FOR WHICH f DOES $A - B \in S_p$ IMPLY THAT $f(A) - f(B) \in S_p$?

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This note is closely related to the paper [12] where the Hankel operators are applied to some problems of the perturbation theory of self-adjoint and unitary operators. Namely, in [12] the following properties of functions f on the real line are considered:

a) $\|f(A) - f(B)\| \leq \text{const} \| A - B\|$ for any self-adjoint operators A and B;

b) the mapping $A \mapsto f(A)$ defined on the set of all bounded self-adjoint operators is Fréchet differentiable;

c) if $A - B$ belongs to the trace class S_1 then $f(A) - f(B) \in S_1$ and Krein's trace formula [8] holds for $f(A) - f(B)$.

The same questions can be considered for unitary operators and functions defined on the unit circle $\mathbf{T}$. These questions were previously considered in [3], [4], [8]. The problem to characterize functions having property b) was also posed in [16].

In [12] both necessary and sufficient conditions are found in order that f satisfy properties a)–c). The results of [12] strengthen some results previously obtained in [3], [4], [8].

Here we consider a more general property than c). Namely, we deal with the case of other Schatten-von Neumann classes S_p (see the definition of S_p in [7], [15]). The aim of this note is to present some results in the case $0 < p < 1$ and to pose some problems in the case $1 < p < \infty$. We consider here the case of functions on the unit circle $\mathbf{T}$ and unitary operators. Similar results can be stated for functions on the real line and self-adjoint operators. In conclusion we consider the case of contractions.

Recall (see [3]) that for unitary operators U, V with spectral measures E, F and a function ϕ on $\mathbf{T}$ the following formula holds

$$\phi(U) - \phi(V) = \int_{\mathbf{T}} \int_{\mathbf{T}} [(\phi(\zeta) - \phi(\tau))/(\zeta - \tau)]\, dF(\tau)(U - V)dE(\zeta),$$

if $A - B \in S_p$ and the function $\check{\phi}(\zeta, \tau) \overset{\text{def}}{=} (\phi(\zeta) - (\phi(\tau))/(\zeta - \tau)$, $\zeta, \tau \in \mathbf{T}$, generates bounded transform on S_p

$$T \mapsto \int_{\mathbf{T}} \int_{\mathbf{T}} \check{\phi}(\zeta, \tau)\, dF(\tau)\, T\, dE(\zeta) \tag{1}$$

(see basic facts on double operator integrals in [1], [2]).

Denote by M_p the set of functions ψ on $\mathbf{T} \times \mathbf{T}$ such that the operator

$$T \mapsto \int_{\mathbf{T}} \int_{\mathbf{T}} \psi(\zeta, \tau)\, dF(\tau)\, T\, dE(\zeta)$$

is bounded on $\boldsymbol{S}_p$ for any Borel spectral measures E, F on $\mathbf{T}$.

Clearly, if $\check{\phi} \in M_p$ then $\phi(U) - \phi(V) \in \boldsymbol{S}_p$ whenever $U - V \in \boldsymbol{S}_p$.

For $1 < p < \infty$ a function ϕ satisfies the property in question if and only if the same is true for $p' = p/(p-1)$. So, we can restrict ourselves to the case $0 < p \leq 2$. Next, the case $p = 2$ is trivial and well-investigated in [3]. In this case $\phi(U) - \phi(V) \in \boldsymbol{S}_2$ for any U, V satisfying $U - V \in \boldsymbol{S}_2$ if and only if $\phi' \in L^\infty$.

As in the case $p = 1$ it can be shown that if the operator (1) is bounded then $\check{\phi}$ is a multiplier of the class of kernels of $\boldsymbol{S}_p$ operators, i.e. if λ, μ are positive scalar measures absolutely continuous with respect to F, E and k is a kernel of an integral operator of class $\boldsymbol{S}_p$ from $L^2(\lambda)$ to $L^2(\mu)$ then the function $(\zeta, \tau) \mapsto \check{\phi}(\zeta, \tau) k(\zeta, \tau)$, $\zeta, \tau \in \mathbf{T}$, is also a kernel of an integral operator from $L^2(\lambda)$ to $L^2(\mu)$.

Denote now by A_p the class of functions ϕ on $\mathbf{T}$ such that $\phi(U) - \phi(V) \in \boldsymbol{S}_p$ for any unitary operators U, V satisfying $U - V \in \boldsymbol{S}_p$ and put

$$\|\phi\|_{A_p} = \sup_{U,V} \left(\|\phi(U) - \phi(V)\|_{\boldsymbol{S}_p} \right) / \left(\|U - V\|_{\boldsymbol{S}_p} \right).$$

As in the case $p = 1$ it is not difficult to show that if $\phi \in A_p$ then the operator (1) is bounded on $\boldsymbol{S}_p$ with $E = F$ for any spectral measure E. It follows that if $\phi \in A_p$ then $\check{\phi}$ is a multiplier of the class of kernels of $\boldsymbol{S}_p$ operators on $L^2(\lambda)$ for any positive finite Borel measure λ on $\mathbf{T}$.

In [8] it was conjectured that for $p = 1$ the condition $\phi' \in L^\infty$ is sufficient for $\phi \in A_1$. But in [5] an explicit counterexample was constructed. Nevertheless, until [12] there was no necessary condition on a function for being in A_1 which singles out a function class whose elements ϕ not necessarily satisfy the condition $\phi' \in L^\infty$.

In [12] the trace ideal criterion for Hankel operators [10] was used to obtain such a necessary condition. (Recall that for $\phi \in L^\infty$ the Hankel operator H_ϕ on the Hardy class H^2 is defined by

$$H_\phi f = (I - \mathbf{P}_+)\phi f, \quad f \in H^2,$$

where $\mathbf{P}_+$ is the orthogonal projection from L^2 onto H^2.) It was proved in [12] that if $\phi \in A_1$ then ϕ belongs to the Besov class B^1_{11} (see the definition and properties of the Besov clases B^s_{pq} in [9]). Another (stronger) necessary condition obtained in [12] is that the Hankel operators H_ϕ and $H_{\bar\phi}$ map the Hardy class H^1 into B^1_{11}.

In [12] a sufficient condition is obtained which claims that if ϕ belongs to the Besov class $B^1_{\infty 1}$ then $\phi \in A_1$. True, this condition is not sufficient but it is shown in [12] that they are very closed to each other.

Here we obtain similar results for $0 < p < 1$ which are however less satisfactory than in the case $p = 1$.

THEOREM 1. *Let* $0 < p < 1$ *and* $\phi \in B^{1/p}_{\infty p}$. *Then* $\phi \in A_p$.

It follows from (1) that in order that $\check\phi \in M_p$ it is sufficient to represent $\check\phi$ in the from

$$(\phi(\zeta) - \phi(\tau))/(\zeta - \tau) = \sum_{n\geq 0} f_n(\zeta) g_n(\tau),$$

where $\sum_{n\geq 0} \|f\|^p_{L^\infty} \|g_n\|^p_{L^\infty} < \infty$.

The proof of the existence of such a representation is similar to the case $p = 1$ (see [12]).

THEOREM 2. *Let* $0 < p < 1$ *and* $\phi \in A_p$. *Then* $\phi \in B^{1/p}_{pp}$.

This follows from the $\boldsymbol{S}_p$ criterion for Hankel operators (see [11], [14]). Indeed, as noticed above the kernel $\check\phi$ must be a multiplier of the class of kernels of $\boldsymbol{S}_p$ operators. Pick $k(\zeta, \tau) \equiv 1$. Then $\check\phi(\zeta, \tau) = (\phi(\zeta) - \phi(\tau))/(\zeta - \tau)$ must be a kernel of an operator of class $\boldsymbol{S}_p$. But it is well known (see e.g. [12]) that this operator up to a constant factor is equal to the orthogonal sum of two operators of Hankel type (H_ϕ and $f \mapsto \mathbf{P}_+ \phi(I - \mathbf{P}_+)f$). This yields $\phi \in B^{1/p}_{pp}$.

The following result yields another necessary condition.

THEOREM 3. *Let* $0 < p < 1$ *and* $\phi \in A_p$. *Then the Hankel operators* H_ϕ *and* $H_{\bar\phi}$ *map* H^1 *into* $B^{1/p}_{pp}$.

The proof of this fact is similar to the case $p = 1$ (see [12]) and uses the $\boldsymbol{S}_p$ criterion for Hankel operators. But in contrast to the case $p = 1$ it is impossible to apply duality arguments to characterize the space of functions ϕ such that H_ϕ, $H_{\bar\phi}$ map H^1

into $B_{pp}^{1/p}$. So this condition is not transparent and it is not easy to compare it with the above sufficient condition $\phi \in B_{\infty p}^{1/p}$.

Let us now proceed to the case $1 < p < 2$ (as noticed above the case $2 < p < \infty$ reduces to this one). Of course, by interpolation reasons the condition $\phi \in B_{\infty 1}^{1}$ is also sufficient for $\phi \in A_p$. But it would be more interesting to find an individual sufficient condition for a fixed p. Again by interpolation arguments it is easy to see that the condition $\phi \in (B_{\infty 1}^{1}, \mathbf{Lip}\, 1)_{\theta,p}$, where $\theta = 2(1 - 1/p)$, is also sufficient. The same is true for the complex interpolation spaces $(B_{\infty 1}^{1}, \mathbf{Lip}\, 1)_{\theta}$. Unfortunately, I do not know an explicit description of these interpolation spaces. The following problem seems very interesting.

PROBLEM. Find an explicit description of the spaces $(B_{\infty 1}^{1}, \mathbf{Lip}\, 1)_{\theta,p}$, $(B_{\infty 1}^{1}, \mathbf{Lip}\, 1)_{\theta}$, $0 < \theta < 1$, or $(B_{\infty 1}^{1}, C^1)_{\theta,p}$, $(B_{\infty 1}^{1}, C^1)_{\theta}$, $0 < \theta < 1$.

As for necessary conditions the situation is the following. There is an explicit example [6] which shows that the condition $\phi' \in L^\infty$ (and even $\phi' \in C^1$) is not sufficient for $p < 2$. But unfortunately there is no necessary condition which would single out a function class strictly more narrow than **Lip** 1.

CONJECTURE. Let $1 \le p \le 2$ and $\phi \in A_p$. Then the lacunary Fourier coefficients of ϕ' satisfy

$$\{\phi'(2^n)\}_{n \in \mathbb{Z}_+} \in \ell^p .$$

This assertion is obviously valid for $p = 2$, it is also valid for $p = 1$ because in this case $\phi \in B_{11}^{1}$ (see [12]). It is easy to see from the description of the lacunary Fourier coefficients of continuous functions (see e.g. [13]) that if the conjecture is true then it yields a necessary condition which shows that the condition $\phi \in C^1$ is not sufficient.

It is interesting to notice that the spaces A_p with $1 \le p < \infty$ are fairly close to each other (all of them are between $B_{\infty 1}^{1}$ and **Lip** 1). In particular, $\| z^n \|_{A_p}$ has order n as $n \to \infty$. But as to the case $p < 1$, $\| z^n \|_{A_p}$ has order $n^{1/p}$.

Now we consider the case of contractions. Let T, R be contractions on Hilbert space and ϕ be a function analytic in the unit disc (we suppose for the sake of simplicity that ϕ is continuous on the closed disc). Then the problem is to find conditions on ϕ under which the fact that $T - R \in \boldsymbol{S}_p$ implies that $\phi(T) - \phi(R) \in \boldsymbol{S}_p$.

It is well known that T and R can be represented as integrals over semi-spectral measures, say E and F. That is E and F are countably additive measures taking values in the set of self-adjoint operators A such that $0 \leq A \leq I$, and $T = \int_{\mathbf{T}} \zeta \, dE(\zeta)$, $R = \int_{\mathbf{T}} \zeta \, dF(\zeta)$. It is easy to show that if a function $\overset{\vee}{\phi}$ on $\mathbf{T} \times \mathbf{T}$ belongs to M_p then as in the case of unitary operators the following elementary formula holds

$$\phi(T) - \phi(R) = \int_{\mathbf{T}} \int_{\mathbf{T}} [(\phi(\zeta) - \phi(\tau))/(\zeta - \tau)] dF(\tau)(T - R) dE(\zeta).$$

Thus, using the same arguments as in the case of unitary operators, we can show that all facts valid for unitary operators are also valid for contractions if we deal with functions analytically extended to the unit disc. As to necessary conditions this simply follows from the corresponding necessary conditions for the case of unitary operators if we take for T, R unitary operators.

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