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MATHEMATICS
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Free Compact Convex Sets

by

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The notion of an infinite-dimensional simplex has been introduced by G. Choquet (for the definition and main properties see [6]) and it turns out to be quite adequate in certain problems, particularly in those concerning the uniqueness of various integral representations like the Riesz—Herglotz—Martin representation of positive harmonic functions, [5], or the Lévy—Khintchine representation of logarithms of non-zero infinitely divisible Laplace transforms of probabilistic Borel measures on the compactified half-line $[0, \infty]$, [14].

The purpose of this Note is to show that general arguments concerning the category of compact convex sets and continuous affine transformations yield some new information about simplexes. We shall show, in particular, that the free objects in this category are just the simplexes whose extreme boundaries are compact and extremally disconnected, and affine retracts and free joins of simplexes are simplexes. Combining our Theorem 5 with a theorem of Lindenstrauss ([16], p. 62) we get 17 new conditions necessary and sufficient in order that a compact convex set be a simplex. Proofs are outlined.

1. Preliminaries on categories. The terminology follows that of [15] and [18]. Two morphisms α and β in a category $\mathfrak{K}$ are called *isomorphic* if there exist isomorphisms γ and δ in $\mathfrak{K}$ such that $\alpha = \gamma\beta\delta$. Two categories $\mathfrak{K}$ and $\mathfrak{H}$ will be called *almost isomorphic* (*almost dual*) if there exist one-one covariant (contravariant) functors $\Phi : \mathfrak{K} \to \mathfrak{H}$ and $\Psi : \mathfrak{H} \to \mathfrak{K}$ such that every morphism β in $\mathfrak{H}$ is isomorphic to $\Phi\Psi(\beta)$ and every morphism α in $\mathfrak{K}$ is isomorphic to $\Psi\Phi(\alpha)$.

E.g., the category of Abelian groups and homomorphisms is almost dual to the category of compact Abelian groups and continuous homomorphisms (Pontriagin's theory does not give a duality in the strict sense, because the second dual of a group G is not identical with G but is merely isomorphic to G). Similarly, Stone's representation theorem states that the category of Boolean algebras and homomorphisms is almost dual to the category of 0-dimensional compact spaces and continuous maps.

THEOREM 1. *Let $\mathfrak{K}$ and $\mathfrak{H}$ be almost dual with Φ and Ψ as above. Then* (i) *α is an epimorphism in $\mathfrak{K}$ iff $\Phi(\alpha)$ is a monomorphism in $\mathfrak{H}$*, (ii) *F is a basic free object in $\mathfrak{K}$ iff $\Phi(F)$ is a basic direct object in $\mathfrak{H}$*, (iii) *given a family $\sigma_t : A_t \to A$ of morphisms in $\mathfrak{K}$ $(t \in T)$, the pair $(A, \{\sigma_t\})$ is a free join of $\{A_t\}_{t \in T}$ (in $\mathfrak{K}$) iff the pair $(\Phi(A), \{\Phi(\sigma_t)\})$ is a direct join of $\{(A_t)\}_{t \in T}$ (in $\mathfrak{H}$)*, (iv) *A is a retract of B in $\mathfrak{K}$ iff $\Phi(A)$ is a retract of $\Phi(B)$ in $\mathfrak{H}$, and a morphism $\beta : A \to B$ is a cross-section of a retraction $\alpha : B \to A$ iff $\Phi(\alpha) : \Phi(A) \to \Phi(B)$ is a cross-section of $\Phi(\beta) : \Phi(B) \to \Phi(A)$.*

2. Preliminaries on compact convex sets. By a *compact convex set* we mean a compact convex subset K of a locally convex Hausdorff real linear topological space; ∂K denotes the set of extreme points of K. A function α from a convex set K_1 into a convex set K_2 is called *affine* iff $\alpha(c_1 x + c_2 y) = c_1 \alpha(x) + c_2 \alpha(y)$ whenever $x \in K_1$, $y \in K_1$, $c_1 \geqslant 0$, $c_2 = 1 - c_1 \geqslant 0$. If K is a convex subset of a linear topological space, then $\mathcal{A}(K)$ will denote the space of all continuous affine functionals on K. If K is convex and compact, then $\mathcal{A}(K)$ is a closed linear subspace of the space $\mathcal{C}(K)$ of all continuous real-valued functions on K (provided with the supremum norm) and satisfies the following condition in which $X = K$ and $H = \mathcal{A}(K)$:

(*) $\quad$ H separates the points of X and $1 \in H$.

We recall several known facts which will be needed in the sequel.

(I). Let V be an Archimedean ordered vector space with a (strong) order unit e and with the norm $\|v\| = \inf\{c : v \leqslant ce,\ -v \leqslant ce\}$. Then V is linearly, isometrically and isotonically equivalent to a subspace H of a space $\mathcal{C}(X)$ satisfying (*) such that e corresponds to 1 in H ([12], p. 7).

(II). Let H be any linear subspace of $\mathcal{C}(X)$ satisfying (*) and let

$$K = \mathcal{K}(H) = \{\xi \in H^* : \quad \xi \geqslant 0,\ \|\xi\| = \xi(1) = 1\}.$$

Then K is convex and compact in the *-weak topology of the space H^* conjugate to H. If $h \in H$, let $h^\wedge(\xi) = \xi(h)$ for $\xi \in K$. Then the function $h \to h^\wedge$ is a linear order-preserving isometry from H onto the set $\mathcal{A}(H^*)|K$ of all restrictions (to K) of continuous affine functionals on H^*, which is dense in the space $\mathcal{A}(K)$. If H is closed in $\mathcal{C}(X)$, then the function $h \to h^\wedge$ maps H onto $\mathcal{A}(K)$. (Isometry is due to Kadison, [12]; the fact that this map is onto $\mathcal{A}(H^*) \mid K$ is due to Alfsen, [1]).

(III). Let K be any compact convex set, let $H = \mathcal{A}(K)$ and let $K^\wedge = \mathcal{K}(H)$. If $x \in K$, let $x^\wedge(h) = h(x)$ for h in H. Then the function $x \to x^\wedge$ is an affine homeomorphism from K onto $K^\wedge$ (cf. [2], p. 122 and [1]).

(IV). Let X be compact, let H be a closed linear subspace of $\mathcal{C}(X)$ satisfying (*) and let X_0 be the Šilov boundary of H, [2]. If H is a vector lattice in the ordering induced by $\mathcal{C}(X)$ (but not necessarily a sublattice of $\mathcal{C}(X)$), and if H is an M-space with unit in the sense of Kakutani [13], then $(f \vee_0 g)(x) = \max[f(x), g(x)]$ for every f and g in H and x in X_0, where $f \vee_0 g$ denotes the relative supremum of f and g in H. This means that the restriction $f \to f \mid X_0$ is a lattice isomorphism from H onto a sublattice H_0 of $\mathcal{C}(X_0)$, and hence H_0 must be identical with $\mathcal{C}(X_0)$ by

the Stone—Weierstrass theorem. (This theorem has been proved by Gęba and Semadeni [8], [9] and, independently and in a quite different form, by H. Bauer [2]).

(V). Let H be a linear subspace of a vector lattice E, and let $\beta : E \to H$ be a non-negative projection onto H. Then H is also a vector lattice; if $E = \mathcal{C}(X)$, and H satisfies (*), then H satisfies the assumptions of (IV) ([9], p. 314).

(VI). Let K be compact and convex. Then $\mathcal{A}(K)$ is a vector lattice iff K is a simplex and ∂K is closed ([2], p. 120, [3], [6], [7]).

(VII). Let X be any compact space and let $\mathcal{S}_X = \mathcal{K}(\mathcal{C}(X))$, i.e., $\mathcal{S}_X$ is the set of all probabilistic Radon measures on X. Then $\mathcal{S}_X$ is a simplex, the extreme points of $\mathcal{S}_X$ are precisely the Dirac measures on X, and $\partial \mathcal{S}_X$ is closed and homeomorphic to X. Moreover, if S is any simplex and ∂S is closed, then S can be obtained in the way described above, and $\mathcal{A}(S)$ may be identified with $\mathcal{C}(\partial S)$. For every compact convex set K and every continuous map $\alpha : \partial \mathcal{S}_X \to K$ there exists a unique extension of α to a continuous affine map from $\mathcal{S}_X$ into K, [3].

3. Results. We shall deal with the category in which the class of objects is the class $\mathfrak{K}_0$ of compact convex sets and the class of morphisms is the class $\mathfrak{K}$ of continuous affine maps.

Let $\mathfrak{H}_0$ be the class of all Archimedean ordered vector spaces with distinguished order units and complete with respect to the norm described in (I). Let $\mathfrak{H}$ be the category whose objects are the elements of $\mathfrak{H}_0$ and the morphisms are the non-negative linear operators transforming the distinguished order units onto the distinguished order units. This category is almost isomorphic to the category of all closed subspaces H of spaces $\mathcal{C}(X)$ satisfying (*) and linear operators $\alpha : H_1 \to H_2$ such that $f \geqslant 0$ implies $\alpha(f) \geqslant 0$ and $\alpha(1) = 1$.

We are now going to define two functors $\mathcal{A} : \mathfrak{K} \to \mathfrak{H}$ and $\mathcal{K} : \mathfrak{H} \to \mathfrak{K}$. The object functions have been defined in (I), (II) and (III). If $\alpha : K_1 \to K_2$ is a morphism in $\mathfrak{K}$, then $\beta = \mathcal{A}(\alpha) : \mathcal{A}(K_2) \to \mathcal{A}(K_1)$ is the linear operator β adjoint to α, defined as $\beta h_2(x_1) = h_2[\alpha(x_1)]$ for h_2 in $\mathcal{A}(K_2)$ and x_1 in K_1. If $\beta : H_1 \to H_2$ is any non-negative linear operator and $\beta(1) = 1$, then $\alpha = \mathcal{K}(\beta) : \mathcal{K}(H_2) \to \mathcal{K}(H_1)$ is the continuous affine map adjoint to β, defined as $\alpha\xi_2(h_1) = \xi_2[\beta(h_1)]$ for h_1 in H_1, ξ_2 in $\mathcal{K}(H_2)$.

THEOREM 2. *The categories $\mathfrak{K}$ and $\mathfrak{H}$ are almost dual and this relation is determined by the contravariant functors $\mathcal{A}$ and $\mathcal{K}$. Moreover, each of functors $\mathcal{K}\mathcal{A} : \mathfrak{K} \to \mathfrak{K}$ and $\mathcal{A}\mathcal{K} : \mathfrak{H} \to \mathfrak{H}$ is naturally equivalent to the corresponding identity functor.*

THEOREM 3. *The Cartesian product $K = \Pi K_t$ of a family of compact convex sets (together with the coordinate projections $\pi_t : K \to K_t$) is a direct join in the category $\mathfrak{K}$. The interval $\mathbf{I} = [0, 1]$ is a basic direct object, and the Tychonoff cubes $\mathbf{I}^{\mathfrak{m}}$ are the direct objects in $\mathfrak{K}$.*

THEOREM 4. *Let H be a closed linear subspace of $\mathcal{C}(X)$ satisfying (*). Then every continuous affine functional on $\mathcal{K}(H)$ can be extended to a continuous affine functional on the whole space H^*.*

This follows immediately from (II).

THEOREM 5. *Let K be a compact convex set. Then the following conditions are equivalent*:

(i) *K is a simplex.*

(ii) *The space H^* conjugate to $H = \mathcal{A}(K)$ is a vector lattice.*

(iii) *H^* is an L-space* (in the sense of [13]).

(iv) *H^{**} has the property of Nachbin* (cf. [11]).

(v) *H has the Riesz decomposition property: if $f, g, h \in H$, $f \geqslant 0$, $g \geqslant 0$, $h \geqslant 0$ and $f+g \geqslant h$, then there exist f_0, g_0 in H such that $0 \leqslant f_0 \leqslant f$, $0 \leqslant g_0 \leqslant g$ and $f_0+g_0 = h$.*

Proof. (i) $\Leftrightarrow$ (ii) follows from Choquet's theorem, because $\mathcal{K}(H)$ is a base of the positive cone of H^*. (iii) obviously implies (ii) and (iv); (iv) implies (iii) by a theorem of Grothendieck [11], and (iii) is equivalent to (v) by a theorem of Lindenstrauss ([16], p. 62). Finally, assume (ii). Let $\xi, \eta, \zeta \in H^*$ and $\xi \geqslant 0$, $\eta \geqslant 0$. There exists a Radon measure μ on K such that $\mu(h) = \zeta(h)$ for h in H and $\|\mu\| = \|\zeta\|$. Since $\zeta_+ \leqslant \mu_+|H$ and $\zeta_- \leqslant \mu_-|H$ (by the minimality of the decomposition $\zeta = \zeta_+ - \zeta_-$ in H^*),

$$\|\zeta\| \leqslant \|\zeta_+\|+\|\zeta_-\| \leqslant \|\mu_+\|+\|\mu_-\| = \|\mu\| = \|\zeta\|;$$

moreover, $\|\xi+\eta\| = (\xi+\eta)(1) = \xi(1)+\eta(1) = \|\xi\|+\|\eta\|$, and hence H^* is an L-space. (of [20]).

THEOREM 6. *Let K be an affine retract of a simplex S* (i.e., a retract in $\mathfrak{K}$). *Then K is also a simplex, and if ∂S is closed, then so is ∂K.*

Proof. Let $\beta: K \to S$ be a cross-section of $\alpha: S \to K$ ($\alpha \in \mathfrak{K}$, $\beta \in \mathfrak{K}$), i.e., $\alpha\beta = \varepsilon_K$ (identity on K). Then $\mathcal{A}(\beta)\mathcal{A}(\alpha) = \varepsilon_{\mathcal{A}(K)}$, i.e., $\mathcal{A}(\beta)$ is a non-negative linear projection from $\mathcal{A}(S)$ onto a subspace equivalent to $\mathcal{A}(K)$. If ∂S is closed, we apply (V) and (VI), if not, we pass to the conjugate spaces and apply (V) and Theorem 5.

THEOREM 7. *Every (indexed) family $\{K_t\}_{t \in T}$ of compact convex sets has a free join in $\mathfrak{K}$.*

Proof. If the family is finite, a free join can be constructed as follows. We may assume that $K_i \subset H_i$, where H_i is a hyperplane in a locally convex space E_i and $0 \notin H_i$ ($i = 1, \ldots, n$). Let $E = E_1 + \ldots + E_n$ and let K be the convex hull of the union of images of $K_1, \ldots, K_n$ in E. Then K is compact and (together with the obvious embeddings) is a free join of $\{K_1, \ldots, K_n\}$ in $\mathfrak{K}$.

If the family is infinite, we can exploit Theorems 1 and 2. Let H be the direct join of the spaces $\mathcal{A}(K_t)$ in the category of Banach spaces and linear contractions, i.e., let H be the set of all functions $h = \{h_t\}_{t \in T}$ with $h_t \in \mathcal{A}(K_t)$ and $\|h\| = \sup\{\|h_t\| : t \in T\} < \infty$. Then H is also a direct join in the category $\mathfrak{H}$, hence $K = \mathcal{K}(H)$ is a free join of $\{K_t\}_{t \in T}$ in $\mathfrak{K}$.

THEOREM 8. *A free join of any family of simplexes is a simplex.*

Proof. Assume that the sets K_t in the preceding proof are simplexes. Then, by Theorem 5, each space $\mathcal{A}(K_t)$ has the Riesz decomposition property, and a straight-forward argument shows that H has also this property. Hence, K is a simplex.

THEOREM 9. *A one-point set is a basic free object in $\mathfrak{K}$. An object is free iff it is a simplex whose extreme boundary is a free compact space. A compact convex set is projective in $\mathfrak{K}$ iff it is a simplex whose extreme boundary is a projective compact space, i.e., is compact and extremally disconnected.*

Proof. A free compact convex space K is a free join (in the sense of Theorem 7) of a set of copies of a one-point set; moreover, $\mathcal{A}(K)$ is a direct Banach space (i.e., the space of bounded functions on an isolated set T) and $K = \mathcal{S}_{\beta T}$.

An object is projective iff it is a retract of a free object (cf. [18] and [19]). If X is extremally disconnected, then it is a retract of a space βT with T isolated, [10], [17], and, by (VII), this retraction can be extended to a continuous affine retraction from $\mathcal{S}_{\beta T}$ onto $\mathcal{S}_X$. Conversely, if K is an affine retract of $\mathcal{S}_{\beta T}$, then, by Theorem 6, $K = \mathcal{S}_X$ for a certain compact X and there exists a non-negative projection α from $\mathcal{A}(\mathcal{S}_{\beta T}) = \mathcal{C}(\beta T)$ onto $\mathcal{A}(\mathcal{S}_X) = \mathcal{C}(X)$ such that $\alpha(1) = 1$, hence $\|\alpha\| = 1$ and $\mathcal{C}(X)$ has the property of Nachbin, and X is extremally disconnected.

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