Notes on Conjugate Connections

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The purpose of this paper is to give an overall view of the topics that center around the notion of conjugate connection and its applications within the geometry of affine connections including affine differential geometry. We owe a lot to the book by Schirokow and Schirokow [Sch]. Much of what we write is by now part of the standard background for the subject, but some is relatively new. As general references, see [N2], [N-P1], [Si1], [S-S-V].

1. Conjugate connection – definition

In the following let M be a differentiable manifold of dimension n, h a nondegenerate metric on M, and ∇ a linear connection. We define the *conjugate connection of* ∇ *relative to* h as the linear connection determined by the equation

$$(1.1)\qquad Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z), \text{ where } X, Y, Z \in \mathbf{X}(M),$$

where $\mathbf{X}(M)$ denotes the space of all vector fields on M.

Obviously, the conjugate connection of ∇^* relative to h coincides with ∇. So we can say that ∇ and ∇^* are conjugate relative to h. A standard example for conjugate connection comes from a nondegenerate hypersurface with relative normalization in affine space, as will be seen in Section 7.

When a given connection ∇ has torsion 0: $\nabla_X Y - \nabla_Y X - [X,Y] = 0$, does its conjugate connection ∇^* have torsion 0? The answer to this question is given as follows. We denote by ∇h the covariant differential defined by

$$\nabla h(X,Y,Z) = (\nabla_X h)(Y,Z) = Xh(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

where X, Y, Z are vector fields on M. Since h is symmetric, $\nabla h(X,Y,Z)$ is symmetric in Y and Z. Especially, ∇^* coincides with ∇ if and only if $\nabla h = 0$, that is, ∇ is the Levi-Civita connection for h.

Proposition 1.1. *The conjugate connection ∇^* has torsion 0 if and only if the pair (∇, h) satisfies Codazzi's equation:*

$$(1.2)\qquad (\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z).$$

that is, if and only if ∇h is symmetric in all three variables.

Proof. Using (1.1) we obtain

$$\nabla h(X,Y,Z) = \nabla h(X,Z,Y) = Xh(Z,Y) - h(\nabla_X Z, Y) - h(Z, \nabla_X Y)$$
$$= h(Z, \nabla^*_X Y) - h(Z, \nabla_X Y).$$

Similarly we obtain

$$\nabla h(Y,X,Z) = h(Z, \nabla^*_Y X) - h(Z, \nabla_Y X).$$

Subtract the second equation from the first and use the fact that the torsion of ∇ is 0. We obtain

$$\nabla h(X,Y,Z) - \nabla h(Y,X,Z) = h(Z, \nabla^*_X Y - \nabla^*_Y X - [X,Y]).$$

Since h is nondegenerate, (1.2) holds if and only if the torsion of ∇^* is 0.

We shall say that a torsion-free connection ∇ is compatible with a nondegenerate metric h if (∇, h) satisfies Codazzi's equation. In this case, $C = \nabla h$ is called the cubic form for (∇, h). From (1.1) we get

$$\nabla h + \nabla^* h = 0. \tag{1.3}$$

It now follows that the connection $\hat{\nabla}$ given by

$$\hat{\nabla}_X Y = \frac{1}{2}(\nabla_X Y + \nabla^*_X Y) \tag{1.4}$$

has torsion 0 and is metric, that is, $\hat{\nabla} h = 0$. This means that $\hat{\nabla}$ is the Levi-Civita connection for h. We state

Proposition 1.2. *If a torsion-free connection ∇ is compatible with a nondegenerate metric h, then the conjugate connection ∇^* has torsion 0 and the average of ∇ and ∇^* coincides with the Levi-Civita connection for h.*

Corollary 1.3. *Suppose (M, h) is a semi-Riemannian manifold and ∇ and ∇^* two connections on M. Assume that ∇ has torsion 0 and is compatible with h. If $(\nabla + \nabla^*)/2$ coincides with the Levi-Civita connection $\hat{\nabla}$ of h, then ∇ and ∇^* are conjugate relative to h.*

Proof. Let ∇' be the conjugate connection of ∇ relative to h. Since ∇ has torsion 0 and is compatible with h, Proposition 1.2 implies that $(\nabla + \nabla')/2 = \hat{\nabla}$. Thus $\nabla^* = \nabla'$, that is, ∇^* is conjugate to ∇.

2. Conjugate connection – interpretations

At this point we give two interpretations of conjugate connection. The first is geometric. Let $x_t, 0 \le t \le 1$ be a curve in M. We denote by τ and τ^* the parallel displacement along the curve relative to ∇ and ∇^*, respectively. Then for any $X, Y \in T_x(M)$, we have

$$h(\tau X, \tau^* Y) = h(X, Y). \tag{2.1}$$

Another interpretation of conjugate connection is based on the relationship between the tangent bundle $T(M)$ and the cotangent bundle $T^*(M)$. A linear connection ∇ on M is a linear connection in the vector bundle $T(M)$, and as such, it defines the dual connection ${}^*\nabla$ in $T^*(M)$ as follows. For any section α of $T^*(M)$, we define the covariant derivative ${}^*\nabla_X \alpha$ by

$$< Y, {}^*\nabla_X \alpha >= X(\alpha(Y)) - \alpha(\nabla_X Y), \tag{2.2}$$

for any vector field Y, where $< \, , \, >$ denotes the pairing of vectors and covectors. Now given a nondegenerate metric h, we have an isomorphism $\Phi : T(M) \to T^*(M)$ as vector bundles:

$$< X, \Phi(Z) >= h(X, Z) \quad \text{for all} \quad X \in T(M). \tag{2.3}$$

If we now transfer the dual connection ${}^*\nabla$ to a connection in $T(M)$ by means of Φ, then what we get is precisely the conjugate connection ∇^*, namely,

$$\nabla^*_X Y = \Phi^{-1}({}^*\nabla_X \Phi(Y)), \tag{2.4}$$

which is equivalent to (1.1), as is easily verified. We may thus interpret the conjugate connection ∇^* as the dual connection ${}^*\nabla$ in $T^*(M)$ through the identification Φ by means of h.

3. More relationships

[DNV] contains several fundamental relations for conjugate connections; some more come from [SSV]. We continue from Section 1 and henceforth assume that ∇ is a torsion-free connection compatible with h (namely, the pair (∇, h) satisfies Codazzi's equation). Let ∇^* be the conjugate connection of ∇ relative to h. Let K be the difference tensor between ∇ and the Levi-Civita connection $\hat{\nabla}$ for h. More precisely, we have

$$K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y \quad \text{for all} \quad X, Y \in \mathfrak{X}(M). \tag{3.1}$$

We also write $K(X, Y)$ as $K_X Y$. Since ∇ and $\hat{\nabla}$ have torsion 0, we get $K(X, Y) = K(Y, X)$, or equivalently, $K_X Y = K_Y X$. From (1.4) we obtain

$$K(X, Y) = \hat{\nabla}_X Y - \nabla^*_X Y. \tag{3.2}$$

By applying K_X to h as derivation we obtain

$$\begin{aligned}\nabla h(X,Y,Z) &= (K_X \cdot h)(Y,Z)\\ &= -h(K_X Y, Z) - h(Y, K_X Z).\end{aligned}$$

Since $\nabla h(X,Y,Z)$ and $h(K_X Y, Z)$ are symmetric in X, Y, it follows that $h(Y, K_X Z)$ is symmetric in X, Y as well as in X, Z. Thus $h(Y, K_X Z)$ is symmetric in X, Y, Z and is equal to $h(K_X Y, Z)$. Thus from the equation above we get

$$\nabla h(X,Y,Z) = -2h(K_X Y, Z) = -2h(Y, K_X Z). \tag{3.3}$$

We have

Proposition 3.1. *The curvature tensors R and R^* of ∇ and ∇^*, respectively, are related to each other by*

$$h(R(X,Y)Z,U) = -h(Z, R^*(X,Y)U), \tag{3.4}$$

that is, $R^*(X,Y)$ is the skew-adjoint of $R(X,Y)$ relative to h.

Proof. From $Yh(Z,U) = h(\nabla_Y Z) + h(Z, \nabla^*_Y U)$ we obtain

$$XYh(Z,U) = h(\nabla_X \nabla_Y Z, U) + h(\nabla_Y Z, \nabla_X U) + h(\nabla_X Z, \nabla^*_Y U) + h(Z, \nabla^*_X \nabla^*_Y U).$$

Subtracting from this equation the one obtained by interchanging X and Y as well as the equation

$$[X,Y]h(Z,U) = h(\nabla_{[X,Y]} Z, U) + h(Z, \nabla^*_{[X,Y]} U),$$

we obtain the formula.

Corollary. $R = 0$ *if and only if* $R^* = 0$.

Further relations between R, R^* and the curvature tensor $\hat{R}$ of $\hat{\nabla}$ can be found as follows:

Proposition 3.2.

$$R(X,Y) = \hat{R}(X,Y) + (\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X + [K_X, K_Y], \tag{3.5}$$

which we may also write

$$R(X,Y) = \hat{R}(X,Y) + (\nabla_X K)_Y - (\nabla_Y K)_X. \tag{3.6}$$

$$R^*(X,Y) = \hat{R}(X,Y) - (\hat{\nabla}_X K)_Y + (\hat{\nabla}_Y K)_X + [K_X, K_Y]. \tag{3.7}$$

$$R(X,Y) - R^*(X,Y) = 2[(\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X]. \tag{3.8}$$

$$\mathrm{Ric}(Y,Z) - \mathrm{Ric}^*(Y,Z) = 2\hat{L}(Y,Z) - 2(\hat{\nabla}_Y \hat{T})(Z), \tag{3.9}$$

where Ric and Ric* are the Ricci tensors for ∇ and ∇^*, respectively,

$$\hat{L}(Y,Z) = \mathrm{tr}[X \mapsto (\hat{\nabla}_X K)(Y,Z)] \tag{3.10}$$

and $\hat{T}$ is a 1-form, called the Tchebychev form, defined by

$$\hat{T}(Z) = (\mathrm{tr} K_Z)/n. \tag{3.11}$$

Proof. To obtain (3.5) we compute $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ by using (3.1) and noting

$$(\hat{\nabla}_X K)_Y = [\hat{\nabla}_X, K_Y] - K_{\hat{\nabla}_X Y}.$$

To obtain (3.6) we may observe

$$(\nabla_X K)_Y = (\hat{\nabla}_X K)_Y + (K_X \cdot K)_Y = (\hat{\nabla}_X K)_Y + K_X K_Y - K_{K_X Y}.$$

(3.7) can be obtained in the same way as (3.5) by changing $K_X = \nabla_X - \hat{\nabla}_X$ to $-K_X = \hat{\nabla}_X - \nabla^*_X$ in the computation. (3.8) follows immediately from (3.5) and (3.7).

Take the trace of the linear map $X \mapsto R(X,Y)Z - R^*(X,Y)Z$. Then (3.9) follows from (3.8), (3.10) and the identity:

$$\mathrm{tr}[X \mapsto (\hat{\nabla}_Y K)_X Z] = n(\hat{\nabla}_Y \hat{T})(Z),$$

which can be shown as follows. First note that $(\hat{\nabla}_Y K)_X Z = (\hat{\nabla}_Y K)_Z X$. Now we have $(\hat{\nabla}_Y K)_Z = \hat{\nabla}_Y (K_Z) - K(\hat{\nabla}_Y Z)$, where Z is extended to a vector field for this computation. Thus

$$\mathrm{tr}\hat{\nabla}_Y(K_Z) = Y\mathrm{tr}(K_Z) = nY\hat{T}(Z) \quad \text{and} \quad \mathrm{tr}K_{\hat{\nabla}_Y Z} = nT(\hat{\nabla}_Y Z),$$

which implies $\mathrm{tr}(\hat{\nabla}_Y K)_Z = n(\hat{\nabla}_Y \hat{T})(Z)$.

Proposition 3.4. Let ∇, ∇^* be torsion-free and conjugate relative to h. Let S be a (1,1) tensor field and $\hat{S}(X,Y) := h(SX,Y)$. Then the following equations are equivalent:

(i) $(\nabla_X S)(Y) = (\nabla_Y S)(X)$;

(ii) $(\nabla^*_X \hat{S})(Y,Z) = (\nabla^*_Y \hat{S})(X,Z)$.

Proof. Straightforward verification.

4. Equiaffine connection

A torsion-free connection ∇ on a differentiable manifold M is said to be equiaffine if there is a parallel volume element ω, that is, a non-vanishing n-form ω such that $\nabla\omega = 0$. If such ω exists in a neighborhood of each point of M, we say that ∇ is locally equiaffine. When M is simply connected, ∇ is equiaffine if it is locally equiaffine.

We have

Proposition 4.1. *The following conditions are mutually equivalent:*

(i) ∇ *is locally equiaffine;*

(ii) *The Ricci tensor* Ric *is symmetric;*

(iii) *For any local coordinate system* $\{x^1, \cdots, x^n\}$ *there exists a positive differentiable function* $\phi(x^1, \cdots, x^n)$ *such that*

$$\partial \ln \phi / \partial x^i = \Gamma^k_{ik},$$

where Γ^k_{ij} *are the Christoffel symbols and we use the summation convention.*

Proof. Assume (i). Then the homogeneous holonomy group of ∇ at a point $x \in M$ leaves a non-zero n-form ω_x invariant and is thus contained in $SL(n, R)$. Hence for any $X, Y \in T_x(M), R(X,Y)$, which is contained in its Lie algebra, has trace 0. From the first Bianchi identity: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, we get

$$\mathrm{trace}\{X \mapsto R(X,Y)Z\} + \mathrm{trace}R(Y,Z) + \mathrm{trace}\{X \mapsto R(Z,X)Y\} = 0,$$

thus

$$\mathrm{Ric}(Y,Z) = \mathrm{Ric}(Z,Y).$$

Conversely, assume that (ii) holds at every point $x \in M$. Then trace $R(X,Y) = 0$ as is clear from the arguments above. Since this holds at every point of M, it follows from a well-known theorem that the Lie algebra of the homogeneous holonomy group is generated by all the endomorphisms of the form $\tau R(X,Y)$, where X, Y are arbitrary tangent vectors at an arbitrary point $y \in M$ and τ denotes the parallel displacement along any arbitrary curve τ from y to x. This means that the restricted homogeneous holonomy group at x is contained in $SL(n, R)$, that is to say, that it leaves a certain non-zero n-form ω_x invariant. By displacing ω_x to each point y along any arbitrary curve from x to y, we can well define a parallel n-form ω. Hence ∇ is locally equiaffine. Thus we have proved the equivalence of (i) and (ii).

To show the equivalence of (i) and (iii), suppose ω is a local parallel volume element. For any local coordinate system $\{x^1, \cdots, x^n\}$, we write $\phi = \omega(X_1, \cdots, X_n)$, where $X_i = \partial/\partial x^i$. Then we have

$$(\nabla_{X_i}\omega)(X_1, \cdots, X_n) = X_i\phi - \omega(\nabla_{X_i}X_1, \cdots, X_n) - \cdots - \omega(X_1, ..., X_{n-1}, \nabla_{X_i}X_n),$$

which implies

$$\frac{1}{\phi}\partial \ln \phi/\partial x^i = \sum_{k=1}^{n}(\Gamma_{ik}^k),$$

namely, (iii). Conversely, the formula (iii) implies that the local n-form ω defined by $\omega(X_1, \cdots, X_n) = \phi$ is parallel, completing the proof of Proposition 4.1.

We denote by ω_h the volume element for h. Suppose ω and ω^* are two volume elements such that

$$\omega = \phi\omega_h \qquad \text{and} \qquad \omega^* = \psi\omega_h.$$

Then we have

$$(\nabla_X \omega)/\phi + (\nabla^*_X \omega^*)/\psi = [X \ln(\phi\psi)]\omega_h, \tag{4.1}$$

for any tangent vector X. This easily follows from

$$\nabla_X \omega = (X\phi)\omega_h + \nabla_X \omega_h \qquad \text{and} \qquad \nabla^*_X \omega^* = (X\psi)\omega_h + \nabla^*_X \omega^*.$$

We obtain

Proposition 4.2. Suppose ∇ and ∇^* are conjugate relative to h.

(1) *∇ is equiaffine if and only if ∇^* is. ∇ has symmetric Ricci tensor if and only if ∇^* does.*

(2) *If $\nabla\omega = 0$ and $\nabla^*\omega^* = 0$, then $\omega \cdot \omega^* = c{\omega_h}^2$, where c is a positive constant.*

(3) *$\nabla\omega_h = 0$ if and only if $\nabla^*\omega_h = 0$.*

Proof.

(1) If $\nabla\omega = 0$, set $\omega = \phi\omega_h$, where ϕ is a certain function > 0. Let $\omega^* = \omega_h/\phi$. From (4.1) we obtain $\nabla^*\omega^* = 0$. The converse is obvious. The second assertion follows from Proposition 4.1.

(2) Let $\omega = \phi\omega_h$ and $\omega^* = \psi\omega_h$. From (4.1) and the assumption we get $X[\ln(\phi\psi)] = 0$ for every tangent vector X. Hence $\phi\psi = c > 0$ (constant) .

(3) Suppose $\nabla\omega = 0$. Then in the proof of (1) we get $\phi = 1$ so $\omega^* = \omega_h$ satisfies $\nabla^*\omega_h = 0$.

The condition $\nabla\omega_h = 0$ that appeared above is called apolarity. Geometrically, it means that ω_h is invariant by parallel displacement relative to ∇. If $\nabla h = 0$, then, of course, we have $\nabla\omega_h = 0$. In particular, note that $\hat{\nabla}\omega_h = 0$.

Propostion 4.3. *The following conditions are equivalent:*

(1) $\nabla\omega_h = 0$;

(2) $\mathrm{tr}K_X = 0$ *for all $X \in T(M)$; in other words, the Tchebychev form $\hat{T}$ is identically 0;*

(3) $\mathrm{tr}_h K = 0$;

(4) $\mathrm{tr}_h(\nabla_X h) = 0$ *for all* $X \in T(M)$.

Proof. Recall that $K_X = \nabla_X - \hat{\nabla}_X$. By applying these derivations on ω_h and noting $\hat{\nabla}\omega_h = 0$ we have $\nabla_X \omega_h = K_X \cdot \omega_h = -\mathrm{tr} K_X \omega_h$. Thus (1) and (2) are equivalent.

We show that (2) and (3) are equivalent. For this purpose, let $\{X_1, \cdots, X_n\}$ be an orthonormal basis in $T_x(M)$, that is, $h(X_i, X_j) = \epsilon_i \delta_{ij}$. Then $\mathrm{tr}_h K = \sum_{i=1}^n \epsilon_i K(X_i, X_i)$ and for any $Y \in T_x(M)$ we have

$$\begin{aligned} h(\mathrm{tr}_h K, Y) &= \sum \epsilon_i h(K(X_i, X_i), Y) \\ &= \sum \epsilon_i h(K_{X_i} X_i, Y) = \sum \epsilon_i h(X_i, K_{X_i} Y) \\ &= \sum \epsilon_i h(X_i, K_Y X_i) = \mathrm{tr} K_Y, \end{aligned}$$

which implies that $\mathrm{tr}_h K = 0$ if and only if $\mathrm{tr} K_Y = 0$ for every $Y \in T_x(M)$. The equivalence of (2) and (4) is similarly proved.

As a consequence of (3.1-2), the definition (3.11) of the Tchebychev form and of Proposition 4.1 (iii) we get a geometric interpretation of the Tchebychev form as a measure for deviation of the three volume forms ω, ω, ω_h:

$$n\hat{T} = \frac{1}{2} d \ln \frac{|\omega(X_1, \cdots, X_n)|}{|\omega^*(X_1, \cdots, X_n)|} = d \ln \frac{|\omega(X_1, \cdots, X_n)|}{|\omega_h(X_1, \cdots, X_n)|} = d \ln \frac{|\omega_h(X_1, \cdots, X_n)|}{|\omega^*(X_1, \cdots, X_n)|}.$$

Following the definition given after Proposition 1.1, we shall say that a torsion-free connection ∇ is strongly compatible with a nondegenerate metric h if it is compatible with h and if, furthermore, apolarity condition $\nabla \omega_h = 0$ holds.

Proposition 4.4. *∇ is locally equiaffine if and only if $d\hat{T} = 0$.*

Proof. Suppose ω is a volume element defined on an open subset U of M and write $\omega = \lambda \omega_h$, where λ is a positive function on U. Then

$$\nabla_X(\lambda \omega_h) = (X\lambda)\omega_h + \lambda(\nabla_X \omega_h) = [(X\lambda + n\lambda \hat{T}(X)]\omega_h.$$

Thus $\nabla_X \omega = 0$ if and only if $\hat{T} = -d \ln \lambda$ on U. This proves Propostion 4.4.

5. Two pairs of conjugate connections

The following construction of two pairs of conjugate connections seems new.

Given ∇ compatible with h, assume that there exists a tensor field L of type (1,1) that satisfies the equations of Ricci and of Codazzi, namely,

$$h(LX, Y) = h(X, LY) \text{ and } (\nabla_X L)(Y) = (\nabla_Y L)(X). \tag{5.1}$$

Furthermore, assume that L is nonsingular. We define a new torsion-free connection ∇ by

$$\nabla'_X Y = L^{-1} \nabla_X (LY); \tag{5.2}$$

$$\nabla^{*\prime}_X Y = L^{-1}\nabla^*_X(LY), \tag{5.3}$$

where Y is a vector field and X a tangent vector. Note that these can also be written as

$$\nabla'_X Y = \nabla_X Y + L^{-1}(\nabla_X L)Y$$

$$\nabla^{*\prime}_X Y = \nabla_X Y + L^{-1}(\nabla^*_X L)Y.$$

Then

Proposition 5.1.
(1) ∇' *and* ∇^* *are conjugate relative to* h^*, *where* $h^*(Y,Z) := h(LY,Z)$.
(2) ∇ *and* $\nabla^{*\prime}$ *are conjugate relative to* h^*.

Proof. We compute:

$$\begin{aligned} Xh^*(Y,Z) &= Xh(Y,LZ) = h(\nabla^*_X Y, LZ) + h(Y, \nabla_X(LZ)) \\ &= h^*(\nabla^*_X Y, Z) + h(Y, L\nabla_X Z) + h(Y, (\nabla_X L)Z) \\ &= h^*(\nabla^*{}_X Y, Z) + h^*(Y, \nabla_X Z) + h^*(Y, L^{-1}(\nabla_X L)Z) \\ &= h^*(\nabla^*_X Y, Z) + h^*(Y, \nabla'_X Z), \end{aligned}$$

and hence

$$Xh^*(Y,Z) = h^*(\nabla^*_X Y, Z) + h^*(Y, \nabla'_X Z),$$

proving (1). The computation for (2) is similar.

6. Metrics for conjugate connections

Using an interpretation of conjugate connection given by (2.4), we shall provide a conceptual proof of the following result which was useful in proving the Cartan-Norden theorem in [N-P1].

Proposition 6.1. *Suppose two linear connections* ∇ *and* ∇^* *are conjugate to each other relative to a nondegenerate metric* h. *If* $\nabla g = 0$ *for some nondegenerate metric* g, *then* $\nabla^* g^* = 0$ *for some nondegenerate metric* g^*.

Proof. Recall that (2.4) gives the relationship between ∇^* and the dual connection ${}^*\nabla$ in the cotangent bundle. Now if $\nabla g = 0$, it follows from the lemma below that ${}^*\nabla^* g = 0$, where ${}^*g$ is the the fiber metric in the cotangent bundle ${}^*T(M)$ that is dual to g. We define $g^*(Y,Z) = {}^*g(\Phi Y, \Phi Z)$ for all $Y, Z \in T_x(M)$. Then using (2.4) we can verify that ${}^*\nabla^* g = 0$ implies $\nabla^* g^* = 0$.

Lemma. *Let* g *be a nondegenerate metric on a manifold* M *and let* $\Psi : T(M) \to T^*(M)$ *be the linear isomorphism defined by* g. *Then we have:*
(1) *A linear connection* ∇ *is metric relative to* g *(i.e.* $\nabla g = 0$*) if and only if* ${}^*\nabla_X(\Psi Y) = \Psi(\nabla_X Y)$, *where* ${}^*\nabla$ *is the dual connection in* $T^*(M)$ *of* ∇.

(2) *If a linear connection ∇ is metric relative to g, then $^*\nabla$ is metric relative to the dual metric $^*g$.*

Proof of the lemma. For any vector field Y on M, let $\alpha = \Psi(Y)$ be the corresponding 1-form. For any vector field Z, we have

$$\begin{aligned}(^*\nabla_X\Psi(Y))(Z) &= (^*\nabla_X\alpha(Z)) = X\alpha(Z) - \alpha(\nabla_X Z)\\ &= Xg(Y,Z) - g(Y,\nabla_X Z) + (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z)\\ &= (\nabla_X g)(Y,Z) + \Psi(\nabla_X Y).\end{aligned}$$

Thus $(\nabla_X g)(Y,Z) = 0$ for all X, Y and Z if and only if $^*\nabla_X(\Psi Y) = \Psi(\nabla_X Y)$ for all X, Y, proving (1). If $\nabla g = 0$, then we get

$$\begin{aligned}(^*\nabla_X{}^*g)(\Psi Y, \Psi Z) &= X^*g(\Psi Y,\Psi Z) - {}^*g(^*\nabla_X(\Psi Y),\Psi Z) - {}^*g(\Psi Y, {}^*\nabla_X(\Psi Z))\\ &= Xg(Y,Z) - {}^*g(\Psi(\nabla_X Y),\Psi Z) - {}^*g(\Psi Y,\Psi(\nabla_X Z))\\ &= Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y,\nabla_X Z) = (\nabla_X g)(Y,Z) = 0.\end{aligned}$$

This proves (2).

Remark. If we write $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$ in terms of a local coordinate system, then we have $^*g(dx^i, dx^j) = g^{ij}$, where the matrix $[g^{ij}]$ is inverse to the matrix $[g_{ij}]$ (as in the classical notation.)

Hicks [H] and later Wegner [We], p.64, generalized certain constructions from Euclidean hypersurface theory coming from the three fundamental forms and their Levi-Civita connections, under the assumption of regular Weingarten operator. The notion of conjugate connections seems to give an adequate setting for all this. To be more precise, let M be an hypersurface immersed in Euclidean space E^{n+1}. Let g, h, g^* and A be the induced metric (the first fundamental form I), the second fundamental form II, the third fundamental form, and the shape operator, thus $h(X,Y) = g(AX,Y), g^*(X,Y) = h(AX,Y) = g(AX,AY)$. Denote by ∇ the Levi-Civita connection for g. Assuming A is nonsingular, define ∇^* by

$$\nabla^*_X Y = A^{-1}\nabla_X(AY),$$

where X, Y are vector fields on M. Then we have

Proposition 6.2.

(1) ∇^* *has torsion 0;*

(2) ∇^* *is conjugate to* ∇ *relative to* h*;*

(3) ∇^* *is the Levi-Civita connection for the metric* g^*.

(4) A^{-1} *satisfies Codazzi's equation relative to* ∇^*.

Proof. Straightforward computation.

Thus the situation in Proposition 6.1 is realized in (2) and (3) of Proposition 6.2.

7. Hypersurface theory

In this section we provide a quick introduction to the theory of hypersurfaces in the affine space A^{n+1}. We shall mostly concentrate on relative geometry, which is more general than the classical Blaschke theory.

We denote by A^{n+1} an $(n+1)$-dimensional (real) affine space. As a manifold, it has a natural torsion-free flat affine connection D. Let $f : M \to A^{n+1}$ be an immersion of an n-dimensional differentiable manifold M into A^{n+1}. Suppose we are given a transversal vector field ξ, namely, a vector field $x \mapsto \xi_x$ along f such that, for each $x \in M$, ξ_x and $f_*(X)$ are linearly independent for each $X \neq 0$ in $T_x(M)$. Thus we may write

$$T_{f(x)}(A^{n+1}) = f_*(T_x(M)) + \{\xi_x\},$$

where $\{\xi_x\}$ denotes the span of ξ_x. In the following, we develop a local theory in a domain where ξ is defined.

For any vector fields X, Y on M, we write

$$D_X(f_*(Y)) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{7.1}$$

which defines a torsion-free affine connection ∇ on M and a bilinear, symmetric (0,2)-tensor h. We call ∇ and h the induced connection and the fundamental form, respectively, corresponding to (f, ξ). We may also write

$$D_X\xi = -f_*(SX) + \tau(X)\xi, \tag{7.2}$$

where S is a (1,1)-tensor called the shape operator and τ is a l-form called the transversal connection form.

Proposition 7.1 *If we change ξ to*

$$\bar{\xi} = (\xi + f_*(Z))/\lambda,$$

where Z is a a tangent vector field on M and λ a non-vanishing function, then the corresponding objects change as follows:

(1) $\bar{h} = \lambda h$;

(2) $\bar{\nabla}_X Y = \nabla_X Y - h(X,Y)Z$;

(3) $\bar{\tau} = \tau + \eta - d(\ln \lambda)$,

where η is the 1-form defined by $\eta(X) = h(X,Z)$ for every X;

(4) $\bar{S}X = [SX - \nabla_X Z + \tau(X)Z + h(X,Z)Z]/\lambda$.

From (1) we see that the rank of h remains the same. This number, independent of the choice of ξ, is called the rank of f. If rank $f = n$, f is said to be nondegenerate. In this case, the fundamental form h may be considered as a nondegenerate metric.

We also define an n-form ω by

$$(7.3) \qquad \omega(X_1, \cdots, X_n) = \det[f_*(X_1), \cdots, f_*(X_n), \xi],$$

where det denotes a fixed non-trivial determinant function regarded as a volume element on A^{n+1}. We call ω the induced volume element. By using (7.1) and (7.2) we find

$$(7.3) \qquad \nabla_X \omega = \tau(X)\omega.$$

It follows that $\nabla\omega = 0$ if and only if $\tau = 0$, in other words, $D_X\xi$ is tangential for every $X \in T_x(M)$. In this case, we say that ξ is equiaffine ([N-1], [N-P1]). It is also called a relative normalization ([Si1], [S-S-V]).

In what follows, we shall consider (f, ξ), where f is nondegenerate and ξ is equiaffine. The geometry resulting from such (f, ξ) is called relative geometry of hypersurfaces. The fundamental equations for this geometry are the following.

Proposition 7.2 *For relative geometry we have*

(1) $R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$ - Gauss-

(2) $(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$ - Codazzi for h-

(3) $(\nabla_X S)(Y) = (\nabla_Y S)(X)$ - Codazzi for S -

(4) $h(SX,Y) = h(X,SY)$ - Ricci-.

In the first equation R denotes the curvature tensor of ∇.

The classical Blaschke theory can now be introduced as follows.

Propostion 7.3 *Let $f : M \to A^{n+1}$ be a nondegenerate immersion. There there exists a transversal vector field, uniqe up to sign, satisfying the following two conditions:*

(1) *ξ is equiaffine, that is, $\tau = 0$;*

(2) *the volume element ω_h for the nondegenerate metric h coincides with ω.*

Proof. We only sketch the proof. We start with an equiaffine transversal vector field (for example, pick an arbitrary euclidean metric in A^{n+1} and choose a unit normal vector field as ξ). Let $\{X_1, \cdots, X_n\}$ be a basis such that $\omega(X_1, \cdots, X_n) = 1$ and set $h_{ij} = h(X_i, X_j)$. Then $H := \det[h_{ij}]$ is well-defined independently of the choice of $\{X_1, \cdots, X_n\}$. Let $\lambda = |H|^{-1/(n+2)}$ and let $\bar{\xi} = (\xi + f_*(Z))/\lambda$, where Z is to be chosen so that the form $\bar{\tau}$ in (3) of Proposition 7.1 should become 0; that is, since $\tau = 0$ by assumption, we pick Z determined by grad $Z = d(\ln\lambda)$, i.e., $h(Z,Y) = Y(\ln\lambda)$ for every tangent vector Y.

The transversal vector field ξ as in Proposition 7.3 is called the affine normal or the Blaschke normal. A nondegenerate immersion with this choice of ξ is called a Blaschke immersion ([N-1], [N-P1]). In [Si1], [S-S-V] it is called a nondegenerate immersion with equiaffine normalization. The corresponding h is called the affine metric. In this case, we have $\nabla\omega_h = 0$, which is the apolarity condition already mentioned in Section 4.

8. Conormal and normal maps

We now define the notion of conormal map for a given $f : M \to A^{n+1}$ equipped with ξ. We consider the dual space R_{n+1} of the vector space R^{n+1} associated to A^{n+1}. For each $x \in M$, let v_x be a covector, namely, an element of R_{n+1}, uniquely determined by the conditions:

$$(8.1) \qquad v_x(\xi_x) = 1 \quad \text{and} \quad v_x(f_*(Y)) = 0 \text{ for every } Y \in T_x(M).$$

The map $v : x \in M \mapsto v_x \in R_{n+1}$ is called the conormal map of f. From (8.1) we can obtain

$$(8.2) \qquad v_*(Y)(\xi) = 0 \quad \text{and} \quad v_*(Y)(f_*Z) = -h(Y, Z).$$

From (8.2) we see that the conormal map is an immersion $M \to R_{n+1} - \{0\}$, as f is nondegenerate. Now we consider a centro-affine normalization as follows. We take the vector equal to $-v$, namely, the position vector with its direction reversed as a transversal vector field. Then (8.1) and (8.2) also show that v and $v_x(X)$ are linearly independent for each $X \neq 0$ in $T(M)$. This is a special kind of relative normalization.

We can then write the equation

$$(8.3) \qquad D_X(v_*(Y)) = v_*(\nabla^*_X Y) + h^*(X, Y)(-v),$$

where ∇^* is the induced connection on M and h^* the fundamental form (which may be degenerate).These structures are related to the old ones by

Proposition 8.1

$$(8.4) \qquad h^*(X, Y) = h(SX, Y) \text{ for all } X, Y \in T(M);$$

$$(8.5) \qquad Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z) \text{ for any vector fields } X, Y, Z \text{ on } M.$$

Proof. By computation based on (8.1) and (8.2).

The equation (8.5) appeared as (1.1) in the beginning of Section 1. We have just shown how this equation arises in hypersurface theory.

We now turn to the discussion of the normal map for $f : M \to A^{n+1}$ with relative normalization ξ. For each $x \in M$, let $\phi(x)$ be the end point of the vector ξ_x when it is parallelly displaced so as to have the initial point at the origin. We obtain the normal map: $x \in M \mapsto \phi(x) \in R^{n+1}$. If we take the differential of ϕ, we get

$$(8.6) \qquad \phi_*(Y) = D_Y(\xi) = -f_*(SY) \quad \text{for each} \quad Y \in T(M).$$

Thus ϕ is an immersion if and only if S is nonsingular. In this case, we can take a centro-affine normalization $-\phi$.

We have

Propostion 8.2. *Assume that S is nonsingular. Then for the immersion $\phi : M \to R^{n+1}$ with centro-affine normalization $-\phi$, we write*

$$D_X \phi_*(Y) = \phi_*(\nabla'_X Y) + h'(X,Y)(-\phi). \tag{8.7}$$

Then we have

$$\nabla'_X Y = S^{-1}\nabla_X(SY), \tag{8.8}$$

$$h'(X,Y) = h(SX,Y). \tag{8.9}$$

Proof. Using (8.6) we have

$$\begin{aligned} D_X\phi_*(Y) = D_X f_*(-SY) &= f_*(-(\nabla_X(SY)) - h(X,SY)\xi \\ &= \phi_*(S^{-1}\nabla_X(SY)) + h(X,SY)(-\xi). \end{aligned}$$

Comparing this with (8.7) we get (8.8) and (8.9). We can say that the connection ∇^*, induced by the conormal map, and the connection ∇', induced by the normal map, are conjugate relative to the metric $h^* = h'$. This is formally a special case of Proposition 5.1 (1). A geometric reason can be given as follows. The immersions f and ϕ are in so-called Peterson correspondence, that is, $f_*(T_x(M))$ and $\phi_*(T_x(M))$ are parallel and their transversal vectors ξ_x and $-\phi_x$ are also parallel (modulo sign). Thus the conormal map v for f and the conormal map, say, w for ϕ coincide (up to sign). Now the centro-affine immersion v induces ∇^* on M, and the centro-affine immersion $-w$ induces the connection conjugate to ∇' relative to h' (this is Proposition 8.1 applied to ϕ and w). It follows that ∇^* and ∇' are conjugate relative to h'.

Proposition 8.2 can be found in [N-O].

9. Projective flatness

In this section we discuss the notion of projective flatness and its geometric meaning. See [N-P2].

Let M be an n-dimensional differentiable manifold. We consider torsion-free connections that are locally equiaffine, or equivalently, that have symmetric Ricci tensors (see Proposition 4.1).

Two such connections ∇ and $\overline{\nabla}$ are said to be projectively equivalent if

$$\overline{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X, \tag{9.1}$$

where ρ is a certain 1-form and X and Y are arbitrary vector fields.

We may also speak of a projective change from ∇ to $\overline{\nabla}$ when (9.1) holds.

Proposition 9.1 *Suppose (9.1) holds. Then*

(1) *The pregeodesics (that is geodesics up to parametrization) for ∇ and $\overline{\nabla}$ coincide.*

(2) *The 1-form ρ is closed: $d\rho = 0$.*

Proof.

(1) A curve x_t with an arbitrary parameter t is a pregeodesic for ∇ if and only if $\nabla_t(dx/dt) = k(t)(dx/dt)$, where dx/dt denotes the tangent vector field of the curve and $k(t)$ is a function. Now one can verify from (9.1) that a curve x_t is a pregeodesic for $\overline{\nabla}$ if and only if it is a pregeodesic for ∇.

(2) Since ∇ and $\overline{\nabla}$ are locally equiaffine, there exist local volume elements ω and $\bar{\omega}$ such that $\nabla\omega = 0$ and $\overline{\nabla}\bar{\omega} = 0$. We may assume that $\bar{\omega} = \lambda\,\omega$ for some positive function λ. Now using (9.1) we can compute:

$$(9.2)\quad (\overline{\nabla}_X\bar{\omega})(Y_1,\cdots,Y_n) = \lambda(\nabla_X\omega)(Y_1,\cdots,Y_n) + [X\lambda - (n+1)\lambda\rho(X)]\omega(Y_1,\cdots,Y_n).$$

Since $\nabla\omega = 0$ and $\overline{\nabla}\bar{\omega} = 0$, we conclude that $\rho = d\ln\lambda/(n+1)$. Hence $d\rho = 0$.

Given an affine connection ∇, we define the projective curvature tensor W by

$$(9.3)\qquad W(X,Y)Z = R(X,Y)Z - [\gamma(Y,Z)X - \gamma(X,Z)Y],$$

where γ is the normalized Ricci tensor, namely, $\gamma(Y,Z) = \mathrm{Ric}(Y,Z)/(n-1)$.

We also need to consider the Codazzi equation for the Ricci tensor:

$$(9.4)\qquad (\nabla_X\mathrm{Ric})(Y,Z) = (\nabla_Y\mathrm{Ric})(X,Z).$$

An affine connection ∇ is said to be projectively flat if around each point there is a projective change of ∇ to a flat affine connection. Now the following result is well-known (see [E], Section 32, including a more general case where the Ricci tensor is not symmetric).

Theorem 9.2. *Let M be an n-dimensional manifold with a torsion-free affine connection that has symmetric Ricci tensor. Then*

(1) *If dim $M \geq 3$, then ∇ is projectively flat if and only if $W = 0$. In this case, (9.4) also holds.*

(2) *If dim $M = 2$, then W is identically 0. For ∇ to be projectively flat, it is necessary and sufficient that (9.4) holds.*

Now we shall discuss centro-affine immersions in general to illustrate projective flatness and projective change of a connection.

Consider an immersion f of a differentiable manifold M of dimension n into $A^{n+1} - \{o\}$ (affine space with a point o removed, which we can identify with the vector space

$R^{n+1} - \{0\}$ with the zero vector removed). We say that f is centro-affine if for each $x \in M$ the position vector $f(x)$ and $f_*(X)$ for each nonzero vector X in $T_x(M)$ are linearly independent. In this case, f is an immersion and we consider a centro-affine normalization, that is, $\xi_x = -f(x)$. For vector fields X, Y on M we have

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)(-f), \tag{9.5}$$

where ∇ is the induced connection.

We shall consider a radial change of f, that is, let us consider a new immersion $\bar{f}$ given by

$$\bar{f}(x) = \lambda f(x), \tag{9.6}$$

where λ is an arbitrary positive function on M. We consider $\bar{f}$ also as a centro-affine immersion. Now we have

Proposition 9.3.

(1) *The connection $\overline{\nabla}$ induced by the centro-affine immersion $\bar{f}$ is projectively equivalent to ∇:*

$$\overline{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X,$$

where $\rho = d \ln \lambda$.

(2) *By choosing λ suitably, we can make $\overline{\nabla}$ flat. Thus ∇ is projectively flat.*

Proof. From $\bar{f} = \lambda f$ we obtain for vector fields X, Y on M

$$\bar{f}_*(Y) = (Y\lambda)f + \lambda f_*(Y)$$

$$\begin{aligned} D_X \bar{f}_*(Y) &= (XY\lambda)f + (Y\lambda)f_*(Y) + (X\lambda)f_*(Y) + \lambda D_X f_*(Y) \\ &= f_*((Y\lambda)X + (X\lambda)Y + \lambda(\nabla_X Y)) + (XY\lambda + \lambda h(X,Y))f. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} D_X \bar{f}_*(Y) &= \bar{f}_*(\overline{\nabla}_X Y) + \bar{h}(X,Y)\bar{f} \\ &= \lambda f_*(\overline{\nabla}_X Y) + [(\overline{\nabla}_X Y)\lambda + \lambda\bar{h}(X,Y)]f. \end{aligned}$$

Comparing the two equations above mod f, we obtain the desired relation in (1).

(2) Geometrically, we choose λ in such a way that $\bar{f}(M)$ is part of a hyperplane. Note that the correspondence $f(x) \mapsto \bar{f}(x)$ takes pregeodesics of $f(M)$ onto lines on $\bar{f}(M)$.

Corollary. *For a nondegenerate immersion $M \to A^{n+1}$ with relative normalization, the connection ∇^* induced by the conormal map is projectively flat. If the shape operator S for f is nonsingular, the connection ∇' induced by the normal map is also projectively flat.*

We prove the converse of Proposition 9.3 (2), namely,

Proposition 9.4 *Let M be an n-dimensional manifold with a torsion-free, projectively flat connection ∇ that has symmetric Ricci tensor. Then around each point x of M there is a centro-affine imbedding f which induces ∇.*

Proof. Let $\overline{\nabla}$ be a flat connection on a neighborhood U of x which is related to ∇ by (9.1). Obviously, a neighborhood U has a centro-affine imbedding $\bar{f}$ onto part of a hyperplane in $A^{n+1} - \{o\}$. Since ρ is closed, we can find a suitable function by which we can radially change $\bar{f}$ to obtain a centro-affine imbedding f which induces the given connection ∇.

10. Radon's theorem and Norden's theorems.

The classical theorem of Radon for Blaschke surfaces states: Given a simply connected 2-dimensional differentiable manifold M^2 with a nondegenerate metric h and a cubic form satisfying apolarity condition between them, we can realize M^2 as a Blaschke surface in A^3 in such a way that h and C become the affine metric and the cubic form, provided a certain integrability condition is satisfied. This condition is rather complicated and difficult to understand. Now we have a modern version for any dimensions in the the frame work of relative geometry, which can be specialized to the Blaschke immersions. See [D-N-V].

Theorem 10.1. *Let M be a simply connected n-manifold, $n \geq 2$, admitting a torsion-free connection ∇ with symmetric Ricci tensor and a nondegenerate metric h. Assume that the Codazzi equation is satisfied: $(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$. Then a necessary and sufficient condition for (∇, h) to be realized as the induced connection and the fundamental form for a nondegenerate immersion $f : M \to A^{n+1}$ is that the conjugate connection ∇^* of ∇ relative to h is projectively flat. Such f is unique up to affine congruence.*

Corollary. *In Theorem 10.1, assume furthermore the apolarity condition (cf. Proposition 4.3):*

$$tr_h \nabla_X h = 0 \text{ for every } X \in T(M).$$

Then there is a Blaschke immersion $f : M \to A^{n+1}$, where A^{n+1} is provided with a suitable parallel volume $(n+1)$-element, which realizes ∇ and h.

Our final goal is to prove the following two results (see [Sch], Appendix by Norden, p.231-2). One gives a different formulation of the existence theorem, and the other is concerned with the determination of (0,2)-tensors satisfying Codazzi's equation on a manifold with a projectively flat connection.

Let us recall the notion of a support function. Let $f : M \to A^{n+1}$ be a nondegenerate immersion with relative normalization ξ. Let $v : M \to R_{n+1}$ be its conormal map. Given a point c in A^{n+1}, we define the support function $\rho_c(x)$ on M by setting

$$\rho_c(x) = v(c - f(x)), \text{ where } x \in M.$$

Then we can write

$$c - f(x) = \rho(x)\xi_x + f_*(Z_x),$$

where Z is a certain tangent vector at x.

For simplicity, take $c = 0$ and write ρ for this support function. We have

Proposition 10.2. *The support function ρ satisfies the equation*

$$h(X,Y) = \rho\gamma^*(X,Y) + \mathrm{Hess}^*_\rho(X,Y), \tag{10.1}$$

where γ^* is the normalized Ricci tensor of the connection ∇^* induced by v and Hess* *is the Hessian relative to ∇^*, i.e.*

$$\mathrm{Hess}^*_\rho(X,Y) = XY\rho - (\nabla^*_X Y)\rho. \tag{10.2}$$

Proof. From $v(f) = -\rho$, we obtain

$$(D_Y v)(f) + v(f_*Y) = -Y\rho, \qquad \text{i.e.} \qquad v_*(Y)(f) = -Y\rho.$$

Using $-f = \rho\xi + f_*(Z)$, where Z is a vector field, we get

$$(v_*(Y))(-\rho\xi - f_*(Z)) = h(Y,Z),$$

hence

$$h(Y,Z) = -Y\rho. \tag{10.3}$$

Now differentiating $v_*(Y)(f) = -Y\rho$ we obtain

$$(D_X v_*(Y))(f) + v_*(Y)(f_*(X)) = -XY\rho,$$

hence

$$(v_*(\nabla^*_X Y) + h^*(X,Y)(-v))(f) - h(X,Y) = -XY\rho,$$

that is,

$$h(\nabla^*_X Y, Z) + \rho h^*(X,Y) - h(X,Y) = -XY\rho.$$

On the other hand, we have

$$h(\nabla^*_X Y, Z) = -(\nabla^*_X Y)\rho.$$

By taking the difference and by observing that $\gamma^* = h^*$ from the Gauss equation for the conormal map V, we obtain (10.1).

Remark. Actually, any support function ρ_c satisfies (10.1), as can be similarly proved. The converse also holds (see Proposition 10.6 below). We now state

Proposition 10.3. *Let ∇ be a torsion-free, projectively flat affine connection with symmetric Ricci tensor. Let γ^* be its normalized Ricci tensor. If ρ is any differentiable function on M, then the (0,2)-tensor of the form*

$$h(X,Y) = \mathrm{Hess}^*_\rho(X,Y) + \rho\gamma^*(X,Y), \tag{10.4}$$

satisfies Codazzi's equation relative to ∇^*.

Proof. Tedious but straightforward computation using $R^*(X,Y)Z = \gamma^*(Y,Z)X - \gamma^*(X,Z)Y$, $(\nabla^*_X\gamma^*)(Y,Z) = (\nabla^*_Y\gamma^*)(X,Z)$, and (10.2)

We can now prove

Theorem 10.4. *Let ∇^* be a torsion-free, projectively flat affine connection with symmetric normalized Ricci tensor γ^* on a simply connected manifold M of dimension n. If h is a nondegenerate, symmetric (0,2) tensor field satisfying Codazzi's equation*

$$(\nabla^*_X h)(Y,Z) = (\nabla^*_Y h)(X,Z),$$

then there is a function ρ on M such that h is given in the form (10.1).

Proof. Let ∇ be the conjugate connection of ∇^* relative to the nondegenerate metric h. Since (∇^*, h) satisfies Codazzi's equation, it follows from Proposition 1.1 (reverse the roles of ∇ and ∇^*) that ∇ has torsion 0. The pair (∇, h) also satisfies Codazzi's equation. By Proposition 4.2, we also know that ∇ has symmetric Ricci tensor. By Theorem 10.1, we can find an affine immersion f such that ∇^* is the connection induced by the conormal map v of f. Now we can take any support function ρ for f (see Proposition 10.2).

Remark. Ferus [F] proved Theorem 10.4 in local form in the case of a Riemannian manifold (M, ∇^*) of constant sectional curvature with Levi-Civita connection ∇^*; in this case, however, h is not asssumed to be nondegenerate. See also [O-S] for related results.

We shall now prove

Theorem 10.5. *Let ∇^* be a torsion-free, projectively flat affine connection with symmetric normalized Ricci tensor γ^* on a simply connected manifold M of dimension n. For any differentiable function ρ on M such that the tensor h given by*

$$h(X,Y) = \rho\gamma^*(X,Y) + \mathrm{Hess}^*_\rho(X,Y)$$

is nondegenerate, there is an affine immersion $f : M \to A^{n+1}$ such that ∇^* is the connection induced by the conormal map v for f and the given function ρ is a support function. Such f is determined uniquely up to affine congruence.

Proof. By Proposition 10.3, h satisfies Codazzi's equation relative to ∇^*. By repeating the same arguments as in Theorem 10.4, we know that there is an affine immersion $f : M \to A^{n+1}$ such that ∇^* is the connection induced by v. What remains to be shown is that ρ is a support function (for some choice of the point o). This will follow from Proposition 10.6 below. The uniqueness part of Theorem 10.5 can be proved as follows. If the given function ρ is to be a support function, then by Proposition 10.2 we know that h is uniquely determined by ρ and ∇^*. Thus the uniqueness of f follows from the uniqueness part of Theorem 10.1.

We now prove

Proposition 10.6. *Let $f : M \to A^{n+1}$ be a nondegenerate immersion with relative normalization, and let $v : M \to R_{n+1}$ be the conormal immersion with induced connection ∇^*. Then a solution of the equation (10.1) must be of the form*

$$\rho = v(c - f),$$

where c is a constant vector in R^{n+1}.

Proof. Set

$$c = \rho\xi - f_*(Z) + f,$$

where Z =grad ρ, that is, $h(Z, Y) = Y\rho$ for all tangent vectors Y. Then for any tangent vector X we get $D_X c = (X\rho)\xi - \rho f_*(SX) - f_*(\nabla_X Z) - h(X, Z)\xi + f_*(X)$.
By virtue of $X\rho = h(X, Z)$ we get

$$\begin{aligned} h(Y, D_X c) &= -\rho h(SX, Y) + h(X, Y) - h(\nabla_X Z, Y) \\ &= \rho\gamma^*(X, Y) - h(X, Y) + h(\nabla_X Z, Y). \end{aligned}$$

Now from

$$(\nabla_X h)(Y, Z) = Xh(Z, Y) - h(\nabla_X Z, Y) - h(Z, \nabla_X Y)$$

$$(\nabla_X h)(Y, Z) = -2h(K_X Y, Z),$$

we get

$$h(\nabla_X Z, Y) = XY\rho - (\nabla_X Y)\rho + 2(K_X Y)\rho.$$

But we have

$$-\nabla_X Y + 2K_X Y = -\nabla^*_X Y,$$

hence we get

$$h(Y, D_X c) = -\rho\gamma^*(X, Y) + h(X, Y) - \mathrm{Hess}^*_\rho(X, Y) = 0,$$

for all X, Y, showing that c is constant and $v(c) = -\rho - v(f)$, namely, $\rho = v(c - f)$, as desired.

This proposition is simplified from [Pe-Si].

As a concluding remark, we state that the global uniqueness question for Codazzi tensors and applications are treated in [Si3]. We also note that the notion of conjugate connection (or dual connection) has beeen found useful in mathematical statistics. See [A1] and [A2]. The terminology of statistical manifold is used there as well as in [K4].

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