0$ for every $\bar{x} \in \bar{A}$. For such $x_0 \in R$, as $\bar{a}_\nu \in \bar{A}$ $(\nu = 1,$

$2, \ldots)$, we have $\|a_\nu - x_0\| \geqq |\bar{a}_\nu(a_\nu) - \bar{a}_\nu(x_0)| = \bar{a}_\nu(a_\nu) \geqq \frac{1}{2}\|a_\nu\|$. As $\|a_\nu - x_0\| \geqq \|x_0\| - \|a_\nu\|$, we obtain hence for every $\nu = 1, 2, \ldots$

$$\|a_\nu - x_0\| \geqq \operatorname{Max}\left\{ \|x_0\| - \|a_\nu\|, \frac{1}{2}\|a_\nu\| \right\} \geqq \frac{1}{3}\|x_0\|,$$

contradicting that a_ν $(\nu = 1, 2, \ldots)$ is dense in R. Therefore $\bar{A}$ coincides with $\bar{R}$, and hence $x = y$. Accordingly we obtain $\bar{x}(x) = \varphi(\bar{x})$. As $\bar{x} \in \bar{R}$ may be arbitrary, we conclude hence $\bar{x}(x) = \varphi(\bar{x})$ for every $\bar{x} \in \bar{R}$. Thus R is regular by definition.

§76 Uniformly convex norms

Let R be a normed space. If for any two positive numbers $\varepsilon > \varepsilon'$ we can find $\delta > 0$ such that $\|x\|, \|y\| \leqq \frac{1}{\varepsilon'}$, $|\|x\| - \|y\|| \leqq \varepsilon'$, $\|x - y\| \geqq \varepsilon$ implies $\|x\| + \|y\| \geqq \|x + y\| + \delta$, then we shall say that R is uniformly convex, or that the norm of R is uniformly convex.

Theorem 1. In order that the norm of R be uniformly convex, it is necessary and sufficient that for any $\varepsilon > 0$ we can find $\delta > 0$ such that $\|x\| = \|y\| = 1$, $\|x - y\| \geqq \varepsilon$ implies $\|x + y\| \leqq 2 - \delta$.

Proof. As the necessity is trivial, we shall prove the sufficiency. For two positive numbers $\varepsilon > \varepsilon'$ we can find by assumption $\delta_0 > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x - y\| \geqq \varepsilon'(\varepsilon - \varepsilon')$$

implies $\|x + y\| \leqq 2 - \delta_0$. If $\|y\| - \varepsilon' \leqq \|x\| \leqq \|y\| \leqq \frac{1}{\varepsilon'}$, $\|x - y\| \geqq \varepsilon$, then

$$\left\| \frac{1}{\|x\|} x - \frac{1}{\|y\|} y \right\| = \left\| \frac{1}{\|x\|}(x - y) + \frac{\|y\| - \|x\|}{\|x\|\|y\|} y \right\|$$

$$\geqq \frac{1}{\|x\|}\|x - y\| - \frac{\|y\| - \|x\|}{\|x\|} \geqq \frac{1}{\|x\|}(\varepsilon - \varepsilon') \geqq \varepsilon'(\varepsilon - \varepsilon'),$$

and hence we obtain $\left\| \frac{1}{\|x\|} x + \frac{1}{\|y\|} y \right\| \leqq 2 - \delta_0$. On the other hand,

$$\left\| \frac{1}{\|x\|} x + \frac{1}{\|y\|} y \right\| = \left\| \frac{1}{\|x\|}(x + y) - \frac{\|y\| - \|x\|}{\|x\|\|y\|} y \right\|$$

$$\geqq \frac{1}{\|x\|}\|x + y\| - \frac{\|y\| - \|x\|}{\|x\|}.$$

Consequently we have $\|x\| + \|y\| \geqq \|x + y\| + \delta_0\|x\|$. As

$$\|x\| + \|y\| \geqq \|x - y\| \geqq \varepsilon > \varepsilon' \geqq \|y\| - \|x\|,$$

we have $2\|x\| \geqq \varepsilon - \varepsilon'$, and hence we obtain

$$\|x\| + \|y\| \geqq \|x + y\| + \delta_0 \frac{\varepsilon - \varepsilon'}{2}$$

Theorem 2. Let R be uniformly convex. For a manifold $\bar{A}$ of the adjoint space $\bar{R}$, if $\lim_{\nu\to\infty}\bar{x}(a_\nu)=\bar{x}(a_0)$ for every $\bar{x}\in\bar{A}$

$$\overline{\lim_{\nu\to\infty}}\|a_\nu\|\leq\|a_0\|, \qquad \|a_0\|=\sup_{\bar{x}\in\bar{A}}\frac{|\bar{x}(a_0)|}{\|\bar{x}\|},$$

then we have $\lim_{\nu\to\infty} a_\nu = a_0$.

Proof. We can assume obviously that $a_0 \neq 0$. For each $\bar{x}\in\bar{A}$ we have by assumption

$$|\bar{x}(a_0)| = \lim_{\nu\to\infty}|\bar{x}(a_\nu)| \leq \underline{\lim}_{\nu\to\infty}\|\bar{x}\|\|a_\nu\|,$$

and hence $\|a_0\| = \sup_{\bar{x}\in\bar{A}}\frac{|\bar{x}(a_0)|}{\|\bar{x}\|} \leq \underline{\lim}_{\nu\to\infty}\|a_\nu\|$. Accordingly we conclude by assumption

$$\lim_{\nu\to\infty}\|a_\nu\| = \|a_0\|.$$

If we have not $\lim_{\nu\to\infty} a_\nu = a_0$, then we can find $\varepsilon > 0$ and a subsequence a_{ν_μ} $(\mu = 1, 2, \ldots)$ such that

$$\|a_{\nu_\mu} - a_0\| \geq \varepsilon \qquad \text{for every } \mu = 1, 2, \ldots$$

As R is uniformly convex by assumption, for a positive number

$$\varepsilon' < \operatorname{Min}\left\{\frac{1}{\|a_0\|}, \varepsilon\right\}$$

we can find by definition $\delta > 0$ such that $\|x\|, \|y\| \leq \frac{1}{\varepsilon'}$, $|\|x\|-\|y\|| \leq \varepsilon'$, $\|x-y\| \geq \varepsilon$ implies $\|x\|+\|y\| \geq \|x+y\| + \delta$. As $\|a_0\| < \frac{1}{\varepsilon'}$, and $\lim_{\mu\to\infty}\|a_{\nu_\mu}\| = \|a_0\|$, we can find μ_0 such that $\|a_{\nu_\mu}\| < \frac{1}{\varepsilon'}$, $|\|a_{\nu_\mu}\| - \|a_0\|| \leq \varepsilon'$ for every $\mu \geq \mu_0$, and hence we have for every $\mu \geq \mu_0$

$$\|a_{\nu_\mu} + a_0\| \leq \|a_{\nu_\mu}\| + \|a_0\| - \delta.$$

Accordingly we have for every $\mu \geq \mu_0$ and $\bar{x}\in\bar{R}$

$$|\bar{x}(a_{\nu_\mu}) + \bar{x}(a_0)| \leq \|\bar{x}\|(\|a_{\nu_\mu}\| + \|a_0\| - \delta).$$

As $\lim_{\mu\to\infty}\bar{x}(a_{\nu_\mu}) = \bar{x}(a_0)$ for every $\bar{x}\in\bar{A}$ by assumption, we obtain hence

$$2|\bar{x}(a_0)| \leq \|\bar{x}\|(2\|a_0\| - \delta) \qquad \text{for every } \bar{x}\in\bar{A},$$

and consequently by assumption

$$\|a_0\| = \sup_{\bar{x}\in\bar{A}}\frac{|\bar{x}(a_0)|}{\|\bar{x}\|} \leq \|a_0\| - \frac{1}{2}\delta,$$

contradicting $\delta > 0$. Therefore we have $\lim_{\nu\to\infty} a_\nu = a_0$.

Recalling the formula §72(3), as a special case of Theorem 2 we have

Theorem 3. If R is uniformly convex, then

$$\text{w-}\lim_{\nu\to\infty} a_\nu = a, \qquad \lim_{\nu\to\infty}\|a_\nu\| = \|a_0\|$$

implies $\lim_{\nu\to\infty} a_\nu = a$.

§77 Uniformly even norms

Let R be a normed space. If for any two positive numbers ε, ε' we can find $\delta > 0$ such that $\|x\|, \|y\| \geq \varepsilon'$, $\|x-y\| \leq \delta$ implies

$$\|x\| + \|y\| \leq \|x+y\| + \varepsilon\|x-y\|,$$

then we shall say that R is uniformly even, or that the norm of R is uniformly even.

Theorem 1. In order that the norm of R be uniformly even, it is necessary and sufficient that for any $\varepsilon > 0$ we can find $\delta > 0$ such that $\|x\| = 1$, $\|y\| \leq \delta$ implies

$$\|x+y\| + \|x-y\| \leq 2 + \varepsilon\|y\|.$$

Proof. If the norm of R is uniformly even, then for any positive number $\varepsilon < \frac{1}{2}$ we can find by definition a positive number $\delta < \frac{1}{2}$ such that $\|x\|, \|y\| \geq \varepsilon$, $\|x-y\| \leq 2\delta$ implies

$$\|x\| + \|y\| \leq \|x+y\| + \frac{\varepsilon}{2}\|x-y\|.$$

For such ε, δ, if $\|a\| = 1$, $\|b\| \leq \delta$, then, putting $x = a+b$, $y = a-b$, we have $\|x\|, \|y\| \geq \|a\| - \|b\| > \frac{1}{2}$, $\|x-y\| = 2\|b\| \leq 2\delta$, and consequently $\|a+b\| + \|a-b\| \leq 2 + \varepsilon\|b\|$.

Conversely, if $\|x\| = 1$, $\|y\| \leq \delta$ implies

$$\|x+y\| + \|x-y\| \leq 2 + \varepsilon\|y\|,$$

then, putting $\delta_0 = \operatorname{Min}\{\varepsilon\delta, \varepsilon\}$, we see that for $\|a\|, \|b\| \geq \varepsilon$, $\|a-b\| \leq \delta$, as $\|a+b\| \geq 2\|a\| - \|a-b\| \geq 2\varepsilon - \delta_0 \geq \varepsilon$, putting

$$x = \frac{1}{\|a+b\|}(a+b), \qquad y = \frac{1}{\|a+b\|}(a-b),$$

we have $\|x\| = 1$, $\|y\| \leq \frac{\delta_0}{\varepsilon} \leq \delta$, and hence

$$\frac{2\|a\|}{\|a+b\|} + \frac{2\|b\|}{\|a+b\|} \leq 2 + \varepsilon\frac{\|a-b\|}{\|a+b\|},$$

that is, $\|a\| + \|b\| \leq \|a+b\| + \frac{\varepsilon}{2}\|a-b\|$. Therefore R is uniformly even by definition.

Theorem 2. If R is uniformly convex, then the adjoint space $\bar{R}$ of R is uniformly even.

Proof. If R is uniformly convex, then for any $\varepsilon > 0$ we can find by Theorem 1 in §76 $\delta > 0$ such that $\|x\| = \|y\| = 1$, $\|x-y\| \geq \varepsilon$ implies $\|x+y\| \leq 2 - 2\delta$. For such ε, δ, if $\|\bar{x}\| = 1$, $\|\bar{y}\| \leq \delta$,

$\bar{x}, \bar{y} \in \bar{R}$, then we have by the definition of the adjoint norm

$$\|\bar{x}+\bar{y}\| + \|\bar{x}-\bar{y}\| = \sup_{\|x\|=\|y\|=1} \{(\bar{x}+\bar{y})(x) + (\bar{x}-\bar{y})(y)\}$$

$$= \sup_{\|x\|=\|y\|=1}\{\bar{x}(x+y)+\bar{y}(x-y)\} \leq \sup_{\|x\|=\|y\|=1}\{\|x+y\| + \|\bar{y}\|\|x-y\|\}.$$

Here, if $\|x-y\|\geq\varepsilon$, then, as $\|x-y\|\leq 2$, we have

$$\|x+y\|+\|\bar{y}\|\|x-y\| \leq 2-2\delta+2\delta = 2.$$

Thus we obtain $\|\bar{x}+\bar{y}\|+\|\bar{x}-\bar{y}\| \leq 2+\varepsilon\|\bar{y}\|$. Therefore the adjoint space $\bar{R}$ is uniformly even by Theorem 1.

Theorem 3. If R is uniformly even, then the adjoint space $\bar{R}$ is uniformly convex.

Proof. If R is uniformly even, then for any $\varepsilon>0$ we can find by Theorem 1 $\delta>0$ such that $\|x\|=1$, $\|y\|\leq\delta$, $x, y\in R$ implies

$$\|x+y\|+\|x-y\|\leq 2+\frac{\varepsilon}{4}\|y\|.$$

For such ε, δ, if $\|\bar{x}\|=\|\bar{y}\|=1$, $\|\bar{x}-\bar{y}\|\geq\varepsilon$, $\bar{x}, \bar{y}\in\bar{R}$, then we can find by the formula §72(1) $y\in R$ such that

$$\|y\|=\delta, \qquad (\bar{x}-\bar{y})(y)\geq \frac{1}{2}\varepsilon\delta,$$

and we have then $\|\bar{x}+\bar{y}\| = \sup_{\|x\|=1}\{(\bar{x}+\bar{y})(x)\}$

$$= \sup_{\|x\|=1}\{\bar{x}(x+y)+\bar{y}(x-y)-(\bar{x}-\bar{y})(y)\}$$

$$\leq \sup_{\|x\|=1}\{\|x+y\|+\|x-y\|-\frac{1}{2}\varepsilon\delta\}$$

$$\leq 2+\frac{\varepsilon}{4}\|y\|-\frac{1}{2}\varepsilon\delta = 2-\frac{1}{4}\varepsilon\delta.$$

Therefore the adjoint space $\bar{R}$ is uniformly convex by §76 Theorem 1.

Theorem 4. If R is uniformly convex and complete, then R is regular by the norm topology.

Proof. If R is uniformly convex, then the adjoint space $\bar{R}$ of R is uniformly even by Theorem 2. Thus the adjoint space $\bar{\bar{R}}$ of $\bar{R}$ is uniformly convex by Theorem 3. For an arbitrary $\bar{\bar{x}}\in\bar{\bar{R}}$, we can find by the formula §72(1) a sequence $\bar{x}_\nu\in\bar{R}$ $(\nu=1, 2, \dots)$ such that

$$\bar{\bar{x}}(\bar{x}_\nu)\geq\|\bar{\bar{x}}\|-\frac{1}{\nu}, \qquad \|\bar{x}_\nu\|=1 \qquad (\nu=1, 2, \dots).$$

For such $\bar{x}_\nu\in\bar{R}$ $(\nu=1, 2, \dots)$, as

$$\Big|\sum_{\nu=1}^{\kappa}\xi_\nu\bar{\bar{x}}(\bar{x}_\nu)\Big| = \Big|\bar{\bar{x}}\Big(\sum_{\nu=1}^{\kappa}\xi_\nu\bar{x}_\nu\Big)\Big| \leq \|\bar{\bar{x}}\|\Big\|\sum_{\nu=1}^{\kappa}\xi_\nu\bar{x}_\nu\Big\|$$

for every finite number of real numbers ξ_ν $(\nu=1, 2, \dots, \kappa)$, we can find by §52 Theorem 8 a sequence $x_\rho\in R$ $(\rho=1, 2, \dots)$ such that

$$\bar{x}_\nu(x_\rho)=\bar{\bar{x}}(\bar{x}_\nu) \qquad \text{for } \nu=1, 2, \dots, \rho,$$

$$\|x_\rho\|\leq\|\bar{\bar{x}}\|+\frac{1}{\rho} \qquad \text{for } \rho=1, 2, \dots.$$

Then we have obviously for every $\nu=1, 2, \dots$

$$\lim_{\rho\to\infty}\bar{x}_\nu(x_\rho)=\bar{\bar{x}}(\bar{x}_\nu),$$

and further

$$\overline{\lim_{\rho\to\infty}}\|x_\rho\|\leq\|\bar{\bar{x}}\|,$$

$$\|\bar{\bar{x}}\| = \sup_{\nu=1,2,\dots}|\bar{\bar{x}}(\bar{x}_\nu)|.$$

Thus, considering R as a subspace of $\bar{\bar{R}}$ by the relation:

$$x(\bar{x})=\bar{x}(x) \qquad \text{for every } \bar{x}\in\bar{R},$$

we obtain by §76 Theorem 2

$$\lim_{\rho\to\infty}\|x_\rho-\bar{\bar{x}}\|=0.$$

As R is complete by assumption, we conclude hence $\bar{\bar{x}}\in R$. Therefore R is regular by definition.

Theorem 5. If R is uniformly even and complete, then R is regular by the norm topology.

Proof. If R is uniformly even, then the adjoint space $\bar{R}$ of R is uniformly convex by Theorem 3 and complete by §72 Theorem 1. Thus $\bar{R}$ is regular by Theorem 4. As R is complete by assumption, we conclude hence by §75 Theorem 4 that R is regular.

CHAPTER XI

MODULARED SPACES

§78 Modular conditions

Let R be a linear space. A mapping m of R into $\{\xi : 0 \leqq \xi \leqq +\infty\}$ is said to be a modular on R, if m satisfies the modular conditions:

1) $m(0) = 0$,

2) $m(-x) = m(x)$ for every $x \in R$,

3) for any $x \in R$ we can find $\lambda > 0$ such that $m(\lambda x) < +\infty$,

4) $m(\xi x) = 0$ for all $\xi > 0$ implies $x = 0$,

5) $\alpha + \beta = 1$, $\alpha, \beta \geqq 0$ implies $m(\alpha x + \beta y) \leqq \alpha m(x) + \beta m(y)$,

6) $m(x) = \sup_{0 \leqq \xi < 1} m(\xi x)$ for every $x \in R$.

A linear space R associated with a modular m is called a modulared space, and $m(x)$ the modular of an element $x \in R$.

Let R be a modulared space in the sequel. We see easily by 5), 6) that $m(\xi x)$ is a convex non-decreasing function of $\xi \geqq 0$ and left hand continuous. Thus we see further by 2) that if $m(\alpha x) < +\infty$ for some $\alpha > 0$, then $m(\xi x)$ is a continuous convex function of ξ for $|\xi| \leqq \alpha$.

From 2), 5) we conclude easily

(1) $$m(x + y) \leqq \frac{1}{2} m(2x) + \frac{1}{2} m(2y),$$

(2) $$m\Big(\sum_{\nu=1}^{\kappa} \alpha_\nu x_\nu\Big) \leqq \sum_{\nu=1}^{\kappa} |\alpha_\nu| \, m(x_\nu) \qquad \text{for } \sum_{\nu=1}^{\kappa} |\alpha_\nu| \leqq 1.$$

As $x = (1-\varepsilon) y + \varepsilon\big(y + \frac{1}{\varepsilon}(x-y)\big)$, we obtain further

(3) $$m(x) \leqq m(y) + \varepsilon\, m\Big(y + \frac{1}{\varepsilon}(x-y)\Big) \qquad \text{for } 0 < \varepsilon \leqq 1.$$

A sequence of elements $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is said to be modular convergent to a limit $a \in R$ and denoted by

$$\text{m-}\lim_{\nu\to\infty} a_\nu = a,$$

if we have $\lim_{\nu\to\infty} m(\xi(a_\nu - a)) = 0$ for every $\xi \geqq 0$. Such a limit a is determined uniquely, if a_ν $(\nu = 1, 2, \ldots)$ is modular convergent. Because, if $\lim_{\nu\to\infty} m(\xi(a_\nu - a)) = \lim_{\nu\to\infty} m(\xi(a_\nu - b)) = 0$ for every $\xi \geqq 0$, then, as $m(\frac{1}{2}\xi(a - b)) \leqq \frac{1}{2} m(\xi(a_\nu - a)) + \frac{1}{2} m(\xi(a_\nu - b))$ by (1), we conclude $m(\xi(a - b)) = 0$ for every $\xi \geqq 0$, and hence $a - b = 0$ by 4).

By virtue of the formula (1), we can prove easily

Theorem 1. $\text{m-}\lim_{\nu\to\infty} a_\nu = a$, $\text{m-}\lim_{\nu\to\infty} b_\nu = b$ implies

$$\text{m-}\lim_{\nu\to\infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b.$$

$\text{m-}\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} \alpha_\nu = \alpha$ implies $\text{m-}\lim_{\nu\to\infty} \alpha_\nu a_\nu = \alpha a$.

Theorem 2. If $\text{m-}\lim_{\nu\to\infty} a_\nu = a$ and $m(8a_\nu) < +\infty$ for every $\nu = 1, 2, \ldots$, then we have $\lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty$.

Proof. On account of the formula (3) and the modular condition 5), we have obviously for $0 < \varepsilon < 1$

$$m(4a) \leqq m(4a_\nu) + \frac{\varepsilon}{2}\Big\{ m(8a_\nu) + m\Big(\frac{8}{\varepsilon}(a_\nu - a)\Big)\Big\},$$

and hence $m(4a) < +\infty$ by assumption. Furthermore we obtain likewise

$$m(2a_\nu) \leqq m(2a) + \frac{\varepsilon}{2}\Big\{ m(4a) + m\Big(\frac{4}{\varepsilon}(a_\nu - a)\Big)\Big\},$$

and hence we conclude by assumption for every $\varepsilon > 0$

$$\overline{\lim_{\nu\to\infty}}\, m(2a_\nu) \leqq m(2a) + \frac{\varepsilon}{2} m(4a).$$

Thus we have $\overline{\lim}_{\nu\to\infty} m(2a_\nu) \leqq m(2a) < +\infty$. Similarly, from

$$m(a) \leqq m(a_\nu) + \frac{\varepsilon}{2}\Big\{ m(2a_\nu) + m\Big(\frac{2}{\varepsilon}(a_\nu - a)\Big)\Big\}$$

we conclude hence $m(a) \leqq \underline{\lim}_{\nu\to\infty} m(a_\nu)$, and from

$$m(a_\nu) \leqq m(a) + \frac{\varepsilon}{2}\Big\{ m(2a) + m\Big(\frac{2}{\varepsilon}(a_\nu - a)\Big)\Big\}$$

we obtain likewise $\overline{\lim}_{\nu\to\infty} m(a_\nu) \leqq m(a)$. Therefore we have

$$\lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty.$$

Theorem 3. If $\text{m-}\lim_{\mu\to\infty} a_\mu = a$, $\text{m-}\lim_{\nu\to\infty} a_{\mu,\nu} = a_\mu$ $(\mu = 1, 2, \ldots)$, then we can find ν_μ $(\mu = 1, 2, \ldots)$ such that $\text{m-}\lim_{\mu\to\infty} a_{\mu,\nu_\mu} = a$.

Proof. We can find by assumption ν_μ $(\mu = 1, 2, \ldots)$ such that

$$m(\mu(a_{\mu,\nu_\mu} - a_\mu)) \leqq \frac{1}{\mu} \qquad \text{for every } \mu = 1, 2, \ldots .$$

Then we have by the formula (1) for $\mu \geqq 2\lambda > 0$

$$m(\lambda(a_{\mu,\nu_\mu} - a)) \leqq \frac{1}{2} m(2\lambda(a_{\mu,\nu_\mu} - a_\mu)) + \frac{1}{2} m(2\lambda(a_\mu - a))$$
$$\leqq \frac{1}{2\mu} + \frac{1}{2} m(2\lambda(a_\mu - a)),$$

and consequently $\lim_{\mu\to\infty} m(\lambda(a_{\mu,\nu_\mu} - a)) = 0$ for every $\lambda > 0$.

If a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is modular convergent, then we see easily by (1) $\lim_{\mu,\nu\to\infty} m(\lambda(a_\mu - a_\nu)) = 0$ for every $\lambda \geqq 0$. If every sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ subject to the condition:

$$\lim_{\mu,\nu\to\infty} m(\lambda(a_\mu - a_\nu)) = 0 \qquad \text{for every } \lambda \geqq 0,$$

is modular convergent, then we shall say that R is modular complete, or that the modular of R is complete.

For a sequence $a_\nu \in R$ ($\nu = 1, 2, \ldots$), if we have

$$\text{m-}\lim_{\nu\to\infty} (a_1 + a_2 + \cdots + a_\nu) = a,$$

then we shall say that a series $\sum_{\nu=1}^{\infty} a_\nu$ is modular convergent with sum a, and we shall write $a = \sum_{\nu=1}^{\infty} a_\nu$.

Theorem 4. *If R is modular complete, then for every sequence $a_\nu \in R$ ($\nu = 1, 2, \ldots$) subject to $\sup_{\nu=1,2,\ldots} m(a_\nu) < +\infty$, and for every absolutely convergent series $\sum_{\nu=1}^{\infty} \alpha_\nu$, the series $\sum_{\nu=1}^{\infty} \alpha_\nu a_\nu$ is modular convergent.*

Proof. Recalling the formula (2) we have by assumption

$$\lim_{\mu,\rho\to\infty} m\left(\sum_{\nu=\mu}^{\rho} \alpha_\nu a_\nu\right) \leqq \lim_{\mu,\rho\to\infty} \sum_{\nu=\mu}^{\rho} |\alpha_\nu| m(a_\nu) = 0.$$

Accordingly the series $\sum_{\nu=1}^{\infty} \alpha_\nu a_\nu$ is modular convergent, because R is modular complete by assumption.

§79 Modular bounded linear functionals

Let R be a modulared space. A linear functional φ on R is said to be modular bounded, if

$$\sup_{m(x)\leqq 1} |\varphi(x)| < +\infty.$$

Theorem 1. *In order that a linear functional φ on R be modular bounded, it is necessary and sufficient that we can find two positive numbers α, γ such that $\alpha\varphi(x) \leqq \gamma + m(x)$ for every $x \in R$.*

Proof. If φ is modular bounded, then we can find by definition $\alpha > 0$ such that $\sup_{m(x)\leqq 1} |\alpha\varphi(x)| \leqq 1$. If $1 \leqq m(x) < +\infty$, then we can find $\lambda > 0$ such that $m(\lambda x) = 1$, $0 < \lambda \leqq 1$, and we have by the modular condition 5) $1 = m(\lambda x) \leqq \lambda m(x)$. For such λ we have hence

$$\alpha\varphi(x) = \frac{1}{\lambda}\alpha\varphi(\lambda x) \leqq \frac{1}{\lambda} \leqq m(x).$$

Therefore we conclude $\alpha\varphi(x) \leqq 1 + m(x)$ for every $x \in R$. The inverse is evident by definition.

Theorem 2. *Let A be a linear manifold of R. If a linear functional φ on A satisfies*

$$\varphi(x) \leqq \gamma + m(x) \quad \text{for every } x \in A,$$

then we can find a linear functional ψ on R such that

$$\psi(x) = \varphi(x) \quad \text{for every } x \in A,$$

$$\psi(x) \leqq \gamma + m(x) \quad \text{for every } x \in R.$$

Proof. We consider all linear functionals φ_λ on a linear manifold A_λ ($\lambda \in \Lambda$) such that $A \subset A_\lambda$ for every $\lambda \in \Lambda$,

$$\varphi_\lambda(x) = \varphi(x) \quad \text{for } x \in A,$$

$$\varphi_\lambda(x) \leqq \gamma + m(x) \quad \text{for } x \in A_\lambda.$$

For two elements $\lambda_1, \lambda_2 \in \Lambda$, we shall write $\varphi_{\lambda_1} \subset \varphi_{\lambda_2}$, if

$$A_{\lambda_1} \subset A_{\lambda_2}, \quad \varphi_{\lambda_1}(x) = \varphi_{\lambda_2}(x) \quad \text{for every } x \in A_{\lambda_1}.$$

By virtue of Maximal Theorem, we can find a maximal system $\Gamma \subset \Lambda$ such that for any $\lambda_1, \lambda_2 \in \Gamma$ we have $\varphi_{\lambda_1} \subset \varphi_{\lambda_2}$ or $\varphi_{\lambda_2} \subset \varphi_{\lambda_1}$. For such a maximal system Γ, putting $A_0 = \sum_{\lambda\in\Gamma} A_\lambda$,

$$\psi_0(x) = \varphi_\lambda(x) \quad \text{for } x \in A_\lambda, \ \lambda \in \Gamma,$$

we obtain a linear functional ψ_0 on a linear manifold A_0 such that we have $\varphi_\lambda \subset \psi_0$ for every $\lambda \in \Gamma$, and hence there is $\lambda_0 \in \Gamma$ for which we have $\psi_0 = \varphi_{\lambda_0}$, because Γ is a maximal system subject to the indicated condition. For such $\lambda_0 \in \Gamma$ we need only prove $A_{\lambda_0} = R$.

Now we assume that $x_0 \in R$ but $x_0 \overline{\in} A_{\lambda_0}$. For every $x, y \in A_{\lambda_0}$ and positive numbers λ, μ we have by the modular condition 5)

$$\begin{aligned}
&\lambda\{\gamma + m(x + \tfrac{1}{\lambda}x_0) - \psi_0(x)\} + \mu\{\gamma + m(y - \tfrac{1}{\mu}x_0) - \psi_0(y)\} \\
&= \lambda m(x + \tfrac{1}{\lambda}x_0) + \mu m(y - \tfrac{1}{\mu}x_0) + (\lambda+\mu)\gamma - \psi_0(\lambda x + \mu y) \\
&\geqq (\lambda+\mu)\left\{m\left(\frac{\lambda x + \mu y}{\lambda+\mu}\right) + \gamma - \psi_0\left(\frac{\lambda x + \mu y}{\lambda+\mu}\right)\right\} \geqq 0,
\end{aligned}$$

and consequently

$$\lambda\{\gamma + m(x + \tfrac{1}{\lambda}x_0) - \psi_0(x)\} \geqq \mu\{\psi_0(y) - \gamma - m(y - \tfrac{1}{\mu}x_0)\}.$$

Therefore there exists a real number α such that

$$\lambda\{\gamma + m(x + \tfrac{1}{\lambda}x_0) - \psi_0(x)\} \geqq \alpha \geqq \mu\{\psi_0(y) - \gamma - m(y - \tfrac{1}{\mu}x_0)\}$$

for every $x, y \in A_{\lambda_0}$ and positive numbers λ, μ. Putting

$$\psi(x + \xi x_0) = \psi_0(x) + \xi\alpha \quad \text{for } x \in A_{\lambda_0}, \ -\infty < \xi < +\infty,$$

we see easily that ψ is a linear functional on the linear manifold generated by A_{λ_0} and x_0. Furthermore, for $\xi > 0$, putting $\lambda = \frac{1}{\xi}$, $\mu = \frac{1}{\xi}$, we have

$$\psi(x + \xi x_0) = \psi_0(x) + \xi\alpha$$

$$\leqq \psi_0(x) + \{\gamma + m(x+\xi x_0) - \psi_0(x)\} = \gamma + m(x+\xi x_0),$$

$$\psi(x - \xi x_0) = \psi_0(x) - \xi\alpha$$
$$\leqq \psi_0(x) - \{\psi_0(x) - \gamma - m(x - \xi x_0)\}$$
$$= \gamma + m(x + \xi x_0).$$

Thus we have $\psi_0 \subset \psi$ but $\psi_0 \neq \psi$, contradicting that Γ is a maximal system subject to the indicated condition. Therefore we have $A_{x_0} = R$, and hence ψ_0 satisfies our requirement.

__Theorem__ 3. __For an element__ $a \in R$, __if there is__ $\varepsilon > 0$ __such that__ $m((1+\varepsilon)a) < +\infty$, __then, putting__

$$\gamma = \alpha - m(a), \qquad \alpha = \lim_{\varepsilon \to +0} \frac{1}{\varepsilon}\{m((1+\varepsilon)a) - m(a)\},$$

__we can find a linear functional__ φ __on__ R __such that__

$$\varphi(a) = \gamma + m(a),$$

$$\varphi(x) \leqq \gamma + m(x) \quad \text{for every } x \in R.$$

__Proof__. Putting $\varphi_0(\xi a) = \xi\alpha$ $(-\infty < \xi < +\infty)$, we obtain obviously a linear functional φ_0 on the linear manifold generated by a single element a. As $m(\xi a)$ is a convex function of ξ, we have

$$\frac{m(a) - m(\xi a)}{1 - \xi} \begin{cases} \leqq \alpha & \text{for } 0 \leqq \xi < 1, \\ \geqq \alpha & \text{for } \xi > 1, \end{cases}$$

and hence $m(a) - m(\xi a) \leqq \alpha(1 - \xi)$ for every $\xi \geqq 0$. Therefore

$$\varphi_0(\xi a) = \xi\alpha \leqq \gamma + m(\xi a) \quad \text{for every } \xi \geqq 0.$$

Especially we obtain $\gamma \geqq 0$, putting $\xi = 0$. As $\alpha \geqq 0$, we have hence

$$\varphi_0(\xi a) \leqq \gamma + m(\xi a) \quad \text{for every real number } \xi,$$

and further $\varphi_0(a) = \alpha = \gamma + m(a)$. Then we can find by Theorem 2 a linear functional φ on R such that

$$\varphi(a) = \varphi_0(a) = \gamma + m(a),$$
$$\varphi(x) \leqq \gamma + m(x) \quad \text{for every } x \in R.$$

§80 __Modular adjoint spaces__

Let R be a modulared space and $\overline{R}$ the totality of modular bounded linear functionals on R. We see easily by definition that $\overline{R}$ constitutes a linear space. Now, putting

(1) $\quad \overline{m}(\overline{x}) = \sup_{x \in R}\{\overline{x}(x) - m(x)\} \quad$ for every $\overline{x} \in \overline{R}$,

we shall prove that $\overline{m}$ satisfies the modular conditions in §78.

It is obvious by definition

1') $\quad \overline{m}(0) = 0$,

2') $\quad \overline{m}(-\overline{x}) = \overline{m}(\overline{x}) \quad$ for every $\overline{x} \in \overline{R}$.

Recalling §79 Theorem 1, we obtain immediately by definition

3') $\quad$ for any $\overline{x} \in \overline{R}$ we can find $\alpha > 0$ such that $\overline{m}(\alpha\overline{x}) < +\infty$.

If $\overline{m}(\xi\overline{x}) = 0$ for every $\xi > 0$, then we have by definition

$$\xi\overline{x}(x) \leqq m(x) \quad \text{for every } x \in R \text{ and } \xi > 0.$$

On the other hand, for any $x \in R$ we can find $\alpha > 0$ such that $m(\alpha x) < +\infty$, and for such α we have hence for every $\xi > 0$

$$\overline{x}(x) \leqq \frac{m(\alpha x)}{\xi\alpha}, \qquad \overline{x}(-x) \leqq \frac{m(\alpha x)}{\xi\alpha}.$$

Making ξ tend to $+\infty$, we obtain therefore $\overline{x}(x) = 0$. Thus we have

4') $\quad \overline{m}(\xi\overline{x}) = 0$ for every $\xi > 0$ implies $\overline{x} = 0$.

For $\alpha + \beta = 1$, $\alpha, \beta \geqq 0$ we have by definition

$$\overline{m}(\alpha\overline{x} + \beta\overline{y}) = \sup_{x \in R}\{\alpha\overline{x}(x) + \beta\overline{y}(x) - m(x)\}$$
$$\leqq \alpha \sup_{x \in R}\{\overline{x}(x) - m(x)\} + \beta \sup_{x \in R}\{\overline{y}(x) - m(x)\}.$$

Therefore we have

5') $\quad \alpha + \beta = 1$, $\alpha, \beta \geqq 0$ implies $\overline{m}(\alpha\overline{x} + \beta\overline{y}) \leqq \alpha\overline{m}(\overline{x}) + \beta\overline{m}(\overline{y})$.

For every $\overline{x} \in \overline{R}$ we have by definition

$$\overline{m}(\overline{x}) = \sup_{\overline{x}(x) \geqq 0}\left(\sup_{0 \leqq \xi < 1}\{\xi\overline{x}(x) - m(x)\}\right) = \sup_{0 \leqq \xi < 1}\overline{m}(\xi\overline{x}),$$

that is, we have

6') $\quad \overline{m}(\overline{x}) = \sup_{0 \leqq \xi < 1}\overline{m}(\xi\overline{x}) \quad$ for every $\overline{x} \in \overline{R}$.

Thus $\overline{m}$ is by definition a modular on $\overline{R}$. This modular $\overline{m}$ is called the __adjoint modular__ of m, and the linear space $\overline{R}$ associated with the adjoint modular $\overline{m}$ is called the __modular adjoint space__ of R. For the adjoint modular $\overline{m}$ we have obviously by the definition (1)

(2) $\quad |\overline{x}(x)| \leqq \overline{m}(\overline{x}) + m(x) \quad$ for $\overline{x} \in \overline{R}$, $x \in R$.

__Theorem__ 1. __The modular adjoint space__ $\overline{R}$ __of__ R __is modular complete.__

__Proof__. For a sequence $\overline{a}_\nu \in \overline{R}$ $(\nu = 1, 2, \dots)$, if

$$\lim_{\mu,\nu \to \infty}\overline{m}(\xi(\overline{a}_\mu - \overline{a}_\nu)) = 0 \quad \text{for every } \xi \geqq 0,$$

then, since we have by the formula (2) for every $x \in R$ and $\gamma \geqq 0$

$$\xi \bar{a}_\mu(\gamma x) - \xi \bar{a}_\nu(\gamma x) \leqq \bar{m}(\xi(\bar{a}_\mu - \bar{a}_\nu)) + m(\gamma x),$$

and since for any $x \in R$ we can find by the modular condition 3) $\gamma > 0$ such that $m(\gamma x) < +\infty$, we have hence for such $\gamma > 0$

$$\overline{\lim_{\mu,\nu \to +\infty}} |\bar{a}_\mu(x) - \bar{a}_\nu(x)| \leqq \frac{m(\gamma x)}{\xi \gamma} \qquad \text{for every } \xi > 0.$$

Making ξ tend to $+\infty$, we obtain consequently

$$\lim_{\mu,\nu \to +\infty} |\bar{a}_\mu(x) - \bar{a}_\nu(x)| = 0.$$

Therefore, putting $\varphi(x) = \lim_{\nu \to \infty} \bar{a}_\nu(x)$ for every $x \in R$, we obtain a linear functional φ on R and we have for every $\xi > 0$

$$\xi \varphi(x) - \xi \bar{a}_\nu(x) \leqq \underline{\lim}_{\mu \to \infty} \bar{m}(\xi(\bar{a}_\mu - \bar{a}_\nu)) + m(x).$$

Therefore we see easily by §79 Theorem 1 that such φ is modular bounded, that is, $\varphi \in \bar{R}$, and we have by the definition (1) for every $\nu = 1, 2, \dots$

$$\bar{m}(\xi(\varphi - \bar{a}_\nu)) \leqq \underline{\lim}_{\mu \to \infty} \bar{m}(\xi(\bar{a}_\mu - \bar{a}_\nu)).$$

Consequently we obtain $\lim_{\nu \to \infty} \bar{m}(\xi(\varphi - \bar{a}_\nu)) = 0$ for every $\xi > 0$.

<u>Theorem</u> 2. <u>For every</u> $x \in R$ <u>we have</u>

$$m(x) = \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(x) - \bar{m}(\bar{x}) \}.$$

<u>Proof</u>. We have obviously by the formula (2) for every $a \in R$

$$m(a) \geqq \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(a) - \bar{m}(\bar{x}) \}.$$

For an element $a \in R$, if there is $\varepsilon > 0$ such that $m((1+\varepsilon)a) < +\infty$, then we have by §79 Theorem 3

$$(*) \qquad m(a) = \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(a) - \bar{m}(\bar{x}) \}.$$

Furthermore, if $m(\lambda a) < +\infty$ for every positive number $\lambda < 1$, then

$$m(\lambda a) = \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(\lambda a) - \bar{m}(\bar{x}) \}$$

for $0 \leqq \lambda < 1$, as proved just above, and hence we obtain by the modular condition 6)

$$m(a) = \sup_{0 \leqq \lambda < 1} m(\lambda a) = \sup_{0 \leqq \lambda < 1} \sup_{\bar{x}(a) \geqq 0} \{ \bar{x}(\lambda a) - \bar{m}(\bar{x}) \}$$
$$= \sup_{\bar{x}(x) \geqq 0} \{ \bar{x}(a) - \bar{m}(\bar{x}) \} = \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(a) - \bar{m}(\bar{x}) \}.$$

Therefore we need only prove (*) in the case where we can find a positive number $\varepsilon < 1$ such that $m((1-\varepsilon)a) = +\infty$. For this purpose, we set $\varphi(\xi a) = \xi$ $(-\infty < \xi < +\infty)$. Then we have for every $\gamma > 0$

$$\gamma \varphi(\xi a) \leqq \gamma(1-\varepsilon) + m(\xi a) \qquad (-\infty < \xi < +\infty),$$

because $m((1-\varepsilon)a) = +\infty$ by assumption. Accordingly we can find by §79 Theorem 2 $\bar{a}_\gamma \in \bar{R}$ such that

$$\bar{a}_\gamma(\xi a) = \gamma \varphi(\xi a) \qquad (-\infty < \xi < +\infty)$$

$$\bar{a}_\gamma(x) \leqq \gamma(1-\varepsilon) + m(x) \qquad \text{for every } x \in R.$$

For such $\bar{a}_\gamma \in \bar{R}$ we have by the definition (1)

$$\bar{m}(\bar{a}_\gamma) \leqq \gamma(1-\varepsilon),$$

and hence $\bar{a}_\gamma(a) - \bar{m}(\bar{a}_\gamma) \geqq \gamma \varphi(a) - \gamma(1-\varepsilon) = \gamma\varepsilon$. Therefore we have for every $\gamma > 0$

$$\sup_{\bar{x} \in \bar{R}} \{ \bar{x}(a) - \bar{m}(\bar{x}) \} \geqq \gamma \varepsilon.$$

Making γ tend to $+\infty$, we conclude from this relation

$$\sup_{\bar{x} \in \bar{R}} \{ \bar{x}(a) - \bar{m}(\bar{x}) \} = +\infty = m(a),$$

as we wish to prove.

<u>Theorem</u> 3. <u>For a manifold</u> $\bar{A}$ <u>of the modular adjoint space</u> $\bar{R}$ <u>of</u> R, <u>if</u>

$$\lim_{\nu \to \infty} \bar{x}(a_\nu) = \bar{x}(a) \qquad \underline{\text{for every}}\ \bar{x} \in \bar{A},$$
$$m(a) = \sup_{\bar{x} \in \bar{A}} \{ \bar{x}(a) - \bar{m}(\bar{x}) \},$$

<u>then we have</u> $m(a) \leqq \underline{\lim}_{\nu \to \infty} m(a_\nu)$.

<u>Proof</u>. For any $\bar{x} \in \bar{A}$ we have by the formula (2)

$$\bar{x}(a_\nu) - \bar{m}(\bar{x}) \leqq m(a_\nu) \qquad \text{for every } \nu = 1, 2, \dots .$$

From this relation we conclude by assumption

$$\bar{x}(a) - \bar{m}(\bar{x}) \leqq \underline{\lim}_{\nu \to \infty} m(a_\nu) \qquad \text{for every } \bar{x} \in \bar{A},$$

and hence further by assumption

$$m(a) \leqq \underline{\lim}_{\nu \to \infty} m(a_\nu).$$

Recalling Theorem 2, we obtain as a special case of Theorem 3

<u>Theorem</u> 4. <u>If</u> $\lim_{\nu \to \infty} \bar{x}(a_\nu) = \bar{x}(a)$ <u>for every</u> $\bar{x} \in \bar{R}$, <u>then we have</u>

$$m(a) \leqq \underline{\lim}_{\nu \to \infty} m(a_\nu).$$

For a modulared space R, every linear manifold A of R may be considered itself as a modulared space associated with the same modular of R. In this sense, A will be called a <u>subspace</u> of R.

As the modular adjoint space $\bar{R}$ of R also is a modulared space by the adjoint modular $\bar{m}$, we can consider further the modular adjoint space $\bar{\bar{R}}$ of $\bar{R}$ with the adjoint modular $\bar{\bar{m}}$ of $\bar{m}$. Then, by virtue of Theorem 2, R may be considered as a subspace of $\bar{\bar{R}}$ by the relation:

$$x(\bar{x}) = \bar{x}(x) \qquad \text{for every } x \in R \text{ and } \bar{x} \in \bar{R}.$$

If R coincides with the whole $\bar{\bar{R}}$ in this sense, then we shall say that R is <u>regular</u>, or that the modular m of R is <u>regular</u>.

§81 Modular norms

Let R be a modulared space. For every $\lambda > 0$, putting

(1) $U_\lambda = \{ x : m(x) \leqq \lambda \}$,

we obtain a scalar-closed symmetric convex vicinity in R. In fact, for every $x \in R$, as $\lim_{\xi \to 0} m(\xi x) = 0$, we see easily that U_λ is a vicinity in R. It is evident by the modular condition 2) that U_λ is symmetric. For every $x, y \in U_\lambda$, $\alpha + \beta = 1$, $\alpha, \beta \geqq 0$ implies by the modular condition 5) $m(\alpha x + \beta y) \leqq \alpha m(x) + \beta m(y) \leqq \lambda$, that is, $\alpha x + \beta y \in U_\lambda$, and hence U_λ is convex. Furthermore we see at once by the modular condition 6) that U_λ is scalar-closed.

The vicinity U_λ defined by (1) will be called λ sphere of R. Concerning spheres of R we have obviously by definition

(2) $U_\rho \supset U_\lambda$ for $\rho > \lambda > 0$.

For a positive number $\varepsilon < 1$ we have by the modular condition 5)

$$m(\varepsilon x) \leqq \varepsilon m(x) \quad \text{for every } x \in R.$$

Thus we have

(3) $\varepsilon U_\lambda \subset U_{\varepsilon\lambda}$ for $0 < \varepsilon \leqq 1$.

From this relation we conclude easily

(4) $U_{\alpha\lambda} \subset \alpha U_\lambda$ for $\alpha \geqq 1$.

For every $\lambda > 0$, as U_λ is a symmetric convex vicinity, we obtain uniquely a linear topology $\wp^m$ on R such that U_λ is a basis of $\wp^m$. Furthermore we see by the relations (2) and (3) that this linear topology $\wp^m$ is the same for every $\lambda > 0$. This same linear topology $\wp^m$ is called the modular topology of R. Thus we have

Theorem 1. *Every sphere U_λ of R is itself a basis of the modular topology of R for all $\lambda > 0$.*

Theorem 2. *The modular topology of R is of single vicinity, convex, and separative.*

Proof. The modular topology $\wp^m$ is obviously of single vicinity by Theorem 1, and furthermore convex, because U_λ is symmetric and convex. For any element $x \neq 0$, we can find by the modular condition 4) $\alpha > 0$ such that $m(\alpha x) > 0$. Then, for a positive number $\lambda < m(\alpha x)$ we have obviously $\alpha x \bar{\in} U_\lambda$, and hence $x \bar{\in} \frac{1}{\alpha} U_\lambda$. As U_λ is a basis of the modular topology $\wp^m$, we see that $\wp^m$ is separative.

Theorem 3. *We have $m\text{-}\lim_{\nu\to\infty} a_\nu = a$ if and only if $\lim_{\nu\to\infty} a_\nu = a$ by the modular topology $\wp^m$ of R.*

Proof. If $m\text{-}\lim_{\nu\to\infty} a_\nu = a$, then for any $\xi > 0$ we have by definition $\lim_{\nu\to\infty} m(\xi(a_\nu - a)) = 0$, and hence for any $\lambda > 0$ we can find ν_0 such that $\xi(a_\nu - a) \in U_\lambda$ for $\nu \geqq \nu_0$, that is, $a_\nu \in \frac{1}{\xi} U_\lambda + a$ for $\nu \geqq \nu_0$. As U_λ is a basis of $\wp^m$, we have hence $\lim_{\nu\to\infty} a_\nu = a$ by the modular topology $\wp^m$.

Conversely, if $\lim_{\nu\to\infty} a_\nu = a$ by $\wp^m$, then for any two positive numbers ξ, λ we can find ν_0 such that $a_\nu \in \frac{1}{\xi} U_\lambda + a$ for $\nu \geqq \nu_0$, and hence $\xi(a_\nu - a) \in U_\lambda$ for $\nu \geqq \nu_0$, that is, $m(\xi(a_\nu - a)) \leqq \lambda$ for $\nu \geqq \nu_0$. Therefore we have $\lim_{\nu\to\infty} m(\xi(a_\nu - a)) = 0$ for every $\xi > 0$, that is, we have $m\text{-}\lim_{\nu\to\infty} a_\nu = a$.

For every $\lambda > 0$, as U_λ is symmetric and convex, we see easily by Theorems 1 and 2 that the pseudo-norm of U_λ is a norm on R and the modular topology of R coincides with the norm topology by this norm. The pseudo-norm of 1 sphere U_1 of R will be called the modular norm of R and denoted by $|||x|||$ $(x \in R)$.

Concerning the modular norm $|||x|||$ we have obviously by definition

(5) $\{ x : |||x||| \leqq 1 \} = \{ x : m(x) \leqq 1 \}$.

From this relation we conclude easily

(6) $|||x||| = \inf_{m(\xi x) \leqq 1} \frac{1}{|\xi|}$ for every $x \in R$.

Theorem 4. *$|||x||| \leqq 1$ implies $m(x) \leqq |||x|||$, and $|||x||| > 1$ implies $m(x) \geqq |||x|||$.*

Proof. If $0 < |||x||| \leqq 1$, then we can find $\xi \geqq 1$ such that we have $|||\xi x||| = 1$. Then we have by (5) $m(\xi x) \leqq 1$, and hence by the modular condition 5) $\xi m(x) \leqq m(\xi x) \leqq 1$. Consequently we obtain $m(x) \leqq \frac{1}{\xi} = |||x|||$.

If $|||x||| > 1$, then for any ξ subject to $1 < \xi < |||x|||$, we have obviously $|||\frac{1}{\xi} x||| > 1$, and hence by the formula (5) and the modular condition 5) $\frac{1}{\xi} m(x) \geqq m(\frac{1}{\xi}x) > 1$, that is, $1 < \xi < |||x|||$ implies $m(x) > \xi$. Therefore we have $m(x) \geqq |||x|||$, if $|||x||| > 1$.

As the norm topology by the modular norm coincides with the modular topology of R, we have obviously by Theorem 3

Theorem 5. We have $m\text{-}\lim_{\nu\to\infty} a_\nu = a$ if and only if

$$\lim_{\nu\to\infty} |||a_\nu - a||| = 0.$$

Therefore we have further

Theorem 6. A modulared space R is modular complete, if and only if the modular norm of R is complete.

Recalling the definition in §79, we see at once that a linear functional φ on R is modular bounded, if and only if φ is bounded by the modular norm. Therefore we have

Theorem 7. The modular adjoint space of R coincides with the adjoint space of R by the modular norm.

Therefore we have by Theorem 5

Theorem 8. $m\text{-}\lim_{\nu\to\infty} a_\nu = a$ implies $\lim_{\nu\to\infty} \bar{x}(a_\nu) = \bar{x}(a)$ for $\bar{x} \in \bar{R}$.

§82 Quotient spaces

Let R be a modulared space and A a linear manifold of R. For the quotient space R/A, if we set

$$(1) \qquad m(X) = \sup_{0 \leqq \lambda < 1} \{ \inf_{x \in X} m(\lambda x) \} \qquad \text{for } X \in R/A,$$

then we see easily that $m(X)$ satisfies the modular conditions except 4). If A is closed by the modular topology $\mathcal{V}^m$ of R, then $m(X)$ satisfies furthermore the modular condition 4). In fact, if for an element $X_0 \in R/A$, $m(\xi X_0) = 0$ for every $\xi \geqq 0$, then we can find by definition a sequence of elements $x_\nu \in X_0$ $(\nu = 1, 2, \ldots)$ such that

$$m(\nu x_\nu) < \frac{1}{\nu} \qquad \text{for every } \nu = 1, 2, \ldots,$$

and hence $m\text{-}\lim_{\nu\to\infty} x_\nu = 0$ by definition. As A is closed by assumption, every residue class $X \in R/A$ is closed by the modular topology $\mathcal{V}^m$ too, and hence we obtain $0 \in X_0$ by Theorem 2 in §81, that is, $X_0 = A$.

Therefore, if a linear manifold A of R is closed by $\mathcal{V}^m$, then $m(X)$

defined by (1) is a modular on the quotient space R/A. This modular $m(X)$ will be called the relative modular of the modular m of R.

Concerning the relative modular we have obviously by (1)

$$(2) \qquad m(X) \leqq \inf_{x \in X} m(x) \leqq m((1+\varepsilon)X) \qquad \text{for every } \varepsilon > 0,$$

$$(3) \qquad \sup_{m(X) \leqq 1} m(\alpha X) \leqq \sup_{m(x) \leqq 1} m(\beta x) \qquad \text{for } 0 < \alpha < \beta,$$

$$(4) \qquad \inf_{m(X) \geqq 1} m(\alpha X) \geqq \inf_{m(x) \geqq 1} m(\beta x) \qquad \text{for } \alpha > \beta > 0.$$

Furthermore we conclude easily from (2)

Theorem 1. If $m((1+\varepsilon)X) < +\infty$ for some $\varepsilon > 0$, then we have

$$m(X) = \inf_{x \in X} m(x).$$

Theorem 2. The modular norm of the quotient space R/A is the relative norm of the modular norm of R, that is,

$$|||X||| = \inf_{x \in X} |||x||| \qquad \text{for every } X \in R/A.$$

Proof. Putting $\|X\| = \inf_{x \in X} |||x|||$ for $X \in R/A$, we obtain the relative pseudo-norm of $|||x|||$ on R/A, as defined in §51. For any $\lambda > \|X\|$, we have then $1 > \|\frac{1}{\lambda}X\|$, and hence we can find $x \in \frac{1}{\lambda}X$ such that $|||x||| < 1$, which yields $m(x) \leqq 1$ by the formula §81(5). From this relation we conclude by (2) that $m(\frac{1}{\lambda}X) \leqq 1$, and hence $|||\frac{1}{\lambda}X||| \leqq 1$, that is, $\lambda \geqq |||X|||$. Therefore we have

$$\|X\| \geqq |||X||| \qquad \text{for every } X \in R/A.$$

On the other hand, for any $\lambda > |||X|||$, considering λ' subject to $\lambda > \lambda' > |||X|||$, we have $|||\frac{1}{\lambda'}X||| < 1$, and hence we conclude likewise by (1) and §81(5) that we can find $x \in \frac{1}{\lambda}X$ such that $m(x) \leqq 1$, which yields $|||x||| \leqq 1$ by §81 Theorem 4. Thus we have $\|\frac{1}{\lambda}X\| \leqq 1$, and consequently $\|X\| \leqq \lambda$. Therefore we conclude $\|X\| \leqq |||X|||$ for every $X \in R/A$.

Theorem 3. If a linear manifold $\bar{A}$ of the modular adjoint space $\bar{R}$ of R is weakly closed, then, putting

$$A = \{ x : \bar{x}(x) = 0 \quad \text{for every } \bar{x} \in \bar{A} \},$$

$\bar{A}$ coincides with the modular adjoint space of the quotient space R/A as a modulared space.

Proof. Recalling §73 Theorem 3, we see easily by Theorem 2 that $\bar{A}$ coincides with the adjoint space of R/A by the modular norm. Thus we conclude by §81 Theorem 7 that $\bar{A}$ coincides with the modular adjoint space of R/A. Therefore we need only prove that the adjoint modular $\bar{m}$ of m in $\bar{A}$ coincides with the adjoint modular of the relative modular $m(X)$ ($X \in R/A$), that is,

$$\bar{m}(\bar{x}) = \sup_{X \in R/A} \{\bar{x}(X) - m(X)\} \quad \text{for every } \bar{x} \in \bar{A}.$$

In fact, we have by Theorem 1 and the modular condition 6)

$$\sup_{X \in R/A} \{\bar{x}(X) - m(X)\} = \sup_{X \in R/A} \{\bar{x}(X) - \inf_{x \in X} m(x)\}$$
$$= \sup_{X \in R/A} \sup_{x \in X} \{\bar{x}(x) - m(x)\} = \bar{m}(\bar{x}).$$

Theorem 4. *For a finite number of elements $\bar{a}_\nu \in \bar{R}$ and real numbers α_ν ($\nu = 1, 2, \dots, \kappa$), if $\sum_{\nu=1}^{\kappa} \xi_\nu \bar{a}_\nu = 0$ implies $\sum_{\nu=1}^{\kappa} \xi_\nu \alpha_\nu = 0$, and for a positive number γ, if*

$$\sum_{\nu=1}^{\kappa} \xi_\nu \alpha_\nu \leqq \gamma + \bar{m}\Big(\sum_{\nu=1}^{\kappa} \xi_\nu \bar{a}_\nu\Big)$$

for every finite number of real numbers ξ_ν ($\nu = 1, 2, \dots, \kappa$), then for any positive number $\varepsilon < 1$ we can find an element $x \in R$ such that

$$m((1-\varepsilon)x) \leqq \gamma, \qquad \bar{a}_\nu(x) = \alpha_\nu \qquad (\nu = 1, 2, \dots, \kappa).$$

Proof. Putting $A = \{x : \bar{a}_\nu(x) = 0 \text{ for every } \nu = 1, 2, \dots, \kappa\}$, we obtain a closed linear manifold A of R, and the quotient space R/A is by §46 Theorem 3 finite-dimensional, and further the modular adjoint space of R/A is composed of all linear combination from $\bar{a}_\nu$ ($\nu = 1, 2, \dots, \kappa$). As $\sum_{\nu=1}^{\kappa} \xi_\nu \bar{a}_\nu = 0$ implies $\sum_{\nu=1}^{\kappa} \xi_\nu \alpha_\nu = 0$ by assumption, we can find by §45 Theorem 4 an element $X_0 \in R/A$ such that

$$\bar{a}_\nu(X_0) = \alpha_\nu \qquad \text{for every } \nu = 1, 2, \dots, \kappa.$$

For such $X_0 \in R/A$ we have by the second assumption

$$\sum_{\nu=1}^{\kappa} \xi_\nu \bar{a}_\nu(X_0) = \sum_{\nu=1}^{\kappa} \xi_\nu \alpha_\nu \leqq \gamma + \bar{m}\Big(\sum_{\nu=1}^{\kappa} \xi_\nu \bar{a}_\nu\Big)$$

for every finite number of real numbers ξ_ν ($\nu = 1, 2, \dots, \kappa$), and hence we conclude by §80 Theorem 2 that $m(X_0) \leqq \gamma$. Therefore for any positive number $\varepsilon < 1$ we can find by the definition (1) $x_0 \in X_0$ such that

$$m((1-\varepsilon)x_0) \leqq \gamma,$$

and for such $x_0 \in X_0$ we have obviously for every $\nu = 1, 2, \dots,$

$$\bar{a}_\nu(x_0) = \bar{a}_\nu(X_0) = \alpha_\nu.$$

§83 Associated norms

Let R be a modulared space. The modular adjoint space $\bar{R}$ of R is itself a modulared space, and hence the modular norm is defined on $\bar{R}$. We obtain thus a norm on R as the adjoint norm of the modular norm of $\bar{R}$, that is, as

(1) $$\|x\| = \sup_{|||\bar{x}||| \leqq 1} |\bar{x}(x)| \qquad \text{for every } x \in R.$$

This norm $\|x\|$ ($x \in R$) is called the *associated norm* of R. By virtue of the formula §81(5) we also can define the associated norm as

(2) $$\|x\| = \sup_{\bar{m}(\bar{x}) \leqq 1} |\bar{x}(x)| \qquad \text{for every } x \in R.$$

From (1) we conclude immediately

(3) $$|\bar{x}(x)| \leqq \|x\| \, |||\bar{x}||| \qquad (x \in R,\ \bar{x} \in \bar{R}).$$

Theorem 1. *The associated norm is equivalent to the modular norm and we have for every $x \in R$*

$$|||x||| \leqq \|x\| \leqq 2|||x|||.$$

Proof. If $|||x||| \leqq 1$, then we have $m(x) \leqq 1$ by the formula §81 (5), and hence by the formula §80(2)

$$\|x\| = \sup_{\bar{m}(\bar{x}) \leqq 1} |\bar{x}(x)| \leqq 1 + m(x) \leqq 2.$$

Thus we conclude $\|x\| \leqq 2|||x|||$ for every $x \in R$.

We suppose now $\|x\| \leqq 1$. If $1 \leqq \bar{m}(\bar{x}) < +\infty$, then we can find $\lambda > 0$ such that $\bar{m}(\lambda \bar{x}) = 1$, $0 < \lambda \leqq 1$, and then we have by the formula (2) and the modular condition 5)

$$\bar{x}(x) - \bar{m}(\bar{x}) = \frac{1}{\lambda} \lambda \bar{x}(x) - \bar{m}\Big(\frac{1}{\lambda} \lambda \bar{x}\Big) \leqq \frac{1}{\lambda} - \frac{1}{\lambda} \bar{m}(\lambda \bar{x}) = 0.$$

Thus $\bar{m}(\bar{x}) \geqq 1$ implies $\bar{x}(x) - \bar{m}(\bar{x}) \leqq 0$. Accordingly we have by §80 Theorem 2

$$m(x) = \sup_{\bar{m}(\bar{x}) \leqq 1} \{\bar{x}(x) - \bar{m}(\bar{x})\} \leqq \sup_{\bar{m}(\bar{x}) \leqq 1} \bar{x}(x) = \|x\| \leqq 1,$$

and hence $|||x||| \leqq 1$ by the formula §80(2). Therefore we conclude

$$|||x||| \leqq \|x\| \qquad \text{for every } x \in R.$$

Theorem 2. *The adjoint norm of the modular norm coincides with the associated norm of the modular adjoint space $\bar{R}$ of R, that is,*

$$\|\bar{x}\| = \sup_{|||x||| \leqq 1} |\bar{x}(x)| \qquad \text{for every } \bar{x} \in \bar{R}.$$

Proof. Let $\bar{\bar{R}}$ be the modular adjoint space of $\bar{R}$ and $\bar{\bar{m}}$ the modular of $\bar{\bar{R}}$. Then we have by the definition (2)

$$\|\bar{x}\| = \sup_{\bar{\bar{m}}(\bar{\bar{x}}) \leqq 1} |\bar{\bar{x}}(\bar{x})| \qquad \text{for every } \bar{x} \in \bar{R}.$$

By virtue of §80 Theorem 2, R may be considered as a subspace of $\bar{\bar{R}}$ by the relation: $x(\bar{x}) = \bar{x}(x)$ for every $\bar{x} \in \bar{R}$. Thus we have obviously

$$\|\bar{x}\| \geqq \sup_{m(x) \leqq 1} |\bar{x}(x)| \qquad \text{for every } \bar{x} \in \bar{R}.$$

For an element $\bar{x} \in \bar{R}$, we can find obviously a sequence of elements $\bar{\bar{x}}_\nu \in \bar{\bar{R}}$ ($\nu = 1, 2, \dots$) such that

$$\bar{\bar{x}}_\nu(\bar{x}) > \|\bar{x}\| - \frac{1}{\nu}, \qquad \bar{\bar{m}}(\bar{\bar{x}}_\nu) \leqq 1 \qquad (\nu = 1, 2, \dots).$$

Then, as $\xi \bar{\bar{x}}_\nu(\bar{x}) \leqq \bar{\bar{m}}(\bar{\bar{x}}_\nu) + \bar{m}(\xi \bar{x}) \leqq 1 + \bar{m}(\xi \bar{x})$ for every real number ξ, we can find by §82 Theorem 4 $x_\nu \in R$ such that

$$\bar{\bar{x}}_\nu(\bar{x}) = \bar{x}(x_\nu), \qquad m\left(\left(1 - \frac{1}{\nu}\right) x_\nu\right) \leqq 1.$$

For such x_ν ($\nu = 1, 2, \dots$) we have

$$\bar{x}\left(\left(1 - \frac{1}{\nu}\right) x_\nu\right) = \left(1 - \frac{1}{\nu}\right) \bar{\bar{x}}_\nu(\bar{x}) \geqq \left(1 - \frac{1}{\nu}\right)\left(\|\bar{x}\| - \frac{1}{\nu}\right).$$

Thus we have for every $\nu = 1, 2, \dots$

$$\sup_{m(x) \leqq 1} |\bar{x}(x)| \geqq \left(1 - \frac{1}{\nu}\right)\left(\|\bar{x}\| - \frac{1}{\nu}\right),$$

and consequently, making ν tend to ∞, we obtain

$$\sup_{m(x) \leqq 1} |\bar{x}(x)| \geqq \|\bar{x}\|.$$

Therefore we have by the formula §81(5)

$$\|\bar{x}\| = \sup_{m(x) \leqq 1} |\bar{x}(x)| = \sup_{|||x||| \leqq 1} |\bar{x}(x)|.$$

Recalling the formula §72(3) we obtain obviously by Theorem 2

(4) $$|||x||| = \sup_{\|\bar{x}\| \leqq 1} |\bar{x}(x)| \qquad \text{for every } x \in R.$$

Consequently we have naturally

(5) $$|\bar{x}(x)| \leqq |||x||| \, \|\bar{x}\| \qquad (x \in R, \ \bar{x} \in \bar{R}).$$

Concerning the associated norm, I. Amamiya obtained

(6) $$\|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi}.$$

In fact, putting $\alpha = \inf_{\xi > 0} \frac{1}{\xi}(1 + m(\xi x))$, as $\xi \alpha \leqq 1 + m(\xi x)$ for every real number ξ, we can find by §79 Theorem 2 $\bar{x}_0 \in \bar{R}$ such that $\bar{x}_0(x) = \alpha$ and $\bar{m}(\bar{x}_0) \leqq 1$, and consequently $\|x\| \geqq \alpha$ by definition. On the other hand, we have by definition

$$\|\xi x\| = \sup_{\bar{m}(\bar{x}) \leqq 1} |\bar{x}(\xi x)| \leqq 1 + m(\xi x),$$

and hence $\|x\| \leqq \alpha$.

§84 Simpleness

Let R be a modulared space. If $m(x) = 0$ implies $x = 0$, then we shall say that R is simple, or that the modular m of R is simple.

Theorem 1. *In order that the modular adjoint space $\bar{R}$ of R be simple, it is necessary and sufficient that*

$$\lim_{\xi \to 0} \frac{m(\xi x)}{\xi} = 0 \qquad \text{for every } x \in R.$$

Proof. As $m(\xi x)$ is a convex function of ξ, there exists obviously the limit $\lim_{\xi \to +0} \frac{m(\xi x)}{\xi}$. If $\lim_{\xi \to +0} \frac{m(\xi a)}{\xi} \geqq \gamma$ for some $a \in R$ and for some $\gamma > 0$, then we have obviously $\xi \gamma \leqq m(\xi a)$ for every real number ξ, and hence we can find by §79 Theorem 2 $\bar{a} \in \bar{R}$ such that

$$\bar{a}(a) = \gamma, \qquad \bar{a}(x) \leqq m(x) \qquad \text{for every } x \in R.$$

For such $\bar{a} \in \bar{R}$, we have $\bar{m}(\bar{a}) = 0$ by the definition §80(1), but $\bar{a}(a) \neq 0$. Consequently $\bar{R}$ is not simple by definition.

Conversely, if $\bar{m}(\bar{a}) = 0$ and $\lim_{\xi \to 0} \frac{m(\xi x)}{\xi} = 0$ for every $x \in R$, then we have by the definition §80(1) $\bar{a}(x) - m(x) \leqq 0$ for every $x \in R$, and hence for any $x \in R$ we have

$$|\bar{a}(x)| \leqq \frac{m(\xi x)}{\xi} \qquad \text{for every } \xi > 0.$$

Making ξ tend to 0, we obtain hence $\bar{a}(x) = 0$ for every $x \in R$, that is, $\bar{a} = 0$. Therefore $\bar{R}$ is simple by definition.

Theorem 2. *In order that we have*

$$\lim_{\xi \to 0} \frac{\bar{m}(\xi \bar{x})}{\xi} = 0 \qquad \text{for every } \bar{x} \in \bar{R},$$

it is necessary and sufficient that $\lim_{\nu \to \infty} m(x_\nu) = 0$ implies

$$\lim_{\nu \to \infty} \bar{x}(x_\nu) = 0 \qquad \text{for every } \bar{x} \in \bar{R}.$$

Proof. For an element $\bar{x} \in \bar{R}$, if we can find a sequence $x_\nu \in R$ ($\nu = 1, 2, \dots$) and $\gamma > 0$ such that $\lim_{\nu \to \infty} m(x_\nu) = 0$, $\bar{x}(x_\nu) > \gamma$ for every $\nu = 1, 2, \dots$, then there is obviously a sequence $\alpha_\nu > 0$ ($\nu = 1, 2, \dots$) such that $\lim_{\nu \to \infty} \alpha_\nu = 0$, $\frac{1}{\alpha_\nu} m(x_\nu) < \frac{1}{2}\gamma$ for every $\nu = 1, 2, \dots$. Then we have for every $\nu = 1, 2, \dots$

$$\alpha_\nu \bar{x}(x_\nu) - m(x_\nu) > \frac{1}{2} \alpha_\nu \gamma.$$

From this relation we conclude by the formula §80(2)

$$\frac{\bar{m}(\alpha_\nu \bar{x})}{\alpha_\nu} \geqq \frac{1}{2}\gamma \qquad \text{for every } \nu = 1, 2, \dots,$$

but $\lim_{\nu \to \infty} \alpha_\nu = 0$.

Conversely, if we can find $\bar{a} \in \bar{R}$ and $\gamma > 0$ such that

$$\lim_{\xi \to +0} \frac{\overline{m}(\xi \bar{a})}{\xi} > \gamma > 0, \qquad \overline{m}(\bar{a}) < +\infty,$$

then there is by the definition §80(1) a sequence $x_\nu \in R$ $(\nu = 1, 2, \dots)$ such that $\frac{1}{\nu}\bar{a}(x_\nu) - m(x_\nu) > \frac{1}{\nu}\gamma$. From this relation we conclude by the formula §80(2) $\overline{m}(\bar{a}) + m(x_\nu) \geqq \bar{a}(x_\nu) > \gamma + \nu m(x_\nu)$ and hence

$$\overline{m}(\bar{a}) - \gamma > (\nu - 1) m(x_\nu) \qquad \text{for every } \nu = 1, 2, \dots .$$

Making ν tend to ∞, we obtain therefore

$$\lim_{\nu \to \infty} m(x_\nu) = 0, \qquad \text{but} \qquad \lim_{\nu \to \infty} \bar{a}(x_\nu) \geqq \gamma.$$

From Theorems 1 and 2 we conclude immediately

Theorem 3. In order that the modular adjoint space $\bar{\bar{R}}$ of the modular adjoint space $\bar{R}$ of R be simple, it is necessary and sufficinet that $\lim_{\nu\to\infty} m(x_\nu) = 0$ implies $\lim_{\nu\to\infty} \bar{x}(x_\nu) = 0$ for every $\bar{x} \in \bar{R}$.

Theorem 4. If the modular adjoint space $\bar{R}$ of R is simple, then $\lim_{\nu\to\infty} \overline{m}(\bar{x}_\nu) = 0$ implies $\lim_{\nu\to\infty} \bar{x}_\nu(x) = 0$ for every $x \in R$.

Proof. We have by the formula §80(2) for every $x \in R$ and $\xi > 0$

$$|\bar{x}_\nu(\xi x)| \leqq \overline{m}(\bar{x}_\nu) + m(\xi x).$$

Thus we obtain by assumption for $\xi > 0$

$$\overline{\lim_{\nu\to\infty}} |\bar{x}_\nu(x)| \leqq \frac{m(\xi x)}{\xi}.$$

As $\lim_{\xi\to 0} \frac{m(\xi x)}{\xi} = 0$ by Theorem 1, we obtain therefore $\lim_{\nu\to\infty} \bar{x}_\nu(x) = 0$.

Theorem 5. If the modular adjoint space $\bar{R}$ of R is simple, then for any sequence $\bar{x}_\nu \in \bar{R}$ $(\nu = 1, 2, \dots)$ subject to the condition:

$$\lim_{\mu,\nu\to\infty} \overline{m}(\bar{x}_\mu - \bar{x}_\nu) = 0,$$

we can find uniquely $\bar{x} \in \bar{R}$ such that $\lim_{\nu\to\infty} \overline{m}(\bar{x}_\nu - \bar{x}) = 0$.

Proof. If $\lim_{\mu,\nu\to\infty} \overline{m}(\bar{x}_\mu - \bar{x}_\nu) = 0$, then we have by Theorem 4

$$\lim_{\mu,\nu\to\infty} |\bar{x}_\mu(x) - \bar{x}_\nu(x)| = 0 \qquad \text{for every } x \in R.$$

Thus, putting $\varphi(x) = \lim_{\nu\to\infty} \bar{x}_\nu(x)$ for every $x \in R$, we obtain a linear functional φ on R, and we conclude by the formula §80(2)

$$\bar{x}_\nu(x) - \varphi(x) \leqq \underline{\lim}_{\mu\to\infty} \overline{m}(\bar{x}_\mu - \bar{x}_\nu) + m(x)$$

for every $\nu = 1, 2, \dots$. From this relation we conclude further $\varphi \in \bar{R}$ and we have by the definition §80(1) for every $\nu = 1, 2, \dots$

$$\overline{m}(\bar{x}_\nu - \varphi) \leqq \underline{\lim}_{\mu\to\infty} \overline{m}(\bar{x}_\mu - \bar{x}_\nu).$$

Making ν tend to ∞, we obtain therefore by assumption

$$\lim_{\nu\to\infty} \overline{m}(\bar{x}_\nu - \varphi) = 0.$$

If $\lim_{\nu\to\infty} \overline{m}(\bar{x}_\nu - \bar{x}) = 0$, $\lim_{\nu\to\infty} \overline{m}(\bar{x}_\nu - \bar{y}) = 0$, then we have by the formula §78(1) $\overline{m}(\frac{1}{2}(\bar{x} - \bar{y})) = 0$, and hence $\bar{x} - \bar{y} = 0$, because $\bar{R}$ is simple by assumption. Thus such $\bar{x}$ is uniquely determined.

Let R be simple in the sequel. Then we can introduce a new conception of convergence. A sequence of elements $a_\nu \in R$ $(\nu = 1, 2, \dots)$ is said to be conditionally modular convergent to a limit $a \in R$, if we can find $\alpha > 0$ such that

$$\lim_{\nu\to\infty} m(\alpha(a_\nu - a)) = 0,$$

and then we shall write $\text{c-m-}\lim_{\nu\to\infty} a_\nu = a$.

If a sequence $a_\nu \in R$ $(\nu = 1, 2, \dots)$ is conditionally modular convergent, then the limit is determined uniquely. Because, if

$$\lim_{\nu\to\infty} m(\alpha(a_\nu - a)) = 0, \qquad \lim_{\nu\to\infty} m(\beta(a_\nu - b)) = 0$$

for two positive numbers α, β, then, putting $\gamma = \frac{1}{2}\operatorname{Min}\{\alpha, \beta\}$, we conclude by the formula §80(2) $m(\gamma(a - b)) = 0$, and hence $a = b$, as R is simple by assumption.

It is evident by definition that the conditionally modular convergence is weaker than the modular convergence, that is,

$$\text{m-}\lim_{\nu\to\infty} a_\nu = a \qquad \text{implies} \qquad \text{c-m-}\lim_{\nu\to\infty} a_\nu = a.$$

Concerning the conditionally modular convergence, we can prove easily by the formula §80(2)

Theorem 6. If $\text{c-m-}\lim_{\nu\to\infty} a_\nu = a$, $\text{c-m-}\lim_{\nu\to\infty} b_\nu = b$, then we have

$$\text{c-m-}\lim_{\nu\to\infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b,$$

and further $\text{c-m-}\lim_{\nu\to\infty} \alpha_\nu a_\nu = \alpha a$ for $\lim_{\nu\to\infty} \alpha_\nu = \alpha$.

§85 Uniformly simple modulars

Let R be a modulared space. If

$$\inf_{m(x) \geqq 1} m(\xi x) > 0 \qquad \text{for every } \xi > 0,$$

then we shall say that R is uniformly simple, or that the modular m of R is uniformly simple. If R is uniformly simple, then R is simple, because for any element $x \neq 0$, as $m(\xi x)$ is a non-decreasing convex function of $\xi \geqq 0$, we can find by the modular condition 4) $\alpha > 0$ such that $m(\alpha x) > 1$.

Theorem 1. *In order that R be uniformly simple, it is necessary and sufficient that the conditionally modular convergence coincides with the modular convergence, i. e., $\lim_{\nu\to\infty} m(x_\nu) = 0$ implies $\lim_{\nu\to\infty} m(\xi x_\nu) = 0$ for every $\xi > 0$.*

Proof. We suppose firstly that R is uniformly simple. If

$$\lim_{\nu\to\infty} m(x_\nu) = 0, \qquad x_\nu \neq 0 \qquad (\nu = 1, 2, \ldots),$$

then, as $m(\xi x)$ is a non-decreasing convex function of $\xi \geq 0$, we can find a sequence $\lambda_\nu > 0$ $(\nu = 1, 2, \ldots)$ such that

$$m(\xi x_\nu) \begin{cases} > 1 & \text{for } \xi > \lambda_\nu, \\ \leq 1 & \text{for } 0 \leq \xi \leq \lambda_\nu. \end{cases}$$

For such λ_ν $(\nu = 1, 2, \ldots)$, we have obviously for every $\nu = 1, 2, \ldots$

$$m(x_\nu) = m\left(\frac{1}{2\lambda_\nu}(2\lambda_\nu x_\nu)\right), \qquad m(2\lambda_\nu x_\nu) > 1$$

As R is uniformly simple by assumption, if $\overline{\lim}_{\nu\to\infty} \frac{1}{2\lambda_\nu} > \delta$ for some $\delta > 0$, then we have $\overline{\lim}_{\nu\to\infty} m(x_\nu) \geq \inf_{m(x)\geq 1} m(\delta x) > 0$, contradicting the assumption: $\lim_{\nu\to\infty} m(x_\nu) = 0$. Therefore we have $\lim_{\nu\to\infty} \frac{1}{2\lambda_\nu} = 0$. Thus, for any $\xi > 0$ we can find ν_0 such that $\frac{\xi}{\lambda_\nu} < 1$ for every $\nu \geq \nu_0$, and hence by the modular condition 5)

$$m(\xi x_\nu) \leq \frac{\xi}{\lambda_\nu} m(\lambda_\nu x_\nu) < \frac{\xi}{\lambda_\nu} \qquad \text{for } \nu \geq \nu_0.$$

Therefore we have $\lim_{\nu\to\infty} m(\xi x_\nu) = 0$ for every $\xi \geq 0$.

Secondly, if R is not uniformly simple, then we can find by definition $\delta > 0$ such that $\inf_{m(x)\geq 1} m(\delta x) = 0$, and hence we can find a sequence $x_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that $\lim_{\nu\to\infty} m(\delta x_\nu) = 0$ but $m(x_\nu) \geq 1$ for every $\nu = 1, 2, \ldots$.

Recalling §81 Theorem 5, we obtain immediately by Theorem 1

Theorem 2. *If R is uniformly simple, then for any $\varepsilon > 0$ we can find $\delta > 0$ such that $|||x||| \geq \varepsilon$ implies $m(x) \geq \delta$.*

A modulared space R is said to be uniformly monotone, or we shall say that the modular m of R is uniformly monotone, if

$$\lim_{\xi\to 0} \frac{1}{\xi} \sup_{m(x)\leq 1} m(\xi x) = 0.$$

With this definition we have

Theorem 3. *If R is uniformly simple, then the modular adjoint*

space $\bar{R}$ of R is uniformly monotone.

Proof. As $m(\xi x)$ is a non-decreasing convex function of $\xi \geq 0$, corresponding to every element $x \neq 0$ of R, there is by the modular condition 4) $\lambda_x > 0$ such that

$$m(\xi x) \begin{cases} > 1 & \text{for } \xi > \lambda_x, \\ \leq 1 & \text{for } 0 \leq \xi \leq \lambda_x. \end{cases}$$

Then we have naturally $m(2\lambda_x x) > 1$ for every $x \in R$, and hence

$$\inf_{0\neq x\in R} m(2\lambda_x \delta x) > 0 \qquad \text{for every } \delta > 0,$$

because R is uniformly simple by assumption. Thus, putting

$$\varepsilon = \inf_{0\neq x\in R} m(2\lambda_x \delta x),$$

we have by the modular condition 5) and the formula §80(2) that if

$$0 < \xi < \frac{\varepsilon}{4\delta}, \qquad \rho > \delta, \qquad \bar{m}(\bar{x}) \leq 1,$$

then we have

$$\begin{aligned} \bar{x}(2\lambda_x \rho x) - \frac{1}{\xi} m(2\lambda_x \rho x) &\leq 2\rho\, \bar{x}(\lambda_x x) - \frac{\rho}{\xi\delta} m(2\lambda_x \delta x) \\ &\leq 2\rho\{\bar{m}(\bar{x}) + m(\lambda_x x)\} - \frac{\rho\varepsilon}{\xi\delta} \\ &\leq 4\rho\left(1 - \frac{\varepsilon}{4\xi\delta}\right) < 0. \end{aligned}$$

Therefore we conclude by §80(1) that $0 < \xi < \frac{\varepsilon}{4\delta}$, $\bar{m}(\bar{x}) \leq 1$ implies

$$\begin{aligned} \frac{1}{\xi}\bar{m}(\xi\bar{x}) &= \sup_{0<\rho\leq\delta,\, 0\neq x\in R} \left\{ \bar{x}(2\lambda_x \rho x) - \frac{1}{\xi} m(2\lambda_x \rho x) \right\} \\ &\leq \sup_{0<\rho\leq\delta,\, 0\neq x\in R} \bar{x}(2\lambda_x \rho x) \leq \sup_{0\neq x\in R} 2\delta\, |\bar{x}(\lambda_x x)| \\ &\leq 2\delta(\bar{m}(\bar{x}) + 1) \leq 4\delta, \end{aligned}$$

and consequently $\overline{\lim}_{\xi\to +0} \frac{1}{\xi} \sup_{\bar{m}(\bar{x})\leq 1} \bar{m}(\xi\bar{x}) \leq 4\delta$. As $\delta > 0$ may be arbitrary, we obtain hence $\lim_{\xi\to 0} \frac{1}{\xi} \sup_{\bar{m}(\bar{x})\leq 1} \bar{m}(\xi\bar{x}) = 0$.

Theorem 4. *If R is uniformly monotone, then the modular adjoint space $\bar{R}$ of R is uniformly simple.*

Proof. If $\bar{m}(\bar{x}) \geq 1$, $\bar{x} \in \bar{R}$, then we have $|||\bar{x}||| \geq 1$ by §81 Theorem 4, and hence $\|\bar{x}\| \geq |||\bar{x}||| \geq 1$ by §83 Theorem 1. As

$$\|\bar{x}\| = \sup_{m(x)\leq 1} |\bar{x}(x)|$$

by §83 Theorem 2, we can find then $a \in R$ such that $\bar{x}(a) > \frac{1}{2}$, $m(a) \leq 1$. For such $a \in R$ we have by the definition §80(1) for every ξ, $\rho > 0$

$$\bar{m}(\xi\bar{x}) \geq \xi\bar{x}(\rho a) - m(\rho a) \geq \rho\left\{\frac{\xi}{2} - \frac{1}{\rho} m(\rho a)\right\}.$$

As R is uniformly monotone by assumption, for any $\xi > 0$ we can find $\rho > 0$ such that $m(x) \leq 1$ implies $\frac{1}{\rho} m(\rho x) < \frac{1}{4}\xi$, and hence

$$\overline{m}(\xi\bar{x}) \geqq \rho\left\{\frac{\xi}{2} - \frac{1}{\rho}m(\rho a)\right\} \geqq \frac{\xi\rho}{4}.$$

From this relation we conclude

$$\inf_{\overline{m}(\bar{x}) \geqq 1} \overline{m}(\xi\bar{x}) \geqq \frac{\xi\rho}{4} > 0.$$

Thus $\overline{R}$ is uniformly simple by definition.

§86 Finiteness

Let R be a modulared space. An element $a \in R$ is said to be finite, if $m(\xi a) < +\infty$ for every $\xi > 0$.

With this definition we have obviously by §78 Theorem 2

Theorem 1. If $m\text{-}\lim_{\nu\to\infty} a_\nu = a$ and every a_ν $(\nu = 1, 2, \dots)$ is finite, then the limit a also is finite and we have for every ξ

$$m(\xi a) = \lim_{\nu\to\infty} m(\xi a_\nu).$$

As $m(\xi x)$ is a non-decreasing convex function of $\xi \geqq 0$, we have obviously by §81 Theorem 4

Theorem 2. For a finite element $a \in R$ we have $|||a||| = 1$ if and only if $m(a) = 1$.

If every element of R is finite, then we shall say that R is finite, or that the modular m of R is finite. With this definition, we can conclude immediately from Theorem 2

Theorem 3. If R is finite, then we have

$$m\left(\frac{1}{|||x|||}x\right) = 1 \qquad \text{for } 0 \neq x \in R.$$

A modulared space R is said to be uniformly finite, or we shall say that the modular m of R is uniformly finite, if

$$\sup_{m(x) \leqq 1} m(\xi x) < +\infty \qquad \text{for every } \xi > 0.$$

Theorem 4. If R is uniformly finite, then for any $\varepsilon > 0$ we can find $\delta > 0$ such that $|1 - |||x||||| \leqq \delta$ implies $|1 - m(x)| \leqq \varepsilon$.

Proof. As R is uniformly finite by assumption, we can put

$$\alpha = \sup_{m(x) \leqq 1} m(2x) < +\infty.$$

If $m(a) = 1$, then, as $m(\xi a)$ is a non-decreasing convex function of $\xi \geqq 0$, we see easily that we have

$$m(\xi a) \leqq 1 + (\alpha - 1)(\xi - 1) \qquad \text{for } 1 \leqq \xi \leqq 2,$$

$$m(\xi a) \geqq 1 + (\alpha - 1)(\xi - 1) \qquad \text{for } 0 \leqq \xi \leqq 1.$$

Thus $m(a) = 1$, $|\xi - 1| < 1$ implies $|m(\xi a) - 1| \leqq (\alpha - 1)|\xi - 1|$. For an arbitrary element $x \neq 0$, as $m\left(\frac{1}{|||x|||}x\right) = 1$ by Theorem 3, we obtain hence $|m(x) - 1| \leqq (\alpha - 1)\,|\,|||x||| - 1\,|$ for $|\,|||x||| - 1\,| < 1$. From this relation we conclude easily our assertion.

A modulared space R is said to be uniformly increasing, or we shall say that the modular m of R is uniformly increasing, if

$$\lim_{\xi\to\infty} \frac{1}{\xi} \inf_{m(x) \geqq 1} m(\xi x) = +\infty.$$

Theorem 5. If R is uniformly increasing, then for any two positive numbers ε, ε' we can find $\delta > 0$ such that $m(x) \geqq \delta$, $\overline{m}(\bar{x}) \leqq \varepsilon'$ implies $m(x) - \bar{x}(x) \geqq \varepsilon$.

Proof. If $m(x) \geqq 1$, $\overline{m}(\bar{x}) \leqq \varepsilon'$, then we can find α such that

$$m(\alpha x) = 1, \qquad 0 < \alpha \leqq 1,$$

and we have then by the formula §80(2)

$$\bar{x}(x) = \frac{1}{\alpha}\bar{x}(\alpha x) \leqq \frac{1}{\alpha}(\overline{m}(\bar{x}) + m(\alpha x)) \leqq \frac{1}{\alpha}(\varepsilon' + 1).$$

As R is uniformly increasing by assumption, we can find $\lambda > 0$ such that

$$\frac{1}{\xi}\inf_{m(x) \geqq 1} m(\xi x) \geqq \varepsilon + \varepsilon' + 1 \qquad \text{for } \xi \geqq \lambda.$$

For such λ, if $m(x) < \bar{x}(x) + \varepsilon$, then we have

$$\alpha\, m\left(\frac{1}{\alpha}\alpha x\right) < \alpha(\bar{x}(x) + \varepsilon) \leqq \varepsilon' + 1 + \alpha\varepsilon \leqq \varepsilon + \varepsilon' + 1,$$

and hence $\frac{1}{\alpha} < \lambda$. This relation yields

$$m(x) < \frac{1}{\alpha}(\varepsilon + \varepsilon' + 1) < \lambda(\varepsilon + \varepsilon' + 1).$$

Therefore, putting $\delta = \operatorname{Max}\{\lambda(\varepsilon + \varepsilon' + 1), 1\}$, we see that $m(x) \geqq \delta$, $\overline{m}(\bar{x}) \leqq \varepsilon'$ implies $m(x) \geqq \bar{x}(x) + \varepsilon$.

Theorem 6. If R is uniformly increasing, then the modular adjoint space $\overline{R}$ of R is uniformly finite.

Proof. If R is uniformly increasing, then for any $\lambda > 0$ we can find by definition $\rho \geqq 1$ such that

$$\inf_{m(x) \geqq 1} \frac{1}{\xi} m(\xi x) \geqq 2\lambda \qquad \text{for } \xi \geqq \rho.$$

If $\overline{m}(\bar{x}) \leqq 1$, $1 \leqq m(x) < +\infty$, then there is $\xi > 0$ such that

$$m\left(\frac{1}{\xi}x\right) = 1, \qquad 0 < \frac{1}{\xi} \leqq 1,$$

and hence by the formula §80(2)

$$\bar{x}\left(\frac{1}{\xi}x\right) \leqq \overline{m}(\bar{x}) + m\left(\frac{1}{\xi}x\right) \leqq 2.$$

For such ξ, if $\xi \geqq \rho$, then we have

$$\lambda \bar{x}(x) - m(x) = \xi \{ \lambda \bar{x}(\tfrac{1}{\xi} x) - \tfrac{1}{\xi} m(\xi \tfrac{1}{\xi} x)\} \leqq 0,$$

and if $0 < \xi \leqq \rho$, then we have

$$\lambda \bar{x}(x) - m(x) \leqq \xi \lambda \bar{x}(\tfrac{1}{\xi} x) \leqq 2 \rho \lambda .$$

If $\bar{m}(\bar{x}) \leqq 1$, $m(x) \leqq 1$, then we have obviously by §80(2)

$$\lambda \bar{x}(x) - m(x) \leqq \lambda (\bar{m}(\bar{x}) + m(x)) - m(x) \leqq 2\lambda .$$

Furthermore it is evident that if $\bar{m}(\bar{x}) \leqq 1$, $m(x) = +\infty$, then

$$\lambda \bar{x}(x) - m(x) \leqq 0 .$$

Consequently we obtain by the definition §80(1)

$$\sup_{\bar{m}(\bar{x}) \leqq 1} \bar{m}(\lambda \bar{x}) \leqq 2 \rho \lambda .$$

Therefore $\bar{R}$ is uniformly finite by definition.

Theorem 7. *If R is uniformly finite, then the modular adjoint space $\bar{R}$ of R is uniformly increasing.*

Proof. If $\bar{m}(\bar{x}) \geqq 1$, then we have by Theorems 4 in §81 and 1 in §83 $\|\bar{x}\| \geqq |||\bar{x}||| \geqq 1$, and hence we can find by §83 Theorem 2 $x \in R$ such that $\bar{x}(x) > \frac{1}{2}$, $m(x) \leqq 1$. For such x, we have by §80(2)

$$\frac{1}{\xi} \bar{m}(\xi \bar{x}) \geqq \bar{x}(\rho x) - \frac{1}{\xi} m(\rho x) \geqq \frac{1}{2} \rho - \frac{1}{\xi} m(\rho x)$$

for every $\xi, \rho > 0$. As R is uniformly finite by assumption, for any $\rho > 0$, putting $\lambda_\rho = \sup_{m(x) \leqq 1} m(\rho x) < +\infty$, we have

$$\frac{1}{\xi} \bar{m}(\xi \bar{x}) \geqq \frac{1}{2} \rho - 1 \qquad \text{for every } \xi \geqq \lambda_\rho .$$

Therefore we obtain for every $\rho > 0$

$$\lim_{\xi \to \infty} \frac{1}{\xi} \inf_{\bar{m}(\bar{x}) \geqq 1} \bar{m}(\xi \bar{x}) \geqq \frac{1}{2} \rho - 1,$$

and consequently $\lim_{\xi \to \infty} \frac{1}{\xi} \inf_{\bar{m}(\bar{x}) \geqq 1} \bar{m}(\xi \bar{x}) = +\infty$. Thus $\bar{R}$ is uniformly increasing by definition.

§87 Uniformly convex modulars

A modulared space R is said to be *uniformly convex*, or we shall say that the modular m of R is *uniformly convex*, if for any two $\varepsilon, \gamma > 0$ we can find $\delta > 0$ such that $m(x)$, $m(y) \leqq \gamma$, $m(x-y) \geqq \varepsilon$ implies

$$\frac{1}{2}\{m(x) + m(y)\} \geqq m(\tfrac{1}{2}(x+y)) + \delta .$$

With this definition, we will prove firstly

Theorem 1. *Let R be uniformly convex. For a manifold $\bar{A}$ of the modular adjoint space $\bar{R}$ of R, if*

$$\lim_{\nu \to \infty} \bar{x}(a_\nu) = \bar{x}(a) \qquad \text{for every } \bar{x} \in \bar{A},$$

$$m(a) = \sup_{\bar{x} \in \bar{A}} \{\bar{x}(a) - \bar{m}(\bar{x})\},$$

$$\overline{\lim_{\nu \to \infty}}\, m(a_\nu) \leqq m(a) < +\infty,$$

then $\lim_{\nu \to \infty} m(a_\nu - a) = 0$.

Proof. If we can find $\varepsilon > 0$ and a subsequence a_{ν_μ} ($\mu = 1, 2, \ldots$) from a_ν ($\nu = 1, 2, \ldots$) such that

$$m(a_{\nu_\mu} - a) \geqq \varepsilon \qquad \text{for every } \mu = 1, 2, \ldots,$$

then there is by assumption $\delta > 0$ such that we have for every $\mu = 1, 2, \ldots$

$$\frac{1}{2}\{m(a_{\nu_\mu}) + m(a)\} \geqq m(\tfrac{1}{2}(a_{\nu_\mu} + a)) + \delta .$$

Then, as we have by the definition §80(1) for every $\bar{x} \in \bar{R}$

$$m(\tfrac{1}{2}(a_{\nu_\mu} + a)) \geqq \bar{x}(\tfrac{1}{2}(a_{\nu_\mu} + a)) - \bar{m}(\bar{x}),$$

we obtain by assumption for every $\bar{x} \in \bar{A}$

$$m(a) \geqq \overline{\lim_{\mu \to \infty}}\, \frac{1}{2}\{m(a_{\nu_\mu}) + m(a)\} \geqq \bar{x}(a) - \bar{m}(\bar{x}) + \delta,$$

contradicting the assumption: $m(a) = \sup_{\bar{x} \in \bar{A}} \{\bar{x}(a) - \bar{m}(\bar{x})\}$.

Theorem 2. *If R is uniformly convex and uniformly simple, then*

$$\lim_{\nu \to \infty} \bar{x}(a_\nu) = \bar{x}(a) \qquad \text{for every } \bar{x} \in \bar{R},$$

$$\overline{\lim_{\nu \to \infty}}\, m(a_\nu) \leqq m(a) < +\infty$$

implies $\text{m-}\lim_{\nu \to \infty} a_\nu = a$.

Proof. By virtue of §80 Theorem 2, we have for every $a \in R$

$$m(a) = \sup_{\bar{x} \in \bar{R}} \{\bar{x}(a) - \bar{m}(\bar{x})\}.$$

Thus we can conclude by Theorem 1 from our assumption

$$\lim_{\nu \to \infty} m(a_\nu - a) = 0 .$$

As R is uniformly simple by assumption, this relation yields by §85 Theorem 1 $\text{m-}\lim_{\nu \to \infty} a_\nu = a$.

Theorem 3. *If R is uniformly simple, uniformly finite, and uniformly convex, then the modular norm $|||x|||$ of R is uniformly convex.*

Proof. As R is uniformly simple by assumption, for any $\varepsilon > 0$ we can find by §85 Theorem 2 $\varepsilon' > 0$ such that $|||x||| \geqq \varepsilon$ implies $m(x) \geqq \varepsilon'$. For such ε', as R is uniformly convex by assumption, we can find by definition $\delta' > 0$ such that $m(x)$, $m(y) \leqq 1$, $m(x-y) \geqq \varepsilon'$ implies

$$\frac{1}{2}\{m(x)+m(y)\} \geq m\left(\frac{1}{2}(x+y)\right)+\delta'.$$

Furthermore, as R is uniformly finite by assumption, for such δ' we can find by §86 Theorem 4 $\delta>0$ such that

$$|1-|||x|||| \leq \delta \quad \text{implies} \quad |1-m(x)| \leq \frac{1}{2}\delta'.$$

If $|||x|||=|||y|||=1$, $|||x-y||| \geq \varepsilon$, then we have $m(x)=m(y)=1$ by §86 Theorem 2, and $m(x-y) \geq \varepsilon'$ by the definition of ε'. Consequently we obtain $1=\frac{1}{2}\{m(x)+m(y)\} \geq m(\frac{1}{2}(x+y))+\delta$, and hence $1-m(\frac{1}{2}(x+y)) \geq \delta'$. This relation yields $1-|||\frac{1}{2}(x+y)||| \geq \delta$ by the definition of δ. Therefore $|||x|||=|||y|||=1$, $|||x-y||| \geq \varepsilon$ implies $2-2\delta \geq |||x+y|||$, and hence the modular norm is uniformly convex by §76 Theorem 1.

By virtue of §77 Theorem 4, if a normed space is uniformly convex and complete, then it is regular. Therefore, recalling §81, we see by Theorem 3 that if a modulared space R is uniformly simple, uniformly finite, uniformly convex, and modular complete, then R is regular. However we can prove the regularity of R under weaker assumptions:

Theorem 4. If R is uniformly simple, uniformly convex, and modular complete, then R is regular.

Proof. Let $\bar{\bar{R}}$ be the modular adjoint space of the modular adjoint space $\bar{R}$. For any $\bar{\bar{x}} \in \bar{\bar{R}}$ subject to $\bar{\bar{m}}(\bar{\bar{x}})<+\infty$, we can find by the definition §80(1) a sequence $\bar{x}_\nu \in \bar{R}$ $(\nu=1,2,\ldots)$ such that

$$\bar{\bar{x}}(\bar{x}_\nu) \geq \bar{\bar{m}}(\bar{\bar{x}})+\bar{m}(\bar{x}_\nu)-\frac{1}{\nu}.$$

Then, as we have by the formula §80(2)

$$\sum_{\nu=1}^{\kappa}\xi_\nu\bar{\bar{x}}(\bar{x}_\nu) \leq \bar{\bar{m}}(\bar{\bar{x}})+\bar{m}\left(\sum_{\nu=1}^{\kappa}\xi_\nu\bar{x}_\nu\right)$$

for every finite number of real numbers ξ_ν $(\nu=1,2,\ldots,\kappa)$, we can find by §82 Theorem 4 a sequence $x_\rho \in R$ $(\rho=1,2,\ldots)$ such that

$$\bar{x}_\nu(x_\rho)=\bar{\bar{x}}(\bar{x}_\nu) \qquad \text{for every } \nu=1,2,\ldots,\rho,$$

$$m\left(\left(1-\frac{1}{\rho}\right)x_\rho\right) \leq \bar{\bar{m}}(\bar{\bar{x}}) \qquad \text{for every } \rho=1,2,\ldots .$$

For such x_ρ $(\rho=1,2,\ldots)$ we have

$$\lim_{\mu,\nu\to\infty} m\left(\left(1-\frac{1}{\nu}\right)x_\nu-\left(1-\frac{1}{\mu}\right)x_\mu\right)=0.$$

Because, if we can find $\varepsilon>0$ and two subsequences λ_ν, μ_ν $(\nu=1,2,\ldots)$ from $\{1,2,\ldots\}$ such that we have for every $\nu=1,2,\ldots$

$$m\left(\left(1-\frac{1}{\lambda_\nu}\right)x_{\lambda_\nu}-\left(1-\frac{1}{\mu_\nu}\right)x_{\mu_\nu}\right) \geq \varepsilon,$$

then, as R is uniformly convex by assumption, we can find by definition $\delta>0$ such that

$$\frac{1}{2}\left\{m\left(\left(1-\frac{1}{\lambda_\nu}\right)x_{\lambda_\nu}\right)+m\left(\left(1-\frac{1}{\mu_\nu}\right)x_{\mu_\nu}\right)\right\} \geq m\left(\frac{1}{2}\left(\left(1-\frac{1}{\lambda_\nu}\right)x_{\lambda_\nu}+\left(1-\frac{1}{\mu_\nu}\right)x_{\mu_\nu}\right)\right)+\delta$$

for every $\nu=1,2,\ldots$. On the other hand we have

$$\frac{1}{2}\left\{m\left(\left(1-\frac{1}{\lambda_\nu}\right)x_{\lambda_\nu}\right)+m\left(\left(1-\frac{1}{\mu_\nu}\right)x_{\mu_\nu}\right)\right\} \leq \bar{\bar{m}}(\bar{\bar{x}}),$$

and by the definition §80(1) for $\rho \leq \lambda_\nu, \mu_\nu$

$$m\left(\frac{1}{2}\left(\left(1-\frac{1}{\lambda_\nu}\right)x_{\lambda_\nu}+\left(1-\frac{1}{\mu_\nu}\right)x_{\mu_\nu}\right)\right)$$
$$\geq \bar{x}_\rho\left(\frac{1}{2}\left(\left(1-\frac{1}{\lambda_\nu}\right)x_{\lambda_\nu}+\left(1-\frac{1}{\mu_\nu}\right)x_{\mu_\nu}\right)\right)-\bar{m}(\bar{x}_\rho)$$
$$=\left(1-\frac{1}{2\lambda_\nu}-\frac{1}{2\mu_\nu}\right)\bar{\bar{x}}(\bar{x}_\rho)-\bar{m}(\bar{x}_\rho).$$

Thus we obtain for such δ

$$\bar{\bar{m}}(\bar{\bar{x}}) \geq \left(1-\frac{1}{2\lambda_\nu}-\frac{1}{2\mu_\nu}\right)\bar{\bar{x}}(\bar{x}_\rho)-\bar{m}(\bar{x}_\rho)+\delta$$

for every $\rho \leq \lambda_\nu, \mu_\nu$. Making ν tend to ∞, we conclude hence

$$\bar{\bar{m}}(\bar{\bar{x}}) \geq \bar{\bar{x}}(\bar{x}_\rho)-\bar{m}(\bar{x}_\rho)+\delta \geq \bar{\bar{m}}(\bar{\bar{x}})-\frac{1}{\rho}+\delta$$

for every $\rho=1,2,\ldots$, contradicting $\delta>0$.

As R is uniformly simple and modular complete by assumption, there exists by §85 Theorem 1 $x \in R$ such that $m\text{-}\lim_{\nu\to\infty}(1-\frac{1}{\nu})x_\nu=x$, and then we have by Theorems 7 in §81 and 3 in §80

$$m(x) \leq \varliminf_{\nu\to\infty} m\left(\left(1-\frac{1}{\nu}\right)x_\nu\right) \leq \bar{\bar{m}}(\bar{\bar{x}}),$$

$$\bar{x}_\nu(x)=\bar{\bar{x}}(\bar{x}_\nu) \qquad \text{for every } \nu=1,2,\ldots .$$

For an arbitrary $\bar{x} \in \bar{R}$, we also can apply the same process to $\bar{x}$, $\bar{x}_1, \bar{x}_2, \ldots$ instead of $\bar{x}_1, \bar{x}_2, \ldots$, and we obtain likewise $x_0 \in R$ such that

$$m(x_0) \leq \bar{\bar{m}}(\bar{\bar{x}}), \qquad \bar{x}(x_0)=\bar{\bar{x}}(\bar{x}),$$

$$\bar{x}_\nu(x_0)=\bar{\bar{x}}(\bar{x}_\nu) \qquad \text{for every } \nu=1,2,\ldots .$$

For such x_0, if $m(x-x_0)>0$, then, as R is uniformly convex by assumption, we can find $\delta>0$ such that

$$\frac{1}{2}\{m(x)+m(x_0)\} \geq m\left(\frac{1}{2}(x+x_0)\right)+\delta,$$

and then we have by §80(2) for every $\nu=1,2,\ldots$

$$\bar{\bar{m}}(\bar{\bar{x}}) \geq m\left(\frac{1}{2}(x+x_0)\right)+\delta \geq \bar{x}_\nu\left(\frac{1}{2}(x+x_0)\right)-\bar{m}(\bar{x}_\nu)+\delta$$
$$=\bar{\bar{x}}(\bar{x}_\nu)-\bar{m}(\bar{x}_\nu)+\delta \geq \bar{\bar{m}}(\bar{\bar{x}})-\frac{1}{\nu}+\delta,$$

contradicting $\delta>0$. Therefore we have $m(x-x_0)=0$, and hence $x=x_0$, because R is uniformly simple by assumption. Accordingly we have $\bar{x}(x) = \bar{x}(x_0) = \bar{\bar{x}}(\bar{x})$. As $\bar{x} \in \bar{R}$ may be arbitrary, we conclude that $\bar{x}(x) = \bar{\bar{x}}(\bar{x})$ for every $\bar{x} \in \bar{R}$. Thus R is regular.

§88 Uniformly even modulars

A modulared space R is said to be uniformly even, or we shall say that the modular m of R is uniformly even, if for any two positive numbers ε, γ we can find $\delta > 0$ such that $m(x), m(y) \leqq \gamma$, $|||x-y||| \leqq \delta$ implies
$$\frac{1}{2}\{m(x)+m(y)\} \leqq m\left(\frac{1}{2}(x+y)\right) + \varepsilon|||x-y|||.$$

With this definition we have

Theorem 1. *If R is finite and uniformly even, then for any two positive numbers ε, γ we can find $\delta > 0$ such that $m(x)$, $m(y) \leqq \gamma$, $|||x-y||| \leqq \delta$, $\alpha+\beta=1$, $\alpha, \beta \geqq 0$ implies*
$$\alpha m(x) + \beta m(y) \leqq m(\alpha x + \beta y) + \varepsilon|||x-y|||.$$

Proof. For any $\varepsilon, \gamma > 0$ we can find by definition $\delta > 0$ such that $m(x)$, $m(y) \leqq \gamma$, $|||x-y||| \leqq \delta$ implies
$$\frac{1}{2}\{m(x)+m(y)\} \leqq m\left(\frac{1}{2}(x+y)\right) + \frac{\varepsilon}{2}|||x-y|||.$$
For such ε, γ, δ, we shall prove firstly by the induction that
$m(x)$, $m(y) \leqq \gamma$, $|||x-y||| \leqq \delta$ implies
$$\frac{\alpha}{2^\nu}m(x) + \frac{\beta}{2^\nu}m(y) \leqq m\left(\frac{\alpha}{2^\nu}x + \frac{\beta}{2^\nu}y\right) + \varepsilon|||x-y|||$$
for every pair of natural numbers α, β subject to $\alpha+\beta = 2^\nu$.

It is evident in the case: $\nu = 1$. We suppose that it holds in the case ν, and $\alpha+\beta = 2^{\nu+1}$, $\alpha < \beta$. Then, as $\alpha < 2^\nu$, we have
$$\begin{aligned}
&\frac{\alpha}{2^{\nu+1}}m(x) + \frac{\beta}{2^{\nu+1}}m(y) - m\left(\frac{\alpha}{2^{\nu+1}}x + \frac{\beta}{2^{\nu+1}}y\right)\\
&= \frac{1}{2}\left\{\frac{\alpha}{2^\nu}m(x) + \frac{2^\nu-\alpha}{2^\nu}m(y) - m\left(\frac{\alpha}{2^\nu}x + \frac{2^\nu-\alpha}{2^\nu}y\right)\right\}\\
&\quad + \frac{1}{2}\left\{m\left(\frac{\alpha}{2^\nu}x + \frac{2^\nu-\alpha}{2^\nu}y\right) + m(y)\right\} - m\left(\frac{\alpha}{2^{\nu+1}}x + \frac{\beta}{2^{\nu+1}}y\right)\\
&\leqq \frac{1}{2}\varepsilon|||x-y||| + \frac{1}{2}\varepsilon\left|\left|\left|\frac{\alpha}{2^\nu}x + \frac{2^\nu-\alpha}{2^\nu}y - y\right|\right|\right| \leqq \varepsilon|||x-y|||,
\end{aligned}$$
because we have by the modular condition 5)
$$m\left(\frac{\alpha}{2^\nu}x + \frac{2^\nu-\alpha}{2^\nu}y\right) \leqq \frac{\alpha}{2^\nu}m(x) + \frac{2^\nu-\alpha}{2^\nu}m(y) \leqq \gamma,$$
$$\left|\left|\left|\frac{\alpha}{2^\nu}x + \frac{2^\nu-\alpha}{2^\nu}y - y\right|\right|\right| = \frac{\alpha}{2^\nu}|||x-y||| \leqq \delta.$$
Thus the case $\nu+1$ is proved.

In general, if $\alpha+\beta = 1$, $\alpha, \beta \geqq 0$, then we can find obviously

two sequences of natural numbers α_ν, β_ν $(\nu = 1, 2, \ldots)$ such that
$$\alpha_\nu + \beta_\nu = 2^\nu \qquad (\nu = 1, 2, \ldots),$$
$$\lim_{\nu\to\infty}\frac{\alpha_\nu}{2^\nu} = \alpha, \qquad \lim_{\nu\to\infty}\frac{\beta_\nu}{2^\nu} = \beta.$$
Then we have by §78 Theorem 1
$$\text{m-}\lim_{\nu\to\infty}\left(\frac{\alpha_\nu}{2^\nu}x + \frac{\beta_\nu}{2^\nu}y\right) = \alpha x + \beta y.$$
As R is finite by assumption, we obtain hence by §78 Theorem 2
$$\lim_{\nu\to\infty} m\left(\frac{\alpha_\nu}{2^\nu}x + \frac{\beta_\nu}{2^\nu}y\right) = m(\alpha x + \beta y).$$
On the other hand we have for every $\nu = 1, 2, \ldots$
$$\frac{\alpha_\nu}{2^\nu}m(x) + \frac{\beta_\nu}{2^\nu}m(y) \leqq m\left(\frac{\alpha_\nu}{2^\nu}x + \frac{\beta_\nu}{2^\nu}y\right) + \varepsilon|||x-y|||,$$
as proved just above, and consequently we obtain
$$\alpha m(x) + \beta m(y) \leqq m(\alpha x + \beta y) + \varepsilon|||x-y|||.$$

Theorem 2. *If R is finite and uniformly even, then the modular norm of R is uniformly even.*

Proof. As R is finite and uniformly even by assumption, for any $\varepsilon > 0$ we can find by Theorem 1 a positive number $\delta < \frac{1}{2}$ such that
$m(x)$, $m(y) \leqq 1$, $|||x-y||| \leqq 8\delta$, $\alpha+\beta=1$, $\alpha, \beta \geqq 0$ implies
$$\alpha m(x) + \beta m(y) \leqq m(\alpha x + \beta y) + \frac{\varepsilon}{6}|||x-y|||.$$
For such ε, δ, if $|||x||| = 1$, $|||y||| \leqq \delta$, then, putting
$$\alpha = |||x+y|||, \qquad \beta = |||x-y|||,$$
we have by §86 Theorem 3
$$m\left(\frac{1}{\alpha}(x+y)\right) = m\left(\frac{1}{\beta}(x-y)\right) = 1.$$
As $\alpha \geqq |||x||| - |||y||| \geqq 1-\delta > \frac{1}{2}$, $\beta \geqq |||x||| - |||y||| \geqq 1-\delta > \frac{1}{2}$,
and $|\alpha - \beta| \leqq |||(x+y)-(x-y)||| = 2|||y||| \leqq 2\delta$, we have
$$\left|\left|\left|\frac{1}{\alpha}(x+y) - \frac{1}{\beta}(x-y)\right|\right|\right| \leqq \left|\frac{1}{\alpha} - \frac{1}{\beta}\right| |||x+y||| + \frac{2}{\beta}|||y||| \leqq 8\delta.$$
Therefore
$$\frac{\alpha}{\alpha+\beta}m\left(\frac{1}{\alpha}(x+y)\right) + \frac{\beta}{\alpha+\beta}m\left(\frac{1}{\beta}(x-y)\right) \leqq m\left(\frac{2}{\alpha+\beta}x\right) + \frac{\varepsilon}{3}|||y|||,$$
that is, $1 \leqq m\left(\frac{2}{\alpha+\beta}x\right) + \frac{\varepsilon}{3}|||y|||$.

On the other hand we have by the modular condition 5)
$$m\left(\frac{2}{\alpha+\beta}x\right) \leqq \frac{\alpha}{\alpha+\beta}m\left(\frac{1}{\alpha}(x+y)\right) + \frac{\beta}{\alpha+\beta}m\left(\frac{1}{\beta}(x-y)\right) = 1,$$
and hence by Theorem 4 in §81
$$m\left(\frac{2}{\alpha+\beta}x\right) \leqq \left|\left|\left|\frac{2}{\alpha+\beta}x\right|\right|\right|.$$
Thus we obtain $\alpha+\beta \leqq 2|||x||| + (\alpha+\beta)\frac{\varepsilon}{3}|||y||| \leqq 2 + \varepsilon|||y|||$,

because $\alpha+\beta \leqq 2(|||x|||+|||y|||) \leqq 3$. Therefore the modular norm is uniformly even by §77 Theorem 1.

Recalling Theorems 5 in §77 and 6 in §81, we obtain by Theorem 2

Theorem 3. If R is finite, uniformly even, and modular complete, then R is regular.

Theorem 4. If R is uniformly simple, uniformly increasing, and uniformly convex, then the modular adjoint space $\overline{R}$ of R is uniformly even.

Proof. As R is uniformly increasing by assumption, for any $\gamma > 0$ we can find by §86 Theorem 5 $\gamma' > 1$ such that $\overline{m}(\overline{x}) \leqq \gamma$, $m(x) \geqq \gamma'$ implies $\overline{x}(x) - m(x) \leqq 0$. Thus we have by §80(1) that $\overline{m}(\overline{x}) \leqq \gamma$ implies

$$\overline{m}(\overline{x}) = \sup_{m(x)\leqq\gamma'} \{\overline{x}(x) - m(x)\}.$$

Therefore, if $\overline{m}(\overline{x}), \overline{m}(\overline{y}) \leqq \gamma$, then we have

$$\frac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\} = \frac{1}{2}\sup_{m(x),\, m(y)\leqq\gamma'} \{\overline{x}(x) + \overline{y}(y) - m(x) - m(y)\}.$$

On the other hand we have by the formulas §80(2) and §83(3)

$$\overline{x}(x) + \overline{y}(y) = (\overline{x}+\overline{y})(\tfrac{1}{2}(x+y)) + (\overline{x}-\overline{y})(\tfrac{1}{2}(x-y))$$

$$\leqq 2\overline{m}(\tfrac{1}{2}(\overline{x}+\overline{y})) + 2m(\tfrac{1}{2}(x+y)) + \tfrac{1}{2}|||\overline{x}-\overline{y}|||\,\|x-y\|.$$

Consequently we have

$$\frac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\} \leqq \overline{m}(\tfrac{1}{2}(\overline{x}+\overline{y})) + \sup_{m(x),\, m(y)\leqq\gamma'} \{\tfrac{1}{4}|||\overline{x}-\overline{y}|||\,\|x-y\| + m(\tfrac{1}{2}(x+y)) - \tfrac{1}{2}\{m(x)+m(y)\}\}.$$

As R is uniformly simple by assumption, for any $\varepsilon > 0$ we can find by §85 Theorem 2 $\varepsilon' > 0$ such that $|||x||| \geqq 2\varepsilon$ implies $m(x) \geqq \varepsilon'$. For such ε', as R is uniformly convex by assumption, we can find by definition $\delta' > 0$ such that $m(x), m(y) \leqq \gamma'$, $m(x-y) \geqq \varepsilon'$ implies

$$\frac{1}{2}\{m(x) + m(y)\} \geqq m(\tfrac{1}{2}(x+y)) + \delta'.$$

Now, putting $\delta = \frac{\delta'}{\gamma'}$, we assume that

$$\overline{m}(\overline{x}),\ \overline{m}(\overline{y}) \leqq \gamma, \qquad |||\overline{x}-\overline{y}||| \leqq \delta.$$

If $m(x), m(y) \leqq \gamma'$ and $\|x-y\| \geqq 4\varepsilon$, then we have by §83 Theorem 1

$$|||x-y||| \geqq \frac{1}{2}\|x-y\| \geqq 2\varepsilon,$$

and hence $m(x-y) \geqq \varepsilon'$ by the definition of ε'. As we conclude by Theorems 1 in §83 and 4 in §81

$$\|x-y\| \leqq 2|||x-y||| \leqq 2(|||x|||+|||y|||) \leqq 4\gamma',$$

we obtain hence

$$\frac{1}{4}|||\overline{x}-\overline{y}|||\,\|x-y\| + m(\tfrac{1}{2}(x+y)) - \tfrac{1}{2}\{m(x)+m(y)\}$$
$$\leqq \delta\gamma' - \delta' = 0.$$

Theregore we have

$$\frac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\} \leqq \overline{m}(\tfrac{1}{2}(\overline{x}+\overline{y})) + \tfrac{1}{4}|||\overline{x}-\overline{y}|||\,4\varepsilon,$$

because $m(\frac{1}{2}(x+y)) - \frac{1}{2}\{m(x) + m(y)\} \leqq 0$. Accordingly $\overline{R}$ is uniformly even by definition.

Theorem 5. If R is uniformly finite, uniformly increasing, and uniformly even, then the modular adjoint space $\overline{R}$ is uniformly convex.

Proof. As R is uniformly increasing by assumption, for any $\gamma > 0$ we can find by §86 Theorem 5 $\gamma' > 1$ such that $\overline{m}(\overline{x}) \leqq \gamma$, $m(x) \geqq \gamma'$ implies $\overline{x}(x) - m(x) \leqq 0$. Thus we have by §80(1)

$$\overline{m}(\overline{x}) = \sup_{m(x)\leqq\gamma'} \{\overline{x}(x) - m(x)\} \quad \text{for } \overline{m}(\overline{x}) \leqq \gamma.$$

If $\overline{m}(\overline{x}), \overline{m}(\overline{y}) \leqq \gamma$, then we have by the modular condition 5)

$$\overline{m}(\tfrac{1}{2}(\overline{x}+\overline{y})) \leqq \tfrac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\} \leqq \gamma,$$

and hence

$$\overline{m}(\tfrac{1}{2}(\overline{x}+\overline{y})) = \sup_{m(x)\leqq\gamma'} \{\tfrac{1}{2}(\overline{x}(x) + \overline{y}(x)) - m(x)\}.$$

On the other hand we have by §80(2) for every $x, y \in R$

$$\overline{x}(x) + \overline{y}(x) = \overline{x}(x+y) + \overline{y}(x-y) - (\overline{x}-\overline{y})(y)$$

$$\leqq \overline{m}(\overline{x}) + \overline{m}(\overline{y}) + m(x+y) + m(x-y) - (\overline{x}-\overline{y})(y).$$

Thus we obtain for every $y \in R$

$$\frac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\} \geqq \overline{m}(\tfrac{1}{2}(\overline{x}+\overline{y})) + \tfrac{1}{2}(\overline{x}-\overline{y})(y)$$
$$- \sup_{m(x)\leqq\gamma'} \{\tfrac{1}{2}(m(x+y) + m(x-y)) - m(x)\}.$$

For any positive number $\varepsilon < 1$, if $\overline{m}(\overline{x}-\overline{y}) \geqq \varepsilon$, then we have by Theorems 4 in §81 and 1 in §83 $\|\overline{x}-\overline{y}\| \geqq |||\overline{x}-\overline{y}||| \geqq \varepsilon$, and hence we can find by §83 Theorem 2 $y_0 \in R$ such that

$$(\overline{x}-\overline{y})(y_0) > \frac{1}{2}\varepsilon, \qquad |||y_0||| = 1,$$

and consequently $m(y_0) = 1$ by §86 Theorem 2. As R is uniformly finite by assumption, putting

$$\gamma'' = \sup_{m(x)\leqq 1} m(2\gamma' x) < +\infty,$$

we have that $m(x) \leqq \gamma'$ implies $m(2x) \leqq \gamma''$. Because we have by the modular condition 5) $m(\frac{1}{\gamma'}x) \leqq \frac{1}{\gamma'}m(x) \leqq 1$, and

$$m(2x) = m(2\gamma'(\tfrac{1}{\gamma'}x)).$$

For such γ'', as R is uniformly even by assumption, we can find by definition a positive number $\delta < 1$ such that

$$\frac{1}{2}\{m(x)+m(y)\} \leqq m\left(\frac{1}{2}(x+y)\right) + \frac{\varepsilon}{12} |||x-y|||$$

for $m(x),\ m(y) \leqq \gamma''$, $|||x-y||| \leqq 2\delta$. Then $m(x) \leqq \gamma'$ implies

$$m(x \pm \delta y_0) \leqq \frac{1}{2}\{m(2x)+m(2\delta y_0)\} \leqq \gamma'',$$

because $0<\delta<1<\gamma'$, $m(y_0)=1$. As

$$|||(x+\delta y_0)-(x-\delta y_0)||| = 2\delta\, |||y_0||| = 2\delta,$$

we have therefore that $m(x) \leqq \gamma'$ implies

$$\frac{1}{2}\{m(x+\delta y_0)+m(x-\delta y_0)\} \leqq m(x) + \frac{\varepsilon\delta}{6}.$$

Consequently we obtain

$$\sup_{m(x)\leqq\gamma'} \left\{\frac{1}{2}(m(x+\delta y_0)+m(x-\delta y_0)) - m(x)\right\} \leqq \frac{\varepsilon\delta}{6}.$$

Furthermore, as $\frac{1}{2}(\bar{x}-\bar{y})(\delta y_0) > \frac{\varepsilon\delta}{4}$, we conclude hence

$$\frac{1}{2}\{\bar{m}(\bar{x})+\bar{m}(\bar{y})\} \geqq \bar{m}\left(\frac{1}{2}(\bar{x}+\bar{y})\right) + \frac{\varepsilon\delta}{4} - \frac{\varepsilon\delta}{6} = \bar{m}\left(\frac{1}{2}(\bar{x}+\bar{y})\right) + \frac{\varepsilon\delta}{12}.$$

Therefore $\bar{R}$ is uniformly convex by definition.

§89 $L_{p(t)}$ spaces

For two measurable functions $x(t)$ and $y(t)$ on the closed interval $0 \leqq t \leqq 1$, we shall write $x = y$, if $\{t:\ x(t) \neq y(t)\}$ is a point set of measure zero in the sense of Lebesgue. Then we see easily that the totality of measurable functions constitutes a linear space in this sense, namely the quotient space of the linear space composed of all measurable functions by the linear space composed only of all measurable functions which vanish up to a set of measure zero.

Let $p(t)$ be a measurable function subject to the condition:

$$1 \leqq p(t) \leqq +\infty \qquad (0 \leqq t \leqq 1).$$

We admit for $p(t)$ to assume $+\infty$. We shall denote by $L_{p(t)}$ the totality of measurable functions $x(t)$ for which we can find $\lambda > 0$ such that

$$\int_0^1 \frac{1}{p(t)} |\lambda x(t)|^{p(t)} dt < +\infty.$$

Here we adopt the convention: $\frac{1}{+\infty}\alpha^{+\infty} = \lim_{\nu\to\infty} \frac{1}{\nu}\alpha^{\nu}$, that is,

$$\frac{1}{+\infty}\alpha^{+\infty} = \begin{cases} 0 & \text{for } 0 \leqq \alpha \leqq 1 \\ +\infty & \text{for } \alpha > 1. \end{cases}$$

Then, putting

$$m(x) = \int_0^1 \frac{1}{p(t)} |x(t)|^{p(t)} dt,$$

we see easily that $L_{p(t)}$ constitutes a modulared space.

Theorem 1. $L_{p(t)}$ is modular complete.

Proof. If $\lim_{\mu,\nu\to\infty} m(\xi(x_\mu - x_\nu))=0$ for every $\xi > 0$, then we can find a subsequence x_{ν_μ} $(\mu=1, 2, \ldots)$ from x_ν $(\nu=1, 2, \ldots)$ such that $m(2^\mu(x_{\nu_{\mu+1}} - x_{\nu_\mu})) \leqq 1$. Then we have by §78(2)

$$m\left(\sum_{\mu=1}^{\kappa} |x_{\nu_{\mu+1}} - x_{\nu_\mu}|\right) \leqq \sum_{\mu=1}^{\kappa} \frac{1}{2^\mu} < 1$$

for every $\kappa = 1, 2, \ldots$. Accordingly we see that

$$\sum_{\mu=1}^{\kappa} |x_{\nu_{\mu+1}}(t) - x_{\nu_\mu}(t)|$$

is convergent up to a point set of measure zero. Putting

$$x(t) = x_{\nu_1}(t) + \sum_{\mu=1}^{\infty} \{x_{\nu_{\mu+1}}(t) - x_{\nu_\mu}(t)\},$$

we see easily that $m(x - x_{\nu_1}) \leqq 1$, and hence $x \in L_{p(t)}$. Furthermore we have by the formula §78(2)

$$m\left(2^\rho \sum_{\mu=2\rho}^{\kappa} (x_{\nu_{\mu+1}} - x_{\nu_\mu})\right) \leqq \sum_{\mu=2\rho}^{\kappa} \frac{2^\rho}{2^\mu} \leqq \frac{1}{2^{\rho-1}},$$

and hence $m(2^\rho(x - x_{\nu_{2\rho}})) \leqq \frac{1}{2^{\rho-1}}$ for every $\rho = 1, 2, \ldots$. Thus we conclude $\text{m-}\lim_{\mu\to\infty} x_{\nu_\mu} = x$. From this relation we see easily that we have $\text{m-}\lim_{\nu\to\infty} x_\nu = x$.

Theorem 2. If $L_{p(t)}$ is finite, then $p(t)$ is bounded up to a point set of measure zero.

Proof. Let χ_ν be the characteristic function of the point set

$$\{t:\ \nu \leqq p(t) < \nu+1\} \qquad (\nu = 1, 2, \ldots).$$

If there is a sequence ν_μ $(\mu = 1, 2, \ldots)$ such that

$$\int_0^1 \chi_{\nu_\mu}(t)\, dt \neq 0 \qquad \text{for every } \mu = 1, 2, \ldots,$$

then we can find $\alpha_\mu > 0$ $(\mu = 1, 2, \ldots)$ such that

$$\int_0^1 \frac{1}{p(t)} \alpha_\mu^{p(t)} \chi_{\nu_\mu}(t)\, dt = 1 \qquad (\mu = 1, 2, \ldots),$$

and, putting $x(t) = \sum_{\mu=1}^{\infty} \alpha_\mu \chi_{\nu_\mu}(t)$, we see easily that $m(x) = +\infty$ but $m(\frac{1}{2}x) \leqq \sum_{\mu=1}^{\infty} \frac{1}{2^{\nu_\mu}} < +\infty$. Thus we obtain our assertion.

Theorem 3. If $p(t)$ is bounded, then $L_{p(t)}$ is uniformly finite and uniformly simple.

Proof. If $p(t) \leqq \gamma$, then $m(x) \leqq 1$ implies $m(\xi x) \leqq \xi^{\gamma}$ for every $\xi \geqq 1$. Thus $L_{p(t)}$ is uniformly finite. Furthermore $m(x) \geqq 1$ implies $m(\xi x) \geqq \xi^{\gamma}$ for $0 < \xi \leqq 1$. Hence $L_{p(t)}$ is uniformly simple by definition.

Theorem 4. *If $L_{p(t)}$ is uniformly simple, then $p(t)$ is bounded up to a point set of measure zero.*

Proof. If $\alpha = \inf_{m(x) \geqq 1} m(\frac{1}{2} x) > 0$, then for a positive number β such that $\frac{1}{2^{\beta}} < \alpha$, denoting by χ the characteristic function of $\{t : p(t) > \beta\}$ we have $\int_0^1 \chi(t)\,dt = 0$. Because, if we have $\int_0^1 \chi(t)\,dt \neq 0$, then we can find $\xi > 0$ such that

$$\int_0^1 \frac{1}{p(t)} \xi^{p(t)} \chi(t)\,dt = 1,$$

and, putting $x(t) = \xi \chi(t)$, we have $m(x) = 1$ but $m(\frac{1}{2} x) \leqq \frac{1}{2^{\beta}} < \alpha$.

Theorem 5. *$L_{p(t)}$ is simple, if and only if $p(t)$ is finite up to a point set of measure zero.*

Proof. Denoting by χ_{∞} the characteristic function of the point set $\{t : p(t) = +\infty\}$, we have obviously by definition

$$\int_0^1 \frac{1}{p(t)} (\chi_{\infty}(t))^{p(t)}\,dt = 0.$$

Thus, if $L_{p(t)}$ is simple, then $\chi_{\infty}(t) = 0$ up to a point set of measure zero. Conversely, if $1 \leqq p(t) < +\infty$, then we see easily by definition that $m(x) = 0$ implies $x(t) = 0$ up to a point set of measure zero.

The totality of finite elements in $L_{p(t)}$ is called the *finite subspace* of $L_{p(t)}$ and denoted by $L^{f}_{p(t)}$. We see easily that the finite subspace $L^{f}_{p(t)}$ is a linear manifold of $L_{p(t)}$. As $L_{p(t)}$ is modular complete by Theorem 1, we obtain by §86 Theorem 1

Theorem 6. *The finite subspace $L^{f}_{p(t)}$ of $L_{p(t)}$ is finite and modular complete.*

Theorem 7. *For a sequence $x_{\nu} \in L^{f}_{p(t)}$ $(\nu = 1, 2, \dots)$, if*

$$\lim_{\nu \to \infty} x_{\nu}(t) = 0 \qquad (0 \leqq t \leqq 1)$$

and there is $x_0 \in L^{f}_{p(t)}$ such that $|x_{\nu}(t)| \leqq x_0(t)$ $(0 \leqq t \leqq 1)$, then we have $\lim_{\nu \to \infty} |||x_{\nu}||| = 0$.

Proof. On account of Lebesgue's theorem, we conclude from our

assumption that $\lim_{\nu \to \infty} m(\xi x_{\nu}) = 0$ for every $\xi > 0$, and hence by §81 Theorem 5 $\lim_{\nu \to \infty} |||x_{\nu}||| = 0$.

Corresponding to $p(t)$, we define $q(t)$ $(0 \leqq t \leqq 1)$ as

$$\frac{1}{p(t)} + \frac{1}{q(t)} = 1 \qquad (0 \leqq t \leqq 1).$$

Denoting by B^{+} the totality of bounded positive measurable functions on the closed interval: $0 \leqq t \leqq 1$, we have then for every positive measurable function $y(t)$

$$(*) \quad \int_0^1 \frac{1}{q(t)} y(t)^{q(t)}\,dt = \sup_{x \in B^{+}} \{ \int_0^1 x(t) y(t)\,dt - \int_0^1 \frac{1}{p(t)} x(t)^{p(t)}\,dt \}.$$

Because, we have by Young's inequality

$$\int_0^1 x(t) y(t)\,dt \leqq \int_0^1 \frac{1}{p(t)} x(t)^{p(t)}\,dt + \int_0^1 \frac{1}{q(t)} y(t)^{q(t)}\,dt.$$

On the other hand, denoting by χ_{∞} the characteristic function of the point set $\{t : y(t) = +\infty\}$, if

$$\int_0^1 \frac{1}{q(t)} y(t)^{q(t)} \chi_{\infty}(t)\,dt = 0,$$

then we can find a sequence $y_{\nu} \in B^{+}$ $(\nu = 1, 2, \dots)$ such that

$$\lim_{\nu \to \infty} \int_0^1 \frac{1}{q(t)} y_{\nu}(t)^{q(t)}\,dt = \int_0^1 \frac{1}{q(t)} y(t)^{q(t)}\,dt,$$

and, putting

$$x_{\nu} = y_{\nu}(t)^{\frac{q(t)}{p(t)}} (1 - \chi_{\infty}(t)),$$

we have

$$\int_0^1 x_{\nu}(t) y_{\nu}(t)\,dt - \int_0^1 \frac{1}{p(t)} x_{\nu}(t)^{p(t)}\,dt = \int_0^1 \frac{1}{q(t)} y_{\nu}(t)^{q(t)}\,dt.$$

If

$$\int_0^1 \frac{1}{q(t)} y(t)^{q(t)} \chi_{\infty}(t)\,dt = +\infty,$$

then, we have

$$\lim_{\xi \to \infty} \{ \int_0^1 \xi \chi_{\infty}(t) y(t)\,dt - \int_0^1 \frac{1}{p(t)} (\xi \chi_{\infty}(t))^{p(t)}\,dt \}$$
$$= \lim_{\xi \to \infty} \xi \{ \int_0^1 \chi_{\infty}(t) y(t)\,dt - \int_0^1 \chi_{\infty}(t)\,dt \} = +\infty,$$

and $\xi \chi_{\infty} \in B^{+}$ for every $\xi > 0$.

Making use of the notation

$$m(x) = \int_0^1 \frac{1}{p(t)} |x(t)|^{p(t)}\,dt, \qquad \bar{m}(x) = \int_0^1 \frac{1}{q(t)} |x(t)|^{q(t)}\,dt,$$

we see easily by Young's inequality that if $x \in L_{p(t)}$ and $\bar{x} \in L_{q(t)}$, then $\bar{x}(t)\, x(t)$ is integrable and

$$| \int_0^1 \bar{x}(t) x(t)\,dt | \leqq m(x) + \bar{m}(\bar{x}).$$

Thus, putting

$$\bar{x}(x) = \int_0^1 \bar{x}(t) x(t)\,dt,$$

we see that $L_{q(t)}$ is contained in the modular adjoint space of $L_{p(t)}$ as a subspace.

Now we shall consider the case where $L_{p(t)}$ is simple, that is, $1 \leq p(t) < +\infty$ ($0 \leq t \leq 1$). Denoting by χ_ν the characteristic function of $\{t : p(t) \leq \nu\}$, we see easily that $\lim_{\nu\to\infty} \chi_\nu(t) = 1$ ($0 \leq t \leq 1$), and that $x \in L_{p(t)}$ implies $x\chi_\nu \in L_{p(t)}$ for every $\nu = 1, 2, \dots$. For each modular bounded linear functional φ on $L^f_{p(t)}$, as φ is bounded by the modular norm, we see easily by Theorem 7 that for any bounded sequence of measurable functions $x_\mu(t)$ ($\mu = 1, 2, \dots$), $\lim_{\mu\to\infty} x_\mu(t) = 0$ ($0 \leq t \leq 1$) implies

$$\lim_{\mu\to\infty} \varphi(x_\mu \chi_\nu) = 0 \qquad \text{for every } \nu = 1, 2, \dots,$$

because $x_\mu \in L_{p(t)}$ for every $\mu = 1, 2, \dots$. Therefore we obtain by Radon-Nikodym's theorem a measurable function $\bar{x}(t)$ such that for any characteristic function of measurable set χ we have

$$\varphi(\chi\chi_\nu) = \int_0^1 \bar{x}(t)\chi(t)\chi_\nu(t)\,dt \qquad (\nu = 1, 2, \dots).$$

From this relation we conclude easily by Theorem 7 that

$$\varphi(x) = \int_0^1 \bar{x}(t)x(t)\,dt \qquad \text{for every } x \in L^f_{p(t)}.$$

Furthermore we obtain by (*)

$$\bar{m}(\varphi) = \int_0^1 \frac{1}{q(t)}|\bar{x}(t)|^{q(t)}\,dt.$$

Therefore we have

Theorem 8. *If $L_{p(t)}$ is simple, then the modular adjoint space of the finite subspace $L^f_{p(t)}$ coincides with $L_{q(t)}$ for $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$.*

Recalling Theorems 2 and 3, we obtain therefore

Theorem 9. *The modular adjoint space of $L_{p(t)}$ coincides with $L_{q(t)}$ if and only if $\sup_t p(t) < +\infty$ up to a point set of measure zero.*

Since we have $\sup_t q(t) < +\infty$ if and only if $\inf_t p(t) > 1$, we conclude easily from Theorem 9

Theorem 10. *$L_{p(t)}$ is regular, if and only if*

$$1 < \inf_t p(t) \leq \sup_t p(t) < +\infty$$

up to a point set of measure zero.

Recalling Theorems 3 in §85 and 7 in §86, we obtain by Theorem 3

Theorem 11. *If $\inf_t p(t) > 1$, then $L_{p(t)}$ is uniformly mono-*

tone and uniformly increasing.

Theorem 12. *If $1 < \inf_t p(t) \leq \sup_t p(t) < +\infty$, then $L_{p(t)}$ is uniformly convex and uniformly even at the same time; and both the modular norm and the associated norm of $L_{p(t)}$ are uniformly convex and uniformly even at the same time.*

Proof. Let χ_0 be the characteristic function of $\{t : p(t) \geq 2\}$ and χ_1 that of $\{t : p(t) < 2\}$. Then we see by definition that we have for every $x \in L_{p(t)}$

$$m(x\chi_0) + m(x\chi_1) = m(x).$$

For two positive numbers γ, ε, we assume

$$m(a),\ m(b) \leq \gamma, \qquad m(a-b) \geq \varepsilon.$$

Then we have $m((a-b)\chi_0) \geq \frac{1}{2}\varepsilon$ or $m((a-b)\chi_1) \geq \frac{1}{2}\varepsilon$. If $m((a-b)\chi_0) \geq \frac{1}{2}\varepsilon$, then by virtue of the inequality

$$\frac{|\xi|^p + |\eta|^p}{2} \geq \left|\frac{\xi+\eta}{2}\right|^p + \left|\frac{\xi-\eta}{2}\right|^p \qquad \text{for } p \geq 2,$$

we obtain

$$\frac{1}{2}\{m(a\chi_0) + m(b\chi_0)\} \geq m(\tfrac{1}{2}(a+b)\chi_0) + m(\tfrac{1}{2}(a-b)\chi_0).$$

On the other hand, putting $p_0 = \sup_t p(t) < +\infty$, we have

$$m(\tfrac{1}{2}(a-b)\chi_0) \geq \frac{1}{2^{p_0}} m((a-b)\chi_0) \geq \frac{1}{2^{p_0+1}}\varepsilon.$$

Furthermore we have by the modular condition 5)

$$\frac{1}{2}\{m(a\chi_1) + m(b\chi_1)\} \geq m(\tfrac{1}{2}(a+b)\chi_1).$$

Therefore we obtain $\frac{1}{2}\{m(a) + m(b)\} \geq m(\frac{1}{2}(a+b)) + \frac{1}{2^{p_0+1}}\varepsilon$.

Secondly we consider the other case: $m((a-b)\chi_1) \geq \frac{1}{2}\varepsilon$. If we put $\varepsilon' = \mathrm{Min}\{\frac{\varepsilon}{8\gamma}, \frac{1}{2}\}$ and denote by χ_2 the characteristic function of $\{t : |a(t) - b(t)| \geq \varepsilon'(|a(t)| + |b(t)|),\ p(t) < 2\}$, then we have

$$m((a-b)(\chi_1 - \chi_2)) \leq m(\varepsilon'(|a| + |b|))$$
$$\leq \frac{1}{2}\{m(2\varepsilon' a) + m(2\varepsilon' b)\} \leq 2\varepsilon'\gamma \leq \frac{\varepsilon}{4},$$

and hence $m((a-b)\chi_2) \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$. By virtue of the inequality

$$\frac{|\xi|^p + |\eta|^p}{2} \geq \left|\frac{\xi+\eta}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{\xi-\eta}{|\xi|+|\eta|}\right|^{2-p}\left|\frac{\xi-\eta}{2}\right|^p$$

for $1 \leq p \leq 2$, putting $1 + \sigma = \inf_t p(t)$, we obtain

$$\frac{1}{2}\{m(a\chi_2) + m(b\chi_2)\} \geq m(\tfrac{1}{2}(a+b)\chi_2) + \frac{\sigma}{2}\varepsilon' m(\tfrac{1}{2}(a-b)\chi_2).$$

On the other hand, we have

$$m(\tfrac{1}{2}(a-b)\chi_2) \geq \frac{1}{4} m((a-b)\chi_2) \geq \frac{\varepsilon}{16},$$

and by the modular condition 5)

$$\frac{1}{2}\{m(a(1-\chi_2)) + m(b(1-\chi_2))\} \geqq m(\tfrac{1}{2}(a+b)(1-\chi_2)).$$

Therefore we obtain

$$\frac{1}{2}\{m(a) + m(b)\} \geqq m(\tfrac{1}{2}(a+b)) + \frac{\sigma\varepsilon'\varepsilon}{32}.$$

Consequently we have

$$\frac{1}{2}\{m(a) + m(b)\} \geqq m(\tfrac{1}{2}(a+b)) + \operatorname{Min}\{\frac{\varepsilon}{2^{p_0+1}}, \frac{\sigma\varepsilon'\varepsilon}{32}\}.$$

Thus $L_{p(t)}$ is uniformly convex. Accordingly we conclude by §88 Theorem 4 that $L_{p(t)}$ is uniformly even. Furthermore, recalling §87 Theorem 3 and §88 Theorem 4, we see easily that the modular norm of $L_{p(t)}$ is uniformly convex and uniformly even at the same time; and hence further by Theorems 3 and 4 in §77 that the associated norm of $L_{p(t)}$ also is so.

Finally we consider the special case where $p(t)$ is a constant. In this case, putting $p = p(t)$ $(0 \leqq t \leqq 1)$, we see easily that the modular $m(x)$, the modular norm $|||x|||$, and the associated norm $\|x\|$ are given as follows:

$$m(x) = \int_0^1 \frac{1}{p}|x(t)|^p dt,$$

$$|||x||| = \{\int_0^1 \frac{1}{p}|x(t)|^p dt\}^{\frac{1}{p}} \qquad \text{for } p < +\infty,$$

$$|||x||| = \lim_{\lambda\to+\infty}\{\int_0^1 |x(t)|^\lambda dt\}^{\frac{1}{\lambda}} \qquad \text{for } p = +\infty,$$

$$\|x\| = p^{\frac{1}{p}} q^{\frac{1}{q}} |||x||| \qquad \text{for } 1 < p < +\infty, \ \frac{1}{p} + \frac{1}{q} = 1,$$

$$\|x\| = |||x||| \qquad \text{for } p = 1 \text{ or } +\infty.$$

§90 L_{p-0}, L_{p+0} spaces

For measurable functions $x(t)$ on the closed interval $0 \leqq t \leqq 1$, we shall make use of the notations:

$$\|x\|_p = \{\int_0^1 |x(t)|^p dt\}^{\frac{1}{p}} \qquad \text{for } 1 \leqq p < +\infty,$$

$$\|x\|_\infty = \lim_{p\to+\infty}\{\int_0^1 |x(t)|^p dt\}^{\frac{1}{p}}.$$

On account of Hölder's inequality, we have then

(1) $$\int_0^1 |x(t)y(t)|dt \leqq \|x\|_p \|y\|_q \qquad \text{for } \frac{1}{p} + \frac{1}{q} = 1.$$

Furthermore, denoting by B the totality of bounded measurable functions,

we can prove easily for every measurable function $y(t)$

(2) $$\|y\|_q = \sup_{\|x\|_p \leqq 1,\ x\in B} \int_0^1 |x(t)y(t)|dt.$$

By virtue of the inequality

$$\{\frac{1}{n}\sum_{\nu=1}^{n}|\xi_\nu|^\rho\}^{\frac{1}{\rho}} \geqq \{\frac{1}{n}\sum_{\nu=1}^{n}|\xi_\nu|^\sigma\}^{\frac{1}{\sigma}} \qquad \text{for } \rho > \sigma > 0,$$

we obtain easily

(3) $$\|x\|_\rho \geqq \|x\|_\sigma \qquad \text{for } \rho > \sigma \geqq 1.$$

We see easily that $\|x\|_p$ is a norm on L_p and it is evident by definition that $\|x\|_p$ is equivalent to the modular norm of L_p. Thus, denoting by $\mathcal{V}^p$ the modular topology of L_p, we see that $\mathcal{V}^p$ is the induced linear topology of L_p by the norm $\|x\|_p$.

We have obviously by definition

(4) $$L_\rho \subset L_\sigma \qquad \text{for } \rho > \sigma \geqq 1.$$

For any $p > 1$, putting

$$L_{p-0} = \prod_{1<\xi<p} L_\xi,$$

we obtain a linear space L_{p-0}. Furthermore we define a linear topology $\mathcal{V}^{p-0}$ on L_{p-0} as

$$\mathcal{V}^{p-0} = \bigcup_{1<\xi<p} \mathcal{V}^\xi,$$

considering every $\mathcal{V}^\xi$ as the relative linear topology of $\mathcal{V}^\xi$ in L_{p-0}. Then $\mathcal{V}^{p-0}$ is sequential, because we can find a sequence $p_1 < p_2 < \dots$ such that $\lim_{\nu\to\infty} p_\nu = p$, and we have by (3)

$$\mathcal{V}^{p-0} = \bigcup_{\nu=1}^{\infty} \mathcal{V}^{p_\nu}.$$

But $\mathcal{V}^{p-0}$ is not of single vicinity. Because, putting

(5) $$U_p = \{x : \|x\|_p \leqq 1\},$$

we have by the relation (3) that $U_{p_1} \supset U_{p_2} \supset \dots$ and $U_{p_\nu} L_{p-0}$ $(\nu = 1, 2, \dots)$ constitutes a basis of $\mathcal{V}^{p-0}$. Thus, if $\mathcal{V}^{p-0}$ is of single vicinity, then we can find ν_0 such that $U_{p_{\nu_0}} L_{p-0}$ is a basis of $\mathcal{V}^{p-0}$, and hence the norm $\|x\|_{p_{\nu_0}}$ must be equivalent to the norm $\|x\|_{p_\nu}$ for every $\nu \geqq \nu_0$ in L_{p-0}. However we have obviously $B \subset L_p$ for every $p \geqq 1$, and $\|x\|_\rho$ is not equivalent to $\|x\|_\sigma$ in B for $\rho \neq \sigma$. In fact, if $\rho > \sigma$, then for every positive number $\varepsilon < 1$, denoting by χ_ε the characteristic function of the interval $0 < t < \varepsilon$, we see easily that $\|\frac{1}{\varepsilon^{\frac{1}{\rho}}}\chi_\varepsilon\|_\rho = 1$ but $\|\frac{1}{\varepsilon^{\frac{1}{\rho}}}\chi_\varepsilon\|_\sigma = \varepsilon^{\frac{1}{\sigma}-\frac{1}{\rho}}$.

As every φ^p is convex, φ^{p-0} also is convex by §66 Theorem 3. As every φ^p is separative, φ^{p-0} also is so obviously by definition. Therefore φ^{p-0} is reflexive by Theorems 1 in §58 and 4 in §68.

For $1 \leq p < +\infty$, putting

$$L_{p+0} = \sum_{\xi > p} L_\xi ,$$

we see easily by (4) that L_{p+0} is a linear space. If

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q < +\infty ,$$

then for any $\bar{x} \in L_{q+0}$, we can find $\eta > q$ such that $\bar{x} \in L_\eta$, and, putting $\frac{1}{\xi} + \frac{1}{\eta} = 1$, we have $\xi < p$ and $\bar{x}$ is a continuous linear functional on L_ξ by the relation

$$\bar{x}(x) = \int_0^1 \bar{x}(t)\, x(t)\, dt \qquad \text{for } x \in L_\xi .$$

Consequently every $\bar{x} \in L_{q+0}$ may be considered as a continuous linear functional on L_{p-0} by φ^{p-0}. Furthermore, as L_{p-0} is standard, for each bounded linear functional φ on L_{p-0} we can find by §64 Theorem 2 $\xi < p$ such that φ is continuous by φ^ξ in L_{p-0}. As L_{p-0} is dense in L_ξ by φ^ξ, there exists uniquely $\bar{x} \in L_\eta$ such that

$$\bar{x}(x) = \varphi(x) \qquad \text{for every } x \in L_{p-0} .$$

Therefore L_{q+0} coincides with the adjoint space of L_{p-0} by φ^{p-0}. Thus we can introduce the adjoint topology of φ^{p-0} into L_{q+0}, which will be denoted by φ^{q+0}.

If a linear functional φ on L_{q+0} is bounded by φ^{q+0}, then for any $\xi < p$, putting $\frac{1}{\xi} + \frac{1}{\eta} = 1$, as U_η is by §64 Theorem 6 a bounded manifold of L_{q+0} by φ^{q+0}, there exists uniquely $x_\xi \in L_\xi$ such that

$$\varphi(\bar{x}) = \bar{x}(x_\xi) \qquad \text{for every } \bar{x} \in L_\eta ,$$

because L_ξ coincides by §89 Theorem 9 with the adjoint space of L_η. Furthermore, for any other $\xi' < p$, such $x_{\xi'}$ coincides with x_ξ. Because, if $\xi < \xi'$, then, putting $\frac{1}{\xi'} + \frac{1}{\eta'} = 1$, we have $\eta' < \eta$, and hence we have by the relation (4)

$$\bar{x}(x_\xi) = \bar{x}(x_{\xi'}) \qquad \text{for every } \bar{x} \in L_\eta .$$

As $B \subset L_\eta \subset L_{\eta'}$ and B is dense in $L_{\eta'}$ by $\varphi^{\eta'}$, we conclude hence

$$\bar{x}(x_\xi) = \bar{x}(x_{\xi'}) \qquad \text{for every } \bar{x} \in L_{\eta'} .$$

Consequently we obtain $x_\xi \in L_{\xi'}$ and $x_\xi = x_{\xi'}$. Therefore we conclude

that $x_\xi \in L_{p-0}$ and

$$\varphi(\bar{x}) = \bar{x}(x_\xi) \qquad \text{for every } \bar{x} \in L_{q+0} .$$

Thus L_{p-0} is regular.

Now we can state: L_{p-0} <u>is sequential and regular, but not of single vicinity.</u> L_{q+0} <u>is the adjoint space of</u> L_{p-0} <u>for</u>

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q < +\infty .$$

Accordingly, L_{q+0} also is regular and has a sequential root by §69 Theorem 2. By virtue of Theorems 1 in §67 and 1 §59, we see that L_{p-0} is complete. We have obviously by definition that $L_{p-0} > L_p$ but L_{p-0} does not coincides with L_p. Because, as $\varphi^{p-0} < \varphi^p$ in L_p by definition, if L_{p-0} coincides with L_p, then we obtain by §59 Theorem 4 that $\varphi^{p-0} = \varphi^p$, contradicting that φ^{p-0} is not of single vicinity. Therefore we see further that $L_{q+0} < L_q$ but L_{q+0} does not coincides with L_q.

CHAPTER XII

COMBINATION OF LINEAR TOPOLOGICAL SPACES

§91 Product spaces

Let R and S be two linear topological spaces with linear topologies $\mathfrak{U}$ and $\mathfrak{V}$ respectively. For a basis $\mathfrak{A}$ of $\mathfrak{U}$ and a basis $\mathfrak{B}$ of $\mathfrak{V}$, we obtain by §56 Theorem 7 a basis (U, V) $(U \in \mathfrak{A}, V \in \mathfrak{B})$ of the product $(\mathfrak{U}, \mathfrak{V})$ which is the linear topology of the product space (R, S). In this § we shall consider the adjoint space of the product space (R, S).

Theorem 1. For a bounded manifold A of (R, S) we can find a bounded manifold B of R and a bounded manifold C of S such that

$$A \subset (B, C).$$

Proof. Putting $B = \{x : (x, y) \in A\}$, $C = \{y : (x, y) \in A\}$, we have obviously $A \subset (B, C)$. For any $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$, as A is bounded by assumption, we can find $\alpha > 0$ such that $A \subset (\alpha U, \alpha V)$, and hence we have $B \subset \alpha U$, $C \subset \alpha V$. Therefore B and C are bounded by definition.

Let $\bar{R}$ be the adjoint space of R and $\bar{S}$ that of S. For any $\bar{x} \in \bar{R}$ and $\bar{y} \in \bar{S}$, putting

$$(\bar{x}, \bar{y})(x, y) = \bar{x}(x) + \bar{y}(y) \qquad (x \in R,\ y \in S)$$

we obtain obviously a linear functional $(\bar{x}, \bar{y})$ on the product space (R, S). Furthermore $(\bar{x}, \bar{y})$ is bounded. Because, for any bounded manifold A of (R, S), we can find by Theorem 1 a bounded manifold B of R and C of S such that $A \subset (B, C)$. For such B, C, we have by definition $\sup_{x \in B} |\bar{x}(x)| < +\infty$, $\sup_{y \in C} |\bar{y}(y)| < +\infty$, and hence

$$\sup_{(x,y) \in A} |(\bar{x}, \bar{y})(x, y)| \leq \sup_{x \in B, y \in C} |\bar{x}(x) + \bar{y}(y)| < +\infty.$$

Conversely, for any bounded linear functional φ on (R, S), putting

$$\bar{x}(x) = \varphi(x, 0), \qquad \bar{y}(y) = \varphi(0, y) \qquad (x \in R,\ y \in S),$$

we obtain a linear functional $\bar{x}$ on R and $\bar{y}$ on S. For any bounded manifold B of R, as $(B, \{0\})$ is a bounded manifold of (R, S), we have

$$\sup_{x \in B} |\bar{x}(x)| = \sup_{(x,y) \in (B, \{0\})} |\varphi(x, y)| < +\infty,$$

and hence $\bar{x}$ is bounded, that is, we have $\bar{x} \in \bar{R}$. We also can prove

likewise that $\bar{y} \in \bar{S}$. Furthermore we have

$$(\bar{x}, \bar{y})(x, y) = \varphi(x, 0) + \varphi(0, y) = \varphi(x, y).$$

Therefore the product space $(\bar{R}, \bar{S})$ may be considered as the adjoint space of (R, S) merely as a linear space.

Now we shall prove that the linear topology of $(\bar{R}, \bar{S})$ coincides with the adjoint topology of the product $(\mathfrak{U}, \mathfrak{V})$. Let $\bar{\mathfrak{U}}$ be the adjoint topology of $\mathfrak{U}$ and $\bar{\mathfrak{V}}$ that of $\mathfrak{V}$. For any bounded manifold A of (R, S) we can find by Theorem 1 a bounded manifold B of R and C of S such that $A \subset (B, C)$. For such (B, C), putting

$$\bar{U} = \{\bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in B\},$$
$$\bar{V} = \{\bar{y} : |\bar{y}(y)| \leq 1 \quad \text{for every } y \in C\},$$

we have by definition $(\bar{U}, \bar{V}) \in (\bar{\mathfrak{U}}, \bar{\mathfrak{V}})$, and

$$\begin{aligned}
\{(\bar{x}, \bar{y}) : |(\bar{x}, \bar{y})(x, y)| \leq 1 \quad &\text{for every } (x, y) \in A\} \\
\supset \{(\bar{x}, \bar{y}) : |\bar{x}(x) + \bar{y}(y)| \leq 1 \quad &\text{for every } x \in B,\ y \in C\} \\
\supset \{(\bar{x}, \bar{y}) : |\bar{x}(x)| \leq \tfrac{1}{2},\ |\bar{y}(y)| \leq \tfrac{1}{2} \quad &\text{for } x \in B,\ y \in C\} \\
= (\tfrac{1}{2}\bar{U}, \tfrac{1}{2}\bar{V}).&
\end{aligned}$$

Thus the adjoint topology of $(\mathfrak{U}, \mathfrak{V})$ is weaker than $(\bar{\mathfrak{U}}, \bar{\mathfrak{V}})$. On the other hand, for any $\bar{U} \in \bar{\mathfrak{U}}$ and $\bar{V} \in \bar{\mathfrak{V}}$, we can find a bounded manifold B of R and C of S such that

$$\bar{U} \supset \{\bar{x} : |\bar{x}(x)| \leq 1 \quad \text{for every } x \in B\},$$
$$\bar{V} \supset \{\bar{y} : |\bar{y}(y)| \leq 1 \quad \text{for every } y \in C\};$$

Here we can assume obviously that B and C are symmetric, that is, we have $B = (-1)B$, $C = (-1)C$. Then (B, C) is obviously a bounded manifold of (R, S) and we have

$$\begin{aligned}
\{(\bar{x}, \bar{y}) : |\bar{x}(x) + \bar{y}(y)| \leq 1 \quad &\text{for every } x \in B,\ y \in C\} \\
\subset \{(\bar{x}, \bar{y}) : |\bar{x}(x)| \leq 1,\ |\bar{y}(y)| \leq 1 \quad &\text{for every } x \in B,\ y \in C\} \\
= (\bar{U}, \bar{V}).&
\end{aligned}$$

Thus the adjoint topology of $(\mathfrak{U}, \mathfrak{V})$ is stronger than $(\bar{\mathfrak{U}}, \bar{\mathfrak{V}})$.

Consequently we obtain

Theorem 2. The product space $(\bar{R}, \bar{S})$ coincides with the adjoint space of the product space (R, S) by the relation:

$$(\bar{x}, \bar{y})(x, y) = \bar{x}(x) + \bar{y}(y) \qquad (\bar{x} \in \bar{R},\ \bar{y} \in \bar{S},\ x \in R,\ y \in S),$$

for the adjoint space $\bar{R}$ of R and that $\bar{S}$ of S.

Theorem 3. If both R and S are standard, then the product space (R, S) also is standard.

Proof. For any bounded manifold $\bar{A}$ of $(\bar{R}, \bar{S})$ we can find by Theorem 1 a bounded manifold $\bar{B}$ of $\bar{R}$ and $\bar{C}$ of $\bar{S}$ such that $\bar{A} \subset (\bar{B}, \bar{C})$. For such $\bar{B}$, $\bar{C}$, as both R and S are standard, putting

$$U = \{ x : |\bar{x}(x)| \leqq 1 \quad \text{for every} \quad \bar{x} \in \bar{B} \},$$

$$V = \{ y : |\bar{y}(y)| \leqq 1 \quad \text{for every} \quad \bar{y} \in \bar{C} \},$$

we have by §64 Theorem 8 that $U \in \mathfrak{U}$, $V \in \mathfrak{V}$, and

$$\sup_{(x, y) \in (U, V),\ (\bar{x}, \bar{y}) \in \bar{A}} |(\bar{x}, \bar{y})(x, y)| \leqq \sup_{x \in U,\ y \in V,\ \bar{x} \in \bar{B},\ \bar{y} \in \bar{C}} |\bar{x}(x) + \bar{y}(y)| \leqq 2 .$$

Thus $\bar{A}$ is uniformly bounded by definition, and hence (R, S) is standard by §64 Theorem 10, because (R, S) is obviously convex by §56 Theorem 7.

If both R and S are separated, then (R, S) is obviously separated by definition. Therefore we obtain by Theorem 3 and §68 Theorem 4

Theorem 4. If both R and S are reflexive, then the product space (R, S) also is reflexive.

By virtue of Theorem 2, we conclude immediately

Theorem 5. If both R and S are regular, then the product space (R, S) also is regular.

§92 Product norms

Let R and S be two normed spaces. A norm $\|(x, y)\|$ on the product space (R, S) is said to be a product norm, if

$$\|(x, 0)\| = \|x\| \qquad \text{for every } x \in R,$$

$$\|(0, y)\| = \|y\| \qquad \text{for every } y \in S,$$

$$\|(x, y)\| = \|(x, -y)\| \qquad \text{for every } x \in R,\ y \in S.$$

Putting

(1) $\|(x, y)\|_M = \|x\| + \|y\| \qquad (x \in R,\ y \in S),$

we obtain obviously a product norm $\|(x, y)\|_M$. This product norm is

called the maximum norm. And, putting

(2) $\|(x, y)\|_m = \operatorname{Max}\{\|x\|, \|y\|\} \qquad (x \in R,\ y \in S),$

we obtain a product norm $\|(x, y)\|_m$. This norm is called the minimum norm. For these two norms we have obviously for every $x \in R$, $y \in S$

(3) $\|(x, y)\|_m \leqq \|(x, y)\|_M \leqq 2\,\|(x, y)\|_m .$

For every product norm $\|(x, y)\|$ we conclude obviously by definition

$$\|(x, y)\| = \|(x, 0) + (0, y)\| \leqq \|x\| + \|y\|,$$

$$\|(x, y)\| = \|(x, -y)\| = \tfrac{1}{2}\{\|(x, y)\| + \|(x, -y)\|\} \geqq \tfrac{1}{2}\|(2x, 0)\| = \|x\|,$$

and similarly $\|(x, y)\| \geqq \|y\|$. Thus we have for every $x \in R$, $y \in S$

(4) $\|(x, y)\|_m \leqq \|(x, y)\| \leqq \|(x, y)\|_M .$

Recalling §71 Theorem 7, we see by the relations (3) and (4) that the norm topology of the product space (R, S) is the same for every product norm. Furthermore we have

Theorem 1. The norm topology of (R, S) by a product norm coincides with the product of the norm topologies of R and S.

Proof. For the unit sphere U of R and V of S, (U, V) is by §56 Theorem 7 a basis of the product of the norm topologies of R and S, and we have obviously

$$(U, V) = \{ (x, y) : \|(x, y)\|_m \leqq 1 \}.$$

Thus the product of the norm topologies is the norm topology of (R, S) by the minimum norm. Consequently we obtain our assertion by the relations (3) and (4).

Let $\bar{R}$ be the adjoint space of R and $\bar{S}$ that of S. Then the product space $(\bar{R}, \bar{S})$ is by §91 Theorem 2 the adjoint space of the product space (R, S).

Theorem 2. For a product norm of (R, S), its adjoint norm is a product norm of $(\bar{R}, \bar{S})$.

Proof. For a product norm $\|(x, y)\|$ of (R, S) we have

$$\|(\bar{x}, 0)\| = \sup_{\|(x, y)\| \leqq 1} |\bar{x}(x)| \leqq \sup_{\|x\| \leqq 1} |\bar{x}(x)| = \|\bar{x}\|,$$

because $\|x\| \leqq \|(x, y)\|$ by the relation (4). On the other hand, as $\|x\| \leqq 1$ implies $\|(x, 0)\| \leqq 1$, we have

$$\|\bar{x}\| = \sup_{\|x\| \le 1} |\bar{x}(x)| \le \sup_{\|(x, y)\| \le 1} |\bar{x}(x)| = \|(\bar{x}, 0)\|.$$

Thus we have $\|\bar{x}\| = \|(\bar{x}, 0)\|$. We can prove likewise $\|\bar{y}\| = \|(0, \bar{y})\|$ for every $\bar{y} \in \bar{S}$. Furthermore we have for every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$

$$\|(\bar{x}, \bar{y})\| = \sup_{\|(x, y)\| \le 1} |(\bar{x}, \bar{y})(x, y)| = \sup_{\|(x, -y)\| \le 1} |(\bar{x}, -\bar{y})(x, -y)|$$
$$\sup_{\|(x, y)\| \le 1} |(\bar{x}, -\bar{y})(x, y)| = \|(\bar{x}, -\bar{y})\|.$$

Therefore $\|(\bar{x}, \bar{y})\|$ is a product norm on $(\bar{R}, \bar{S})$ by definition.

Theorem 3. *For the maximum norm of the product space (R, S), its adjoint norm is the minimum norm of $(\bar{R}, \bar{S})$. For the minimum norm of (R, S), its adjoint norm is the maximum norm of $(\bar{R}, \bar{S})$.*

Proof. For the adjoint norm $\|(\bar{x}, \bar{y})\|$ of the minimum norm, we have by definition

$$\|(\bar{x}, \bar{y})\| = \sup_{\|x\| \le 1,\ \|y\| \le 1} \{\bar{x}(x) + \bar{y}(y)\} = \|\bar{x}\| + \|\bar{y}\|.$$

For the adjoint norm $\|(\bar{x}, \bar{y})\|$ of the maximum norm, we have further by definition

$$\|(\bar{x}, \bar{y})\| = \sup_{\|x\| + \|y\| \le 1} \{\bar{x}(x) + \bar{y}(y)\}$$
$$= \sup_{0 \le \xi \le 1} \Big(\sup_{\|x\| \le \xi,\ \|y\| \le 1 - \xi} \{\bar{x}(x) + \bar{y}(y)\} \Big)$$
$$= \sup_{0 \le \xi \le 1} \{\xi \|\bar{x}\| + (1 - \xi)\|\bar{y}\|\} = \mathrm{Max}\{\|\bar{x}\|, \|\bar{y}\|\}.$$

§93 Product of modulared spaces

Let R and S be two modulared spaces. For the product space (R, S), putting

$$m(x, y) = m(x) + m(y) \qquad \text{for } x \in R,\ y \in S,$$

we see easily that $m(x, y)$ is a modular on (R, S). This modulared space (R, S) is called the *product space* of two modulared spaces R and S.

Theorem 1. *The modular norm of the product space (R, S) is a product norm on (R, S) for the modular norms of R and S.*

Proof. We obtain by the formula §81(6) for every $x \in R$

$$|||(x, 0)||| = \inf_{m(\xi x) \le 1} \frac{1}{|\xi|} = |||x|||,$$

and likewise $|||(0, y)||| = |||y|||$ for every $y \in S$. Furthermore, for every $(x, y) \in (R, S)$ we have

$$|||(x, y)||| = \inf_{m(\xi x) + m(\xi y) \le 1} \frac{1}{|\xi|} = |||(x, -y)|||.$$

Therefore we obtain our assertion by definition.

Let $\bar{R}$ be the modular adjoint space of R and $\bar{S}$ that of S. Then we have

Theorem 2. *The product space $(\bar{R}, \bar{S})$ coincides with the adjoint space of the product space (R, S) by the relation:*

$$(\bar{x}, \bar{y})(x, y) = \bar{x}(x) + \bar{y}(y)$$

for $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$, $x \in R$, $y \in S$.

Proof. Recalling §81 Theorem 6, we see at once by Theorem 1 and §91 Theorem 2 that $(\bar{R}, \bar{S})$ coincides with the modular adjoint space of (R, S) as a linear space. Furthermore, we have by definition for every $(\bar{x}, \bar{y}) \in (\bar{R}, \bar{S})$

$$\bar{m}(\bar{x}, \bar{y}) = \sup_{x \in R,\ y \in S} \{\bar{x}(x) + \bar{y}(y) - m(x) - m(y)\}$$
$$= \bar{m}(\bar{x}) + \bar{m}(\bar{y}).$$

Thus we obtain our assertion by definition.

As the associated norm of (R, S) is by definition the adjoint norm of the modular norm of $(\bar{R}, \bar{S})$, we obtain hence by §92 Theorem 2

Theorem 3. *The associated norm of the product space (R, S) is a product norm on (R, S) for the associated norms of R and S.*

A modulared space R is said to be *isometric* to a modulared space S, if there is a transformation a from R to S such that

$$m(a(x)) = m(x) \qquad \text{for every } x \in R.$$

With this definition we have

Theorem 4. *The product space $(L_{p_1(t)}, L_{p_2(t)})$ is isometric to $L_{p_3(t)}$, if we put*

$$p_3(t) = \begin{cases} p_1(2t) & \text{for } 0 < t < \frac{1}{2} \\ p_2(2t - 1) & \text{for } \frac{1}{2} < t < 1. \end{cases}$$

Proof. For every $x \in L_{p_1(t)}$, $y \in L_{p_2(t)}$, putting

$$(x, y)(t) = \begin{cases} \dfrac{1}{2^{\frac{1}{p_1(2t)}}}\, x(2t) & \text{for } 0 < t < \frac{1}{2} \\ \dfrac{1}{2^{\frac{1}{p_2(2t-1)}}}\, y(2t-1) & \text{for } \frac{1}{2} < t < 1, \end{cases}$$

we obtain a measurable function $(x, y)(t)$ for $0 < t < 1$, and we have

$$\int_0^1 \frac{1}{p_3(t)} |(x, y)(t)|^{p_3(t)} dt = 2 \int_0^{\frac{1}{2}} \frac{1}{p_1(2t)} |x(2t)|^{p_1(2t)} dt$$
$$+ 2 \int_{\frac{1}{2}}^1 \frac{1}{p_2(2t-1)} |y(2t-1)|^{p_2(2t-1)} dt = m(x) + m(y).$$

Thus we see easily by definition that $(L_{p_1(t)}, L_{p_2(t)})$ is isometric to $L_{p_3(t)}$.

§94 Bilinear functionals

Let R and S be two linear spaces. A functional φ on the product space (R, S) is said to be *bilinear*, if

$$\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \qquad \text{for } x_1, x_2 \in R,\ y \in S,$$

$$\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \qquad \text{for } x \in R,\ y_1, y_2 \in S,$$

and for every real number ξ

$$\varphi(\xi x, y) = \varphi(x, \xi y) = \xi \varphi(x, y) \qquad \text{for } x \in R,\ y \in S.$$

For a linear functional $\tilde{x}$ on R and $\tilde{y}$ on S, we define $\tilde{x}\tilde{y}$ to mean a bilinear functional on (R, S) by the relation

$$\tilde{x}\tilde{y}(x, y) = \tilde{x}(x)\tilde{y}(y) \qquad \text{for } x \in R,\ y \in S.$$

For two bilinear functionals φ, ψ on (R, S), we define $\alpha\varphi + \beta\psi$ for every real numbers α, β to mean a bilinear functional on (R, S) by the relation:

$$(\alpha\varphi + \beta\psi)(x, y) = \alpha\varphi(x, y) + \beta\psi(x, y) \qquad \text{for } x \in R,\ y \in S.$$

With this definition, we see easily that the totality of bilinear functionals on (R, S) constitutes a linear space. This linear space is called the *biassociated space* of (R, S) and denoted by $\widetilde{RS}$.

For the associated space $\tilde{R}$ of R and that $\tilde{S}$ of S, we have then obviously that $\tilde{x} \in \tilde{R}$, $\tilde{y} \in \tilde{S}$ implies $\tilde{x}\tilde{y} \in \widetilde{RS}$ and for every real number ξ

$$(\xi\tilde{x})\tilde{y} = \tilde{x}(\xi\tilde{y}) = \xi\tilde{x}\tilde{y} \qquad \text{for } \tilde{x} \in \tilde{R},\ \tilde{y} \in \tilde{S},$$

$$(\tilde{x}_1 + \tilde{x}_2)\tilde{y} = \tilde{x}_1\tilde{y} + \tilde{x}_2\tilde{y} \qquad \text{for } \tilde{x}_1, \tilde{x}_2 \in \tilde{R},\ \tilde{y} \in \tilde{S},$$

$$\tilde{x}(\tilde{y}_1 + \tilde{y}_2) = \tilde{x}\tilde{y}_1 + \tilde{x}\tilde{y}_2 \qquad \text{for } \tilde{x} \in \tilde{R},\ \tilde{y}_1, \tilde{y}_2 \in \tilde{S}.$$

Furthermore we see at once

Theorem 1. *If $\tilde{x}_\nu \in \tilde{R}$ $(\nu = 1, 2, \ldots, \kappa)$ are linearly independent, then $\sum_{\nu=1}^{\kappa} \tilde{x}_\nu \tilde{y}_\nu = 0$, $\tilde{y}_\nu \in \tilde{S}$ $(\nu = 1, 2, \ldots, \kappa)$ implies $\tilde{y}_\nu = 0$ for every $\nu = 1, 2, \ldots, \kappa$. If $\tilde{y}_\nu \in \tilde{S}$ $(\nu = 1, 2, \ldots, \kappa)$ are linearly independent, then $\sum_{\nu=1}^{\kappa} \tilde{x}_\nu \tilde{y}_\nu = 0$, $\tilde{x}_\nu \in \tilde{R}$ $(\nu = 1, 2, \ldots, \kappa)$ implies $\tilde{x}_\nu = 0$ for every $\nu = 1, 2, \ldots, \kappa$.*

Every element $(x, y) \in (R, S)$ may be considered as a linear functional on the biassociated space $\widetilde{RS}$ by the relation:

$$(x, y)(\tilde{z}) = \tilde{z}(x, y) \qquad \text{for every } \tilde{z} \in \widetilde{RS}.$$

Thus we can introduce a weak linear topology into $\widetilde{RS}$ by the system of linear functionals (x, y) for all $x \in R$ and $y \in S$. This weak linear topology is called the *weak topology* of the biassociated space $\widetilde{RS}$.

On account of §35 Theorem 8, we can prove likewise as §67 Theorem 1

Theorem 2. *The weak topology of the biassociated space $\widetilde{RS}$ is complete.*

Now let R and S be linear topological spaces with linear topologies $\mathfrak{U}$ and $\mathfrak{V}$ respectively.

Theorem 3. *In order that a bilinear functional φ on the product space (R, S) be continuous by the product $(\mathfrak{U}, \mathfrak{V})$, it is necessary and sufficient that we can find $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$ such that*

$$\sup_{x \in U,\ y \in V} |\varphi(x, y)| < +\infty.$$

Proof. If φ is continuous by $(\mathfrak{U}, \mathfrak{V})$, then we can find by §20 Theorem 2 $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$ such that

$$(U, V) \subset \{(x, y) : |\varphi(x, y)| < 1\}.$$

Conversely, we assume for some $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$

$$\gamma = \sup_{x \in U,\ y \in V} |\varphi(x, y)| < +\infty,$$

and V_1 be a star vicinity of $\mathfrak{V}$ such that $V_1 \times V_1 \subset V$. Then for any $x_0 \in R$, $y_0 \in S$ we can find $\alpha > 0$ such that $x_0 \in \alpha U$, $y_0 \in \alpha V_1$, and for any positive number $\varepsilon < \alpha$, if $(x, y) - (x_0, y_0) \in \varepsilon(U, V_1)$, then we have $\frac{1}{\varepsilon}(x - x_0) \in U$, $\frac{1}{\varepsilon}(y - y_0) \in V_1$, $\frac{1}{\alpha}x_0 \in U$, $\frac{1}{\alpha}y_0 \in V_1$, and hence $\frac{1}{\alpha}y \in \frac{\varepsilon}{\alpha}V_1 \times V_1 \subset V$. These relations yield by assumption

$$|\varphi(x, y) - \varphi(x_0, y_0)| \leqq \varepsilon\alpha \left|\varphi\left(\tfrac{1}{\varepsilon}(x - x_0), \tfrac{1}{\alpha}y\right)\right| + \varepsilon\alpha \left|\varphi\left(\tfrac{1}{\alpha}x_0, \tfrac{1}{\varepsilon}(y - y_0)\right)\right| \leqq 2\varepsilon\alpha\gamma.$$

Therefore φ is continuous by the product $(\mathfrak{U}, \mathfrak{V})$.

A bilinear functional φ on the product space (R, S) is said to be *bounded*, if we have for every bounded manifold A of (R, S)

$$\sup_{(x, y) \in A} |\varphi(x, y)| < +\infty.$$

Recalling §91 Theorem 1, we have then obviously that φ is bounded, if and only if we have for every bounded manifolds B of R and C of S

$$\sup_{x\in B,\, y\in C} |\varphi(x,y)| < +\infty .$$

<u>Theorem</u> 4. <u>If both R and S are sequential, then every bounded bilinear functional φ on the product space (R, S) is continuous by the product $(\mathcal{U}, \mathcal{V})$.</u>

<u>Proof</u>. Putting $W = \{(x,y) : |\varphi(x,y)| \leqq 1\}$, we see at once that W is a vicinity in (R, S). Furthermore, for any bounded manifold A of (R, S), we can find $\lambda > 0$ such that

$$\sup_{(x,y)\in\frac{1}{\lambda}A} |\varphi(x,y)| = \sup_{(x,y)\in A} \left|\frac{1}{\lambda^2}\varphi(x,y)\right| \leqq 1,$$

and hence $\frac{1}{\lambda}A \subset W$, namely $A \subset \lambda W$. As (R, S) is sequential by §58 Theorem 8, we obtain $W \in (\mathcal{U}, \mathcal{V})$ by §58 Theorem 1. Therefore φ is continuous by Theorem 3.

§95 <u>Biadjoint spaces</u>

Let R and S be two linear topological spaces with linear topologies $\mathcal{U}$ and $\mathcal{V}$ respectively. The totality of bounded bilinear functionals on the product space (R, S) is called the <u>biadjoint space</u> of (R, S) and denoted by $\overline{RS}$. Let $\mathfrak{A}$ be the totality of bounded manifolds of (R, S). We see easily by §53 Theorem 3 that, putting

$$\overline{U}_A = \{\overline{Z} : |\overline{Z}(x,y)| \leqq 1 \quad \text{for } (x,y)\in A,\ \overline{Z}\in\overline{RS}\}$$

corresponding to every $A \in \mathfrak{A}$, we obtain a symmetric convex vicinity $\overline{U}_A$ in $\overline{RS}$, and there exists uniquely by §53 Theorem 3 a linear topology $\overline{\mathcal{N}\mathcal{V}}$ on $\overline{RS}$ such that the system $\overline{U}_A$ $(A \in \mathfrak{A})$ is a basis of $\overline{\mathcal{N}\mathcal{V}}$. This linear topology $\overline{\mathcal{N}\mathcal{V}}$ is called the <u>biadjoint topology</u> of the product $(\mathcal{U}, \mathcal{V})$ and denoted by $\overline{\mathcal{U}\mathcal{V}}$. The biadjoint space $\overline{RS}$ is defined as a linear topological space by the biadjoint topology $\overline{\mathcal{U}\mathcal{V}}$.

<u>Theorem</u> 1. <u>For any</u> $U \in \mathcal{U}$ <u>and</u> $V \in \mathcal{V}$,

$$\{\overline{Z} : |\overline{Z}(x,y)| \leqq 1 \quad \text{for every } x\in U,\ y\in V\}$$

<u>is a bounded manifold of the biadjoint space</u> $\overline{RS}$.

<u>Proof</u>. For any bounded manifold A of the product space (R, S) we can find by §91 Theorem 1 a bounded manifolds B of R and C of S

such that $A \subset (B, C)$. For such B, C we can find $\alpha > 0$ such that $\alpha B \subset U$, $\alpha C \subset V$. Then we have obviously

$$\{\overline{Z} : |\overline{Z}(x,y)| \leqq 1 \quad \text{for every } x\in U,\ y\in V\}$$
$$\subset \{\overline{Z} : |\overline{Z}(x,y)| \leqq 1 \quad \text{for every } x\in\alpha B,\ y\in\alpha C\}$$
$$= \{\overline{Z} : |\alpha^2\overline{Z}(x,y)| \leqq 1 \quad \text{for every } x\in B,\ y\in C\}$$
$$\subset \{\tfrac{1}{\alpha^2}\overline{Z} : |\overline{Z}(x,y)| \leqq 1 \quad \text{for every } (x,y)\in A\}.$$

Therefore $\{\overline{Z} : |\overline{Z}(x,y)| \leqq 1$ for every $x\in U$, $y\in V\}$ is bounded by the definition of the biadjoint topology $\overline{\mathcal{U}\mathcal{V}}$.

Let $\overline{R}$ be the adjoint space of R and $\overline{S}$ that of S. For any $\overline{x}\in\overline{R}$ and $\overline{y}\in\overline{S}$, $\overline{x}\,\overline{y}$ is obviously by definition a bounded bilinear functional on the product space (R, S).

<u>Theorem</u> 2. <u>If both R and S are equivalently strongest or standard, then for any bounded manifolds $\overline{B}$ of $\overline{R}$ and $\overline{C}$ of $\overline{S}$, putting</u>

$$\overline{A} = \{\overline{x}\,\overline{y} : \quad \overline{x}\in\overline{B},\ \overline{y}\in\overline{C}\},$$

<u>we obtain a bounded manifold $\overline{A}$ of the biadjoint space $\overline{RS}$</u>.

<u>Proof</u>. By virtue of §64 Theorem 8, as both R and S are by assumption equivalently strongest or standard, putting

$$U = \{x : |\overline{x}(x)| \leqq 1 \quad \text{for every } \overline{x}\in\overline{B}\},$$
$$V = \{y : |\overline{y}(y)| \leqq 1 \quad \text{for every } \overline{y}\in\overline{C}\},$$

we have $U \in \mathcal{U}$ and $V \in \mathcal{V}$. For such U, V, we have obviously

$$\overline{A} \subset \{\overline{Z} : |\overline{Z}(x,y)| \leqq 1 \quad \text{for every } x\in U,\ y\in V\},$$

because $\overline{x}\,\overline{y}(x,y) = \overline{x}(x)\,\overline{y}(y)$. Thus we see by §95 Theorem 1 that $\overline{A}$ is a bounded manifold of $\overline{RS}$.

For any $x\in R$, $y\in S$, putting

$$xy(\overline{Z}) = \overline{Z}(x,y) \qquad \text{for every } \overline{Z}\in\overline{RS},$$

we obtain obviously a linear functional xy on the biadjoint space $\overline{RS}$. Thus, recalling §94, we obtain the weak linear topology of $\overline{RS}$ by the system xy $(x\in R,\ y\in S)$. This weak linear topology of $\overline{RS}$ is called the <u>weak topology</u> of $\overline{RS}$. As we have by definition

$$\{\overline{Z} : |xy(\overline{Z})| = |\overline{Z}(x,y)| \leqq 1\} \in \overline{RS}$$

for every $x\in R$ and $y\in S$, xy is a continuous linear functional on the biadjoint space $\overline{RS}$. Thus we have

Theorem 3. The weak topology of the biadjoint space $\overline{RS}$ is weaker than the biadjoint topology $\overline{\mathfrak{U}\mathfrak{V}}$.

For every $x \in R$, $y \in S$, we have obviously by definition

$$xy(\bar{x}\bar{y}) = \bar{x}\bar{y}(x, y) = \bar{x}(x)\bar{y}(y)$$

for every $\bar{x} \in \bar{R}$, $\bar{y} \in \bar{S}$. Thus xy may be considered as a bilinear functional on the product space $(\bar{R}, \bar{S})$ by the relation:

$$xy(\bar{x}, \bar{y}) = \bar{x}(x)\bar{y}(y) \qquad (\bar{x} \in \bar{R}, \bar{y} \in \bar{S}).$$

Then we have

Theorem 4. For every $x \in R$, $y \in S$, xy is a continuous bilinear functional on the product space $(\bar{R}, \bar{S})$.

Proof. On account of §64 Theorem 4, for any $x \in R$ and $y \in S$, we can find $\bar{U} \in \bar{\mathfrak{U}}$ and $\bar{V} \in \bar{\mathfrak{V}}$ for the adjoint topology $\bar{\mathfrak{U}}$ of $\mathfrak{U}$ and that $\bar{\mathfrak{V}}$ of $\mathfrak{V}$ such that $\sup_{\bar{x} \in \bar{U}} |\bar{x}(x)| < +\infty$, $\sup_{\bar{y} \in \bar{V}} |\bar{y}(y)| < +\infty$, and hence

$$\sup_{\bar{x} \in \bar{U}, \bar{y} \in \bar{V}} |xy(\bar{x}, \bar{y})| < +\infty .$$

Thus xy is continuous by Theorem 3 in §94.

Theorem 5. If both R and S have sequential roots, then the biadjoint space $\overline{RS}$ is sequential.

Proof. Let B_ν $(\nu = 1, 2, \dots)$ be a sequential root of R and C_ν $(\nu = 1, 2, \dots)$ that of S. Then we see easily by definition that

$$\{\bar{z} : |\bar{z}(x, y)| \leq 1 \text{ for } x \in B_\nu,\ y \in C_\nu\} \qquad (\nu = 1, 2, \dots)$$

is a basis of the biadjoint topology $\overline{\mathfrak{U}\mathfrak{V}}$.

Finally we shall prove

Theorem 6. The biadjoint topology is complete.

Proof. Let $\bar{A}_\lambda$ $(\lambda \in \Lambda)$ be a Cauchy system by the biadjoint topology $\overline{\mathfrak{U}\mathfrak{V}}$. Since the weak topology of the biadjoint space $\overline{RS}$ is by Theorem 3 weaker than $\overline{\mathfrak{U}\mathfrak{V}}$, $\bar{A}_\lambda$ $(\lambda \in \Lambda)$ also is a Cauchy system by the weak topology. As the biassociated space $\widetilde{RS}$ is complete by the weak topology by §94 Theorem 2, there exists hence a limit $\tilde{z}_0 \in \widetilde{RS}$ of $\bar{A}_\lambda$ $(\lambda \in \Lambda)$ for the weak topology.

For every bounded manifolds B of R and C of S, as

$$\{\bar{z} : |\bar{z}(x, y)| \leq 1 \text{ for every } x \in B,\ y \in C\} \in \overline{\mathfrak{U}\mathfrak{V}},$$

we can find $\lambda_0 \in \Lambda$ such that $\bar{u}, \bar{v} \in \bar{A}_{\lambda_0}$ implies

$$|\bar{u}(x, y) - \bar{v}(x, y)| \leq 1 \qquad \text{for every } x \in B,\ y \in C,$$

because $\bar{A}_\lambda$ $(\lambda \in \Lambda)$ is a Cauchy system by $\overline{\mathfrak{U}\mathfrak{V}}$. For any $x \in B$, $y \in C$, and $\varepsilon > 0$, we can find $\bar{v} \in \bar{A}_{\lambda_0}$ such that

$$|\bar{v}(x, y) - \tilde{z}_0(x, y)| < \varepsilon,$$

because $\tilde{z}_0$ is a limit of $\bar{A}_\lambda$ $(\lambda \in \Lambda)$ by the weak topology. Then we have $|\bar{u}(x, y) - \tilde{z}_0(x, y)| < 1 + \varepsilon$ for every $\bar{u} \in \bar{A}_{\lambda_0}$. As $x \in B$, $y \in C$, and $\varepsilon > 0$ may be arbitrary, we conclude hence

$$|\bar{u}(x, y) - \tilde{z}_0(x, y)| \leq 1 \qquad \text{for every } \bar{u} \in \bar{A}_{\lambda_0},\ x \in B,\ y \in C.$$

From this relation we conclude that $\tilde{z}_0 \in \overline{RS}$ and

$$\bar{A}_{\lambda_0} \subset \{\bar{z} : |\bar{z}(x, y) - \tilde{z}_0(x, y)| \leq 1 \text{ for every } x \in B,\ y \in C\}.$$

Thus $\tilde{z}_0$ is a limit of $\bar{A}_\lambda$ $(\lambda \in \Lambda)$ by the biadjoint topology $\overline{\mathfrak{U}\mathfrak{V}}$. Consequently $\overline{\mathfrak{U}\mathfrak{V}}$ is complete by definition.

§96 Cross spaces

Let R and S be two linear spaces. For any $x \in R$ and $y \in S$, putting

(1) $\qquad xy(\tilde{z}) = \tilde{z}(x, y) \qquad (\tilde{z} \in \widetilde{RS}),$

we obtain a linear functional xy on the biassociated space $\widetilde{RS}$ of the product space (R, S). For such linear functionals xy $(x \in R, y \in S)$, we have obviously by the definition (1)

(2) $\qquad (\xi x)y = x(\xi y) = \xi xy \qquad$ for every real number ξ,

(3) $\qquad (x_1 + x_2)y = x_1 y + x_2 y \qquad (x_1, x_2 \in R,\ y \in S)$

(4) $\qquad x(y_1 + y_2) = xy_1 + xy_2 \qquad (x \in R,\ y_1, y_2 \in S)$

Thus, if we denote by $\widetilde{\widetilde{RS}}$ the associated space of $\widetilde{RS}$, then we have $xy \in \widetilde{\widetilde{RS}}$ for every $x \in R$ and $y \in S$. The linear manifold of $\widetilde{\widetilde{RS}}$ generated by the system xy $(x \in R, y \in S)$ is called the cross space of R and S, and denoted by RS.

Let $\tilde{R}$ be the associated space of R and $\tilde{S}$ that of S. For every $\tilde{x} \in \tilde{R}$ and $\tilde{y} \in \tilde{S}$, as $\tilde{x}\tilde{y} \in \widetilde{RS}$, putting

(5) $\qquad z(\tilde{x}, \tilde{y}) = z(\tilde{x}\tilde{y}) \qquad (z \in RS),$

every $z \in RS$ may be considered as a bilinear functional on the product space $(\tilde{R}, \tilde{S})$. Especially we have by (1) and (5)

(6) $$xy(\tilde{x}, \tilde{y}) = \tilde{x}(x)\tilde{y}(y) \quad (x \in R,\ y \in S,\ \tilde{x} \in \tilde{R},\ \tilde{y} \in \tilde{S}).$$

<u>Theorem 1.</u> <u>If a linear manifold $\tilde{A}$ of $\tilde{R}$ is fundamental in R and a linear manifold $\tilde{B}$ of $\tilde{S}$ is fundamental in S, then $z(\tilde{x}, \tilde{y}) = 0$ for every $\tilde{x} \in \tilde{A}$, $\tilde{y} \in \tilde{B}$ implies $z = 0$ for $z \in RS$.</u>

<u>Proof.</u> On account of the definition of the cross space RS, every $z \in RS$ may be represented as a linear combination from xy $(x \in R,\ y \in S)$. Furthermore, for any $z \in RS$ we can find $x_\nu \in R$ $(\nu = 1, 2, \ldots, \varkappa)$ and $y_\nu \in S$ $(\nu = 1, 2, \ldots, \varkappa)$ such that

$$z = \sum_{\nu=1}^{\varkappa} x_\nu y_\nu$$

and x_ν $(\nu = 1, 2, \ldots, \varkappa)$ are linearly independent. Because, if

$$x_\varkappa = \sum_{\nu=1}^{\varkappa-1} \alpha_\nu x_\nu,$$

then we have by the relations (1), (3), and (4)

$$\sum_{\nu=1}^{\varkappa} x_\nu y_\nu = \sum_{\nu=1}^{\varkappa-1} x_\nu y_\nu + \Big(\sum_{\nu=1}^{\varkappa-1} \alpha_\nu x_\nu\Big) y_\varkappa = \sum_{\nu=1}^{\varkappa-1} x_\nu (y_\nu + \alpha_\nu y_\varkappa).$$

If x_ν $(\nu = 1, 2, \ldots, \varkappa)$ are linearly independent, then x_ν $(\nu = 1, 2, \ldots, \varkappa)$ also are linearly independent as linear functionals on $\tilde{A}$ by the relation:

$$x(\tilde{x}) = \tilde{x}(x) \qquad \text{for every } \tilde{x} \in \tilde{A},$$

because $\tilde{A}$ is fundamental in R by assumption. Thus if

$$\sum_{\nu=1}^{\varkappa} x_\nu(\tilde{x}) y_\nu(\tilde{y}) = z(\tilde{x}, \tilde{y}) = 0 \qquad \text{for every } \tilde{x} \in \tilde{A},\ \tilde{y} \in \tilde{B},$$

then we have by §94 Theorem 1 for every $\tilde{y} \in \tilde{B}$

$$\tilde{y}(y_\nu) = y_\nu(\tilde{y}) = 0 \qquad (\nu = 1, 2, \ldots, \varkappa),$$

and hence $y_\nu = 0$ $(\nu = 1, 2, \ldots, \varkappa)$, because $\tilde{B}$ is fundamental in S by assumption. Therefore $z(\tilde{x}, \tilde{y}) = 0$ for every $\tilde{x} \in \tilde{A}$, $\tilde{y} \in \tilde{B}$ implies $z = 0$.

<u>Theorem 2.</u> <u>The biassociative space $\widetilde{RS}$ of the product space (R, S) coincides with the associative space of the cross space RS by the relation:</u> $\tilde{z}(z) = z(\tilde{z})$ <u>for</u> $z \in RS$, $\tilde{z} \in \widetilde{RS}$.

<u>Proof.</u> Every $\tilde{z} \in \widetilde{RS}$ may be considered obviously as a linear functional on the cross space RS by the relation:

$$\tilde{z}(z) = z(\tilde{z}) \qquad \text{for every } z \in RS.$$

Furthermore, if $\tilde{z}(z) = 0$ for every $z \in RS$, then we have naturally

$$\tilde{z}(x, y) = xy(\tilde{z}) = \tilde{z}(xy) = 0 \qquad \text{for every } x \in R,\ y \in S,$$

and hence $\tilde{z} = 0$. On the other hand, for any linear functional φ on RS, putting

$$\varphi(x, y) = \varphi(xy) \qquad \text{for every } x \in R,\ y \in S,$$

we obtain obviously a bilinear functional φ on (R, S). Thus, we can find uniquely $\tilde{z} \in \widetilde{RS}$ such that

$$\varphi(x, y) = \tilde{z}(x, y) \qquad \text{for every } x \in R,\ y \in S,$$

and hence $\varphi(xy) = \tilde{z}(xy)$ for every $x \in R$, $y \in S$. Putting

$$A = \{ z :\ \varphi(z) = \tilde{z}(z),\ z \in RS \},$$

we obtain obviously a linear manifold A of RS, which contains all xy $(x \in R,\ y \in S)$. As RS is the linear manifold generated by xy $(x \in R,\ y \in S)$ by definition, we conclude hence $A = RS$, and consequently $\varphi(z) = \tilde{z}(z)$ for every $z \in RS$. Therefore we obtain our assertion.

By virtue of Theorem 1, we can state

<u>Theorem 3.</u> <u>The cross space RS is a subspace of the biassociated space $\widetilde{\tilde{R}\tilde{S}}$ of the product space $(\tilde{R}, \tilde{S})$ by the relation:</u>

$$z(\tilde{x}, \tilde{y}) = z(\tilde{x}\tilde{y}) \qquad \text{for } z \in RS,\ \tilde{x} \in \tilde{R},\ \tilde{y} \in \tilde{S}.$$

On account of Theorem 3, for the cross space $\tilde{R}\tilde{S}$, every $\tilde{z} \in \tilde{R}\tilde{S}$ may be considered as a linear functional on the cross space RS.

In this sense we have

<u>Theorem 4.</u> <u>The cross space $\tilde{R}\tilde{S}$ of the associated spaces $\tilde{R}$ and $\tilde{S}$ is a subspace of the biassociated space $\widetilde{RS}$, that is, for $\tilde{z} \in \tilde{R}\tilde{S}$,</u>

$\tilde{z}(z) = 0$ <u>for every</u> $z \in RS$ <u>implies</u> $\tilde{z} = 0$.

<u>Proof.</u> Let $\tilde{\tilde{R}}$ be the associated space of $\tilde{R}$ and $\tilde{\tilde{S}}$ that of $\tilde{S}$. Considering every $x \in R$ as a linear functional on $\tilde{R}$ by the relation:

$$x(\tilde{x}) = \tilde{x}(x) \qquad \text{for every } \tilde{x} \in \tilde{R},$$

R is obviously a subspace of $\tilde{\tilde{R}}$ and furthermore fundamental in $\tilde{R}$. Similarly, S is a subspace of $\tilde{\tilde{S}}$ and fundamental in $\tilde{S}$. If $\tilde{z}(z) = 0$ for every $z \in RS$, then we have naturally for every $x \in R$ and $y \in S$

$$\tilde{z}(x, y) = \tilde{z}(xy) = 0,$$

and hence we obtain $\tilde{z} = 0$ by Theorem 1.

<u>Theorem 5.</u> <u>For a subspace B of R and a subspace C of S, the</u>

<u>cross space $\mathfrak{B}\mathfrak{C}$ is a subspace of the cross space RS.</u>

<u>Proof</u>. We need by definition only prove that for $x_\nu \in \mathfrak{B}$, $y_\nu \in \mathfrak{C}$ ($\nu = 1, 2, \dots, \varkappa$), we have $\sum_{\nu=1}^{\varkappa} x_\nu y_\nu = 0$ in $\mathfrak{B}\mathfrak{C}$, if and only if we have $\sum_{\nu=1}^{\varkappa} x_\nu y_\nu = 0$ in RS. For any $x_\nu \in \mathfrak{B}$, $y_\nu \in \mathfrak{C}$ ($\nu = 1, 2, \dots, \varkappa$), we can find by the method applied in Proof of Theorem 1, linearly independent $x'_\nu \in \mathfrak{B}$ ($\nu = 1, 2, \dots, \varkappa'$) such that we have

$$\sum_{\nu=1}^{\varkappa} x_\nu y_\nu = \sum_{\nu=1}^{\varkappa'} x'_\nu y'_\nu$$

for some $y'_\nu \in \mathfrak{C}$ ($\nu = 1, 2, \dots, \varkappa'$) in $\mathfrak{B}\mathfrak{C}$ as well as in RS. Thus, if $\sum_{\nu=1}^{\varkappa'} x'_\nu y'_\nu = 0$ in $\mathfrak{B}\mathfrak{C}$, then we have $y'_\nu = 0$ for every $\nu = 1, 2, \dots, \varkappa'$, and hence $\sum_{\nu=1}^{\varkappa'} x'_\nu y'_\nu = 0$ in RS. Conversely, if $\sum_{\nu=1}^{\varkappa'} x'_\nu y'_\nu = 0$ in RS, then we conclude likewise $\sum_{\nu=1}^{\varkappa'} x'_\nu y'_\nu = 0$ in $\mathfrak{B}\mathfrak{C}$.

§97 <u>Cross topologies</u>

Let R and S be two linear topological spaces with linear topologies $\mathfrak{U}$ and $\mathfrak{V}$ respectively. The adjoint space $\overline{R}$ of R is a subspace of the associated space $\widetilde{R}$ of R, and the adjoint space $\overline{S}$ of S is a subspace of the associated space $\widetilde{S}$ of S. Thus, for every $\bar{x} \in \overline{R}$, $\bar{y} \in \overline{S}$, $\bar{x}\bar{y}$ may be considered by §96 Theorem 2 a linear functional on the cross space RS.

A linear topology $\mathfrak{W}$ on the cross space RS is said to be a <u>cross topology</u>, if

1) $\mathfrak{W}$ is standard,

2) for every bounded manifolds $\mathfrak{B}$ of R and $\mathfrak{C}$ of S, $\{xy : x \in \mathfrak{B}, y \in \mathfrak{C}\}$ is bounded by $\mathfrak{W}$,

3) for every bounded manifolds $\overline{\mathfrak{B}}$ of $\overline{R}$ and $\overline{\mathfrak{C}}$ of $\overline{S}$, $\{\bar{x}\bar{y} : \bar{x} \in \overline{\mathfrak{B}}, \bar{y} \in \overline{\mathfrak{C}}\}$ is uniformly bounded by $\mathfrak{W}$.

Let $\bar{\mathfrak{A}}$ be the totality of bounded manifolds of the biadjoint space $\overline{R}\,\overline{S}$. As every $z \in RS$ is by definition a linear functional on $\widetilde{R}\widetilde{S}$, putting for each $\bar{A} \in \bar{\mathfrak{A}}$

$$W_{\bar{A}} = \{z : |z(\bar{z})| \leqq 1 \text{ for every } \bar{z} \in \bar{A}\},$$

we obtain a convex vicinity $W_{\bar{A}}$ in the cross space RS. Furthermore

we see easily by §53 Theorem 3 that there exists uniquely a linear topology $\mathfrak{W}_\Delta$ on RS such that $W_{\bar{A}}$ ($\bar{A} \in \bar{\mathfrak{A}}$) is a basis of $\mathfrak{W}_\Delta$. It is obvious by definition that $\mathfrak{W}_\Delta$ is convex. Recalling §95 Theorem 2, we see by definition that $\mathfrak{W}_\Delta$ satisfies the condition 3). For every bounded manifolds $\mathfrak{B}$ of R and $\mathfrak{C}$ of S, putting

$$\overline{W} = \{\bar{z} : |\bar{z}(x, y)| \leqq 1 \text{ for every } x \in \mathfrak{B}, y \in \mathfrak{C}\},$$

we have $\overline{W} \in \overline{\mathfrak{U}}\,\overline{\mathfrak{V}}$ by definition. Thus, for each bounded manifold $\bar{A}$ of $\overline{R}\,\overline{S}$, we can find $\alpha > 0$ such that $\bar{A} \subset \alpha\overline{W}$, and then we have

$$\{xy : x \in \mathfrak{B}, y \in \mathfrak{C}\} \subset \{z : |z(\bar{z})| \leqq 1 \text{ for every } \bar{z} \in \overline{W}\}$$
$$= \alpha\{z : |z(\bar{z})| \leqq 1 \text{ for every } \bar{z} \in \alpha\overline{W}\} \subset \alpha W_{\bar{A}}.$$

Consequently $\{xy : x \in \mathfrak{B}, y \in \mathfrak{C}\}$ is a bounded manifold of RS by $\mathfrak{W}_\Delta$. Therefore $\mathfrak{W}_\Delta$ satisfies the condition 2).

$\mathfrak{W}_\Delta$ is stronger than every other convex linear topology on RS subject to the conditions 2) and 3). Because, for a convex linear topology $\mathfrak{W}$ on RS subject* to the conditions 2) and 3), if we denote by $\overline{RS}^{\mathfrak{W}}$ the adjoint space of RS by $\mathfrak{W}$, then we see by §96 Theorem 2 that $\overline{RS}^{\mathfrak{W}}$ is a subspace of the biassociated space $\widetilde{R}\widetilde{S}$. Furthermore, as we see by the condition 2) that every $\bar{z} \in \overline{RS}^{\mathfrak{W}}$ is a bounded bilinear functional on (R, S), $\overline{RS}^{\mathfrak{W}}$ is a subspace of the biadjoint space $\overline{R}\,\overline{S}$. For any closed convex vicinity $W \in \mathfrak{W}$, as W is scalar-closed by §54 Theorem 2, putting

$$\bar{A} = \{\bar{z} : |\bar{z}(z)| \leqq 1 \text{ for every } z \in W, \ \bar{z} \in \overline{RS}^{\mathfrak{W}}\},$$

we have by §52 Theorem 3

$$W = \{z : |\bar{z}(z)| \leqq 1 \text{ for every } \bar{z} \in \bar{A}\}.$$

Furthermore $\bar{A}$ is a bounded manifold of $\overline{R}\,\overline{S}$. Because, for any bounded manifolds $\mathfrak{B}$ of R and $\mathfrak{C}$ of S, we can find by the condition 2) $\alpha > 0$ such that $\{xy : x \in \mathfrak{B}, y \in \mathfrak{C}\} \subset \alpha W$, and then

$$\bar{A} \subset \{\bar{z} : |\bar{z}(z)| \leqq 1 \text{ for every } z \in W, \ \bar{z} \in \widetilde{R}\widetilde{S}\}$$
$$= \alpha\{\bar{z} : |\bar{z}(z)| \leqq 1 \text{ for every } z \in \alpha W, \ \bar{z} \in \widetilde{R}\widetilde{S}\}$$
$$\subset \alpha\{\bar{z} : |\bar{z}(x, y)| \leqq 1 \text{ for every } x \in \mathfrak{B}, y \in \mathfrak{C}\}.$$

Thus $\bar{A}$ is by definition a bounded manifold of $\overline{R}\,\overline{S}$, and hence we have $W \in \mathfrak{W}_\Delta$ by definition. Therefore $\mathfrak{W}_\Delta$ is stronger than $\mathfrak{W}$.

By virtue of §57 Theorem 8, there exists the standard linear topology $\mathfrak{M}$ which is equivalent to $\mathfrak{M}_\Delta$. Such $\mathfrak{M}$ satisfies obviously the conditions 2) and 3) too, and hence $\mathfrak{M}_\Delta$ is stronger than $\mathfrak{M}$, as proved just above. Thus $\mathfrak{M}_\Delta$ is standard, and consequently $\mathfrak{M}_\Delta$ is a cross topology on RS. Therefore we can state

Theorem 1. There exists the strongest cross topology $\mathfrak{M}_\Delta$ on the cross space RS, and for a root $\bar{\mathfrak{A}}$ of the biadjoint space $\overline{RS}$

$$\{z : |z(\bar{z})| \leq 1 \quad \text{for every } \bar{z} \in \bar{A}\} \qquad (\bar{A} \in \bar{\mathfrak{A}})$$

is a basis of $\mathfrak{M}_\Delta$.

Theorem 2. There exists the weakest cross topology $\mathfrak{M}_w$ on the cross space RS, and $\mathfrak{M}_w$ is equivalent to the linear topology on RS, of which for a root $\bar{\mathfrak{B}}$ of $\bar{R}$ and a root $\bar{\mathfrak{C}}$ of $\bar{S}$

$$\{z : |z(\bar{x}\bar{y})| \leq 1 \quad \text{for every } \bar{x} \in \bar{B}, \bar{y} \in \bar{C}\} \qquad (\bar{B} \in \bar{\mathfrak{B}}, \bar{C} \in \bar{\mathfrak{C}})$$

is a basis.

Proof. By virtue of §53 Theorem 3, we can introduce a linear topology $\mathfrak{M}_0$ into the cross space RS such that

$$\{z : |z(\bar{x}\bar{y})| \leq 1 \quad \text{for every } \bar{x} \in \bar{B}, \bar{y} \in \bar{C}\} \qquad (\bar{B} \in \bar{\mathfrak{B}}, \bar{C} \in \bar{\mathfrak{C}})$$

is a basis of $\mathfrak{M}_0$. Then $\mathfrak{M}_0$ satisfies obviously the condition 3). For every bounded manifold B of R and C of S, and for every $\bar{B} \in \bar{\mathfrak{B}}$, $\bar{C} \in \bar{\mathfrak{C}}$, we have by §95 Theorem 2

$$\sup_{x \in B,\, y \in C,\, \bar{x} \in \bar{B},\, \bar{y} \in \bar{C}} |\bar{x}\bar{y}(xy)| < +\infty .$$

Consequently $\mathfrak{M}_0$ satisfies the condition 2) too. As $\mathfrak{M}_0$ is convex, there exists by §57 Theorem 8 a standard linear topology $\mathfrak{M}_w$ on RS, which is equivalent to $\mathfrak{M}_0$. Then $\mathfrak{M}_w$ also satisfies obviously the conditions 2) and 3), and hence $\mathfrak{M}_w$ is a cross topology on RS.

For any cross topology $\mathfrak{M}$ on RS, every bounded manifold of RS by $\mathfrak{M}$ is obviously by definition bounded by $\mathfrak{M}_0$ too, and hence naturally bounded by $\mathfrak{M}_w$. Accordingly $\mathfrak{M} \smile \mathfrak{M}_w$ is equivalent to $\mathfrak{M}$. As $\mathfrak{M}$ is standard, we obtain hence $\mathfrak{M} \smile \mathfrak{M}_w \subset \mathfrak{M}$. This relation yields by definition $\mathfrak{M}_w \subset \mathfrak{M}$. Therefore $\mathfrak{M}_w$ is the weakest cross topology on RS.

By virtue of §96 Theorem 2, the biassociated space $\widetilde{RS}$ coincides

with the associated space of the cross space RS. For the strongest cross topology $\mathfrak{M}_\Delta$ on the cross space RS, if a linear functional φ on RS is bounded by $\mathfrak{M}_\Delta$, then φ is by definition a bounded bilinear functional on the product space (R, S), that is, $\varphi \in \overline{RS}$, because for any bounded manifolds B of R and C of S, $\{xy : x \in B, y \in C\}$ is a bounded manifold of RS. Conversely, every $\varphi \in \overline{RS}$ is a continuous linear functional on RS by $\mathfrak{M}_\Delta$, as we see at once by the definition of $\mathfrak{M}_\Delta$. Thus the adjoint space $\overline{RS}^{\mathfrak{M}_\Delta}$ of RS by $\mathfrak{M}_\Delta$ coincides with $\overline{RS}$ as a linear space. And we see further that the adjoint topology of $\mathfrak{M}_\Delta$ is stronger than the biadjoint topology $\overline{\mathfrak{M}\mathfrak{P}}$. If a manifold $\bar{A}$ of $\overline{RS}$ is bounded by $\overline{\mathfrak{M}\mathfrak{P}}$, then $\bar{A}$ is uniformly bounded in RS by $\mathfrak{M}_\Delta$, on account of the definition of $\mathfrak{M}_\Delta$. Thus we have

Theorem 3. The adjoint space $\overline{RS}^{\mathfrak{M}_\Delta}$ of the cross space RS by the strongest cross topology $\mathfrak{M}_\Delta$ coincides with the biadjoint space $\overline{RS}$ of the product space (R, S) except for linear topologies. The adjoint topology $\mathfrak{M}_\Delta$ is stronger than the biadjoint topology but equivalent to this.

§98 Cross norms

Now let R and S be normed spaces. A bilinear functional φ on the product space (R, S) is said to be norm bounded, if

$$\sup_{\|x\| \leq 1,\, \|y\| \leq 1} |\varphi(x, y)| < +\infty \qquad (x \in R,\ y \in S).$$

With this definition, it is obvious that φ is norm bounded if and only if φ is bounded for the norm topologies of R and S. Thus for the biadjoint space $\overline{RS}$ of R and S by the norm topologies, we can define a norm on $\overline{RS}$ by the relation:

$$(1) \qquad \|\bar{z}\| = \sup_{\|x\| \leq 1,\, \|y\| \leq 1} |\bar{z}(x, y)| \qquad (x \in R,\ y \in S).$$

This norm on $\overline{RS}$ is called the biadjoint norm of the norms on R and S. We see easily by definition that the norm topology of $\overline{RS}$ by the biadjoint norm coincides with the biadjoint topology of the norm topologies on R and S. Thus we have by §95 Theorem 6 that the biadjoint norm is complete. For the adjoint space $\bar{R}$ of R and that $\bar{S}$ of S we have obviously by the definition (1)

(2) $$\|\bar{x}\bar{y}\| = \|\bar{x}\|\|\bar{y}\| \qquad (\bar{x}\in\bar{R},\ \bar{y}\in\bar{S}).$$

As the cross space RS is by definition a subspace of the adjoint space of the biadjoint space $\overline{RS}$, we obtain a norm on RS as the adjoint norm of the biadjoint norm. This norm on RS is called the maximum norm on the cross space RS and denoted by $\|z\|_{max}$ ($z\in RS$), that is,

(3) $$\|z\|_{max} = \sup_{\|\bar{z}\|\le 1} |z(\bar{z})| \qquad (\bar{z}\in\overline{RS},\ z\in RS).$$

With this definition we have

(4) $$\|xy\|_{max} = \|x\|\|y\| \qquad (x\in R,\ y\in S).$$

Because, $\|x\|\le 1$, $\|y\|\le 1$ implies by the definition (1)

$$\|xy\|_{max} = \sup_{\|\bar{z}\|\le 1} |\bar{z}(x,y)| \le 1,$$

and hence $\|xy\|_{max}\le\|x\|\|y\|$. On the other hand we have by the formula (2) and §72(3)

$$\|xy\|_{max} \ge \sup_{\|\bar{x}\|\le 1,\ \|\bar{y}\|\le 1} |\bar{x}(x)\bar{y}(y)| = \|x\|\|y\|.$$

We have further obviously by §97 Theorem 1

Theorem 1. The norm topology of the cross space RS by the maximum norm is the strongest cross topology.

Recalling §58 Theorem 1, we obtain by §97 Theorem 3

Theorem 2. The adjoint space of the cross space RS by the maximum norm coincides with the biadjoint space $\overline{RS}$, and we have

(5) $$\|\bar{z}\| = \sup_{\|z\|_{max}\le 1} |z(\bar{z})| \qquad (\bar{z}\in\overline{RS},\ z\in RS)$$

for the biadjoint norm $\|\bar{z}\|$ ($\bar{z}\in\overline{RS}$).

Because we have by the definition (3)

$$\|\bar{z}\| \ge \sup_{\|z\|_{max}\le 1} |z(\bar{z})|,$$

On the other hand we have by the formula (4)

$$\|\bar{z}\| = \sup_{\|x\|\le 1,\ \|y\|\le 1} |\bar{z}(x,y)| \le \sup_{\|z\|_{max}\le 1} |z(\bar{z})|.$$

Concerning the maximum norm on the cross space RS we have further

(6) $$\|z\|_{max} = \inf_{z=\sum_{\nu=1}^{\kappa} x_\nu y_\nu} \sum_{\nu=1}^{\kappa} \|x_\nu\|\|y_\nu\|.$$

Because, putting $\|z\|_1 = \inf_{z=\sum_{\nu=1}^{\kappa} x_\nu y_\nu} \sum_{\nu=1}^{\kappa} \|x_\nu\|\|y_\nu\|$, we have by (3)

$$\|z\|_{max} \le \inf_{z=\sum_{\nu=1}^{\kappa} x_\nu y_\nu} \sum_{\nu=1}^{\kappa} \|x_\nu y_\nu\|_{max} = \|z\|_1.$$

Thus, we see easily that $\|z\|_1$ ($z\in RS$) satisfies the norm conditions. Furthermore we have by this relation

$$\|x\|\|y\| = \|xy\|_{max} \le \|xy\|_1 \le \|x\|\|y\|,$$

and hence we obtain

$$\|xy\|_1 = \|x\|\|y\| \qquad (x\in R,\ y\in S).$$

Thus for the biassociated space $\widetilde{RS}$ we have

$$\sup_{\|z\|_1\le 1} |\bar{z}(z)| \ge \sup_{\|x\|\le 1,\ \|y\|\le 1} |\bar{z}(x,y)| \qquad (x\in R,\ y\in S,\ \bar{z}\in\widetilde{RS}).$$

Accordingly we see by §96 Theorem 2 that the adjoint space $\overline{RS}^1$ of the cross space RS by the norm $\|z\|_1$ ($z\in RS$) is contained in the adjoint space of RS by the maximum norm and

$$\|\bar{z}\|_1 \ge \|\bar{z}\| \qquad (\bar{z}\in\overline{RS}^1)$$

for the biadjoint norm $\|\bar{z}\|$ ($\bar{z}\in\overline{RS}$). On the other hand we have by Theorem 2

$$\|\bar{z}\| = \sup_{\|z\|_{max}\le 1} |\bar{z}(z)| \ge \sup_{\|z\|_1\le 1} |\bar{z}(z)|.$$

Therefore $\overline{RS}^1$ coincides with $\overline{RS}$ and we have

$$\|\bar{z}\|_1 = \|\bar{z}\| \qquad (\bar{z}\in\overline{RS}).$$

Consequently we have by the formula §72(3)

$$\|z\|_1 = \sup_{\|\bar{z}\|_1\le 1} |\bar{z}(z)| = \sup_{\|\bar{z}\|\le 1} |\bar{z}(z)| = \|z\|_{max}.$$

As the biadjoint norm of the adjoint norm $\|\bar{x}\|$ ($\bar{x}\in\bar{R}$) and $\|\bar{y}\|$ ($\bar{y}\in\bar{S}$), we obtain a norm on the cross space RS. This norm is called the minimum norm of the cross space RS and denoted by $\|z\|_{min}$ ($z\in RS$), that is,

(7) $$\|z\|_{min} = \sup_{\|\bar{x}\|\le 1,\ \|\bar{y}\|\le 1} |z(\bar{x},\bar{y})| \qquad (\bar{x}\in\bar{R},\ \bar{y}\in\bar{S}).$$

With this definition we have obviously by the formula §72(3)

(8) $$\|xy\|_{min} = \|x\|\|y\| \qquad (x\in R,\ y\in S).$$

Theorem 3. For a norm $\|z\|$ on the cross space RS, in order that the norm topology of RS be a cross topology, it is necessary and sufficient that we can find two positive numbers α, β such that

$$\|xy\| \le \alpha\|x\|\|y\| \qquad (x\in R,\ y\in S),$$

$$|z(\bar{x},\bar{y})| \le \beta\|z\|\|\bar{x}\|\|\bar{y}\| \qquad (z\in RS,\ \bar{x}\in\bar{R},\ \bar{y}\in\bar{S}).$$

Proof. If the norm topology of RS is a cross topology, then we have by the condition 2) in §97 that $\|x\|\le 1$, $\|y\|\le 1$ implies

$\|xy\| \leqq \alpha$ for some $\alpha > 0$, and hence $\|xy\| \leqq \alpha \|x\|\|y\|$; and by the condition 3) in §97 that $\|z\| \leqq 1$, $\|\bar{x}\| \leqq 1$, $\|\bar{y}\| \leqq 1$ implies for some $\beta > 0$, $|z(\bar{x}\bar{y})| \leqq \beta$, and hence $|z(\bar{x}\bar{y})| \leqq \beta \|z\|\|\bar{x}\|\|\bar{y}\|$.

Conversely, if a norm $\|z\|$ ($z \in RS$) satisfies the indicated conditions, then we see easily that the norm topology satisfies the conditions 2) and 3) in §97. As the norm topology is standard, the norm topology is by definition a cross topology of RS.

A norm $\|z\|$ on the cross space RS is called a cross norm, if

(9) $\|xy\| = \|x\|\|y\|$ $\quad (x \in R,\ y \in S)$

(10) $|z(\bar{x}\bar{y})| \leqq \|z\|\|\bar{x}\|\|\bar{y}\|$ $\quad (z \in RS,\ \bar{x} \in \bar{R},\ \bar{y} \in \bar{S})$.

With this definition, we see easily that both the maximum norm and the minimum norm on the cross space RS are cross norms. Furthermore we have

Theorem 4. In order that a norm $\|z\|$ ($z \in RS$) be a cross norm on the cross space RS, it is necessary and sufficient that we have

$$\|z\|_{min} \leqq \|z\| \leqq \|z\|_{max} \quad \text{for every } z \in RS.$$

Proof. If $\|z\|$ ($z \in RS$) is a cross norm, then we have by (10) and (7) for every $z \in RS$

$$\|z\| \geqq \sup_{\|\bar{x}\| \leqq 1,\ \|\bar{y}\| \leqq 1} |z(\bar{x}\bar{y})| = \|z\|_{min},$$

and further by (9) and (6)

$$\|z\| \leqq \inf_{z = \sum_{\nu=1}^{\kappa} x_\nu y_\nu} \sum_{\nu=1}^{\kappa} \|x_\nu y_\nu\| = \inf_{z = \sum_{\nu=1}^{\kappa} x_\nu y_\nu} \sum_{\nu=1}^{\kappa} \|x_\nu\|\|y_\nu\| = \|z\|_{max}.$$

Conversely, if $\|z\|_{min} \leqq \|z\| \leqq \|z\|_{max}$ for every $z \in RS$, then we have by (4) and (8) $\|xy\| = \|x\|\|y\|$ ($x \in R$, $y \in S$), and by (7)

$$|z(\bar{x}\bar{y})| \leqq \|z\|_{min}\|\bar{x}\|\|\bar{y}\| \leqq \|z\|\|\bar{x}\|\|\bar{y}\|.$$

Consequently $\|z\|$ ($z \in RS$) is a cross norm by definition.

§99 Biadjoint modulars

Let R and S be modulared spaces. A bilinear functional φ on the product space (R, S) is said to be modular bounded, if

$$\sup_{m(x) \leqq 1,\ m(y) \leqq 1} |\varphi(x, y)| < +\infty \quad (x \in R,\ y \in S).$$

With this definition we have

Theorem 1. In order that a bilinear functional φ on the product space (R, S) be modular bounded, it is necessary and sufficient that we can find two positive numbers α and γ such that

$$\alpha\varphi(x, y) \leqq \gamma + \frac{1}{2}(m(x) + m(y))^2$$

for every $x \in R$ and $y \in S$.

Proof. If φ is modular bounded, then we can find by definition $\alpha > 0$ such that

$$\sup_{m(x) \leqq 1,\ m(y) \leqq 1} |\alpha\varphi(x, y)| \leqq \frac{1}{2}.$$

If $1 < m(x) < +\infty$, $1 < m(y) < +\infty$, then we can find positive numbers $\xi, \eta < 1$ such that $m(\xi x) = 1$, $m(\eta y) = 1$. Then we have by the modular condition 5) $\xi m(x) \geqq 1$, $\eta m(y) \geqq 1$, and hence

$$\alpha\varphi(x, y) = \frac{1}{\xi\eta}\alpha\varphi(\xi x, \eta y) \leqq \frac{1}{2\xi\eta} \leqq m(x)m(y).$$

If $1 < m(x) < +\infty$, $m(y) \leqq 1$, then we can find a positive number $\xi < 1$ such that $m(\xi x) = 1$. Then we have $\xi m(x) \geqq 1$, and hence

$$\alpha\varphi(x, y) = \frac{1}{\xi}\alpha\varphi(\xi x, y) \leqq \frac{1}{2\xi^2} \leqq \frac{1}{2}m(x)^2.$$

We conclude likewise that $m(x) \leqq 1$, $1 < m(y) < +\infty$ implies

$$\alpha\varphi(x, y) \leqq \frac{1}{2}m(y)^2.$$

Therefore we have for every $x \in R$ and $y \in S$

$$\alpha\varphi(x, y) \leqq \frac{1}{2} + \frac{1}{2}(m(x) + m(y))^2.$$

Conversely, if $\alpha\varphi(x, y) \leqq \gamma + \frac{1}{2}(m(x) + m(y))^2$ for every $x \in R$ and $y \in S$, then we have obviously

$$\sup_{m(x) \leqq 1,\ m(y) \leqq 1} |\varphi(x, y)| \leqq \gamma + 2.$$

Recalling §81 Theorem 3, we see at once by definition that a bilinear functional φ on the product space (R, S) is modular bounded, if and only if φ is norm bounded by the modular norms of R and S, and hence, if and only if φ belongs to the biadjoint space $\overline{RS}$ of R and S by the norm topologies for the modular norms. Thus the biadjoint space $\overline{RS}$ will be called the modular biadjoint space of R and S.

Now, putting

(1) $\quad \bar{m}(\bar{z}) = \sup_{x \in R,\ y \in S} \{\bar{z}(x, y) - \frac{1}{2}(m(x) + m(y))^2\}$ $\quad (\bar{z} \in \overline{RS})$,

we shall prove that $\bar{m}$ satisfies the modular conditions.

It is evident by definition that

1') $\overline{m}(0) = 0$,

2') $\overline{m}(-\overline{z}) = \overline{m}(\overline{z})$.

Recalling Theorem 1 we obtain immediately by definition

3') for any $\overline{z} \in \overline{RS}$ we can find $\alpha > 0$ such that $\overline{m}(\alpha \overline{z}) < +\infty$.

If $\overline{m}(\xi \overline{z}) = 0$ for every $\xi > 0$, then we have by definition

$$\xi \overline{z}(x, y) \leqq \frac{1}{2}(m(x) + m(y))^2$$

for every $x \in R$, $y \in S$, and $\varepsilon > 0$. On the other hand, for any $x \in R$ and $y \in S$ we can find two positive numbers ξ, η such that $m(\xi x) < +\infty$, $m(\eta y) < +\infty$, and then

$$|\overline{z}(x, y)| \leqq \frac{1}{2\xi\xi\eta}\{m(\xi x) + m(\eta y)\}^2$$

for every $\xi > 0$. Consequently we obtain $\overline{z}(x, y) = 0$. Thus we have

4') $\overline{m}(\xi \overline{z}) = 0$ for every $\xi > 0$ implies $\overline{z} = 0$.

Furthermore we can prove easily by definition

5') $\alpha + \beta = 1$, $\alpha, \beta \geqq 0$ implies for every $\overline{z}_1, \overline{z}_2 \in \overline{RS}$

$$\overline{m}(\alpha \overline{z}_1 + \beta \overline{z}_2) \leqq \alpha \overline{m}(\overline{z}_1) + \beta \overline{m}(\overline{z}_2),$$

6') $\overline{m}(\overline{z}) = \sup_{0 \leqq \xi < 1} \overline{m}(\xi \overline{z})$.

This modular $\overline{m}$ on the modular biadjoint space $\overline{RS}$ is called the <u>biadjoint modular</u> of the modulars of R and S.

Concerning the biadjoint modular $\overline{m}(\overline{z})$ ($\overline{z} \in \overline{RS}$), we have obviously by the definition (1)

(2) $\overline{z}(x, y) \leqq \overline{m}(\overline{z}) + \frac{1}{2}(m(x) + m(y))^2$ ($x \in R$, $y \in S$).

Let $\|\overline{z}\|$ ($\overline{z} \in \overline{RS}$) be the biadjoint norm of the modular norm $|||x|||$ ($x \in R$) and $|||y|||$ ($y \in S$), that is,

$$\|\overline{z}\| = \sup_{m(x) \leqq 1,\, m(y) \leqq 1} |\overline{z}(x, y)|$$

Then, we have obviously by the formula (2)

(3) $\overline{m}(\overline{z}) \leqq 1$ implies $\|\overline{z}\| \leqq 3$.

On the other hand, we have already proved in Proof of Theorem 1 that $\|\overline{z}\| \leqq 1$ implies for $m(x) > 1$ or $m(y) > 1$

$$\overline{z}(x, y) \leqq \frac{1}{2}(m(x) + m(y))^2.$$

Thus we obtain by the definition (1) that $\|\overline{z}\| \leqq 1$ implies

$$\overline{m}(\overline{z}) = \sup_{m(x) \leqq 1,\, m(y) \leqq 1} \{\overline{z}(x, y) - \frac{1}{2}(m(x) + m(y))^2\}.$$

Therefore we have

(4) $\|\overline{z}\| \leqq 1$ implies $\overline{m}(\overline{z}) \leqq 1$.

By virtue of the relations (3) and (4), we conclude that the modular topology of $\overline{RS}$ coincides with the norm topology of $\overline{RS}$ by the biadjoint norm for the modular norms of R and S. Therefore we have by Theorem 6 in §95

<u>Theorem 2.</u> <u>The biadjoint modular of the modulars of R and S is complete, and the modular topology of the modular biadjoint space $\overline{RS}$ coincides with the biadjoint topology of the modular topologies of R and S.</u>

Recalling §81 Theorem 3, we obtain by (3) and (4) relation of the biadjoint norm and the modular norm of $\overline{RS}$:

(5) $|||\overline{z}||| \leqq \|\overline{z}\| \leqq 3\,|||\overline{z}|||$ ($\overline{z} \in \overline{RS}$).

For the associated norms $\|\overline{x}\|$ ($\overline{x} \in \overline{R}$) and $\|\overline{y}\|$ ($\overline{y} \in \overline{S}$), we have obviously by definition

(6) $\|\overline{x}\,\overline{y}\| = \|\overline{x}\|\,\|\overline{y}\|$ ($\overline{x} \in \overline{R}$, $\overline{y} \in \overline{S}$)

for the modular adjoint spaces $\overline{R}$ of R and $\overline{S}$ of S.

For every $\overline{x} \in \overline{R}$ and $\overline{y} \in \overline{S}$ we have by definition

$$\overline{m}(\overline{x}\,\overline{y}) = \sup_{x \in R,\, y \in S} \{\overline{x}(x)\overline{y}(y) - \frac{1}{2}(m(x) + m(y))^2\}$$

$$\leqq \sup_{x \in R,\, y \in S} \{(\overline{m}(\overline{x}) + m(x))(\overline{m}(\overline{y}) + m(y)) - \frac{1}{2}(m(x) + m(y))^2\}$$

$$\leqq \frac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\}^2.$$

Accordingly we have

(7) $\overline{m}(\overline{x}\,\overline{y}) \leqq \frac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\}^2$ ($\overline{x} \in \overline{R}$; $\overline{y} \in \overline{S}$).

From (5), (6) we conclude immediately by §83 Theorem 1

$$|||\overline{x}\,\overline{y}||| \leqq \|\overline{x}\|\,\|\overline{y}\| \leqq 4\,|||\overline{x}|||\,|||\overline{y}|||,$$

$$3\,|||\overline{x}\,\overline{y}||| \geqq \|\overline{x}\|\,\|\overline{y}\| \geqq |||\overline{x}|||\,|||\overline{y}|||.$$

Therefore we obtain by §81 Theorem 4 for every $\overline{x} \in \overline{R}$, $\overline{y} \in \overline{S}$

(8) $\frac{1}{3}|||\overline{x}|||\,|||\overline{y}||| \leqq |||\overline{x}\,\overline{y}||| \leqq \|\overline{x}\|\,\|\overline{y}\| \leqq 4\,|||\overline{x}|||\,|||\overline{y}|||$.

§100 <u>Cross modulars</u>

Let R and S be modulared spaces. We can define a modular on the cross space RS as the adjoint modular of the biadjoint modular $\overline{m}(\overline{z})$ ($\overline{z} \in \overline{RS}$) for the modular biadjoint space $\overline{RS}$. This modular on the cross space RS is called the strong modular of RS, and denoted by m_Δ, that is,

$$m_\Delta(z) = \sup_{\overline{z} \in \overline{RS}} \{ \overline{z}(z) - \overline{m}(\overline{z}) \} \qquad (z \in RS)$$

for the biadjoint modular

$$\overline{m}(\overline{z}) = \sup_{x \in R,\, y \in S} \{ \overline{z}(x, y) - \tfrac{1}{2}(m(x) + m(y))^2 \} \qquad (\overline{z} \in \overline{RS}).$$

As $\overline{z}(x, y) \leq \overline{m}(\overline{z}) + \frac{1}{2}(m(x)+m(y))^2$, we obtain immediately

(1) $$m_\Delta(xy) \leq \tfrac{1}{2}\{m(x) + m(y)\}^2 \qquad (x \in R,\ y \in S).$$

Furthermore, recalling the formula §99(7), we have

$$m_\Delta(z) \geq z(\overline{x}\overline{y}) - \overline{m}(\overline{x}\overline{y}) \geq z(\overline{x}\overline{y}) - \tfrac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\}^2,$$

that is,

(2) $$z(\overline{x}\overline{y}) \leq m_\Delta(z) + \tfrac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\}^2 \qquad (\overline{x} \in \overline{R},\ \overline{y} \in \overline{S}).$$

We shall denote by $|||z|||_\Delta$ ($z \in RS$) the modular norm for the strong modular m_Δ. Then $|||z|||_\Delta \leq 1$, $|||\overline{x}||| \leq 1$, $|||\overline{y}||| \leq 1$ implies $|z(\overline{x}\overline{y})| \leq 2$, because we have $m_\Delta(z) \leq 1$, $\overline{m}(\overline{x}) \leq 1$, $\overline{m}(\overline{y}) \leq 1$ by §81 Theorem 4. Therefore we conclude

(3) $$|z(\overline{x}\overline{y})| \leq 2\, |||z|||_\Delta\, |||\overline{x}|||\, |||\overline{y}||| \qquad (z \in RS,\ \overline{x} \in \overline{R},\ \overline{y} \in \overline{S}).$$

This relation yields by the formula §83(4) and §83 Theorem 1

$$|||x|||\,|||y||| \leq 2\, |||xy|||_\Delta .$$

On the other hand, if $|||x||| \leq 1$, $|||y||| \leq 1$, then we have $m_\Delta(xy) \leq 2$ by (1), and hence $m_\Delta(\frac{1}{2}xy) \leq \frac{1}{2}m_\Delta(xy) \leq 1$. Thus we obtain

$$\tfrac{1}{2}|||xy|||_\Delta \leq |||x|||\,|||y|||$$

by §81 Theorem 4. Consequently we have

(4) $$\tfrac{1}{2}|||x|||\,|||y||| \leq |||xy|||_\Delta \leq 2\, |||x|||\,|||y||| \qquad (x \in R,\ y \in S).$$

By virtue of Theorems 4 and 5 in §96, we see that the cross space RS may be considered as a subspace of the biassociated space of the product space $(\overline{R}, \overline{S})$. Thus we can define a modular on RS as the biadjoint modular of the adjoint modulars $\overline{m}(\overline{x})$ ($\overline{x} \in \overline{R}$) and $\overline{m}(\overline{y})$ ($\overline{y} \in \overline{S}$). This modular on the cross space RS is called the weak modular of RS and denoted by $m_w(z)$ ($z \in RS$), that is,

(5) $$m_w(z) = \sup_{\overline{x} \in \overline{R},\, \overline{y} \in \overline{S}} \{ z(\overline{x}\overline{y}) - \tfrac{1}{2}(\overline{m}(\overline{x}) + \overline{m}(\overline{y}))^2 \}.$$

With this definition we have obviously

(6) $$z(\overline{x}\overline{y}) \leq m_w(z) + \tfrac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\}^2 \qquad (\overline{x} \in \overline{R},\ \overline{y} \in \overline{S}).$$

Recalling the formula §99(7), we obtain furthermore

(7) $$m_w(xy) \leq \tfrac{1}{2}\{m(x) + m(y)\}^2 \qquad (x \in R,\ y \in S).$$

Denoting by $|||z|||_w$ ($z \in RS$) the modular norm for the weak modular m_w, we have by the formula §99(5)

(8) $$|||z|||_w \leq \|z\| \leq 3\, |||z|||_w \qquad (z \in RS)$$

for the biadjoint norm

$$\|z\| = \sup_{|||\overline{x}||| \leq 1,\, |||\overline{y}||| \leq 1} |z(\overline{x}\overline{y})| = \sup_{\overline{m}(\overline{x}) \leq 1,\, \overline{m}(\overline{y}) \leq 1} |z(\overline{x}\overline{y})|.$$

Consequently we have

(9) $$|z(\overline{x}\overline{y})| \leq 3\, |||z|||_w\, |||\overline{x}|||\, |||\overline{y}|||. \qquad (z \in RS,\ \overline{x} \in \overline{R},\ \overline{y} \in \overline{S}).$$

Furthermore we obtain by the formula §99(8)

(10) $$\tfrac{1}{3}|||x|||\,|||y||| \leq |||xy|||_w \leq 4\, |||x|||\,|||y||| \qquad (x \in R,\ y \in S).$$

By virtue of Theorems 1 and 2 in §97, we obtain by definition

Theorem 1. The modular topology of the cross space RS by the strong modular m_Δ is the strongest cross topology, and that by the weak modular m_w is the weakest cross topology.

A modular m on the cross space RS is called a cross modular, if

(11) $$m(xy) \leq \tfrac{1}{2}\{m(x) + m(y)\}^2 \qquad \text{for } x \in R,\ y \in S,$$

(12) $$z(\overline{x}\overline{y}) \leq m(z) + \tfrac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\}^2 \qquad \text{for } z \in RS,\ \overline{x} \in \overline{R},\ \overline{y} \in \overline{S}.$$

With this definition we see at once that both the strong modular and the weak modular are cross modulars. We can prove further

Theorem 2. In order that a modular m on the cross space RS be a cross modular, it is necessary and sufficient that we have

$$m_w(z) \leq m(z) \leq m_\Delta(z) \qquad \text{for every } z \in RS$$

for the strong modular m_Δ and the weak modular m_w on RS.

Proof. If m_1 is a cross modular on the cross space RS, then for any element $\widetilde{z}$ of the biassociated space $\widetilde{RS}$ we have by (11)

$$\sup_{m(x) \leq 1,\, m(y) \leq 1} |\widetilde{z}(x, y)| \leq \sup_{m_1(z) \leq 2} |\widetilde{z}(z)|.$$

Thus we see by definition that the modular adjoint space $\overline{RS}^{m_1}$ of RS by the modular m_1 is contained in the biadjoint space $\overline{RS}$ of (R, S).

For the adjoint modular $\overline{m}_1(\overline{z})$ $(\overline{z} \in \overline{RS}^{m_1})$ of m_1, we have by (11)

$$\overline{m}_1(\overline{z}) = \sup_{z \in RS} \{\overline{z}(z) - m_1(z)\}$$
$$\geqq \sup_{x \in R,\, y \in S} \{\overline{z}(x, y) - \frac{1}{2}(m(x) + m(y))^2\},$$

and hence we obtain by the definition §99(1)

$$\overline{m}_1(\overline{z}) \geqq \overline{m}(\overline{z}) \qquad \text{for every } \overline{z} \in \overline{RS}^{m_1}$$

for the biadjoint modular $\overline{m}$ of R and S. Accordingly we have by §80 Theorem 2 for every $z \in RS$

$$m_1(z) = \sup_{\overline{z} \in \overline{RS}^{m_1}} \{\overline{z}(z) - \overline{m}_1(\overline{z})\}$$
$$\leqq \sup_{\overline{z} \in \overline{RS}} \{\overline{z}(z) - \overline{m}(\overline{z})\} = m_s(z).$$

Recalling the definition (5), we obtain furthermore by (12)

$$m_1(z) \geqq \sup_{\overline{x} \in \overline{R},\, \overline{y} \in \overline{S}} \{z(\overline{x}\,\overline{y}) - \frac{1}{2}(\overline{m}(\overline{x}) + \overline{m}(\overline{y}))^2\} = m_w(z)$$

for every $z \in RS$.

Conversely, for a modular m_1 on the cross space RS, if

$$m_w(z) \leqq m_1(z) \leqq m_s(z) \qquad \text{for every } z \in RS,$$

then we have by the formular (1) for every $x \in R$ and $y \in S$

$$m_1(xy) \leqq m_s(xy) \leqq \frac{1}{2}\{m(x) + m(y)\}^2,$$

and furthermore by the formula (6) for every $\overline{x} \in \overline{R}$ and $\overline{y} \in \overline{S}$

$$z(\overline{x}\,\overline{y}) \leqq m_1(z) + \frac{1}{2}\{\overline{m}(\overline{x}) + \overline{m}(\overline{y})\}^2.$$

Therefore m_1 is a cross modular on the cross space RS by definition.

NOTE I

Definition of linear topologies

Linear topologies were defined first by Kolmogoroff as follows. Let R be a linear space. A topology $\mathcal{T}$ on R is said to be a linear topology on R, if

1) the operation $x + y$ $(x, y \in R)$ is continuous by $\mathcal{T}$, that is, for any a, $b \in R$ and $a + b \in U \in \mathcal{T}$, we can find $a \in V \in \mathcal{T}$, and $b \in W \in \mathcal{T}$ such that $V \times W \subset U$;

2) the operation ξx $(x \in R)$ is continuous by the number topology and $\mathcal{T}$, that is, for any $a \in R$ and real number α, if $\alpha a \in U \in \mathcal{T}$, then we can find $a \in V \in \mathcal{T}$ and $\varepsilon > 0$ such that

$$\xi x \in U \qquad \text{for } x \in V,\ |\xi - \alpha| < \varepsilon.$$

Let $\mathcal{T}$ be a linear topology on R in this sense. From the condition 1) we conclude that for every $a \in R$, the transformation

$$\mathfrak{a}(x) = x - a$$

from R to R itself is continuous by $\mathcal{T}$, and hence $A \in \mathcal{T}$ implies $A + a \in \mathcal{T}$ for every $a \in R$. From the condition 2) we conclude likewise that $A \in \mathcal{T}$ imlies $\alpha A \in \mathcal{T}$ for every real number $\alpha \neq 0$. Furthermore we see by 2) that for any $0 \in A \in \mathcal{T}$ and $a \in R$, as $0a = a$, we can find $\varepsilon > 0$ such that $\xi a \in A$ for $|\xi| \leqq \varepsilon$, and that for any $0 \in A \in \mathcal{T}$ we can find $\varepsilon > 0$ and $0 \in B \in \mathcal{T}$ such that $\xi \varepsilon B \subset A$ for $|\xi| \leqq 1$. Finally we see by 1) that for any $0 \in A \in \mathcal{T}$ we can find $0 \in B \in \mathcal{T}$ such that $B \times B \subset A$. Therefore there exists uniquely by §54 Theorem 8 a linear topology $\mathcal{U}$ in the sense of the text such that the induced topology by $\mathcal{U}$ coincides with $\mathcal{T}$.

Conversely it is obvious by definition that for a linear topology $\mathcal{U}$ in the sense of the text, the induced topology $\mathcal{T}^{\mathcal{U}}$ by $\mathcal{U}$ satisfies the conditions 1) and 2). The definition of linear topologies in the text is due to von Neumann.

NOTE II

Linear quasi-metrics

A quasi-metric $m(x, y)$ ($x, y \in R$) on a linear space R is said to be a linear quasi-metric, if

1) for any $\varepsilon > 0$ we can find $\delta > 0$ such that $m(x, y) \leqq \delta$ implies $m(x+z, y+z) \leqq \varepsilon$ for every $x, y, z \in R$,

2) $\lim_{\nu\to\infty} m(a_\nu, 0) = 0$ implies $\lim_{\nu\to\infty} m(\xi a_\nu, 0) = 0$ for every real number ξ,

3) $\lim_{\nu\to\infty} \alpha_\nu = 0$ implies $\lim_{\nu\to\infty} m(\alpha_\nu x, 0) = 0$ for every $x \in R$.

With this definition it is obvious that the induced quasi-metric by a quasi-norm is a linear quasi-metric. Now let $m(x, y)$ ($x, y \in R$) be a linear quasi-metric in the sequel.

From the condition 1) we conclude that $\lim_{\nu\to\infty} m(a_\nu, b_\nu) = 0$ implies

$$\lim_{\nu\to\infty} m(a_\nu + x, b_\nu + x) = 0 \qquad \text{for every } x \in R.$$

Therefore we obtain by 3) that $\lim_{\nu\to\infty} \alpha_\nu = \alpha$ implies

$$\lim_{\nu\to\infty} m(\alpha_\nu x, \alpha x) = 0 \qquad \text{for every } x \in R,$$

and hence, from the relation

$$| m(\alpha_\nu x, 0) - m(\alpha x, 0) | \leqq m(\alpha_\nu x, \alpha x),$$

we conclude that $\lim_{\nu\to\infty} \alpha_\nu = \alpha$ implies

$$\lim_{\nu\to\infty} m(\alpha_\nu x, 0) = m(\alpha x, 0) \qquad \text{for every } x \in R,$$

that is, $m(\xi x, 0)$ is a continuous function of ξ for every $x \in R$.

For an arbitrary $\varepsilon > 0$, putting

$$\varphi(\xi) = \inf_{m(\xi x) > \varepsilon} m(x, 0) \qquad \text{for } \xi > 0,$$

we have obviously

$$\varphi(\xi) = \inf_{m(x,0) > \varepsilon} m\left(\frac{1}{\xi} x, 0\right) \qquad \text{for } \xi > 0.$$

As $m(\frac{1}{\xi} x, 0)$ is a continuous function of $\xi > 0$ for every $x \in R$, $\varphi(\xi)$ is upper semi-continuous for $\xi > 0$. Thus there exists a continuous point $\xi_0 > 0$ of φ. For such ξ_0, we have $\varphi(\xi_0) > 0$ by the condition 2), and hence for a positive number $\delta < \varphi(\xi_0)$, we can find $\delta' > 0$ such that $|\xi - \xi_0| \leqq \delta'$ implies $\varphi(\xi) > \delta$. For such δ and δ', if $|\eta| \leqq \delta'$, $m(x, 0) \leqq \delta$, then we have naturally

$$|(\eta + \xi_0) - \xi_0| \leqq \delta',$$

and hence we obtain

$$m((\eta + \xi_0)x, 0) \leqq \varepsilon, \qquad m(\xi_0 x, 0) \leqq \varepsilon.$$

For any $\varepsilon' > 0$ we can find by the condition 1) $\varepsilon > 0$ such that

$$m(x, y) \leqq \varepsilon \quad \text{implies} \quad m(x+z, y+z) < \frac{1}{2}\varepsilon'.$$

For such $\varepsilon > 0$ we obtain thus

$$m(\eta x, 0) \leqq m((\eta + \xi_0)x, \eta x) + m((\eta + \xi_0)x, 0) \leqq \varepsilon + \frac{1}{2}\varepsilon'.$$

Therefore we conclude that for any $\varepsilon > 0$ we can find $\delta > 0$ and $\delta' > 0$ such that $|\xi| \leqq \delta$, $m(x, 0) \leqq \delta'$ implies $m(\xi x, 0) \leqq \varepsilon$.

Now, putting

$$U_\varepsilon = \{ x : m(x, 0) \leqq \varepsilon \} \qquad (\varepsilon > 0),$$

we see easily by 3) that every U_ε ($\varepsilon > 0$) is a vicinity in R, and it is obvious that we have

$$U_\varepsilon U_{\varepsilon'} = U_\delta \qquad \text{for } \delta = \mathrm{Min}\{\varepsilon, \varepsilon'\}.$$

For any $\varepsilon > 0$, we can find $\delta > 0$ and $\delta' > 0$ such that

$$\xi U_{\delta'} \subset U_\varepsilon \qquad \text{for } |\xi| \leqq \delta,$$

as proved just above. Furthermore for any $\varepsilon > 0$ we can find by 1) a positive number $\delta < \frac{1}{2}\varepsilon$ such that $m(x, y) \leqq \delta$ implies

$$m(x+z, y+z) \leqq \frac{1}{2}\varepsilon \qquad \text{for every } x, y, z \in R,$$

and hence we have that $m(a, 0) \leqq \delta$, $m(b, 0) \leqq \delta$ implies

$$m(a+b, 0) \leqq m(a+b, b) + m(b, 0) \leqq \frac{1}{2}\varepsilon + \delta \leqq \varepsilon,$$

that is, $U_\delta \times U_\delta \subset U_\varepsilon$.

Therefore, by virtue of §53 Theorem 3 there exists uniquely a linear topology $\mathfrak{P}$ on R, such that U_ε ($\varepsilon > 0$) is a basis of $\mathfrak{P}$. Such a linear topology $\mathfrak{P}$ is obviously sequential, and we see easily by 1) that the induced uniformity by $\mathfrak{P}$ coincides with the induced uniformity by the quasi-metric $m(x, y)$ ($x, y \in R$).

NOTE III

(C) spaces

Let S be a topological space. The totality of bounded continuous functions on S constitutes obviously a linear space. This linear space is called a (C) space. We can introduce into the (C) space C on a topological space S a norm by

$$\|\varphi\| = \sup_{x \in S} |\varphi(x)| \qquad (\varphi \in C).$$

Then we see easily that C is a complete normed space by this norm.

Let R be a normed space and $\overline{R}$ the adjoint space of R. By vitue of §72 Theorem 5, the unit sphere $\overline{U}$ of $\overline{R}$ is weakly compact, and we have by the formula §72 (5)

$$\|x\| = \sup_{\overline{x} \in \overline{U}} |\overline{x}(x)| \qquad \text{for every } x \in R.$$

Furthermore, it is obvious by definition that for every $x \in R$, putting

$$\varphi_x(\overline{x}) = \overline{x}(x) \qquad (\overline{x} \in \overline{U}),$$

we obtain a continuous function φ_x on the unit sphere $\overline{U}$ by the weak topology and

$$\varphi_{\alpha x + \beta y} = \alpha \varphi_x + \beta \varphi_y \qquad \text{for every } x, y \in R,$$

$$\|x\| = \sup_{\overline{x} \in \overline{U}} |\varphi_x(\overline{x})| \qquad \text{for every } x \in R.$$

Therefore we can state that for any normed space R there is a compact topological space $\overline{U}$ such that R is isometric to a linear manifold of the (C) space on $\overline{U}$ as a normed space.

NOTE IV

Inequalities

We have employed several inequalities without proof in §89 and §90. Now we shall prove them in the sequel.

As $\frac{d^2}{dt^2}(-\log t) \geqq 0$ for $t > 0$, $-\log t$ is a convex function of $\xi > 0$, and hence $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \geqq 1$, $\xi, \eta > 0$ implies

$$-\log\left(\frac{1}{p}\xi + \frac{1}{q}\eta\right) \leqq -\frac{1}{p}\log \xi - \frac{1}{q}\log \eta,$$

that is, $\frac{1}{p}\xi + \frac{1}{q}\eta \geqq \xi^{\frac{1}{p}}\eta^{\frac{1}{q}}$. Therefore we obtain Young's inequality

$$\xi\eta \leqq \frac{1}{p}\xi^p + \frac{1}{q}\eta^q$$

for $\xi, \eta \geqq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \geqq 1$.

For positive measurable functions $x(t)$ and $y(t)$ $(0 \leqq t \leqq 1)$, putting

$$\alpha = \left\{\int_0^1 x(t)^p dt\right\}^{\frac{1}{p}}, \qquad \beta = \left\{\int_0^1 y(t)^q dt\right\}^{\frac{1}{q}},$$

we obtain by the Young's inequality

$$\frac{1}{\alpha\beta} x(t) y(t) \leqq \frac{1}{p}\left(\frac{1}{\alpha} x(t)\right)^p + \frac{1}{q}\left(\frac{1}{\beta} y(t)\right)^q,$$

and hence

$$\frac{1}{\alpha\beta}\int_0^1 x(t) y(t) dt \leqq \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore we obtain Hölder's inequality

$$\int_0^1 x(t) y(t) dt \leqq \left\{\int_0^1 x(t)^p dt\right\}^{\frac{1}{p}} \left\{\int_0^1 y(t) dt\right\}^{\frac{1}{q}}$$

for $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \geqq 1$.

Putting

$$\varphi(\xi, \eta) = (\xi + \eta)^p + (\xi - \eta)^p \qquad \text{for } \xi \geqq \eta \geqq 0,$$

we have

$$\varphi_\xi(\xi, \eta) = \frac{\partial}{\partial \xi}\varphi(\xi, \eta) = p\{(\xi+\eta)^{p-1} + (\xi - \eta)^{p-1}\}$$

$$\varphi_{\xi\eta}(\xi, \eta) = \frac{\partial}{\partial \eta}\varphi_\xi(\xi, \eta) = p(p-1)\{(\xi+\eta)^{p-1} - (\xi - \eta)^{p-1}\}.$$

Thus, if $p \geqq 2$, then we have

$$\varphi_{\xi\eta}(\xi, \eta) \geqq 0 \qquad \text{for every } \xi \geqq \eta \geqq 0.$$

Therefore we have for $\xi \geqq \eta \geqq 0$

$$\varphi_\xi(\xi, \eta) \geqq \varphi_\xi(\xi, 0) = 2p\xi^{p-1},$$

and hence $\varphi(\xi, \eta) - \varphi(\eta, \eta) \geqq 2\xi^p - 2\eta^p$. As $p \geqq 2$, we obtain

$$\varphi(\xi, \eta) \geqq 2\xi^p + (2^p - 2)\eta^p \geqq 2(\xi^p + \eta^p),$$

that is, we have

$$|\xi + \eta|^p + |\xi - \eta|^p \geqq 2(|\xi|^p + |\eta|^p) \qquad \text{for } p \geqq 2.$$

From this relation we conclude

$$\frac{|\xi|^p + |\eta|^p}{2} \geqq \left|\frac{\xi + \eta}{2}\right|^p + \left|\frac{\xi - \eta}{2}\right|^p \qquad \text{for } p \geqq 2.$$

It is well known:

$$(1 + \xi)^p = \sum_{\nu=0}^{\infty} \frac{p(p-1)\cdots(p-\nu+1)}{\nu!}\xi^\nu \qquad \text{for } |\xi| < 1.$$

If $1 < p < 2$, then, putting $\alpha_0 = 1$,

$$\alpha_\nu = \frac{p(p-1)\cdots(p-\nu+1)}{\nu!} \qquad (\nu = 1, 2, \ldots),$$

we have obviously

$$\alpha_{2\nu} > 0, \qquad \alpha_{2\nu+1} < 0 \qquad \text{for } \nu = 0, 1, 2, \ldots,$$

and henc for $0 \leqq \xi < 1$

$$(1+\xi)^p + (1-\xi)^p = 2\sum_{\nu=0}^{\infty} \alpha_{2\nu}\,\xi^{2\nu} \geqq 2 + p(p-1)\,\xi^2.$$

Therefore we obtain for $1 \leqq p \leqq 2$, $|\xi| \geqq |\eta|$

$$|\xi+\eta|^p + |\xi-\eta|^p \geqq 2|\xi|^p + p(p-1)\,|\xi|^{p-2}\,|\eta|^2.$$

From this relation we conclude for $1 \leqq p \leqq 2$

$$\frac{|\xi|^p + |\eta|^p}{2} \geqq \left|\frac{\xi+\eta}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{\xi-\eta}{|\xi|+|\eta|}\right|^{2-p}\left|\frac{\xi-\eta}{2}\right|^p.$$

As $\frac{d^2}{dt^2}t^p = p(p-1)t^{p-2} \geqq 0$ for $p > 1$, $t \geqq 0$, $t^p\ (p > 1)$ is a convex function of $t \geqq 0$, and hence we have

$$\frac{1}{n}\sum_{\nu=1}^{n} \xi_\nu^{\,p} \geqq \left(\frac{1}{n}\sum_{\nu=1}^{n} \xi_\nu\right)^p$$

for every finite number of $\xi_\nu \geqq 0\ (\nu = 1, 2, \ldots, n)$. Putting $p = \frac{\rho}{\sigma}$ in this inequality, we obtain immediately

$$\left\{\frac{1}{n}\sum_{\nu=1}^{n} |\xi_\nu|^\rho\right\}^{\frac{1}{\rho}} \geqq \left\{\frac{1}{n}\sum_{\nu=1}^{n} |\xi_\nu|^\sigma\right\}^{\frac{1}{\sigma}} \qquad \text{for } \rho > \sigma > 0.$$

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