CHAPTER V

STATISTICAL MANIFOLD

March 1998

1. Introduction

2. Young's functions

2.1. Convex functions. let us first recall some properties of the convex fuctions.

DEFINITION 5.1. *A real function ϕ on $\mathbb{R}$ is convex if the Janssen inequality*

$$\phi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\phi(x_1) + (1-\lambda)\phi(x_2)\,, \quad \forall\, \lambda \in [0,1]\,, \tag{5.1}$$

holds $\forall\, x_1, x_2 \in \mathbb{R}$.

PROPOSITION 5.1. *A convex function ϕ on an open interval $]a,b[$ is continuous on this interval.*

But a convex function has stronger interesting properties. We recall only the important one's in view of the following

LEMMA 5.1. *Let ϕ a convex real function on the open interval $]a,b[$*

i - ϕ has a right derivative f_+ and a left derivative f_- at every point and

$$f_-(t) \leq f_+(t)\,, t \in]a,b[. \tag{5.2}$$

ii - f_+ and f_- are non-decreasing

$$f_+(t_1) \leq f_+(t_2)\,, \quad f_-(t_1) \leq f_-(t_2)\,, \quad \text{if} \quad t_1 \leq t_2\,.$$

Moreover, f_+ is continuous from the right and f_- is continuous from the left

$$\lim_{t \downarrow t_0} f_+(t) = f_+(t_0)\,. \tag{5.3}$$

$$\lim_{t \uparrow t_0} f_-(t) = f_-(t_0)\,. \tag{5.4}$$

The right and left deivatives are equal except perhaps for at most a countable number of points.

iii - ϕ is absolutely continuous and satisfies the Lipshitz condition in every finite interval,

$$\left|\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}\right| < K_{ab} < +\infty\,, \quad a < x_1 < x_2 < b\,, \tag{5.5}$$

where K_{ab} is a positive constant depending on the boundary of the interval $]a,b[$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

THEOREM 5.1. *Let ϕ :$]a,b[\to \mathbb{R}$be a function. Then ϕ is convex iff foe each closed subinterval $[c,d] \in]a,b[$ we have $]a,b[$ can be represented in the form:*

$$\phi(x) = \phi(c) + \int_a^x f(t)\,dt\,, \quad x \in [c,d]\,, \tag{5.6}$$

where $f(t)$ is a non-increasing left continuous function.

Remark: In the previous theorem the function f can be choosen Right continuous.

2.2. The Young functions.

2.2.1. A class of conjugate convex functions. In his stuties on Fourier series, W. H. Young has analysed certain convex function $\Phi : \mathbb{R} \to \mathbb{R}^+$ which satisfy the conditions:

$$\begin{aligned} \Phi(-x) &= \Phi(x) \\ \Phi(0) &= 0 \\ \lim_{x\to\infty} \Phi(x) &= +\infty\,. \end{aligned} \tag{5.7}$$

With each such function Φ, one can associate another convex function $\Psi : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\Psi(x) = \sup\{x|y| - \Psi(x) \,:\, x \geq 0\}, \quad y \in \mathbb{R}\,. \tag{5.8}$$

DEFINITION 5.2. *A convex function $\Phi : \mathbb{R} \to \mathbb{R}^+$ which satisfies the conditions (5.7) is called a Young function.*

The function $\Psi : \mathbb{R} \to \mathbb{R}^+$ defined by the equation (5.8) is called the the conjugate (or complementary) function to the Young function Φ.

It follows from the definition that the convex Ψ

$$\begin{aligned} \Psi(0) &= 0\,, \\ \Psi(-x) &= \Psi(x)\,, \\ \lim_{y\to\infty} \Psi(y) &= +\infty\,, \end{aligned}$$

and Ψ is a Young function.

From (5.8) the pair (Φ, Ψ) satisfies the Young's inequality:

$$xy \leq \Phi(x) + \Psi(y)\,. \quad x, y \in \mathbb{R} \tag{5.9}$$

Exemples of Young functions:

i) Let $\Phi(x) = |x|^p, p \geq 1$. Then Φ is a continuous Young function such that $\Phi(x) = 0$ iff $x = 0$, and $\lim_{x\to\infty} \Phi(x) = +\infty$ while $\Phi(x) < \infty$ for all $x \in \mathbb{R}$.

ii) Let $\Phi(x) = O$, $0 \leq |x| \leq a$; $\Phi'(x) > 0$, $a < |x| < b$, and $\Phi(x) = +\infty$ for $x \geq b$, where Φ' is a continuous increasing convex function on $]a, b[$. Then Φ is a Young function which is continuous on $]-b, b[$, and jumps to $+\infty$ at $|x| = b$.

iii) Let $\Psi(y) = 0$ for $0 \leq |y| < 1$; $\Psi(y) = +\infty$ for $|y| > 1$. Then Ψ is the conjugate to $\Phi(x) = |x|$, implying that that the conjugate function of a continuous function Φ on $\mathbb{R}$ can be a jump function.

Remark: A young function $\Phi : \mathbb{R} \to \mathbb{R}^+$ est convex and $\Phi(0) = O$, but which may jump to $+\infty$ at finite point. If $\Phi(a) = +\infty$ for some $a > 0$, then $\Phi(x) = +\infty$ for $x > a$.

THEOREM 5.2. *Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a Young function. Then it can be represented as:*

$$\Phi(x) = \int_0^{|x|} f(t)\, dt\,, \quad x \in \mathbb{R}\,, \tag{5.10}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$is nondecreasing left continuous; $\phi(0) = 0$ and if $f(x) = +\infty$ for $x \geq a$ then $\Phi(x) = +\infty$, $x \geq a \geq 0$.

Under some continuity conditions the conjugate pair of Young functions present interesting nontrivial properties and ordering relation. conditions.

2.2.2. The N-functions. We are now interested by a usefull class of nice continuous Young functions increasing on $\mathbb{R}^+$.

DEFINITION 5.3. *A continuous convex function $\Phi : \mathbb{R} \to \mathbb{R}^+$ is called N-function if:*

$$\begin{aligned} &i\,) \quad \Phi(x) = \Phi(-x)\,, \\ &ii\,) \quad \Phi(x) = 0\,, \quad \textit{iff } x = 0\,, \\ &iii\,) \quad \lim_{x \uparrow +\infty} \frac{\Phi(x)}{x} = +\infty\,, \\ &iv\,) \quad \lim_{x \downarrow 0} \frac{\Phi(x)}{x} = 0\,. \end{aligned} \tag{5.11}$$

PROPOSITION 5.2. *A N-function Φ is a Young function*

PROPOSITION 5.3. *Let Φ be a N-function*

$$i\,) \quad \Phi(x) > 0\,, \forall\, x \neq 0\,. \tag{5.12}$$

$$ii\,) \quad \Phi(x_1) < \Phi(x_2) \quad \textit{if} \quad x_1 < x_2\,. \tag{5.13}$$

$$iii\,) \quad \Phi(\alpha x) < \alpha \Phi(x)\,, \quad 0 < \alpha < 1\,. \tag{5.14}$$

or equivalently $\dfrac{\Phi(x)}{x}$ *is strictly increasing for* $x > 0$.

iv) The restriction $\Phi_+ = \Phi\,\lfloor_{\mathbb{R}^+}$ of Φ on $\mathbb{R}^+$ has a concave inverse Φ_+^{-1}, $\Phi_+^{-1}(\Phi_+(x)) = 1$.

vi) The composition $\Phi = \Phi_1 \circ \Phi_2$ of two Young functions is also a Young function.

THEOREM 5.3. *A N-function Φ has an integral representation*

$$\Phi(x) = \int_0^{|x|} f(t)\, dt\,, \tag{5.15}$$

where f is a nondecreasing left continuous positive function on $\mathbb{R}^+$ which satisfies

$$\begin{aligned} f(0) &= 0\,, \\ \lim_{t\uparrow+\infty} f(t) &= +\infty\,. \end{aligned} \tag{5.16}$$

Remark: If Φ is a N-function the function f can be choosen nondecreasing right continuous positive function on $\mathbb{R}^+$.

2.3 Conjugate Young N-functions. Let Φ a N-function, then there exist a nondecreasing right continuous real function f on $\mathbb{R}^+$ such that $f(0) = 0$ and $\lim_{t\uparrow+\infty} f(t) = +\infty$. Let us introduce the non-negative function h on $\mathbb{R}^+$

$$h(s) = \sup_{f(t)\leq s} t\,. \tag{5.17}$$

PROPOSITION 5.4. *The function h defined in (5.17) is a non-decreasing right continuous real function on $\mathbb{R}^+$ such that*

$$h(0) = 0\,, \qquad \lim_{s\uparrow+\infty} h(s) = +\infty\,. \tag{5.18}$$

Moreover

$$(h\circ f)(t) \geq t\,, \quad (f\circ h)(s) \geq s\,. \tag{5.19}$$

and for $\varepsilon > 0$:

$$h(f(t)-\varepsilon) \leq t\,, \quad f(h(s)-\varepsilon) \leq s\,. \tag{5.20}$$

Then h is the right inverse of f.

Remark: h is the inverse of f if f increase monotonically.

THEOREM 5.4. *The two N-functions Φ and Ψ defined by*

$$\Phi(x) = \int_0^{|x|} f(t)\, dt\,, \quad \Psi(y) = \int_0^{|y|} h(s)\, ds\,, \ \text{ with } h(s) = \sup_{f(t)\leq s} t\,,$$

are called conjugate.

The pair of Young function (Φ, Ψ) satisties the relation (5.8)

$$\Psi(y) = \sup_{x\geq 0}[x|y| - \Phi(x)]\,.$$

PROPOSITION 5.5. *Two conjugate N-functions Φ and Ψ verify the Young's inequality*

$$xy \leq \Phi(x) + \Psi(y)\,, \tag{5.21}$$

the equality being reached by

i) $y = \operatorname{sgn}(x)\, f(|x|)$ for x given,
ii) $x = \operatorname{sgn}(y)\, h(|y|)$ for y given.

Examples of conjugate N-functions

(1)

$$\Phi(x) = \frac{|x|^\alpha}{\alpha}\,,\ \alpha > 1\,, \qquad f(x) = x^{\alpha-1}\,,\ x \geq 0\,,$$
$$\Psi(y) = \frac{|y|^\beta}{\beta}\,,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1\,, \qquad h(y) = y^{\beta-1}\,,\ y \geq 0\,.$$

(2)

$$\Phi(x) = e^{|x|} - |x| - 1\,, \qquad f(x) = e^x - 1\,,\ x \geq 0\,,$$
$$\Psi(y) = (1+|y|)\ln(1+|y|) - |y|\,, \quad h(y) = \ln(y_1)\,,\ y \geq 0\,.$$

(3)

$$\Phi(x) = e^{x^2} - 1\,, \qquad f(x) = 2xe^{x^2}\,,\ x \geq 0\,,$$
$$\Psi(y) \text{ no explicit form}\,, \quad h(y) \text{ no explicit form}\,.$$

THEOREM 5.5. *Let the inequality $\Phi_1(x) \geq \Phi_2(x)$ for the Young functions Φ_1 and Φ_2 for $x \leq x_0$. Then the inequality $\Psi_2(y) \geq \Psi_1(y)$ holds for the conjugate functions Ψ_1 and Ψ_2 for $y \leq y_0 = f_2(x)$, where f_2 is the rigtht derivative of Φ_2.*

2.4. Equivalent Young (N-)functions. It is possible to define a partial order on the family of Young N-functions in the following way.

Let Φ_1 and Φ_2 be two N-functions, we shall write

$$\Phi_1 \prec \Phi_2\,, \tag{5.22}$$

if there exist positive constant x_0 and k such that

$$\Phi_1(x) \leq \Phi_2(kx)\,, \tag{5.23}$$

This inequality compare the rapidity of growth of the Young functions for large values of x.

DEFINITION 5.5. *Two N-function Φ_1 and Φ_2 are equivalent and write $\Phi_1 \sim \Phi_2$ if $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$.*

Example: $\Phi_1(x) = \cosh x - 1$, and $\Phi_2(x) = e^{|x|} - |x| - 1$ are equivalent ($\Phi_1 \sim \Phi_2$).

LEMMA 5.2. *let Φ_1 and Φ_2 two N-functions with conjugate Ψ_1 and Ψ_2 respectively*

i - $\Phi_1 \sim \Phi_2 \iff \exists\, k, k' > 0,\ x_0 > 0$ *such that:*

$$\Phi_1(kx) \leq \Phi_2(x) \leq \Phi_1(k'x), \quad x > x_0 .$$

ii - $\Phi_2(x) = \Phi_1(kx),\ k > 0 \implies \Phi_1 \sim \Phi_2 .$

iii - $\Phi_1 \prec \Phi_2 \iff \Psi_2 \prec \Psi_1 .$

THEOREM 5.6. *let Φ_1 and Φ_2 two Young functions with conjugate Ψ_1 and Ψ_2 respectively, then*

$$\Phi_1 \sim \Phi_2 \iff \Psi_1 \sim \Psi_2 .$$

Let us now give some equivalence criterion

PROPOSITION 5.6. *Let Φ_1 and Φ_2 be two N-functions with integral representation*

$$\Phi_1(x) = \int_0^{|x|} f_1(t)\,dt, \quad \Phi_1(x) = \int_0^{|x|} f_1(t)\,dt .$$

and with conjugate Ψ_1 and Ψ_2 respectively. Then

i - Let b a finite positive real number

$$\lim_{t\uparrow+\infty} \frac{f_1(t)}{f_2(t)} = b > 0 \Longrightarrow \Phi_1 \sim \Phi_2 . \tag{5.24}$$

ii - If we denote by $h_2(s) = \sup_{f_2(t)\leq s} t$ and by b a finite positive real number, then

$$\lim_{s\uparrow+\infty} \frac{f_1(h_2(s))}{s} = b > 0,\ a.e. \Longrightarrow \Phi_1 \sim \Phi_2 . \tag{5.25}$$

2.5 The Δ_2 and ∇_2-conditions. The cmoparisons of N-functions given in the preceeding section become more useful in the theory when a corresponding classification based on the rapidity of their growth is added

DEFINITION 5.6. *The Young function Φ satisfies the Δ_2-condition (globaly) if there exists constants $k > 0$, $x_0 \geq 0$ such that*

$$\Phi(2x) \leq k\Phi(x),\ x \geq x_0 > 0\ (x_0 = 0) . \tag{5.26}$$

Remark: The Δ_2-condition imply

$$\Phi(\ell x) \leq k_\ell \Phi(x),\ x > x_0,\ \ell > 1 . \tag{5.27}$$

DEFINITION 5.7. *The Young function Φ satisfies the ∇_2-condition (globaly) if there exists constants $\ell > 1$, $x_0 \geq 0$ such that*

$$\Phi(x) \leq \frac{1}{2\ell}\Phi\ell(x)\,, \;\; x \geq x_0 > 0 \; (x_0 = 0)\,. \tag{5.28}$$

PROPOSITION 5.7. *Let Φ be a N-function and Ψ its conjugate and f and h the left derivative of Φ and ψ respectively.*

i) If Φ satisfies the Δ_2-condition (∇_2-condition) any Young function equivalent to Φ also satisfies the Δ_2-condition (∇_2-condition).

ii) Φ satisfies the Δ_2-condition iff it exists constants $\alpha > 1$, $x_0 \geq 0$ such that, for $x \geq x_0$

$$\frac{xf(x)}{\Phi(x)} < \alpha\,. \tag{5.29}$$

iii) Φ satisfies the Δ_2-condition if its conjugate Ψ has a convex right derivative.

iv) Φ satisfies the Δ_2-condition iff its conjugate Ψ satisfies the ∇_2-condition.

v) Φ satisfies the Δ_2-condition iff it exists constants $\beta > 1$, $y_0 \geq 0$ such that, for $y \geq y_0$

$$\frac{yh(y)}{\Psi(y)} > \beta\,, \tag{5.30}$$

with h the right derivative of the conjugate Ψ of Φ.

Examples

1) $\Phi(x) = a\,|x|^{\alpha}$, $\alpha > 1$ satisfy the Δ_2 condition.

2) $\Phi(x) = e^{|x|} - |x| - 1$, does not satisfies the Δ_2 condition (because it increase more quikely than any power).

3) $\Psi(x) = (1+|y|)\ln(1+|y|) - |y|$, (the conjugate of the Φ defined in Example 2) satisfies the Δ_2 condition (the right derivative of Φ is $e^x - 1$ for $x > 0$ and it is convex).

3. Orlicz spaces

3.1 The Orlicz class $\mathcal{L}^{\Phi}(\mathcal{X}, P)$. Let $(\mathcal{X}, \mathcal{F}, P)$ be a probability space where P is a continuous probability[1].

Remark: In the case where P is replaced by a σ-finite continuous measure μ we impose to $\mathcal{X}$ to be a bounded closed subset of a finite-dimensional Euclidean space.

[1] By continuous probability we understand the existence of a subset A of every set B such that $P[A] = \frac{1}{2}P[B]$.

DEFINITION 5.8. *Let Φ be a Young function, then the class of real-function defined on $\mathcal{X}$ such that*

$$r_P(u,\Phi) = \mathbb{E}_P[\Phi \circ u] = \int_{\mathcal{X}} \Phi((u(t))\, dP(t) < +\infty\,, \tag{5.31}$$

is called the Orlicz class $\mathcal{L}^\Phi(\mathcal{X}, P)$.

Remark: If every function in the class $\mathcal{L}^\Phi(\mathcal{X}, P)$ is summable on $\mathcal{X}$, not all summable function belong to $\mathcal{L}^\Phi(\mathcal{X}, P)$, however,

LEMMA 5.3. *Let us consider the family of all Orlicz classes on $(\mathcal{X}, \mathcal{F}, P)$*
i - All bounded function on $\mathcal{X}$ belong to $\mathcal{L}^\Phi(\mathcal{X}, P)$, $\forall\,\Phi$.
ii - Every summable function on $\mathcal{X}$ belongs to some Orlicz class:

$$L^1(\mathcal{X}, P) \in \bigcup_\Phi \mathcal{L}^\Phi(\mathcal{X}, P)\,.$$

THEOREM 5.7. *The Orlicz class $\mathcal{L}^\Phi(\mathcal{X}, P)$, is a convex set for every Young function Φ. Moreover the class is linear if and only if Φ satisfies the Δ_2-condition.*

LEMMA 5.4. *Let $u \in \mathcal{L}^\Phi(\mathcal{X}, P)$, then the following Janssen integral inequality holds*

$$\Phi\left(\int_{\mathcal{X}} u(x)\, dP(x)\right) \leq \int_{\mathcal{X}} \Phi(u(x))\, dP(x)\,, \quad or \quad \Phi\left(\mathbb{E}_P[u]\right) \leq \mathbb{E}_P[\Phi \circ u]\,. \tag{5.32}$$

Remark: In the case of a σ-finite measure μ on $\mathcal{X}$ the equation (5.32) take the form:

$$\Phi\left(\int_G \frac{u(x)}{\mu[G]}\, d\mu(x)\right) \leq \frac{1}{\mu[G]} \int_G \Phi(u(x)\, d\mu(x)\,,$$

for closed bounded set $G \in \mathcal{X}$.

THEOREM 5.8. *Let Φ_1 and Φ_2 two Young functions,*
i - The inclusion $\mathcal{L}^{\Phi_1} \subset \mathcal{L}^{\Phi_2}$ holds if, and only if, there exists positive constants k and x_0 such that $\Phi_2(x) \leq k\,\Phi_1(x)$, $x \geq x_0$.
ii - The two Young function Φ_1 and Φ_2 determine the same Orlicz class if, and only if, there exists positive constants k, k' and x_0 such that :

$$k\,\Phi_2(x) \leq \Phi_1(x) \leq k'\,\Phi_2(x)\,. \tag{5.33}$$

3.2 The Orlicz linear space $L^\Phi(\mathcal{X}, P)$. We have just seen that the Orlicz class $\mathcal{L}^\Phi(\mathcal{X}, P)$ of real functions on the probability space $(\mathcal{X}, \mathcal{F}, P)$, associated to the Young function Φ, is a linear space only if the Δ_2-condition is satisfied. Let us now introduce a linear space associated to an Orlicz class.

DEFINITION 5.9. *Let Φ and Ψ two conjugate Young functions. We shall denote by $L^{\Phi}(\mathcal{X},P)$ the set of real function u on $\mathcal{X}$ such that*

$$L^{\Phi}(\mathcal{X},P)=\left\{u\,\Big|\,(u,v)=\mathbb{E}_P[uv]<+\infty\,,\ \forall v\in\mathcal{L}^{\Psi}(\mathcal{X},P)\right\}. \tag{5.34}$$

As usually we identify fonction which differ by a set of P-measure zero.

PROPOSITION 5.8. *Let Φ and Ψ two conjugate Young functions,*
i - The sets $L^{\Phi}(\mathcal{X},P)$ and $L^{\Psi}(\mathcal{X},P)$ are linear spaces.
ii - For every pair of function $u\in\mathcal{L}^{\Phi}(\mathcal{X},P)$ and $v\in\mathcal{L}^{\Psi}(\mathcal{X},P)$,

$$(u,v)=r_P(u,\Phi)+r_P(v,\Psi)\,,\ u\in\mathcal{L}^{\Phi}(\mathcal{X},P)\ v\in\mathcal{L}^{\Psi}(\mathcal{X},P)\,. \tag{5.35}$$

iii - Moreover $\mathcal{L}^{\Phi}(\mathcal{X},P)\subset L^{\Phi}(\mathcal{X},P)$.
iv - Let $u\in L^{\Phi}(\mathcal{X},P)$. Then

$$\sup_{r_P(v,\Psi)\le 1}\mid(u,v)\mid=\sup_{r_P(v,\Psi)\le 1}\Big|\mathbb{E}_P[uv]\Big|<+\infty\,. \tag{5.36}$$

3.3 The Orlicz norm. Let us define the quantity

$$\|u\|^{O}_{\Phi;P}=\sup_{r_P(v,\Psi)\le 1}\mid(u,v)\mid=\sup_{r_P(v,\Psi)\le 1}\Big|\mathbb{E}_P[uv]\Big|\,. \tag{5.37}$$

The previous lemma and Eq(5.36) allows to define a norm on $L^{\Phi}(\mathcal{X},P)$. Indeed

1) $\|u\|^{O}_{\Phi;P}=0$ if, and only if, $u=0$ a.e.
2) $\|\alpha\,u\|^{O}_{\Phi;P}=|\alpha|\,\|u\|^{O}_{\Phi;P}$.
3) $\|u_1+u_2\|^{O}_{\Phi;P}\le\|u_1\|^{O}_{\Phi}+\|u_2\|^{O}_{\Phi;P}$.

DEFINITION 5.9. *The norm $u\to\|u\|^{O}_{\Phi;P}$ is called the Orlicz norm and the normed linear space $L^{\Phi}(\mathcal{X},P)$ the Orlicz space.*

Example: Let $\Phi(u(x))=\frac{|u(x)|^{\alpha}}{\alpha}$, $\alpha>1$. Then norm $\|u\|_{\alpha}$ in the linear space $L^{\alpha}(\mathcal{X},P)$ is connected to the Orlicz norm by:

$$\|u\|^{O}_{\Phi;P}=\beta^{\frac{1}{\beta}}\|u\|_{\alpha}\,,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\,. \tag{5.38}$$

THEOREM 5.7. *An Orlicz space is complete, so it is a Banach space.*

LEMMA 5.11. *Let Φ and Ψ two conjugate Young functions*

i - For every $u \in \mathcal{L}^\Phi(\mathcal{X}, P)$, we have,

$$\|u\|^O_{\Phi;P} \leq r_P(u, \Phi) + 1\,. \tag{5.39}$$

ii - Let f be the right derivative of Φ. Suppose $u \in L^\Phi(\mathcal{X}, P)$ and $\|u\|^O_{\Phi;P} \leq 1$. Then the function $v_0(x) = f(|u(x)|)$ belongs to $\mathcal{L}^\phi(\mathcal{X}, P)$ and $r_P(v_0, \Psi) \leq 1$.

iii - Suppose $\|u\|^O_{\Phi;P} \leq 1$. Then $u \in \mathcal{L}^\Phi(\mathcal{X}, P)$ and

$$r_P(\mathcal{X}, \Phi) \leq \|u\|^O_{\Phi;P}\,. \tag{5.40}$$

Moreover

$$\Phi\left(\frac{u(x)}{\|u\|^O_{\Phi;P}}\right) dP(x) \leq 1\,. \tag{5.41}$$

THEOREM 5.8. *Let Φ and Ψ two conjugate Young function. Then*

$$\left|\int_{\mathcal{X}} u(x)v(x)\,dP(x)\right| \leq \|u\|^O_{\Phi;P}\ \|v\|^O_{\Psi;P}\,,\ \ \forall u \in L^\Phi(\mathcal{X}, P)\,,\ v \in L^\Psi(\mathcal{X}, P)\,. \tag{5.42}$$

DEFINITION 5.10. *We say that a sequence of function $u_n \in L^\Phi(\mathcal{X}, P)$ is mean convergent to the function $u_0 \in L^\Phi(\mathcal{X}, P)$ if*

$$\lim_{n\uparrow+\infty} \int_{\mathcal{X}} \Phi(u_n(x) - u_0(x))\,dP(x) = 0\,. \tag{5.43}$$

THEOREM 5.9. *Let the Young function Φ satisfy the Δ_2-condition. The convergence in Orlicz norm is equivalent to the mean convergence.*

LEMMA 5.12. *In the sense of mean convergence, the set of bounded functions is everywhere dense in the class $\mathcal{L}^\Phi(\mathcal{X}, P)$ i.e. for every function $u(x) \in \mathcal{L}^\Phi(\mathcal{X}, P)$ we can construct a sequence of bounded functions u_n such that*

$$\lim_{n\uparrow+\infty} \int_{\mathcal{X}} \Phi(u_n(x) - u(x))\,dP(x) = 0\,.$$

LEMMA 5.13. *Every set $\mathcal{N} \subset \mathcal{L}^\Phi(\mathcal{X}, P)$, which are bounded in mean, i.e. such that*

$$\int_{\mathcal{X}} \Phi(u(x))\,dP(x) \leq k\,,\ \ k > 0\,, u \in \mathcal{N}\,,$$

will also be bounded in Orlicz norm: $\|u\|^O_{\Phi;P} \leq K(k)\,,\ \forall u \in \mathcal{N}$, where K depends only on k.

Remark: The converse in not true in general, because $L^\Phi(\mathcal{X}, P) \not\subset \mathcal{L}^\Phi(\mathcal{X}, P)$. However if Φ satisfies the Δ_2-condition then every set $\mathcal{N} \subset L^\Phi(\mathcal{X}, P)$ which is bounded in Orlicz-norm, will also be bounded in mean.

3.4 The Luxemburg norm. The space $L^{\Phi}(\mathcal{X}, P)$ can also be equipped with a different norm from the orlicz norm.

DEFINITION 5.11. *Let $u \in L^{\Phi}(\mathcal{X}, P)$ and define the quantity*

$$\|u\|^L_{\Phi;P} = \inf_{r_P(\frac{u}{k},\Phi)\leq 1} k\,, \quad k > 0\,. \tag{5.44}$$

$\|u\|^L_{\Phi;P}$ is called the luxemburg norm.

Indeed $\|u\|^L_{\Phi;P}$ verifies the usual axioms:

1) $\|u\|^L_{\Phi;P} = 0$ if, and only if, $u = 0$ a.e.
2) $\|\alpha u\|^L_{\Phi;P} = |\alpha|\,\|u\|^L_{\Phi;P}$.
3) $\|u_1 + u_2\|^L_{\Phi;P} \leq \|u_1\|^L_{\Phi;P} + \|u_2\|^L_{\Phi;P}$.

LEMMA 5.14. *Let $u \in L^{\Phi}(\mathcal{X}, P)$. Then*

$$r_P\left(\frac{u}{\|u\|^L_{\Phi;P}}, \Phi\right) = \int_{\mathcal{X}} \Phi\left(\frac{u(x)}{\|u\|^L_{\Phi;P}}\right) dP(x) \leq 1\,. \tag{5.45}$$

THEOREM 5.10. *The unit sphere of the space $L^{\Phi}(\mathcal{X}, P)$ with respect to the norm $\|u\|^L_{\Phi;P}$ coincides with the set of functions $u \in \mathcal{L}^{\Phi}(\mathcal{X}, P)$ for which $r_P(u, \Phi) \leq 1$. Moreover we have*

$$\begin{aligned} \|u\|^L_{\Phi;P} \leq 1 &\Longrightarrow r_P(u,\Phi) \leq \|u\|^L_{\Phi;P} \\ \|u\|^L_{\Phi;P} \geq 1 &\Longrightarrow r_P(u,\Phi) \geq \|u\|^L_{\Phi;P}\,. \end{aligned} \tag{5.46}$$

THEOREM 5.11. *The Luxemburg and Orlicz norms are equivalent*

$$\|u\|^L_{\Phi;P} \leq \|u\|^O_{\Phi;P} \leq 2\|u\|^L_{\Phi;P}\,. \tag{5.47}$$

It follows that $L^{\Phi}(\mathcal{X}, P)$ equipped with the Luxemburg norm is a Banach space.

Remark: We can give another formula for the definition of the Orlicz norm:

$$\|u\|^O_{\Phi;P} = \sup_{\|v\|^L_{\Psi;P}\leq 1} \Big|\mathbb{E}_P[uv]\Big|\,, \tag{5.48}$$

where Ψ is the conjugate of the Young function Φ.

LEMMA 5.15. *Let Φ and Ψ two conjugate Young functions. We have Hölder like inequalities:*

$$\left|\mathbb{E}_P[uv]\right| \leq \|u\|^O_{\Phi;P}\,\|v\|^L_{\Psi;P}\,,\;\; u \in L^\Phi(\mathcal{X},P)\,,\; v \in L^\Psi(\mathcal{X},P)\,, \tag{5.49}$$

and

$$\left|\mathbb{E}_P[uv]\right| \leq \|u\|^L_{\Phi;P}\,\|v\|^O_{\Psi;P}\,,\;\; u \in L^\Phi(\mathcal{X},P)\,,\; v \in L^\Psi(\mathcal{X},P)\,, \tag{5.50}$$

3.5 Relations between Orlicz classes.

LEMMA 5.16. *Let Φ and Φ' two Young functions. In order that $L^\Phi(\mathcal{X},P) \subset L^{\Phi'}(\mathcal{X},P)$, it is necessary and sufficient that $\Phi \prec \Phi'$ i.e. that there exists a constants u_0 and $k > 0$ such that*

$$\|u\|^O_{\Phi;P} \leq \|ku\|^0_{\Phi';P}\,, \qquad u \geq u_0\,.$$

THEOREM 5.12. *Let Φ and Φ' two Young functions. If $\Phi \sim \Phi'$ then the norms associated to Φ and Φ' are equivalent,that is there exists constants k and k' such that*

$$k\|u\|^O_{\Phi;P} \leq \|u\|^O_{\Phi';P} \leq k'\|u\|^O_{\Phi;P}\,, \quad or \quad k\|u\|^L_{\Phi;P} \leq \|u\|^L_{\Phi';P} \leq k'\|u\|^L_{\Phi;P}\,. \tag{5.51}$$

THEOREM 5.13. *Let Φ and Φ' two Young functions. The the spaces $L^\Phi(\mathcal{X},P)$ and $L^{\Phi'}(\mathcal{X},P)$ consist of the same functions if, and only if, $\Phi \sim \Phi'$. We denote by*

$$L^\Phi(\mathcal{X},P) \simeq L^{\Phi'}(\mathcal{X},P)$$

this property.

The symbol $\simeq$ recall that the two spaces consisit of the same functions and that the norms are equivalent (but not isometric).
Example: $\Phi(x) = \cosh x - 1$ and $\Phi'(x) = e^{|x|} - |x| - 1$ are two equivalent Young functions ($\Phi \sim \Phi'$), then $L^\Phi(\mathcal{X},P) \simeq L^{\Phi'}(\mathcal{X},P)$.

LEMMA 5.17. *Let Φ and Φ' two Young functions. If $L^{\Phi}(\mathcal{X},P) \subset L^{\Phi'}(\mathcal{X},P)$, there exists a constant $k > 0$ such that*

$$\|u\|^O_{\Phi';P} \leq k\|u\|^L_{\Phi;P}\,, \qquad u \in L^\Phi(\mathcal{X},P)\,.$$

3.5. Dual Orlicz spaces. Conjugacy of Young functions Φ and Ψ allows, because of the inequality (5.49) and (5.50) to define the bilinear relation

$$L^\Phi(\mathcal{X},P) \times L^\Psi(\mathcal{X},P) \ni (u,v) \mapsto \mathbb{E}_P[uv] \in \mathbb{R}\,. \tag{5.52}$$

If in the case $\Phi(x) = \frac{|x|^\alpha}{\alpha}$, $\alpha > 1$ ans $\Psi(y) = \frac{|y|^\beta}{\beta}$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ the Orlicz spaces $L^\Phi(\mathcal{X},P)$ and $L^\Psi(\mathcal{X},P)$ are dual Banach spaces, it is not the case in general.

LEMMA 5.18. *Let Φ and Ψ two conjugate Young functions.*
i - The mapping

$$L^{\Phi}(\mathcal{X},P) \ni u \mapsto \mathbb{E}_P[uv]\,, \quad \forall v \in L^{\Psi}(\mathcal{X},P)\,, \tag{5.53}$$

is always defined, linear and continuous; it is an element of the dual $L^{\Phi}(\mathcal{X},P)^*$ of $L^{\Phi}(\mathcal{X},P)$.

ii - The mapping

$$L^{\Psi}(\mathcal{X},P) \ni v \mapsto \mathbb{E}_P[uv]\,, \quad \forall u \in L^{\Phi}(\mathcal{X},P)\,, \tag{5.54}$$

is always defined, linear and continuous; it is an element of the dual space $L^{\Psi}(\mathcal{X},P)^*$ of $L^{\Psi}(\mathcal{X},P)$.

THEOREM 5.14. *Let Φ and Ψ two conjugate Young functions. We have the continuous injection*

$$L^{\Psi}(\mathcal{X},P) \hookrightarrow L^{\Phi}(\mathcal{X},P)^*\,, \quad \text{and} \quad L^{\Phi}(\mathcal{X},P) \hookrightarrow L^{\Psi}(\mathcal{X},P)^*\,, \tag{5.55}$$

where $\hookrightarrow$ is used for the continuous injection.

Moreover if Φ verifies the Δ_2-condition $L^{\Phi}(\mathcal{X},P)^*$ is isometric to $L^{\Phi}(\mathcal{X},P)$, we denote this property by

$$L^{\Phi}(\mathcal{X},P) = L^{\Phi}(\mathcal{X},P)^*\,. \tag{5.56}$$

3.6. The centered Orlicz space $L_0^{\Phi}(\mathcal{X},P)$.

DEFINITION 5.12. *The centered Orlicz space $L_0^{\Phi}(\mathcal{X},P)$ at P is the linear subspace of all random variables $u \in L^{\Phi}(\mathcal{X},P)$ with vanishing expectation,*

$$L_0^{\Phi}(\mathcal{X},P) = \left\{u \in L^{\Phi}(\mathcal{X},P)\,,\ \mathbb{E}_P[u] = 0\right\}\,. \tag{5.57}$$

THEOREM 5.15. *The centered Orlicz space $L_0^{\Phi}(\mathcal{X},P)$ is a closed subspace of $L^{\Phi}(\mathcal{X},P)$.*

In particular $L_0^{\Phi}(\mathcal{X},P)$ is an Orlicz space for the Luxemburg norm (or th Orlicz norm) axssociated to Φ.

LEMMA 5.19. *Let Φ and Ψ two conjugate Young functions. Then*

$$L_0^{\Psi}(\mathcal{X},P)\,, \quad \hookrightarrow L_0^{\Phi}(\mathcal{X},P)^*\,, \quad L_0^{\Phi}(\mathcal{X},P) \hookrightarrow L_0^{\Psi}(\mathcal{X},P)^*\,. \tag{5.58}$$

Let $v \in L_0^{\Psi}(\mathcal{X},P)$ then $v \in L^{\Psi}(\mathcal{X},P)$ then there exists $u^* \in L^{\Phi}(\mathcal{X},P)^*$ such that

$$u^*(v) = \mathbb{E}_P[uv]\,,\ \forall u \in L^{\Phi}(\mathcal{X},P)\,.$$

The restriction $u^*_{\restriction_{L_0^{\Phi}(\mathcal{X},P)}}$ of u^* to $L_0^{\Phi}(\mathcal{X},P)$ is an element of $L_0^{\Phi}(\mathcal{X},P)^*$ such that

$$u^*_{\restriction_{L_0^{\Phi}(\mathcal{X},P)}}(u) = \mathbb{E}_P[uv]\,,\ \forall u \in L_0^{\Phi}(\mathcal{X},P)\,.$$

The mapping $v \mapsto u^*$ is continuous from $L_0^{\Psi}(\mathcal{X},P)$ to $L_0^{\Phi}(\mathcal{X},P)^*$ because the restriction is a contraction.

The same argument applies to show

$$L_0^{\Phi}(\mathcal{X},P) \hookrightarrow L_0^{\Psi}(\mathcal{X},P)^*\,.$$

LEMMA 5.20. *Let Φ and Ψ two conjugate Young functions. Then*

$$L_0^{\Phi}(\mathcal{X},P) \simeq L_0^{\Psi}(\mathcal{X},P)^*, \tag{5.59}$$

if Ψ verifies the Δ_2-condition.

Let $v^* \in L_0^{\Psi}(\mathcal{X},P)^*$. v^* extends to a continuous linear form $\widetilde{v}^*$ on $L^{\Psi}(\mathcal{X},P)$ as follow

$$\widetilde{v}^*(v) = v^*\left(v - \mathbb{E}_P[v]\right), \quad v \in L^{\Psi}(\mathcal{X},P). \tag{5.60}$$

If Ψ verifies the $Delta_2$-condition, $L^{\Phi}(\mathcal{X},P) = L^{\Psi}(\mathcal{X},P)^*$, and let $\tilde{u} \in L^{\Phi}(\mathcal{X},P)$ be the representant of $\widetilde{v}^* \in L^{\Psi}(\mathcal{X},P)^*$; then

$$\widetilde{v}^*(v) = \mathbb{E}_P[\widetilde{u}v], \quad v \in L^{\Psi}(\mathcal{X},P), \tag{5.61}$$

and for $v \in L_0^{\Psi}(\mathcal{X},P)$

$$\widetilde{v}^*(v) = v^*(v) = \mathbb{E}_P[\widetilde{u}v], \quad v \in L_0^{\Psi}(\mathcal{X},P), \tag{5.62}$$

and $\widetilde{u} \in L_0^{\Phi}(\mathcal{X},P)$ and $L_0^{\Psi}(\mathcal{X},P)^* \hookrightarrow L_0^{\Phi}(\mathcal{X},P)$.

3. 7. The space of bounded functions on $\mathcal{X}$.

We shall denote by $L_b^{\Phi}(\mathcal{X},P)$ the closure in $L^{\Phi}(\mathcal{X},P)$ of the set of bounded functions. By Lemma (5.12) the set of bounded functions is everyvhere dense in the Orlicz class $\mathcal{L}^{\Phi}(\mathcal{X},P)$. If Φ satifies the Δ_2-condition, then the set of bounded functions is everywhere dense in the Orlicz space $L^{\Phi}(\mathcal{X},P) = \mathcal{L}^{\Phi}(\mathcal{X},P)$,

LEMMA 5.19. *Let Φ be a Young function. We have the following inclusion*

$$L_b^{\Phi}(\mathcal{X},P) \subseteq \mathcal{L}^{\Phi}(\mathcal{X},P). \tag{5.57}$$

Moreover the equality occure if, and only if, Φ satifies the Δ_2-condition.

LEMMA 5.20. *The set of continuous bounded functions on $\mathcal{X}$ is everywhere dense in the space $L_b^{\Phi}(\mathcal{X},P)$. Moreover the countable set of polynomials on $\mathcal{X}$ with rational coefficients is everywhere dense in $L_b^{\Phi}(\mathcal{X},P)$.*

THEOREM 5.15. *The space $L_b^{\Phi}(\mathcal{X},P)$ is separable.*

Let us assume that the Young function Φ does not satisfies the Δ_2-condition then $L_b^{\Phi}(\mathcal{X},P) \subset \mathcal{L}^{\Phi}(\mathcal{X},P)$. To characterize $L_b^{\Phi}(\mathcal{X},P)$ let us define the positive quantity

$$\delta\left(u, L_b^{\Phi}(\mathcal{X},P)\right) = \inf_{w \in L_b^{\Phi}(\mathcal{X},P)} \|u - w\|_{\Phi;P}^{O}, \ \forall\, u \in L^{\Phi}(\mathcal{X},P), \tag{5.58}$$

and the set

$$\Pi\left(L_b^{\Phi}(\mathcal{X},P), a\right) = \left\{\delta\left(u, L_b^{\Phi}(\mathcal{X},P)\right) \le a, \ a > 0\right\}. \tag{5.59}$$

THEOREM 5.16. *Let us assume that the Young function Φ does not satisfy the Δ_2-condition. Then*

$$\Pi\left(L_b^\Phi(\mathcal{X},P),1\right) \subset \mathcal{L}^\Phi(\mathcal{X},P) \subset \overline{\Pi}\left(L_b^\Phi(\mathcal{X},P),1\right) , \tag{5.60}$$

where $\overline{\Pi}\left(L_b^\Phi(\mathcal{X},P),a\right)$ is the closure in $L^\Phi(\mathcal{X},P)$ of $\Pi\left(L_b^\Phi(\mathcal{X},P),a\right)$.

Remarks: i - The second part of the assertion of the theorem signify that the class $\mathcal{L}^\Phi(\mathcal{X},P)$ is neither an open set nor a closed set in the space $L^\Phi(\mathcal{X},P)$ if the Young function Φ does not satifies the Δ_2-condition.

ii - If the Young function Φ does not satify the Δ_2-condition, the class $\mathcal{L}^\Phi(\mathcal{X},P)$ is not complete in the sense of the mean convergence.

iii - If the Young function Φ does not satifies the Δ_2-condition, the set of bounded functions is nowhere dense in $L^\Phi(\mathcal{X},P)$ inasmuch as all bounded functions are in $L_b^\Phi(\mathcal{X},P)$.

LEMMA 5.21. *The space $L_b^\Phi(\mathcal{X},P)$ is the maximal linear subspace of the space $L^\Phi(\mathcal{X},P)$ which is contained in $\mathcal{L}^\Phi(\mathcal{X},P)$.*

LEMMA 5.22. *The equality*

$$\lim_{n\uparrow+\infty} \|u-u_n\|^O_{\Phi;P} = \delta(u, L_b^\Phi(\mathcal{X},P) > 0\,, \tag{5.61}$$

holds for an arbitrary function $u\in L^\Phi(\mathcal{X},P)$, where

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)|\le n\,,\\ 0 & \text{if } |u(x)|>n\,. \end{cases} \tag{5.62}$$

3.8. Separability of an Orlicz space. As we have shown above, the space $L_b^\Phi(\mathcal{X},P)$ is always separable. This means that the space $L^\Phi(\mathcal{X},P) = \mathcal{L}^\Phi(\mathcal{X},P) = L_b^\Phi(\mathcal{X},P)$ is also separable if the Young function Φ satisfies the Δ_2-condition.

THEOREM 5.16. *The Orlicz space $L^\Phi(\mathcal{X},P)$ is separable if, and only if, the Young function Φ satisfies the Δ_2-condition.*

Example: Let $\Phi(x) = e^{|x|} - |x| - 1$ then its conjugate Young function is $\Psi(x) = (|x|+1)\ln(|x|+1) - 1$. The orlicz space $L^\Phi(\mathcal{X},P)$ is not separable but the orlicz space $L^\Psi(\mathcal{X},P)$ is separable.

4. Cramer classes

4.1. The Cramer class at P.

DEFINITION 5.14. *Let $(\mathcal{X}, \mathcal{F}, P)$ be a probability space. The Cramer class $\mathcal{C}(\mathcal{X}, P)$ at P is the linear space of all random variables u on the mesurable space $(\mathcal{X}, \mathcal{F})$ such that the momentum generating function of u with respect to the probability measure P,*

$$\mathcal{M}_{P,u}(t) = \int_{\mathcal{X}} e^{tu}\, dP = \mathbb{E}_P\left[e^{tu}\right], \ t \in \mathbb{R}, \tag{5.69}$$

is finite in a neighborhood of the origin 0.

Because the interior $\overset{\circ}{D}(\mathcal{M}_{P,u})$ of the domain[2] $D(\mathcal{M}_{P,u})$ of the momentum generating function of $u \in \mathcal{C}(\mathcal{X}, P)$ contain 0, we have the following

LEMMA 5.24. *The momentum generating function of is analytic on a open neighbohood of 0. Then momentums of any order exist.*

$$\mathbb{E}_P\left[|u|^k\right] < +\infty. \tag{5.70}$$

THEOREM 5.17. *The Cramer class $\mathcal{C}(\mathcal{X}, P)$ endowed with the Luxemburg norm*

$$\|u\|_{\Phi_1;P} = \inf\left\{k > 0 \,:\, \mathbb{E}_P\left[\cosh\left(\frac{u}{k}\right) - 1\right] \leq 1\right\}, \tag{5.71}$$

is just the Orlicz space $L^{\Phi_1}(\mathcal{X}, P)$ associated to the Young function $\Phi_1(x) = \cosh x - 1$.

$$\mathcal{C}(\mathcal{X}, P) = L^{\Phi_1}(\mathcal{X}, P), \quad \Phi_1(x) = \cosh x - 1. \tag{5.72}$$

The momemtum generating function $\mathcal{M}_{P,u}(t)$ is finite in an open neighborhood of 0 is if and only if for $k > 0$ sufficiently large $\mathbb{E}_P\left[e^{\frac{u}{k}}\right] < +\infty$ and $\mathbb{E}_P\left[e^{-\frac{u}{k}}\right] < +\infty$, that is $\mathbb{E}_P\left[\cosh\left(\frac{u}{k}\right)\right] < +\infty$. Moreover $\mathbb{E}_P\left[\cosh\left(\frac{u}{k}\right)\right] \to 1$ if $k \to +\infty$. Then (5.71) define a norm on $\mathcal{C}(\mathcal{X}, P)$.

Let us assume $u \in L^{Phi_1}(\mathcal{X}, P)$. Then there exists $k > 0$ such that

$$\mathbb{E}_P\left[e^{\frac{u}{k}} + e^{-\frac{u}{k}}\right] < +\infty.$$

[2]The domain $D(\mathcal{M}_{P,u})$ of the momentum generating function is convex.

Indeed for $\lambda in]0,1[$ and t and t' in $D(\mathcal{M}_{P,u})$

$$\mid e^{\lambda t u + (1-\lambda) t' u} \mid \leq \lambda \mid e^{tu} \mid + (1-\lambda) \mid e^{t' u} \mid.$$

The equality can be satisfied if $t \neq t'$ for e^{tu} and $e^{t'u}$ P-a.s. proportionnal, which is possible only if $P = \delta$ the Dirac probability measure.

By the convexity of the exponential function, the momentum generating function is finite on the interval $]-\frac{1}{k},\frac{1}{k}[$; therefore $u \in \mathcal{C}(\mathcal{X},P)$. Conversely let $u \in \mathcal{C}(\mathcal{X},P)$. Then there exists t such that t and $-t$ are in the domain of the momemtum generating function $\mathcal{M}_{P,u}(t)$, and this means that

$$\mathbb{E}_P\left[e^{tu}+e^{-tu}\right] < +\infty\,,$$

so that $u \in L^{\Phi}(\mathcal{X},P)$.

Remark: 1 - The Orlicz norm being equivalent to the Luxemburg norm in theorem (5.17) we can remplace the Luxemburg norm by the Orlicz norm.

2 - Because the Young function $\Phi_2(x) = e^{|x|} - |x| - 1$ is equivalent to the Young function $\Phi(x) = \cosh x - 1$ we have $L^{\Phi}(\mathcal{X},P) \simeq L^{\Phi_2}(\mathcal{X},P)$ and we can take as conjugate Young function $\Phi_3(x) = (1+|x|)\ln(1+|x|) - |x|$.

LEMMA 5.25. *The cumulant function*

$$K_{P,u}(t) = \ln \mathcal{M}_{P,u}(t) = \ln \mathbb{E}_P\left[e^{tu}\right]\,, \tag{5.73}$$

is a convex function on $D(\mathcal{M}_{P,u})$. Moreover it is strictly convex if P is not a Dirac probability measure.

DEFINITION 5.15. *The centered Cramer class $\mathcal{C}_0(\mathcal{X},P)$ at P is the linear subspace of all random variables $u \in \mathcal{C}(\mathcal{X},P)$ with vanishing expectation,*

$$\mathcal{C}_0(\mathcal{X},P) = \{u \in \mathcal{C}(\mathcal{X},P)\,,\; \mathbb{E}_P[u] = 0\}\,. \tag{5.74}$$

LEMMA 5.26. *The centered Cramer class $\mathcal{C}_0(\mathcal{X},P)$ is a closed subspace of the Orlicz space $L^{\Phi_1}(\mathcal{X},P)$ with $\Phi_1(x) = \cosh x - 1$.*

Assume $u \in L^{\Phi_1}(\mathcal{X},P)$ and $\mathbb{E}_P[u] = 0$. Then by the same argument as previously there exists $k > 0$ such that the momentum generating function is finite on the interval $]-\frac{1}{k},\frac{1}{k}[$; therefore $u \in \mathcal{C}_0(\mathcal{X},P)$. Conversely let $u \in \mathcal{C}_0(\mathcal{X},P)$. Then there exists t such that t and $-t$ are in the domain of the momemtum generating function $\mathcal{M}_{P,u}(t)$, and so that $u \in L^{\Phi_1}(\mathcal{X},P)$. Moreover space $L_0^{\Phi_1}(\mathcal{X},P) = \left\{u \in L^{\Phi_1}(\mathcal{X},P)\,,\mathbb{E}_P[u]=0\right\}$, is a closed subspace of $L^{\Phi_1}(\mathcal{X},P)$.

LEMMA 5.27. *Let $\Phi_1(x) = \cosh x - 1$, $\Phi_2(x) = e^{|x|} - |x| - 1$ and $\Phi_3(x) = (|x|+1)\ln(|x|+1) - |x|$ three Young functions. Then*

$$L^{\Phi_1}(\mathcal{X},P) \simeq L^{\Phi_2}(\mathcal{X},P) = L^{\Phi_3}(\mathcal{X},P)^*\,. \tag{5.75}$$

The relation $L^{\Phi_1}(\mathcal{X},P) \simeq L^{\Phi_2}(\mathcal{X},P)$ is a consequence of $\Phi_1 \sim \Phi_2$. the equality comes from the fact that Φ_2 and Φ_3 are conjugate and that Φ_3 verifies the Δ_2-condition.

LEMMA 5.28. *We have the continuous injection*

$$L_0^{\Phi_3}(\mathcal{X},P) \hookrightarrow L_0^{\Phi_1}(\mathcal{X},P)^* = \mathcal{C}_0(\mathcal{X},P)^*, \tag{5.76}$$

$$\mathcal{C}_0(\mathcal{X},P) = L_0^{\Phi_1}(\mathcal{X},P) \simeq L_0^{\Phi_3}(\mathcal{X},P)^*. \tag{5.77}$$

THEOREM 5.18. *Let Φ_1 and Φ_3 be the Young functions $\Phi_1(x) = \cosh x - 1$ and $\Phi_3(x) = (|x|+1)\ln(|x|+1) - |x|$. The we have the following sequence of continuous embedings*

$$L_0^{\infty}(\mathcal{X},P) \hookrightarrow L_0^{\Phi_1}(\mathcal{X},P) \hookrightarrow \bigcap_{\alpha>1} L_0^{\alpha}(\mathcal{X},P) \hookrightarrow L_0^{\Phi_3}(\mathcal{X},P) \hookrightarrow L_0^{\Phi_1}(\mathcal{X},P)^*. \tag{5.78}$$

LEMMA 5.29. *The multi-linear mappings $(u_1, u_2, \cdots, u_n) \mapsto \mathbb{E}_P[u_1 u_2 \cdots u_n]$ where $u_i \in L_0^{\Phi_1}(\mathcal{X},P)$, are continuous.*

This follows from the continuous inclusion $L_0^{\Phi_1}(\mathcal{X},P) \hookrightarrow L_0^{n}(\mathcal{X},P)$

Remark: In particular the momentum $u \mapsto \mathbb{E}_P[u]$ are continuous.

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