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Some Remarks on Lozanovskyi's Intermediate Normed Lattices

by

Shlomo REISNER

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Summary. Lozanovskyi's "intermediate" normed lattices $\varphi(X, Y)$ are investigated with respect to properties like, e.g. duality and the Fatou property. This is motivated by Lozanowskyi's factorization theorem.

A well-known theorem of G. Ya. Lozanovskyi [4] states the following (the reader who is not acquainted with the concepts involved here will find the definitions following this short introduction):

THEOREM. *Let X be a Köthe function space on the σ-finite (complete) measure space (Ω, Σ, μ) and X' the subspace of the dual space of X, consisting of integrals. For every $0 \leqslant f \in L_1(\mu)$ and $\varepsilon > 0$ there exist $0 \leqslant g \in X$ and $0 \leqslant h \in X'$ such that $f = gh$ and*

$$\|g\|_X \|h\|_{X'} \leqslant (1+\varepsilon)\|f\|_{L_1} \tag{1}$$

If X has the Fatou property the theorem is true for $\varepsilon = 0$ as well.

(The statement in [4] uses different terminology).

Various proofs of this theorem or variations of it have been given since the publication of [4]. We mention in particular Gillespie's proof [1].

Lozanovskyi's original proof has a special elegance and is based on an interpolation construction of Banach lattices. It goes as follows:

Set $Z = X^{1/2}(X')^{1/2}$. Then (by the main theorem of [4])

$$Z'' = (X'')^{1/2}(X''')^{1/2} = (X'')^{1/2}(X')^{1/2} = Z'$$

From the equality $Z'' = Z'$ it follows that $Z' = L_2(\mu)$ so also $Z = L_2(\mu)$, from which the result follows.

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Following the publication of my lecture note [6], which is a presentation of Lozanovskyi's duality theorem for his construction of an intermediate Banach lattice "between" the Banach lattices X and Y, I have received some comments concerning the completeness of the above argument. In particular, concerning the implication: $Z' = L_2(\mu)$ implies $Z = L_2(\mu)$. This implication is a consequence of Lemma 2 in the sequel and is mentioned without proof in Lozanovskyi's original paper [4, Lemma 21]. This implication is, however, clearly wrong in general, if we drop the assumption of norm-completeness of Z. And in fact, Lozanovskyi's theorem as it is stated above, is wrong in general if Z is not norm-complete (cf. Example 10 in the sequel). What is true in this case was stated and proved, in a different way then Lozanovskyi's, by Gillespie [1, Thm 1 (iii)].

This situation leads us to consider what can be said in this, more general situation, about Lozanovskyi's construction and its duality properties. An answer to this is given in Lemma 3 and Theorem 6b), which, in particular, enables one to prove Gillespie's extension of Lozanovskyi's theorem to the non-Banach case, by extending Lozanovskyi's original proof (Corollary 8).

Another question which we treat in this paper is the following: It is known (cf. e.g. [4] or [6]) that if the Köthe function spaces X and Y have the Fatou property then $\varphi(X,Y)$ has the Fatou property as well. One may raise the question whether both X and Y must be assumed to have the Fatou property in order to assure that $\varphi(X,Y)$ will have it. One is not always enough; for example $c_0 = c_0^{1/2}\ell_\infty^{1/2}$ fails to have the Fatou property while ℓ_∞ has it. Also, let M be an Orlicz function and $\varphi(\xi,\eta)$ be defined by

$$\varphi(\xi,\eta) = \begin{cases} 0 & \xi = 0 \\ \xi M^{-1}(\eta/\xi) & \xi > 0. \end{cases} \tag{2}$$

Then $\varphi(c_0,\ell_1) = h_M$ and $\varphi(c_0,\ell_1)'' = \varphi(\ell_\infty,\ell_1) = \ell_M$ where ℓ_M is the Orlicz sequence space associated with M and h_M is the closed span in ℓ_M of the unit vectors (cf. [2]). Now $h_M = \ell_M$ if and only if M satisfies the Δ_2-condition at zero, so if it does not, then $\varphi(c_0,\ell_1)$ fails to have the Fatou property.

We give here a sufficient condition for $\varphi(X,Y)$ to have the Fatou property provided that Y (or X) has it.

Some other results and examples which are connected to the preceding topics are included as well.

We bring now a few definitions and notations, the books [2] and [3] can serve as a standard reference.

A *Köthe function space* on a σ-finite (complete) measure space (Ω,Σ,μ) (cf. [3, Def. 1.b.17]) is a Banach space L consisting of equivalent classes, modulo equality a.e., of locally integrable real (or complex) valued functions on Ω, verifying

(3) $|f| \leqslant |g|$, f measurable and $g \in L$ implies $f \in L$ and $\|f\| \leqslant \|g\|$,

(4) for all $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$ the characteristic function χ_σ is in L.

A space which satisfies all the above axioms except, possibly, norm completeness will be called a *normed function space.*

If L is a normed function space we denote by L' the space of the elements θ in the dual L^* of L of the form $\theta(f) = \int_\Omega fg\,d\mu$ for some measurable g, and we identify θ with g. The space L' with the norm induced from L^* is a Köthe function space. Denote $L_+ = \{f \in L : f \geqslant 0\}$. The norm of f in L is denoted $\|f\|_L$ and in the special case of $L = L_p(\mu)$, the notation is $\|f\|_p$.

Throughout this paper we adopt the convention $0/0 = 0$.

We say that L has the *Fatou property* if $L = L''$. In particular, for every normed function space L, L' has the Fatou property.

We say that $f \in L$ is *norm-absolutely continuous* if $f_n \downarrow 0$ a.e. and $f_n \leqslant f$ for all n, implies $\|f_n\| \to 0$. The space L is *σ-order continuous* if all functions in L are norm-absolutely continuous. L is σ-order continuous if and only if $L^* = L'$.

From now on we assume that all the normed function spaces are defined over the same measure space. We shall make repeated use of the following well-known lemma.

LEMMA 1 cf. [7, pp. 451, 471]. *Let L be a normed function space and f a nonnegative measurable function on Ω. Then $f \in L''$ if and only if there exists a sequence $(f_n)_{n=1}^\infty$ of elements of L, such that $0 \leqslant f_n \uparrow f$ a.e. and $\sup \|f_n\|_L < \infty$. For $f \in L''$ we have*

$$\|f\|_{L''} = \inf\left\{\lim_{n\to\infty} \|f_n\|_L : 0 \leqslant f_n \uparrow f \text{ a.e.}\right\}.$$

LEMMA 2. (a) *Let L be a Köthe function space and let $f \in (L'')_+$ be norm-absolutely continuous. Then $f \in L$ and $\|f\|_L = \|f\|_{L''}$. Hence, if L'' is σ-order continuous then L has the Fatou property.*

(b) *Let L be a normed function space and let $f \in L''$. If f is norm-absolutely continuous then for every $\varepsilon, \delta > 0$ there exists $\Omega_\varepsilon \subset \Omega$ such that $\|f\chi_{\Omega\setminus\Omega_\varepsilon}\| \leqslant \varepsilon$ and $f\chi_{\Omega_\varepsilon} \in L$, $\|f\chi_{\Omega_\varepsilon}\|_L \leqslant (1+\delta)\|f\|_{L''}$.*

P r o o f. Assume that $\|f\|_{L''} = 1$ and $\varepsilon < 1$. By Lemma 1, in both cases (a) and (b) there exists a sequence $f_n \uparrow f$ with $f_n \in L_+$ and $\|f_n\|_L \uparrow 1$.

Assume first that $\mu(\Omega) < \infty$. Then, by Yegorov's theorem, a sequence of measurable sets $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ and a subsequence $(j_k)_{k=1}^\infty$ of $\mathbb{N}$ exist, such that $\mu(\Omega \setminus \Omega_k) \to 0$ and $f_{j_k} \geqslant (1+\delta)^{-1} f$ on Ω_k. By norm-absolute continuity of f, $\|f\chi_{\Omega\setminus\Omega_k}\|_{L''} \to 0$. We find k such that $\|f\chi_{\Omega\setminus\Omega_k}\|_{L''} < \varepsilon$. On Ω_k we have $f \leqslant (1+\delta) f_{j_k}$ hence $f\chi_{\Omega_k} \in L$ and $\|f\chi_{\Omega_k}\|_L \leqslant (1+\delta)\|f_{j_k}\|_L \leqslant$

$(1+\delta)$. Write $A_0 = \Omega_k$. By extraction, we represent Ω as a disjoint union $\Omega = \bigcup_{j=0}^{\infty} A_j$ of measurable sets with $\|f\chi_{A_j}\|_L \leqslant (1+\delta)\varepsilon^j$, $j = 0, 1, \ldots$, and $\|f\chi_{\Omega\setminus\bigcup_{j=0}^{n} A_j}\|_{L''} \leqslant \varepsilon^{n+1}$, $n = 0, 1, \ldots$.

Writing $F_n = f\chi_{\bigcup_{j=0}^{n} A_j}$ we have $F_n \uparrow f$, $\|F_n\|_L \leqslant (1+\delta)(1-\varepsilon)^{-1}$ for all n and (F_n) is a Cauchy sequence in L. If L is a Köthe function space, this completes the proof. Otherwise, it is clear how to complete the proof of case (b).

If $\mu(\Omega) = \infty$, we use norm-absolute continuity of f once again, to represent Ω as a disjoint union $\Omega = \bigcup_{n=1}^{\infty} B_n$ with $\mu(B_n) < \infty$ for all n and $\sum \|f\chi_{B_n}\|_{L''} < \infty$ (in fact, we can make the last sum arbitrarily close to 1). Choose now an appropriate ε for each B_n and complete the proof using the first part. □

Let $\mathcal{U}_2^0$ be the set of all real-valued concave functions φ on $\mathbb{R}_+^2$ which are positive homogeneous and satisfy

$$\forall\, \xi, \eta > 0, \quad \varphi(\xi, 0) = \varphi(0, \eta) = 0, \tag{5}$$

$$\forall\, \xi, \eta > 0, \quad \lim_{\alpha\to\infty} \varphi(\xi, \alpha) = \lim_{\beta\to\infty} \varphi(\beta, \eta) = \infty. \tag{6}$$

Let

$$\widehat{\varphi}(\xi, \eta) = \inf_{\alpha, \beta > 0} \frac{\alpha\xi + \beta\eta}{\varphi(\alpha, \beta)}$$

If $\varphi \in \mathcal{U}_2^0$ then $\widehat{\varphi} \in \mathcal{U}_2^0$ and $\widehat{\widehat{\varphi}} = \varphi$.

Let $\varphi \in \mathcal{U}_2^0$ and let X, Y be two Köthe function spaces on (Ω, Σ, μ). We construct the normed function space $\varphi(X, Y)$ as follows:

$$z \in \varphi(X, Y) \ \text{ iff } \ |z| = \varphi(x, y) \quad \text{for some } x \in X_+,\ y \in Y_+ \tag{7}$$

$$\|z\|_{\varphi(X,Y)} = \inf\{\max(\|x\|_X, \|y\|_Y) : x, y \text{ as above}\} \tag{8}$$

In particular, if $\varphi(\xi, \eta) = \xi^s \eta^{1-s}$ for some $0 < s < 1$, we denote $\varphi(X, Y)$ by $X^s Y^{1-s}$. If X and Y are Köthe function spaces, then so is $\varphi(X, Y)$ and in this case it was proved in [5] (cf. [6]) that $\varphi(X, Y)' = \widehat{\varphi}(X', Y')$ and therefore $\varphi(X, Y)'' = \varphi(X'', Y'')$ (the last identity includes equality of norms, in order to have it in the equality preceding the last one, one should modify appropriately the norm in $\widehat{\varphi}(X', Y')$ — cf. [6]). The following lemma shows that norm-completeness is not needed for the last duality identities to hold.

Lemma 3. *Let X, Y be normed function spaces and let $\varphi \in \mathcal{U}_2^0$. Then $\varphi(X, Y)'' = \varphi(X'', Y'')$ and $\varphi(X, Y)' = \widehat{\varphi}(X', Y')$. The first equality includes equality of norms, the second does so, provided that in the definition of the norm in $\widehat{\varphi}(X, Y)$ by equations (7) and (8) we put $\|x\|_X + \|y\|_Y$ instead of $\max(\|x\|_X, \|y\|_Y)$.*

P r o o f . Let $f \in \varphi(X'', Y'')_+$, $f = \varphi(g, h)$, $g \in (X'')_+$, $h \in (Y'')_+$. By Lemma 1, there exist sequences of nonnegative functions $g_n \uparrow g$, $h_n \uparrow h$ with $\|g_n\|_X \uparrow \|g\|_{X''}$ and $\|h_n\|_Y \uparrow \|h\|_{Y''}$. As $\varphi(g_n, h_n) \uparrow f$ and $\|\varphi(g_n, h_n)\|_{\varphi(X,Y)} \leqslant \max(\|g\|_{X''}, \|h\|_{Y''})$, we conclude by Lemma 1 that $f \in \varphi(X,Y)''$ and $\|f\|_{\varphi(X,Y)''} \leqslant \|f\|_{\varphi(X'',Y'')}$. For the reverse inclusion, we use Lemma 1 again in reverse order. For $f \in (\varphi(X,Y)'')_+$ let $f_n \uparrow f$ be with $\|f_n\|_{\varphi(X,Y)} \uparrow \|f\|_{\varphi(X,Y)''}$. Since $\varphi(X,Y) \subset \varphi(X'',Y'')$ with the obvious norm inequality, and since $\varphi(X'',Y'')$ has the Fatou property (cf. [6]), Lemma 1 yields $f \in \varphi(X'',Y'')$ and $\|f\|_{\varphi(X'',Y'')} \leqslant \|f\|_{\varphi(X,Y)''}$. Thus the first identity of the lemma is established.

Using the duality result for Köthe function spaces and the Fatou property of L' for any normed function space L, we get

$$\begin{aligned}\widehat{\varphi}(X',Y') &= \widehat{\varphi}(X''',Y''') = \varphi(X'',Y'')' \\ &= (\varphi(X,Y)'')' = (\varphi(X,Y)')'' = \varphi(X,Y)'\end{aligned}$$ □

Lemma 2 and the identity $\varphi(X,Y)'' = \varphi(X'',Y'')$ imply easily:

THEOREM 4. *Let $\varphi \in \mathcal{U}_2^0$ and let X, Y be Köthe function spaces. If Y has the Fatou property and $\varphi(X'',Y)$ is σ-order continuous, then $\varphi(X,Y)$ has the Fatou property.*

We say that $\varphi \in \mathcal{U}_2^0$ satisfies the *Right-Δ_2-condition* (R-Δ_2) if there exists a constant $C > 1$ such that for all ξ, η

$$\varphi(2\xi, 2\eta) \leqslant \varphi(\xi, C\eta)$$

The *Left-Δ_2-condition* (L-Δ_2) is defined analogously.

COROLLARY 5. *Let $\varphi \in \mathcal{U}_2^0$ satisfy the R-Δ_2-condition and let X, Y be Köthe function spaces on (Ω, Σ, μ). If Y has the Fatou property and is σ-order continuous then $\varphi(X,Y)$ has the Fatou property (and is σ-order continuous).*

In particular, if φ satisfies the R-Δ_2-condition then $\varphi(X, L_1(\mu))$ has the Fatou property for every Köthe function space X.

P r o o f . By [5] (cf. [6, Prop. 4]) the above assumptions guarantee that $\varphi(W,Y)$ is σ-order continuous for every Köthe function space W. □

If, in the situation of Theorem 4, we do not assume that Y has the Fatou property we can still use similar methods to obtain information about the relations between $\varphi(X,Y)$ and $\varphi(X'',Y'')$ or $\varphi(X'',Y)$.

THEOREM 6. (a) *Let $\varphi \in \mathcal{U}_2^0$ and let X, Y be Köthe function spaces. If $f \in \varphi(X'',Y)$ is norm-absolutely continuous then $f \in \varphi(X,Y)$ and $\|f\|_{\varphi(X,Y)} = \|f\|_{\varphi(X'',Y)}$.*

(b) *Let* $\varphi \in \mathcal{U}_2^0$ *and let* X, Y *be normed function spaces. If* $f \in \widetilde{Z}$ $(\widetilde{Z} = \varphi(X'', Y)$ *or* $\widetilde{Z} = \varphi(X'', Y''))$ *is norm-absolutely continuous in* $\widetilde{Z}$, *then for all* $\delta, \varepsilon > 0$ *there exists a measurable set* $\Omega_\varepsilon \subset \Omega$ *such that*

$$\|f\chi_{\Omega\setminus\Omega_\varepsilon}\|_{\widetilde{Z}} < \varepsilon, \quad f\chi_{\Omega_\varepsilon} \in \varphi(X,Y) \quad \text{and} \quad \|f\chi_{\Omega_\varepsilon}\| \leqslant (1+\delta)\|f\|_{\widetilde{Z}}.$$

We shall not elaborate on the details of the proof of Theorem 6, it applies the same method of the proof of Lemma 2 together with the following lemma (formulated here for the case $\widetilde{Z} = \varphi(X'', Y)$).

LEMMA 7. *Assume* $\mu(\Omega) < \varepsilon$ *and let* $f \in \widetilde{Z} = \varphi(X'', Y)$ *be norm-absolutely continuous. If* $f = \varphi(g, h)$ *with* $g \in X''_+$, $h \in Y_+$ *and* $\max(\|g\|_{X''}, \|h\|_Y) < C$, *then for every* $\varepsilon > 0$ *there exist a measurable set* $\Omega_\varepsilon \subset \Omega$ *and functions* $\widetilde{g} \in X_+$, $\widetilde{h} \in Y_+$, *with supports contained in* Ω_ε, *such that* $\|f\chi_{\Omega\setminus\Omega_\varepsilon}\|_{\widetilde{Z}} < \varepsilon$ *and* $\max(\|\widetilde{g}\|_X, \|\widetilde{h}\|_Y) < C$.

Proof. By Lemma 1 we have a sequence $g_n \uparrow g$ with $g_n \in X_+$ and $\|g_n\| \uparrow \|g\|_{X''}$. By Yegorov's theorem and norm absolute continuity of f, we can find, as in the proof of Lemma 2, a subset Ω_ε of Ω with $\|f\chi_{\Omega\setminus\Omega_\varepsilon}\|_{\widetilde{Z}} < \varepsilon$ and such that $g\chi_{\Omega_\varepsilon} \in X$ and $\|g\chi_{\Omega_\varepsilon}\|_X < (1+\delta)\|g\|_{X''}$. We now take $\delta > 0$ sufficiently small and define $\widetilde{g} = g\chi_{\Omega_\varepsilon}$.

Remark. One should not come to the mistaken conclusion that the last argument actually shows that if $f = \varphi(g, h)$, $g \in X''$, $h \in Y$ (X and Y-Köthe function spaces) is norm-absolutely continuous, then actually $g \in X$. This is wrong in general, as simple examples of Orlicz spaces show. The point is that in the iteration of the application of Lemma 7 throughout the proof of Theorem 6 (a), we should in general take new representations $f\chi_E = \varphi(g_E, h_E)$ in every step (where E is the set of finite measure taking the role of Ω of Lemma 7). In fact, Example 9 in the sequel shows that $\varphi(X, Y)$ may have the Fatou property (and be σ-order continuous) with neither X nor Y having the Fatou property.

COROLLARY 8 (Gillespie [1]). *Let* X *be a normed function space and let* $f \in L_1(\mu)$. *For every* $\varepsilon > 0$ *there exist* $g \in X$, $h \in X'$ *and a measurable set* $\Omega_\varepsilon \subset \Omega$, *such that*

$$f\chi_{\Omega_\varepsilon} = gh, \quad \|g\|_X\|h\|_{X''} \leqslant (1+\varepsilon)\|f\|_1 \quad \text{and} \quad \int\limits_{\Omega\setminus\Omega_\varepsilon} |f|\,d\mu < \varepsilon.$$

Proof. As it was mentioned in the introduction, for $\varphi(\xi, \eta) = \xi^{1/2}\eta^{1/2}$ we have $\varphi(X'', X') = L_2(\mu)$. Theorem 6 (b) now gives a factorization of $L_2(\mu)$ functions whose translation to the above statement is immediate.

Example 9. There exist Köthe function spaces X and Y, both σ-order continuous and without the Fatou property, such that $\varphi(X,Y)$ is σ-order continuous and has the Fatou property.

Let the Köthe sequence space X be defined by:

$$f \in X \text{ if and only if } |f(2k-1)| \to 0 \text{ as } k \to \infty \text{ and } \sum_{k=1}^{\infty} |f(2k)| < \infty.$$

$$\|f\|_X = \max\Big(\max_{1 \leqslant k < \infty} |f(2k-1)|, \sum_{k=1}^{\infty} |f(2k)| \Big)$$

Denote $A_1 = \{2k-1\}_{k=1}^{\infty}$, $A_2 = \{2k\}_{k=1}^{\infty}$. We clearly have

$$X = (c_0(A_1) \oplus \ell_1(A_2))_{\infty}$$

(a direct sum in the ℓ_∞ sense). Let Y be defined in the analogous way, exchanging the roles of A_1 and A_2, i.e.

$$Y = (\ell_1(A_1) \oplus c_0(A_2))_{\infty}.$$

The spaces X and Y are σ-order continuous and fail to have the Fatou property. For $\varphi \in \mathcal{U}_2^0$ let the Orlicz function M_R be associated with φ as in (2) and let M_L be associated with φ by exchanging the roles of ξ and η in (2). It is easy to check that

$$\varphi(X,Y) = (h_{M_R}(A_1) \oplus h_{M_L}(A_2))_{\infty}$$

and

$$\varphi(X'',Y'') = (\ell_{M_R}(A_1) \oplus \ell_{M_L}(A_2))_{\infty}.$$

Hence $\varphi(X,Y)$ has the Fatou property if and only if both M_R and M_L satisfy the Δ_2-condition at zero (such is the case e.g. for $\varphi(\xi,\eta) = \xi^{1-s}\eta^s$, $0 < s < 1$, in this case we have $X^{1-s}Y^s = \big(\ell_{1/s} \oplus \ell_{1/(1-s)}\big)_{\infty}$).

Example 10. There exists a normed (not norm complete), Dedekind complete function space X, for which $X^{1/2}(X')^{1/2} \neq L_2$ but for every $f \in X^{1/2}(X')^{1/2}$ holds $\|f\|_{X^{1/2}(X')^{1/2}} = \|f\|_{L_2}$.

Let $\Omega = \mathbb{R}_+$ equipped with Lebesgue measure μ. For $1 \leqslant p < \infty$ and f measurable, define

$$\|f\|_{X_p} = \Big[\int_0^{\infty} |f(t)|^p \, dt\Big]^{1/p} + \operatorname{esslimsup} |f|,$$

where esslimsup $|f|$ is the essential upper limit of $|f(t)|$ as $t \to \infty$ (that is, esslimsup $|f| = \alpha$ if for all $\gamma > \alpha$, $\mu\{t \geqslant t_0 : |f(t)| \geqslant \gamma\} = 0$ for t_0 big enough, while for all $\beta < \alpha$ and $t_0 \in \mathbb{R}_+$, $\mu\{t \geqslant t_0 : |f(t)| > \beta\} > 0$). Let the space

$$X_p = \{f : \|f\|_{X_p} < \infty\}$$

be equipped with the norm $\|\cdot\|_{X_p}$. Clearly $(X_p)' = L_{p'}(0,\infty)$, $(1/p+1/p'=1)$.

If $1 < p < \infty$, the function

$$f(t) = \begin{cases} n^{1/2} & \text{for } |t-n| \leqslant \frac{1}{2n^{2+1/(p-1)}},\ n = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

can not be represented as $f = g^{1/2}h^{1/2}$ with $g \in X_p$, $h \in L_{p'}$, because if g is essentially bounded for big values of t, it is easily checked that h can not be in $L_{p'}(0,\infty)$. To check that $X_1^{1/2}(L_\infty(0,\infty))^{1/2} \neq L_2(0,\infty)$ is even simpler.

On the other hand, for $1 < p < \infty$, if $f \in L_2(0,\infty)$ is such that $f = g^{1/2}h^{1/2}$ with $g \in X_p$ and $h \in L_{p'}(0,\infty)$ then it is always possible to construct a decreasing function ψ on $\mathbb{R}_+$ such that $|\psi| \leqslant 1$, $\psi(t) \downarrow 0$ as $t \to \infty$ and $\|h/\psi\|_{p'} \leqslant (1+\varepsilon)\|h\|_{p'}$. Defining $\widetilde{g} = \psi g$ and $\widetilde{h} = h/\psi$ we conclude that $\|f\|_{X^{1/2}(X')^{1/2}} = \|f\|_2$.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA, 31905, ISRAEL
e-mail: reisner@mathcs2.haifa.ac.il

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