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MATHEMATICS
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Approximately Tame Algebras of Operators

by

M. M. RAO

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Summary. If $\mathcal{X}$ is a Hilbert space, $\mathcal{X}_\alpha \subset \mathcal{X}$ a finite dimensional subspace, let $B(\mathcal{X})$, $B(\mathcal{X}_\alpha)$ be the B-algebras of bounded operators on $\mathcal{X}$ and $\mathcal{X}_\alpha$, and Q_α a projection on $B(\mathcal{X})$ into $B(\mathcal{X}_\alpha)$. An approximately tame (a.t.) algebra $\mathcal{A} \subset B(\mathcal{X})$ is a B-algebra with a finer topology than the trace of $B(\mathcal{X})$ and verifying (i) $\bigcup_\alpha B(\mathcal{X}_\alpha) \subset \mathcal{A}$, $\mathcal{X}_\alpha \uparrow \mathcal{X}$, (ii) $Q_\alpha(A) \to A$ $(\in \mathcal{A})$ in $\mathcal{A}$. In this note a class of a.t. algebras $L^\Phi(\mathcal{X})$ are constructed using the concepts of Orlicz spaces L^Φ, and it is shown that there are B-algebras $L^\Phi(\mathcal{X}) \subset \mathcal{C}$ ($\mathcal{C}$ = a.t. algebra of compact operators in $B(\mathcal{X})$) that are not a.t. Also if $\mathcal{A} = L^\Phi(\mathcal{X})$ is any a.t. algebra, let $G(\mathcal{A}) = \{I + A \in GL(\mathcal{X}) : A \in \mathcal{A}\}$, $GL(a) = \{I + A \in GL(\mathcal{X}) : A \in B(\mathcal{X}_\alpha)\}$ and $GL(\infty) = \text{ind. lim}\, G(a)$, where $GL(\mathcal{X}) \subset B(\mathcal{X})$ is the group of invertible operators. Then it was noted that the injective map of $GL(\infty)$ into $G(\mathcal{A})$ is a homotopy equivalence. A few related results are discussed.

1. Introduction

Let $\mathcal{X}$ be a complex Hilbert space and $B(\mathcal{X})$ be the Banach (or B-) algebra of bounded operators on $\mathcal{X}$. If $\mathcal{X}_\alpha \subset \mathcal{X}$ is a finite dimensional subspace, let $B(\mathcal{X}_\alpha) \subset B(\mathcal{X})$ be the subalgebra of operators vanishing on the orthogonal complement of $\mathcal{X}_\alpha$. If $P_\alpha : \mathcal{X} \mapsto \mathcal{X}_\alpha$ is the orthogonal projection, with range $\mathcal{X}_\alpha$, let $Q_\alpha : B(\mathcal{X}) \mapsto B(\mathcal{X}_\alpha)$ be the projection defined by $Q_\alpha(A) = P_\alpha A P_\alpha$, $A \in B(\mathcal{X})$. As usual $P_{\alpha_1} \leqslant P_{\alpha_2}$ stands for the ordering: range $(P_{\alpha_1}) \subset$ range (P_{α_2}). The approximation property of the Hilbert space [cf. 2, chap. I, p. 167, and 7, p. 108] implies that there exists an ordered family $\{P_\alpha\}$ of orthogonal projections, with finite dimensional ranges, such that $P_\alpha \to I$, the identity on $\mathcal{X}$, uniformly on every precompact set of $\mathcal{X}$. With this, following [5], the next definition can be introduced:

DEFINITION 1. If $\mathcal{A} \subset B(\mathcal{X})$ is a B-algebra whose topology is at least as strong as the relativized topology of $B(\mathcal{X})$, it is said to be *approximately tame* (a.t.) if (i) $\bigcup_\alpha B(\mathcal{X}_\alpha) \subset \mathcal{A}$ and (ii) $Q_\alpha(A) \to A$, $A \in \mathcal{A}$, in the topology of $\mathcal{A}$.

The purpose of this note is to present a large class of a.t. algebras so that those considered in [5] [cf. also 9, th. 4] are subsumed, which also illuminate their structure and show the extent of such algebras contained in $B(\mathcal{X})$. Then their homotopy types will be considered. Since the property of approximate tameness is shown below to be *not* hereditary, and since such studies are useful in general analysis [cf. 1, p. 763; and 6], the results here may be of some independent interest.

2. A class of B-algebras

Let Φ be a symmetric convex function on the line such that $\Phi(0)=0$ and, if Φ is continuous. $\Phi(x)>0$ for $x>0$. (It is called a Young's function.) Let $A=U[A]$ be the canonical polar decomposition of $A\in B(\mathcal{X})$. Define the positive operator $\Phi([A])$, via the spectral theorem, for a continuous Φ, and let $k=t(\Phi([A]))$ be its trace, so that $0\leqslant k\leqslant\infty$, i.e., $k=\Sigma(\Phi([A])e_i, e_i)$, where $\{e_i\}$ is an orthonormal basis in $\mathcal{X}$ and [cf. 8, p. 37] k is independent of $\{e_i\}$. Let $L^{\Phi}(\mathcal{X})$ be the subset of $B(\mathcal{X})$ such that $A\in L^{\Phi}(\mathcal{X})$ if and only if $\|A\|_{\Phi}<\infty$ where

$$\|A\|_{\Phi}=\inf\left\{k>0: t\left(\Phi\left(\frac{1}{k}[A]\right)\right)\leqslant 1\right\}. \tag{1}$$

Let $\|A\|_{\infty}=\sup\{\|Ax\|:\|x\|\leqslant 1\}$ be the operator norm of $B(\mathcal{X})$. Now $L^{\Phi}(\mathcal{X})$ is clearly linear and normed by (1). It is termed an *Orlicz space* of operators. Their structure is given as follows.

THEOREM 1. *If Φ is continuous and $\Phi(1)=1$, then $L^{\Phi}(\mathcal{X})\subset B(\mathcal{X})$ is a self-adjoint B-algebra under the norm* (1) *and the involution $A\mapsto A^*$ in $L^{\Phi}(\mathcal{X})$, (A^* is the adjoint of A) is an isometry, i.e. $\|A\|_{\Phi}=\|A^*\|_{\Phi}$. Moreover,*

$$\|A\|_{\infty}\leqslant\|A\|_{\Phi},\ \|AB\|_{\Phi}\leqslant\|A\|_{\infty}\|B\|_{\Phi}\leqslant\|A\|_{\Phi}\|B\|_{\Phi}. \tag{2}$$

If Φ_1 and Φ_2 are two continuous (Young's) functions and if $\Phi_1\leqslant\Phi_2$ means $\Phi_1(ax)\leqslant b\Phi_2(x)$, $0\leqslant x\leqslant x_0$, for some fixed positive numbers a, b and x_0, then $L^{\Phi_2}(\mathcal{X})\subset L^{\Phi_1}(\mathcal{X})$ and $\|A\|_{\Phi_2}\leqslant C\|A\|_{\Phi_1}$, where C is a constant depending only on a, b and x_0 (and hence on Φ_1, Φ_2).

If Φ is slightly restricted then the following result holds.

THEOREM 2. *Let Φ be continuous and $\Phi(1)=1$. If there exist positive numbers a, b and x_0 such that $\Phi(ax)\leqslant b\Phi(x)$, $0\leqslant x\leqslant x_0$, then the B-algebra $L^{\Phi}(\mathcal{X})\subset B(\mathcal{X})$ is approximately tame.*

Remark. Taking $\Phi(x)=|x|^p$, $1\leqslant p\leqslant\infty$, and $\mathcal{X}$ separable, these results include those of [5]. From the proof of Theorem 2, it follows that the B-algebra $L^{\Phi}(\mathcal{X})$ fails to be approximately tame if Φ is merely continuous, but does not satisfy the given inequality near the origin.

Proof of Theorem 1. It is sufficient to consider positive $A\in L^{\Phi}(\mathcal{X})$. Then $t\left(\Phi\left(\frac{A}{k}\right)\right)<\infty$ for some $k>0$ so that $\Phi\left(\frac{A}{k}\right)$ is a nuclear operator, [2] (=trace class, [8]). Hence it is compact and there exist $\{\lambda_n\}$, $\lambda_n\downarrow 0$, and $\{e_n\}$ orthonormal vectors (which are eigenvalues and eigenvectors of A), such that

$$\infty>t\left(\Phi\left(\frac{A}{k}\right)\right)=\sum_{n=1}^{\infty}\left(\Phi\left(\frac{A}{k}\right)e_n, e_n\right)=\sum_{n=1}^{\infty}\Phi\left(\frac{\lambda_n}{k}\right). \tag{3}$$

If l^{Φ} is the space of scalar sequences $\{a_n\}$ such that $\sum_{n=1}^{\infty}\Phi\left(\frac{|a_n|}{k}\right)<\infty$ for some $k>0$, with $\|\{a_n\}\|_{\Phi}=\inf\left\{k>0: \sum_{n=1}^{\infty}\Phi\left(\frac{|a_n|}{k}\right)\leqslant 1\right\}$, then it is a B-algebra (since $\Phi(1)=1$), and (3) implies that the mapping $\beta: L^{\Phi}(\mathcal{X}) \mapsto l^{\Phi}$ given by $\beta A = \{\lambda_n\} \in l^{\Phi}$ for positive $A \in L^{\Phi}(\mathcal{X})$, is well-defined and that, with (1), $\|A\|_{\Phi} = \|\{\lambda_n\}\|_{\Phi}$ so that it is an "isometry" between the positive cones of the indicated spaces. It is now clearly possible to *extend* β so as to be a linear "isometry" between $L^{\Phi}(\mathcal{X})$ and l^{Φ}. Using the same symbol for this extended map one has that β to be an isometric isomorphism on all of $L^{\Phi}(\mathcal{X})$ onto l^{Φ}, This implies that $L^{\Phi}(\mathcal{X})$ is a self-adjoint B-algebra in $B(\mathcal{X})$ since $l^{\Phi} \subset l_{\infty}$. Note also that if $A \in L^{\Phi}(\mathcal{X})$, then A is compact. This yields the first inequality of (2). (Here "isometry" has the usual meaning except for the commutativity.)

For the second inequality of (2), assume, for non-triviality, that $B \in L^{\Phi}(\mathcal{X})$, $A \in B(\mathcal{X})$. Let λ_n and μ_n be the eigenvalues of $[B]$ and $[AB]$ respectively. Then by [8, p. 22], $\mu_n \leqslant \|A\|_{\infty} \lambda_n$. Consequently if $k = \|AB\|_{\Phi}$ one has $k < \infty$ and

$$1 = t\left(\Phi\left(\frac{[AB]}{k}\right)\right) = \sum_{n=1}^{\infty} \Phi\left(\frac{\mu_n}{k}\right) \leqslant \sum_{n=1}^{\infty} \Phi\left(\frac{\lambda_n}{k_0}\right), \quad k_0 = k/\|A\|_{\infty}.$$

This means $k_0 \leqslant \|\{\lambda_n\}\|_{\Phi}$ and the second inequality in (2) is an immediate consequence. Finally, the given ordering of Φ_1 and Φ_2 implies $l^{\Phi_2} \subset l^{\Phi_1}$ and the norm inequality holds by [3, ths. 4, 5, pp. 51—52]. The isometry of (3) then gives the corresponding result for $L^{\Phi_i}(\mathcal{X})$, $i = 1, 2$, and the proof of the theorem is complete.

Proof of Theorem 2. Since every operator whose range is finite dimensional is in $L^{\Phi}(\mathcal{X})$, it follows that $\bigcup_{\alpha} B(\mathcal{X}_{\alpha}) \subset L^{\Phi}(\mathcal{X})$ and if $A \in L^{\Phi}(\mathcal{X})$ then $Q_{\alpha}(A) \in B(\mathcal{X}_{\alpha})$ for some a, in the notation of Sec. 1. Again it suffices to consider positive $A \in L^{\Phi}(\mathcal{X})$. If $\beta: L^{\Phi}(\mathcal{X}) \mapsto l^{\Phi}$ is the "isometry" defined above, then the condition on Φ, of the theorem, is sufficient to conclude that simple functions (i.e. all but finitely many terms in the sequences) in l^{Φ} are (norm) dense. This follows from [3, th. 3, p. 58]. Thus if $A = \sum_{n=1}^{\infty} \lambda_n (\cdot, e_n) e_n$ is the representation of A, $\left(\sum_{n=1}^{\infty} \Phi\left(\frac{\lambda_n}{k}\right) < \infty \text{ for a } k > 0, \lambda_n > 0, \text{ and } \{e_n\} \text{ orthonormal}\right)$ and if A_m is a degenerate operator defined as $A_m = \sum_{n=1}^{m} \lambda_n (\cdot, e_n) e_n = \sum_{n=1}^{m} \lambda_n e_n \otimes e_n$, where $(f \otimes g) x = (x, g) f$, [cf. 8, p. 7], then $\|A - A_m\|_{\Phi} \to 0$ as $m \to \infty$, by the isomorphism. Also there exists a monotone family $\{P_{\alpha}\}$, $P_{\alpha}: \mathcal{X} \mapsto \mathcal{X}_{\alpha}$, of orthogonal projections such that $P_{\alpha} \to I$ uniformly on precompact sets since $\mathcal{X}$ is a Hilbert space. Hence for a given $\varepsilon > 0$, there is $n(\varepsilon)$ such that

$$\|(A - A_m) P_{\alpha}\|_{\Phi} \leqslant \|P_{\alpha}\|_{\infty} \|A - A_m\|_{\Phi} \leqslant \|A - A_m\|_{\Phi} < \varepsilon/3 \tag{4}$$

for $m \geqslant n(\varepsilon)$, and where (2) is used. On the other hand, $A_m = \sum_{n=1}^{m} \lambda_n (e_n \otimes e_n)$ so that $A_m (I - P_{\alpha}) = \sum_{n=1}^{m} \lambda_n (e_n \otimes (I - P_{\alpha}) e_n)$, by [8, lemma 2 on p. 7]. Hence

$$\|A_m (I - P_{\alpha})\|_{\Phi} \leqslant \sum_{n=1}^{m} \lambda_n \|e_n \otimes (I - P_{\alpha}) e_n\|_{\Phi}. \tag{5}$$

But by using the method of computation in [8, p. 41] to the above, one sees without difficulty that $\|e_n \otimes (I - P_{\alpha}) e_n\|_{\Phi} = \|(I - P_{\alpha}) e_n\|$. So

(6) $$\|A_m (I-P_\alpha)\|_\Phi \leqslant \sum_{n=1}^{m} \lambda_n \|(I-P_\alpha)\, e_n\|$$

which can be made arbitrarily small by choosing α appropriately (since $P_\alpha \to I$). The estimates (4) and (6) imply

(7) $$\|A\,(I-P_\alpha)\|_\Phi \leqslant \|A-A_m\|_\Phi + \|A_m\,(I-P_\alpha)\|_\Phi + \|(A_m-A)\,P_\alpha\| \to 0\,.$$

Using now the isometry of the involution operation one has $\|(I-P_\alpha)\,A\|_\Phi \to 0$, and finally

$$\|Q_\alpha(A)-A\|_\Phi \leqslant 2\,\|A\,(I-P_\alpha)\|_\Phi \to 0\,.$$

Hence $L^\Phi(\mathcal{X})$ is approximately tame, as was to be proved.

The above proofs yield also the following. (The conclusion about $\mathcal{C}$ was proved in [5], but also follows from the above if an appropriate discontinuous Φ is chosen.)

COROLLARY 1. *If $\mathcal{C} \subset B(\mathcal{X})$ is the set of compact operators, then $L^\Phi(\mathcal{X}) \subset \mathcal{C}$ if Φ is continuous, and moreover, with the operator norm, $\mathcal{C}$ is approximately tame.*

It now follows that, when a continuous $\Phi(\cdot)$ does not satisfy the inequalities of Theorem 2, the a.t. algebra $\mathcal{C}$ which contains self-adjoint B-algebras $L^\Phi(\mathcal{X})$ which are not a.t. In fact, (as is well-known) there exist continuous Φ, violating these inequalities so that 'simple functions' are not dense in l^Φ, and this yields the desired negative result. *So the a.t. property is not hereditary.*

DEFINITION 2. If $GL(\mathcal{X}) \subset B(\mathcal{X})$ is the group of invertible operators, and $GL(\alpha) = \{I+A \in GL(\mathcal{X}) \colon A \in B(\mathcal{X}_\alpha)\}$, then let $GL(\infty) = \varinjlim GL(\alpha)$ be the inductive limit [cf. 7, p. 57 for the latter concept]. Let $G(\mathcal{A}) = \{I+A \in GL(\mathcal{X}) \colon A \in \mathcal{A}\}$ where $\mathcal{A}$ is an a.t. algebra in the sense of Definition 1. $G(\mathcal{A})$ is topologized by the requirement that the map $I+A \mapsto A$ is bicontinuous into A.

Now Theorem 2 together with Theorem B of [5] implies the following:

COROLLARY 2. *If Φ is as in Theorem 2, and $\mathcal{X}$ is separable, then the injection map $i\colon GL(\infty) \mapsto G(\mathcal{A})$ is a homotopy equivalence, where $\mathcal{A} = L^\Phi(\mathcal{X})$.*

3. An extension

A few extensions of the above results will be indicated now. The following concept is given in [2, 9].

DEFINITION 3. If $\mathcal{X}$ is a B-space then it is said to have the approximation property (a.p.) if the identity map I on $\mathcal{X}$ can be approximated uniformly on every precompact set in $\mathcal{X}$, by continuous linear maps π_α of finite rank. (Let $\mathcal{X}_\alpha = \pi_\alpha(\mathcal{X})$.)

THEOREM 3. *If $\mathcal{X}$ is a B-space with the a.p., if $\mathcal{O} \subset \mathcal{X}$ is an open set, and if $\mathcal{O}_\infty = \varinjlim \mathcal{O} \cap X_\alpha$, then the injection map $i\colon \mathcal{O}_\infty \mapsto \mathcal{O}$ is a homotopy equivalence.*

This result is proved by certain modifications of the proof of [5], in which one uses a generalized Urysohn's lemma [4, p. 30] in defining the required homotopy. With this established the next result follows as in [5].

THEOREM 4. *If $\mathcal{A}\subset B(\mathcal{X})$ is an a.t. algebra where $\mathcal{X}$ is a Hilbert space, then the injection map $i: GL(\infty)\mapsto G(\mathcal{A})$ is a homotopy equivalence.*

The following extension of Corollary 2 then holds.

COROLLARY 3. *If $\mathcal{A}=L^{\Phi}(\mathcal{X})$, where $L^{\Phi}(\mathcal{X})$ is as in Theorem 2, then the injection map $i: GL(\infty)\mapsto G(\mathcal{A})$ is a homotopy equivalence.*

Remark. If $\mathcal{A}=\mathcal{C}$, the set of compact operators, and $\mathcal{X}$ is a B-space with a.p,. then in [9] a more general result corresponding to Theorem 4 is given with the injection being a weak homotopy equivalence. The above and related extensions will be considered separately.

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DEPARTMENT OF MATHEMATICS, CARNEGIE-MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA (U.S.A.)

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М. М. Рао, **Аппрiксимативно играноченные алгеьры операторов**

Содержанне. Если $\mathcal{X}$ обозначает Гильбертово пространство, причем $\mathcal{X}_\alpha<\mathcal{X}$-конечно-мерные подпространства положим, что $B(\mathcal{X})$ и $B(\mathcal{X}_\alpha)$ алгебры Буля ограниченных операторов на $\mathcal{X}$ и $\mathcal{X}_\alpha$, а Q_α— проекция $B(\mathcal{X})$ на $B(\mathcal{X}_\alpha)$. Аппроксимативно ограниченные (а. о.) алгебра $\mathcal{A}\subset B(\mathcal{X})$ — это алгебра Буля с топологией более тонкой, чем след $B(\mathcal{X})$ удовлетворяющая следующим формулам:

(i) $\bigcup_\alpha B(\mathcal{X}_\alpha)\subset\mathcal{A}, \mathcal{X}_\alpha\uparrow\mathcal{X}$,

(ii) $Q_\alpha(\mathcal{A}), A\,(\in\mathcal{A})$ в $\mathcal{A}$.

В настоящей заметке построен класс а. о. алгебр $L^{\Phi}(\mathcal{X})$, прибегая к понятиям пространств Орлича L^{Φ}. Показано, что имеются алгебры Буля $L^{\Phi}(\mathcal{X})\subset\mathcal{C}$ ($\mathcal{C}$ обозначает а. о. алгебру компактных операторов в $B(\mathcal{X})$) такие, которые не являются а. о. алгебрами.

Далее, если $\mathcal{A}=L^{\Phi}(\mathcal{X})$ является произвольной а. о. алгеброй, пусть $G(\mathcal{A})=\{I+A\in GL(\mathcal{X}): A\in\mathcal{A}\}$, $GL(a)=\{I+A\in GL(\mathcal{X}): A\in B(\mathcal{X}_\alpha)\}$ и наконец $GL(\infty)=\text{ind.}\lim G(a)$, где $GL(\mathcal{X})\subset B(\mathcal{X})$ является группой обратимых операторов. Замечено, что инъективное отображение $GL(\infty)$ в $G(\mathcal{A})$ является гомотопической эквивалентностью.

Обсуждаются некоторые результаты, связанные с проблемой.