Monads and Their Eilenberg-Moore Algebras in Functional Analysis

by

Zbigniew Semadeni

Institute of Mathematics of the
Polish Academy of Sciences, Warsaw

QUEEN'S PAPERS IN PURE AND
APPLIED MATHEMATICS—NO. 33

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO
CANADA
1973

MONADS AND THEIR EILENBERG-MOORE ALGEBRAS IN FUNCTIONAL ANALYSIS

by

Zbigniew Semadeni

Institute of Mathematics of the Polish Academy of Sciences, Warsaw

QUEEN'S PAPERS IN PURE AND APPLIED MATHEMATICS No. 33

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA

1973

Preface

These notes are based on lectures given at Queen's University during the period October-December 1972 and in August 1973. Their main aim is to show what monads and their algebras mean for categories important in functional analysis. Special emphasis is put on Banach spaces and compact convex sets.

The reader is supposed to be familiar with rudiments of functional analysis. For understanding the categorical core of these notes the knowledge of the concept of an adjoint functor is needed. Some results, however, are stated in the language of functional analysis and are of interest independent of category theory.

These lecture notes are essentially self-contained. Proofs are presented at a pedestrian pace, with all details included. Each section is followed by a set of exercises and bibliographic comments.

§2 contains an exposition of basic facts concerning monads in general; next sections deal with applications to other theories. The overall intent is to translate categorical problems into the language of the specific theory in question; this means that for each given pair of adjoint functors the general form of a T-algebra should be found and the monadicity should be proved (or disproved) by methods of this theory rather than by applying a general monadicity criterion. One such criterion is stated (in §6), namely Linton's criterion, which appears to be the most convenient in applications.

Still, it turns out that in each case considered in these notes, a direct proof is not longer than a verification of Linton's conditions.

The topic discussed in these notes is relatively new. All known results concerning monads in functional analysis seem to have been proved in last three years and only one paper has been published by now.

I take pleasure in acknowledging my debt of gratitude to several mathematicians. At a conference in Oberwolfach in July 1972 Professors John Gray, F.E.J. Linton and F.W. Lawvere directed my attention to monads and encouraged me to work in the topic presented below. During my visit to Wesleyan University in December 1972 Professor Linton generously made his unpublished results available to me, pinpointed an error in an early draft of a part of the notes and helped me greatly with his advice. I am indebted to the audience of my lectures at Queen's University for several valuable comments, especially to Professor Peter Taylor for his new proof of Proposition 7.3 . I also wish to express my gratitude to Professor A.J. Coleman for the invitation to Kingston and to the Department of Mathematics of Queen's University for the hospitality and help while preparing these notes.

Kingston, August 1973 Zbigniew Semadeni

TABLE OF CONTENTS

§1. Preliminaries. All undefined concepts can be found in Semadeni [1971] unless another reference is explicitly stated; a reference of the type #12.2.1 will always refer to that book. Yet, in order to make these note self-contained we recall the terminology and notation.

1.1. $\mathbb{R}$ is the set of reals, $\mathbb{C}$ is the set of complex numbers, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. A linear contraction is a linear operator of norm ≤ 1. If B is a Banach space, B^* is its conjugate space; $\sigma(B,B^*)$ is the weak topology and $\sigma(B^*,B)$ is the *weak topology; κ_B is the canonical map from B into B^{**}.

If X is a topological space, $C(X)$ is the space of bounded continuous scalar-valued functions on X with the supremun norm. If $\phi: X \to Y$ is a continuous map, $C(\phi)$ is the induced linear operator from $C(Y)$ into $C(X)$. If the scalar field is to be specified, we may write $C(X,\mathbb{R})$, $C(X,\mathbb{C})$, $C(\phi,\mathbb{R})$ etc. If X is compact, $M(X)$ is the space of regular Borel scalar-valued measures on X.

1.2. Arbitrary categories will be denoted by capital German letters $\mathfrak{A}$ and $\mathfrak{L}$ (in §3, $\mathfrak{L}_0$ denotes a fixed, specific category). An index of frequently used categories is at the end of these notes. $\mathfrak{L}^0$ is the class of objects of $\mathfrak{L}$; $\mathfrak{L}^*$ is the dual category.

The symbol $\langle A,A'\rangle_{\mathfrak{A}}$ or $\langle A,A'\rangle$ will denote the set of all $\mathfrak{A}$-morphisms from A to A'.

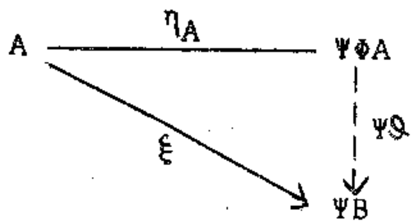
The letter Φ will always denote a covariant functor from $\mathfrak{A}$ to $\mathfrak{B}$ which is a <u>left adjoint</u> of a functor Ψ from $\mathfrak{B}$ to $\mathfrak{A}$; this means that:

(i) there exists a natural transformation

(1.1) $$\eta : \iota_{\mathfrak{A}} \to \Psi\Phi \quad .$$

called the <u>unit</u> of the adjunction, such that for every morphism $\xi : A \to \Psi B$ in $\mathfrak{A}$ there is a unique morphism $\vartheta : \Phi A \to B$ in $\mathfrak{B}$ such that the diagram

(1.2)



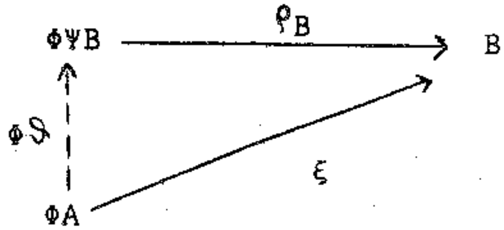
is commutative.

(ii) there exists a natural transformation

(1.3) $$\rho : \Phi\Psi \to \iota_{\mathfrak{B}} \quad ,$$

called the <u>counit</u> of the adjunction, such that for every morphism $\xi : \Phi A \to B$ in $\mathfrak{B}$ there is a unique morphism $\vartheta : A \to \Psi B$ in $\mathfrak{A}$ such that the diagram

(1.4)



is commutative,

(iii) the natural transformations η and ρ satisfy the identities

(1.5) $$\Psi(\rho_B)\eta_{\Psi B} = \iota_{\Psi B} \quad \text{and} \quad \rho_{\Phi A}\Phi(\eta_A) = \iota_{\Phi A} \;,$$

(iv) there is a natural equivalence

(1.6) $$\langle \Phi A, B \rangle_{\mathcal{L}} \cong \langle A, \Psi B \rangle_{\mathfrak{A}}$$

of bifunctors from $\mathfrak{A}^* \times \mathcal{L}$ to Ens given by

(1.7) $$\beta \mapsto \Psi(\beta)\eta_A \quad \text{for} \quad \beta \quad \text{in} \quad \langle \Phi A, B \rangle_{\mathcal{L}}$$

and by the inverse correspondence

(1.8) $$\alpha \mapsto \rho_B \Phi(\alpha) \quad \text{for} \quad \alpha \quad \text{in} \quad \langle A, \Psi B \rangle_{\mathfrak{A}} \quad .$$

Each of the conditions (i) and (ii) may be assumed to be an equivalent definition of Φ being a left adjoint of Ψ ; the existence of two natural transformations (1.1) and (1.3) satisfying (1.5) is also an equivalent definition of adjointness; and the existence of a natural equivalence (1.6) is still another formulation of the same concept. If Φ is a left adjoint of Ψ , Ψ is a <u>right adjoint</u> of Φ .

§2. Monads and their algebras. Let $\mathfrak{A}$ be any category.

2.1 DEFINITION. A monad in $\mathfrak{A}$ is a covariant functor

(2.1) $$T : \mathfrak{A} \longrightarrow \mathfrak{A}$$

together with two natural transformations

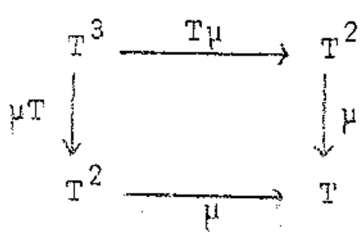
(2.2) $$\eta : \iota_{\mathfrak{A}} \longrightarrow T \quad \text{and} \quad \mu : T^2 \longrightarrow T$$

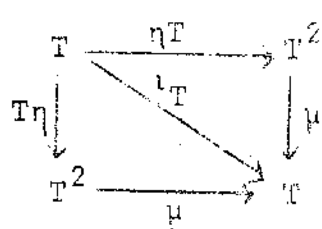
satisfying the following conditions:

(2.3) $$\forall_{A \in \mathfrak{A}^o} \quad \mu_A T(\mu_A) = \mu_A \mu_{T(A)} \quad ,$$

(2.4) $$\forall_{A \in \mathfrak{A}^o} \quad \mu_A \eta_{T(A)} = \iota_{T(A)} = \mu_A T(\eta_A) \quad .$$

The symbol T^2 stands for the composition of T with itself. The conditions (2.3) and (2.4) mean that the diagrams





are commutative. $T\mu$ is the natural transformation

$$T(\mu_A) : T^3(A) \rightarrow T^2(A)$$

obtained from

(2.5) $$\mu_A : T^2(A) \to T(A)$$

by acting with T , while μT is the natural transformation

$$\mu_{T(A)} : T^3(A) \to T^2(A)$$

obtained from (2.5) by substituting $T(A)$ instead of A . Similarly, $T\eta$ and ηT are obtained from

(2.6) $$\eta_A : A \to T(A)$$

by acting with T and substituting $T(A)$, respectively.

In order to simplify the notation we shall often write T instead of (T,η,μ) .

Remark. Monads $(\mathcal{O}\!\!\iota, T, \eta, \mu)$ have been also called "dual standard construction" , "triple" , "triad" . The term "monad" is due to formal resemblence of the definition of a monad to that of a monoid (i.e., a semigroup with unit); a monoid may be regarded as a set T with two maps $\eta : \mathbb{1} \to T$ and $\mu : T\times T \to T$ such that the diagrams

$$\begin{array}{ccc} T\times T\times T & \xrightarrow{\mu\times\iota_T} & T\times T \\ {\scriptstyle \iota_T\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T\times T & \xrightarrow{\mu} & T \end{array} \qquad\qquad \begin{array}{ccc} T & \xrightarrow{(\eta,\iota_T)} & T\times T \\ {\scriptstyle (\iota_T,\eta)}\downarrow & \searrow^{\iota_T} & \downarrow{\scriptstyle \mu} \\ T\times T & \xrightarrow{\mu} & T \end{array}$$

are commutative. Thus, if we call η the <u>unit</u> of the monad and μ the <u>multiplication</u> then (2.3) and (2.4) are to interpreted as the associative law for μ, the left-unit law, and the right-unit law, respectively.

2.2 Proposition. Let a functor $\Phi : \mathcal{A} \to \mathcal{B}$ be a left adjoint of $\Psi : \mathcal{B} \to \mathcal{A}$ and let

(2.7) $$\eta_A : A \to \Psi\Phi(A) \ , \ \rho_B : \Phi\Psi(B) \to B$$

be the corresponding natural transformations. Denote

(2.8) $$T = \Psi\Phi$$

and

(2.9) $$\mu_A = \Psi(\rho_{\Phi A}) \ .$$

Then T together with η and μ is a monad in $\mathcal{A}$.

Proof. T is clearly a functor from $\mathcal{A}$ to $\mathcal{A}$; μ_A is a morphism from $\Psi\Phi\Psi\Phi(A)$ to $\Phi\Psi(A)$, i.e., from $T^2(A)$ to $T(A)$, obtained from

$$\mathcal{A} \xrightarrow{\Phi} \mathcal{B} \ \underset{1_{\mathcal{B}}}{\overset{\Phi\Psi}{\Downarrow \rho}} \ \mathcal{B} \xrightarrow{\Psi} \mathcal{A} \ .$$

Therefore $\mu : T^2 \to T$ is a natural transformation. Condition (2.3) follows from naturality of ρ . Indeed, for each $\beta : B \to B'$ in $\mathcal{B}$ we have

(2.10) $$\beta\varrho_{B'} = \varrho_B\Phi\Psi(\beta) ;$$

substituting $B = \Phi(A)$, $B' = \Phi\Psi\Phi(A)$, $\beta = \varrho_{\Phi A}$, and acting with Ψ we get (2.3). The two conditions in (2.4) mean that

$$\Psi(\varrho_{\Phi A})\eta_{\Psi\Phi A} = \iota_{\Psi\Phi A} \quad \text{and} \quad \Psi(\varrho_{\Phi A})\Psi\Phi(\eta_A) = \iota_{\Psi\Phi A}$$

and they are immediate consequences of the equalities

(2.11) $$\Psi(\varrho_B)\eta_{\Psi B} = \iota_{\Psi B} \quad \text{and} \quad \varrho_{\Phi A}\Phi(\eta_A) = \iota_{\Phi A} ,$$

which hold for every adjunction (Φ,Ψ,η,ρ) . ∎

2.3 DEFINITION. The monad constructed above is the <u>monad determined (or generated) by the adjunction</u> (Φ,Ψ,η,ρ) .

We shall show that, conversely, every monad is determined by some adjunction; this adjunction is not unique but one can distinguish two canonical ways of assigning an adjunction to a given monad and these **two** adjunctions are extreme in the sense which will be explained below.

We begin with the "largest" of all adjunctions determining the given monad.

2.4 DEFINITION. Let (T,η,μ) be a monad in $\mathfrak{A}$. An <u>Eilenberg-Moore algebra</u> of T , shortly: a T-<u>algebra</u>, is a pair

$$(A,\gamma)$$

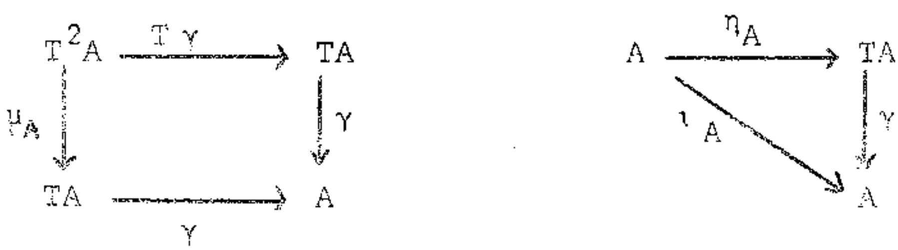
where $A \in \mathfrak{X}^T$ and $\gamma: TA \to A$ is a morphism in $\mathfrak{X}$ such that

(2.12) $$\gamma\mu_A = \gamma T(\gamma)$$

and

(2.13) $$\gamma\eta_A = \iota_A$$

i.e., the diagrams



are commutative. A is the underlying object of the T-algebra (A,γ) and γ is a T-algebra structure on A.

Conditions (2.12) and (2.13) may be regarded as "associative law" and the "unit law" for γ.

2.5 Example. Let G be a fixed semigroup with a unit e. The formulas

$$T(X) = G\times X \;,\quad \eta_X(x) = (e,x) \;,\quad \mu_X(g_1,(g_2,x)) = (g_1g_2,x)$$

determine a monad

$$T: \mathrm{Ens} \to \mathrm{Ens} \;,\quad \eta_X: X \to TX \;,\quad \mu_X: T^2X \to TX \;.$$

Indeed, let $(g_1,(g_2,(g_3,x))) \in T^3X$. Then

$$(g_1,(g_2,(g_3,x))) \underset{T\mu_X}{\longmapsto} (g_1,(g_2g_3,x)) \underset{\mu_X}{\longmapsto} (g_1(g_2g_3),x)$$

while

$$(g_1,(g_2,(g_3,x))) \underset{\mu_{TX}}{\longmapsto} (g_1g_2,(g_3,x)) \underset{\mu_X}{\longmapsto} ((g_1g_2)g_3,x) \quad .$$

This means that the condition $\mu_X T\mu_X = \mu_X\mu_{TX}$ is equivalent to the associativity of the multiplication in G . If $(g,x) \in TX$, then

$$(g,x) \underset{\eta_{TX}}{\longmapsto} (e,(g,x)) \underset{\mu_X}{\longmapsto} (eg,x)$$

and

$$(g,x) \underset{T\eta_X}{\longmapsto} (g,(e,x)) \underset{\mu_X}{\longmapsto} (ge,x) \quad .$$

Thus, the conditon $\mu_X\eta_{TX} = \iota_{TX}$ means that $eg = g$ for all g in G whereas the condition $\mu_X T(\eta_X) = \iota_{TX}$ means that $ge = g$ for all g in G .

Now, a T-algebra is a set X together with a map

$$\gamma\colon G\times X \longrightarrow X$$

satisfying the conditions

$$\gamma(g_1 g_2, x) = \gamma\mu_X(g_1, (g_2, x)) = \gamma T(\gamma)(g_1, (g_2, x))$$
$$= \gamma(g_1, \gamma(g_2, x))$$

and $\gamma(e,x) = x$. If $\gamma(g,x)$ is denoted as $g\cdot x$, then the above conditions can be written in a more familiar form:

$$(g_1 g_2)\cdot x = g_1\cdot(g_2\cdot x) \quad \text{and} \quad e\cdot x = x \ .$$

Such a map γ is called an <u>action</u> of the semigroup G on the set X .

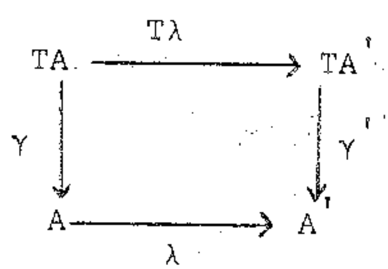
2.6 DEFINITIONS. A <u>morphism</u>

(2.14) $$\lambda^{\nabla}: (A,\gamma) \longrightarrow (A',\gamma')$$

<u>of</u> T-<u>algebras</u> is a triple $(\lambda,(A,\gamma),(A',\gamma'))$ where $\lambda: A \rightarrow A'$ is any $\mathfrak{A}$-morphism such that

(2.15) $$\lambda\gamma = \gamma' T(\lambda)$$

i.e., the diagram



is commutative. $\mathfrak{A}^T$ will denote the category of T-algebras and their morphisms; it is the <u>Eilenberg-Moore</u> <u>category</u> of the monad T . The morphism $\lambda: A \longrightarrow A'$ will be called the

<u>underlying</u> $\mathcal{A}$-morphism of the $\mathcal{A}^T$-morphism (2.14).

We shall now show that for each A the morphism (2.5) determines a T-algebra structure on the object TA .

2.7 Proposition. If $A \in \mathcal{A}^o$, then (TA,μ_A) is a T-algebra. Moreover, if $\alpha: A \to A'$ is a morphism in $\mathcal{A}$, then $T\alpha: TA \to TA'$ is the underlying $\mathcal{A}$-morphism of the morphism

$$(2.16) \qquad (T\alpha)^{\nabla} : (TA,\mu_A) \longrightarrow (TA',\mu_{A'})$$

of T-algebras.

Proof. Let $\gamma = \mu_A$. Then (2.12) becomes the associativity condition (2.3) whereas (2.13) becomes the left-hand side of (2.4). Naturality of μ implies that $T\alpha$ satisfies (2.15). ∎

2.8 DEFINITIONS. (TA,μ_A) is a <u>free</u> T-<u>algebra</u> with the underlying object TA .

The full subcategory of $\mathcal{A}^T$ consisting of all free T-algebras and T-algebra morphisms is called the <u>Kleisli category of the monad</u> T and is denoted by $\mathcal{A}_T$. Thus, the underlying $\mathcal{A}$-morphism of an $\mathcal{A}_T$-morphism

$$\lambda^{\nabla}: (TA,\mu_A) \longrightarrow (TA',\mu_{A'})$$

is a morphism $\lambda: TA \to TA'$ such that

$$(2.17) \qquad \lambda\mu_A = \mu_{A'}T(\lambda) \quad .$$

The covariant functor

$$\Phi_T : \mathfrak{A} \longrightarrow \mathfrak{A}_T \tag{2.18}$$

assigns to each A in $\mathfrak{A}^{\circ}$ the free T-algebra (TA, μ_A) and to each $\alpha: A \to A'$ in $\mathfrak{A}$ the morphism (2.16). The covariant functor

$$\Phi^T : \mathfrak{A} \longrightarrow \mathfrak{A}^T \tag{2.19}$$

is the composition of (2.18) with the embedding functor $\mathfrak{A}_T \longrightarrow \mathfrak{A}^T$. The forgetful functor

$$\Psi^T : \mathfrak{A}^T \longrightarrow \mathfrak{A} \tag{2.20}$$

assigns to each T-algebra (A, γ) its underlying object A and to each $\mathfrak{A}^T$-morphism (2.14) its underlying $\mathfrak{A}$-morphism $\lambda: A \to A'$. The forgetful functor

$$\Psi_T : \mathfrak{A}_T \longrightarrow \mathfrak{A} \tag{2.21}$$

is the restriction of (2.20) to the subcategory $\mathfrak{A}_T$.

2.9. Theorem. Let (T, η, μ) be any monad. Then the functor (2.19) is a left adjoint of (2.20) and the monad determined by this pair of adjoint functors is just the given monad T.
The functor (2.18) is a left adjoint of (2.21) and these functors also determine the same monad T.

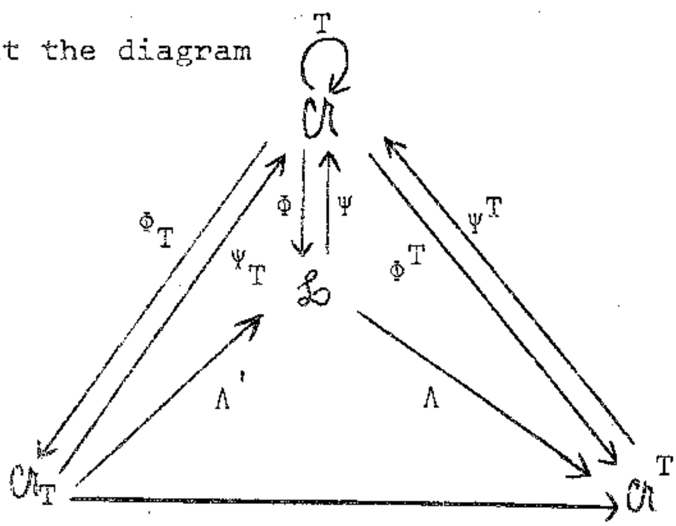
Furthermore, if $\Phi: \mathfrak{A} \longrightarrow \mathfrak{B}$ and $\Psi: \mathfrak{B} \longrightarrow \mathfrak{A}$ is any pair of adjoint functors determining the monad T, then there are unique coviant functors

(2.22) $$\Lambda: \mathfrak{B} \longrightarrow \mathfrak{A}^T$$

and

(2.23) $$\Lambda': \mathfrak{A}_T \longrightarrow \mathfrak{B}$$

such that the diagram



is commutative in the sense that

(2.24) $$\Psi_T \Phi_T = T \quad , \quad \Psi^T \Phi^T = T \quad ,$$

(2.25) $$\Lambda' \Phi_T = \Phi \quad , \quad \Psi \Lambda' = \Psi_T \quad ,$$

(2.26) $$\Lambda \Phi = \Phi^T \quad , \quad \Psi^T \Lambda = \Psi \quad ,$$

and $\Lambda \Lambda'$ is the identical embedding of $\mathfrak{A}_T$ into $\mathfrak{A}^T$.

Proof. The identities (2.24) follow immediately from the definitions of the functors (2.18) - (2.21).

Define an $\mathfrak{A}$-morphism $\eta^T_A : A \to \Psi^T\Phi^T A$ as $\eta^T_A = \eta_A$. It is clear that η^T is a natural transformation from $\iota_{\mathfrak{A}}$ to $\Psi^T\Phi^T$. If (A,γ) is any T-algebra, then $\gamma: TA \longrightarrow A$ is the underlying $\mathfrak{A}$-morphism of an $\mathfrak{A}^T$-morphism

$$(2.28) \qquad \rho^T_{(A,\gamma)} : \Phi^T\Psi^T(A,\gamma) \longrightarrow (A,\gamma) \quad ;$$

indeed, $\Phi^T\Psi^T(A,\gamma) = (TA,\mu_A)$ and if we substitute $\gamma \longmapsto \mu_A$, $\gamma' \longmapsto \gamma$, $\lambda \longmapsto \gamma$ in (2.15) we get (2.12). The naturality of

$$(2.29) \qquad \rho^T : \Phi^T\Psi^T \longrightarrow \iota_{\mathfrak{A}^T}$$

follows from (2.15) and from the definitions of Ψ^T and Φ^T. These two natural transformations satisfy the conditons

$$\rho^T_{\Phi^T A}\, \Phi^T(\eta^T_A) = \iota_{\Phi^T A} \quad \text{and} \quad \Psi^T(\rho^T_{(A,\gamma)})\eta^T_{\Psi^T(A,\gamma)} = \iota_{\Psi^T(A,\gamma)} ,$$

which are, in fact, the conditions $\mu_A T(\eta_A) = \iota_{TA}$ and $\gamma\eta_A = \iota_A$, guaranteed by (2.4) and (2.13). Thus, Φ^T is a left adjoint of Ψ^T, and η^T and ρ^T are the corresponding natural transformations. Since $\mathfrak{A}_T$ is a full subcategory of $\mathfrak{A}^T$ containing the range of the functor Φ^T, the same identities show that Φ_T is a left adjoint of Ψ_T.

Now, suppose that $\Phi: \mathcal{A} \to \mathcal{B}$ is a left adjoint of $\Psi: \mathcal{B} \to \mathcal{A}$ and this pair of functors determine the monad T. We have to prove the existence and the uniqueness of the functors (2.22) and (2.23) satisfying (2.25) and (2.26). We begin with the uniqueness of Λ.

Let $\Lambda: \mathcal{B} \to \mathcal{A}^T$ be any covariant functor satisfying (2.26). Let B be any object in $\mathcal{B}^o$. Then $\Lambda(B)$ must be of the form (A_B, γ_B) with $\gamma_B: TA_B \to A_B$ satisfying (2.12) and (2.13). By (2.26),

$$\Psi(B) = \Psi^T \Lambda(B) = \Psi^T(A_B, \gamma_B) = A_B$$

and hence $A_B = \Psi(B)$. We shall now compute γ_B. The $\mathcal{A}^T$-morphism

$$\rho^T_{(\Psi(B), \gamma_B)} : \Phi^T \Psi^T(\Psi(B), \gamma_B) \longrightarrow (\Psi(B), \gamma_B)$$

(obtained from (2.28) by letting $A = \Psi(B)$, $\gamma = \gamma_B$) satisfies the following equations

(2.30) $$\gamma_B = \Psi^T(\rho^T_{(\Psi B, \gamma_B)}) = \Psi^T(\rho^T_{\Lambda B}) .$$

Naturality of the canonical transformation $\rho: \Phi\Psi \longrightarrow \iota$ (the counit of the adjoint pair Φ, Ψ) implies that the diagram

(2.31) $$\begin{array}{ccc} \Phi\Psi\Phi\Psi B & \xrightarrow{\rho_{\Phi\Psi B}} & \Phi\Psi B \\ {\scriptstyle \Phi\Psi\rho_B}\downarrow & & \downarrow{\scriptstyle \rho_B} \\ \Phi\Psi B & \xrightarrow[\rho_B]{} & B \end{array}$$

is commutative. Applying the functor Λ we get the commutative diagram

(2.32)
$$\begin{array}{ccc}
\Lambda\Phi\Psi\Phi\Psi B & \xrightarrow{\Lambda\rho_{\Phi\Psi B}} & \Lambda\Phi\Psi B \\
{\scriptstyle \Lambda\Phi\Psi\rho_B}\downarrow & & \downarrow{\scriptstyle \Lambda\rho_B} \\
\Lambda\Phi\Psi B & \xrightarrow[\Lambda\rho_B]{} & \Lambda B
\end{array}$$

In turn, the naturality of (2.28) gives rise to a similar commutative diagram

(2.33)
$$\begin{array}{ccc}
\Phi^T\Psi^T\Lambda\Phi\Psi B & \xrightarrow{\rho^T_{\Lambda\Phi\Psi B}} & \Lambda\Phi\Psi B \\
{\scriptstyle \Phi^T\Psi^T\Lambda\rho_B}\downarrow & & \downarrow{\scriptstyle \Lambda\rho_B} \\
\Phi^T\Psi^T\Lambda B & \xrightarrow[\rho^T_{\Lambda B}]{} & \Lambda B
\end{array}$$

From (2.26) it follows that $\Phi^T\Psi^T\Lambda = \Phi^T\Psi = \Lambda\Phi\Psi$. Therefore the objects and vertical morphisms of the diagrams (2.32) and (2.33) are the same. Moreover, for each A in $\mathfrak{A}^{\circ}$, by (2.9) and (2.26),

$$\Psi^T\rho^T_{\Phi^T A} = \Psi^T\rho^T_{(TA,\mu_A)} = \mu_A = \Psi(\rho_{\Phi A}) = \Psi^T\Lambda(\rho_{\Phi A}) \quad .$$

The functor Ψ^T is obviously faithful. Hence $\rho^T_{\Phi^T A} = \Lambda(\rho_{\Phi A})$. Substituting $A = \Psi B$ we get

$$\Lambda\rho_{\Phi\Psi B} = \rho^T_{\Phi^T\Psi B} = \rho^T_{\Lambda\Phi\Psi B} \quad .$$

This means that the upper horizontal morphisms in (2.32) and (2.33) are also identical. Thus,

$$\Lambda\varrho_B\Lambda\Phi\Psi\varrho_B\ \Lambda\varrho_B\Lambda\varrho_{\Phi\Psi B}=\Lambda\varrho_B\rho^T_{\Lambda\Phi\Psi B}=\varrho^T_{\Lambda B}\Phi^T\Psi^T\Lambda\varrho_B=\varrho^T_{\Lambda B}\Lambda\Phi\Psi\varrho_B \quad .$$

Since $\Psi\varrho_B\eta_{\Psi B} = \iota_{\Psi B}$, the morphism $\Psi\varrho_B$ is a retraction and $\Lambda\Phi\Psi\varrho_B$ is also a retraction. Consequently, $\Lambda\Phi\Psi\rho_B$ may be cancelled on the right and we get

$$\Lambda\varrho_B = \varrho^T_{\Lambda B} \quad ;$$

Therefore the equality (2.30) may be written as $\gamma_B=\Psi^T\Lambda\rho_B=\Psi\varrho_B$ and

(2.34) $$\bigwedge_{B\in\mathcal{B}^o} \Lambda(B) = (\Psi(B),\Psi(\rho_B)) \quad .$$

Now, from the condition $\Psi^T\Lambda(\beta) = \Psi(\beta)$ and the definition of Ψ^T if follows that for every $\beta\colon B \longrightarrow B'$ in $\mathcal{B}$ the morphism $\Psi(\beta)$ is the underlying $\mathfrak{A}$-morphism of the $\mathfrak{A}^T$-morphism

(2.35) $$\Lambda(\beta)\colon (\Psi(\beta),\Psi(\rho_B)) \longrightarrow (\Psi(B'),\Psi(\rho_{B'})) \quad .$$

We have shown that if Λ is a functor satisfying (2.26), it must of the form described above; therefore there is at most one such functor.

The last paragraph gives us a method of defining the desired functor Λ . The morphism $\Psi(\rho_B)$ is a T-algebra

structure on ΨB because $\Psi(\rho_B)$ is an $\mathcal{O}\!\mathcal{l}$-morphism from $T(\Psi B)$ to ΨB and the conditions (2.12) and (2.13) follow by applying Ψ to the diagram (2.31) and from (2.11).
If we apply the functor Ψ to the naturality condition for ρ we get an equality which is just the condition (2.15) in the case where $\beta: B \to B'$, $\lambda = \Lambda(\beta)$, $\gamma = \Psi(\rho_B)$ and $\gamma' = \Psi(\rho_{B'})$. This means that (2.35) is a morphism in $\mathcal{O}\!\mathcal{l}^T$
It is obvious that Λ is a covariant functor from $\mathcal{L}$ to $\mathcal{O}\!\mathcal{l}^T$ satisfying (2.26)

We shall now deal with the functor $\Lambda': \mathcal{O}\!\mathcal{l}_T \to \mathcal{L}$ and again begin with the uniqueness. Suppose that Λ' is a functor satisfying (2.25). Since $\Lambda'\Phi_T = \Phi$, for any free T-algebra (TA,μ_A) we get

$$(2.36) \qquad \Lambda'(TA,\mu_A) = \Lambda'\Phi_T(A) = \Phi(A) \quad .$$

Let $\lambda: TA \to TA'$ be the underlying $\mathcal{O}\!\mathcal{l}$-morphism of any $\mathcal{O}\!\mathcal{l}_T$-morphism

$$(2.37) \qquad \lambda^{\nabla}: (TA,\mu_A) \longrightarrow (TA',\mu_{A'}) \quad .$$

Let $\beta = \Lambda'(\lambda^{\nabla})$. By (2.36), β is a morphism from ΦA to $\Phi A'$. Moreover,

$$\Psi(\beta) = \Psi\Lambda'(\lambda^{\nabla}) = \Psi_T(\lambda^{\nabla}) = \lambda \quad .$$

The naturality of ρ and (2.11) imply that

(2.38) $$\beta = \beta\rho_{\Phi A}\,\Phi(\eta_A) = \rho_{\Phi A'}\,\Phi\Psi(\beta)\Phi(\eta_A) = \rho_{\Phi A'}\Phi(\lambda\eta_A) \quad .$$

Thus, the functor Λ' must satisfy (2.36) and (2.38).

Conversely, it is clear that the conditions (2.36) and (2.38) define a functor Λ' satisfying (2.25).

Finally, we shall show that $\Lambda\Lambda'$ is the inclusion functor $\mathfrak{A}_T \to \mathfrak{A}^T$. By (2.36) for every $\mathfrak{A}_T$-object (TA,μ_A)

$$\Lambda\Lambda'(TA,\mu_A) = \Lambda\Lambda'\Phi_T(A) = \Lambda\Phi(A) = \Phi^T(A) = (TA,\mu_A).$$

If λ^∇ is any $\mathfrak{A}_T$-morphism (2.37), then, in virtue of (2.26), (2.38), (2.15), (2.4),

$$\Psi^T\big(\Lambda\Lambda'(\lambda^\nabla)\big) = \Psi^T\Lambda(\rho_{\Phi A'}\Phi(\lambda\eta_A)) = \Psi(\rho_{\Phi A'})\Psi\Phi(\lambda\eta_A)$$

$$= \mu_{A'}T(\lambda)T(\eta_A) = \lambda\mu_A T(\eta_A) = \lambda\iota_{TA} = \lambda = \Psi^T(\lambda^\nabla) \quad .$$

Since the functor Ψ^T is faithful, we get $\Lambda\Lambda'(\lambda^\nabla) = \lambda^\nabla$. ∎

2.10. DEFINITIONS. If $\Phi: \mathfrak{A} \to \mathfrak{B}$ is a left adjoint of $\Psi: \mathfrak{B} \to \mathfrak{A}$, then the functor (2.22) is the <u>Eilenberg-Moore comparison functor</u> , or shortly, a <u>comparison functor</u>. The functor (2.23) is the <u>Kleisli comparison functor</u>.

A functor $\Psi: \mathfrak{B} \to \mathfrak{A}$ is <u>monadic</u> (or <u>tripleable</u>) iff it has a left adjoint $\Phi: \mathfrak{A} \to \mathfrak{B}$ and the corresponding comparison functor $\Lambda: \mathfrak{B} \to \mathfrak{A}^T$ is an isomorphism. Thus, $\Psi: \mathfrak{B} \to \mathfrak{A}$ is monadic iff $\mathfrak{B}$ is isomorphic to the category of T-algebras for some monad T and, under this isomorphism, Ψ becomes the forgetful functor $\Psi^T: \mathfrak{A}^T \to \mathfrak{A}$.

A functor $\Psi: \mathcal{B} \to \mathfrak{A}$ is quasi-monadic iff it has a left adjoint $\Phi: \mathfrak{A} \to \mathcal{B}$ and the corresponding comparison functor $\Lambda: \mathcal{B} \to \mathfrak{A}^T$ is a quasi-isomorphism (i.e., an equivalence) of categories.

A category $\mathcal{B}$ is monadic [quasi-monadic] over $\mathfrak{A}$ iff there exists a monadic [quasi-monadic] functor $\Psi: \mathcal{B} \to \mathfrak{A}$.

2.11. Exercises. (A) Let $\mathfrak{A}$ be an ordered set regarded as a category (#9.2.3). (a) Show that for a given functor $T: \mathfrak{A} \to \mathfrak{A}$ there exist natural transformations η and μ such that (T,η,μ) is a monad if and only if T is a closure operation on $\mathfrak{A}$, i.e., an increasing function satisfying $A \leq T(A)$ and $T(TA) = T(A)$. (b) Show that T-algebras may be identified with closed elements of $\mathfrak{A}$, i.e., such that $T(A) = A$.

(B) Generalize the example 2.5 considering topological spaces rather than sets; what should be assumed about the continuity of the semigroup operation?

(C) Which functors in Theorem 2.9 are faithful for any adjoint pair Φ, Ψ ?

(D) Let (T,η,μ) be a monad in $\mathfrak{A}$. Construct a category $\mathfrak{M}$ in which objects are adjunctions (Φ,Ψ,η,ρ) , with fixed $\mathfrak{A}$ and $\mathcal{B}$ varying, such that the Eilenberg-Moore adjunction (Φ^T,Ψ^T) is terminal and the Kleisli adjunction (Φ_T,Ψ_T) is initial in $\mathfrak{M}$.

2.12. Notes. The notion of a comonad is dual to that of a monad; a comonad consists of a covariant (!) functor S from $\mathfrak{B}$ to $\mathfrak{B}$ together with two natural transformations, one from S to S^2 and the other from S to the identity, subject to axioms dual to (2.3) and (2.4) . If $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a left adjoint of $\Psi : \mathfrak{B} \to \mathfrak{A}$, then $S = \Phi\Psi$ together with suitable natural transformations is a comonad.

Comonads were introduced earlier than monads; they were explicitly defined by R. Godement [1958] under the name "standard construction". P.J. Huber [1961] proved that any pair of adjoint functors generates a comonad. The question whether any comonad can be obtained in this way was solved by S. Eilenberg and J.C. Moore [1965] and H. Kleisli [1965] .

The material above is based mostly on MacLane [1971], Chapter VI; Exercise (D) is from Kleisli [1973].

Monadicity of $\Psi : \mathfrak{B} \to \mathfrak{A}$ is a measure of "algebraicity" of $\mathfrak{B}$ over $\mathfrak{A}$; at any rate it says that $\mathfrak{B}$ can be defined in terms of certain data in $\mathfrak{A}$. For more information in this line, see Linton [1966], [1969a], Schubert[1972], and papers quoted there.

§3. Vector spaces and countably absolutely convex subsets of Banach spaces. The core of this subsection is a discussion of T-algebras determined by the free-Banach-space monad. As an introduction to this topic we shall discuss a simple and well-known example: vector spaces over a field $\mathbb{F}$, where either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

3.1. If X is a set, $V(X)$ will denote the vector space freely generated by X , i.e., the set of all formal linear combinations of elements of X . By formal linear combination

$$\sum_{i=1}^{n}{}^{*} s_i x_i \tag{3.1}$$

(where $s_1, \ldots, s_n \in \mathbb{F}$ and $x_1, \ldots, x_n \in X$) we mean the function $f\colon X \to \mathbb{F}$ defined as

$$f = \sum_{i=1}^{n} s_i \delta_{x_i}^{(X)} \tag{3.2}$$

where $\delta_x^{(X)}(y) = 0$ for y in $X \setminus \{x\}$ and $\delta_x^{(X)}(x) = 1$. Let us note that if $x_i \neq x_j$ for $i \neq j$, then $f(x_i) = s_i$ for $i = 1, \ldots, n$ and $f(x) = 0$ for all other x in X ; if $x_1, \ldots, x_n$ are not different from each other, the numbers s_i should be added accordingly.

V(X) may be described as the space of all functions $f: X \to \mathbb{F}$ such that

$$\{x: f(x) = 0\} \quad \text{is finite;}$$

thus, $V(X) = C_{00}(X)$ where X is regarded as a locally compact space with discrete topology (#7.2.5). The asterisk in (3.1) indicates thet the summation does not refer to a group structure on X ; in fact, the formal sum (3.1) is not an element of X . On the other hand, the sum (3.2) refers to the natural vector-space structure on $V(X)$. The function $\delta_x^{(X)}$ is called the x-th <u>unit vector</u> of $V(X)$. The map

(3.3) $$\delta^{(X)}: \quad X \to V(X)$$

assigning to each x in X the function $\delta_x^{(X)}$ has the following significant property: for any vector space W and any map $\xi: X \to W$ there is a unique linear map $\vartheta: V(X) \to W$ such that the diagram

(3.4) $$\begin{array}{ccc} X & \xrightarrow{\delta^{(X)}} & V(X) \\ & \searrow^{\xi} & \downarrow{\scriptstyle \vartheta} \\ & & W \end{array}$$

is commutative. This unique-factorization property makes V a functor from Ens to Vect; specifically, if $\phi\colon X \to Y$ is any map, then $V(\phi)$ is the unique linear map rendering commutative the diagram

(3.5)
$$\begin{array}{ccc} X & \xrightarrow{\ \delta^{(X)}\ } & V(X) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle V(\phi)} \\ Y & \xrightarrow[\ \delta^{Y}\ ]{} & V(Y) \end{array}$$

It is clear that

(3.6)
$$V\phi\,.\,\Big(\sum_{i=1}^{n}{}^{*}\, s_i x_i\Big) = \sum_{i=1}^{n}{}^{*}\, s_i \phi(x_i) \;;$$

this means that $V\phi.f = g$ $(f \in V(X)\ ,\ g \in V(Y))$ iff

(3.7)
$$g(y) = \sum_{x \in \phi^{\leftarrow}(y)} f(x) \;;$$

The summation in (3.7) is always finite; if $\phi^{\leftarrow}(y)$ is empty, then the corresponding sum is 0.

Thus, the functor $V\colon$ Ens $\to$ Vect is a left adjoint of the forgetful functor $U\colon$ Vect $\to$ Ens. We shall consider the monad (T,δ,μ) determined by the pair

(3.8)
$$\text{Ens} \underset{U}{\overset{V}{\rightleftarrows}} \text{Vect}\ .$$

Thus, T: Ens $\to$ Ens is defined as $T = UV$ and $T(X)$ is the set of all formal linear combinations (3.1) without any vector-space structure on it. The unit of the adjunction (3.8) is given by the maps (3.3). It is easy to verify that the counit

$$\rho_W: \ V(UW) \to W$$

sends formal linear combinations to actual linear combinations in the vector space W, i.e.,

(3.9) $$\rho_W(\textstyle\sum^* s_i w_i) = \sum s_i w_i .$$

Consequently, the map μ_X: $T^2(X) \to T(X)$, defined as $\mu_X = U(\rho_{V(X)})$, also sends formal linear combinations in $UVUV(X)$ to actual linear combinations in $UV(X)$.

A T-algebra is a pair (X,γ), where X is a set and

(3.10) $$\gamma: \ UV(X) \to X$$

is a map such that

(3.11) $$\gamma\delta^{(X)} = 1_X$$

and

(3.12) $$\gamma V(\gamma) = \gamma \mu_X .$$

Any vector space W may be regarded as a T-algebra; in this case γ is the map (3.9) regarded as a map from the set of formal linear combinations $\Sigma^* s_i w_i$ to the underlying set of W. Technically, in virtue of (2.34) and (2.35), the comparison functor

(3.13) $$\Lambda: \quad \text{Vect} \to \text{Ens}^T$$

satisfies $\Lambda(W) = (U(W), U(\rho_W))$ and if $\beta: \; W \to W'$ is a morphism in Vect, then $\Lambda(\beta)$ is the same map regarded as a T-algebra morphism.

3.2. Theorem. The functor (3.13) is an isomorphism of categories.

Before proving the theorem we shall first formulate it in the language of vector spaces. We shall simplify the notation by omitting the letter U in most places; in other words, we will follow the common practice of using the same symbol for a vector space and for its underlying set.

The standard approach is to define a vector space as a pair $(X, \mathfrak{S})$; $\mathfrak{S}$ is here the vector-space structure understood as $\mathfrak{S} = (\mathfrak{S}_1, \mathfrak{S}_2)$ where $\mathfrak{S}_1$ is the addition of elements (a certain function from $X \times X$ to X) and $\mathfrak{S}_2$ is the multiplication by scalars (a function from $\mathbb{F} \times X$ to X), $\mathfrak{S}_1$ and $\mathfrak{S}_2$ being subject to well-known axioms.

The theorem gives another, equivalent definition of a vector space as a pair (X, γ), where X is the underlying set and γ is an axiomatically given structure of linear combinations. Thus, instead of dealing with sums $x_1 + x_2$ and products sx we give at once all linear combinations as

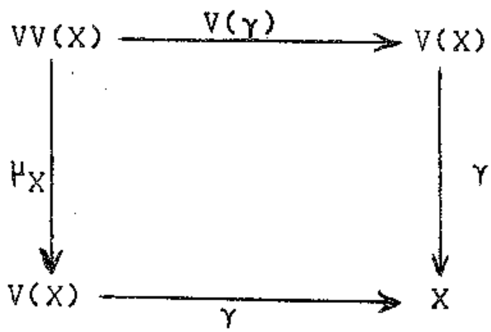
$$\sum s_i x_i = \gamma\left(\sum^* s_i x_i\right) . \tag{3.14}$$

The formal sum (3.1) has a meaning for any set X; therefore, whenever a map (3.10) is given, linear combinations are well defined by (3.14). The point is that the axioms (3.11) and (3.12) are strong enough to guarantee that the linear combinations (3.14) are, in fact, determined by a vector-space structure on X.

Let us examine these axioms. Condition (3.11) has a clear meaning:

$$1 \cdot x = x . \tag{3.15}$$

Indeed, in virtue of (3.14) , $\mathbb{1}\cdot x = \gamma(\delta_x^{(X)}) = \iota_X(x) = x$. We are to show that all other axioms of a vector space (associativity and commutativity of addition, distributivity etc.) can be derived from the assumption that the diagram



is commutative. A generic element of VV(X) can be written as

$$(3.16)\qquad \sum_{j=1}^{m}{}^{*}\, t_j \sum_{i=1}^{n}{}^{*}\, s_{ij}x_{ij} \quad (t_j\in\mathbb{F},\ s_{ij}\in\mathbb{F},\ x_{ij}\in X)$$

which means the same as

$$\sum_j{}^{*}\, t_j \sum_i s_{ij}\delta_{x_{ij}}^{(X)}$$

and the same as

$$\sum_j t_j \delta^{(V(X))}_{\sum_i s_{ij}\delta_{x_{ij}}^{(X)}}$$

The element (3.16) is a function defined on the set $V(X)$ of functions from X to $\mathbb{F}$; the asterisks simplify the notation considerably but we should be aware of the actual meaning of (3.16) . Now, μ_X sends (3.16) to

$$\sum_j t_j \sum_i^* s_{ij} x_{ij} = \sum_j t_j \sum_i s_{ij} \delta_{x_{ij}}^{(X)}$$

(3.17)

$$= \sum_i \sum_j t_j s_{ij} \delta_{x_{ij}}^{(X)}$$

$$= \sum_{i,j}^* t_j s_{ij} x_{ij}$$

The above change of an iterated sum to a double sum is legitimate because it is performed in the vector space $V(X)$. In turn, in virtue of (3.14) γ sends (3.17) to

$$\sum_{i,j} t_j s_{ij} x_{ij} .$$

On the other hand, by (3.6), $V(\gamma)$ sends (3.16) to

(3.18) $$\sum_j^* t_j \sum_i s_{ij} x_{ij} = \sum_j t_j \delta_{\sum_i s_{ij} x_{ij}}^{(X)}$$

(Note that (3.18) is not equal to e.g. $\sum_j^* \sum_i t_j s_{ij} x_{ij}$) .

Finally γ sends (3.18) to

$$\sum_j t_j \sum_i s_{ij} x_{ij} .$$

Thus the condition (3.12) means that

$$(3.19) \qquad \sum_j t_j \sum_i s_{ij} x_{ij} = \sum_{i,j} t_j s_{ij} x_{ij} .$$

It can easily be verified that from (3.19) and (3.15) one can derive all standard axioms of a vector space. For instance, substituting $m=1$, $n=2$, $s_{11}=s_{21}=1$ we get

$$t(x_1+x_2) = tx_1 + tx_2 ;$$

substituting $m=2$, $n=1$, $s_{11}=1$ we get

$$(t_1+t_2)x = t_1 x + t_2 x .$$

This concludes the proof of the theorem. ∎

3.3. The free-Banach-space monad. If B, B' are Banach spaces over the field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, and $\beta : B \to B'$ is a linear contraction, let

$$O\beta : OB \to OB'$$

be the restriction of β to the closed unit ball OB. If X is a set, let $\ell(X)$ be the free Banach space generated by X, i.e., the space of all functions $f:X \to \mathbb{F}$ such that

$$(3.20) \qquad \|f\| = \sum_{x \in X} |f(x)| < \infty$$

(this means that the set $\{x \in X : f(x) \neq 0\}$ is countable and the series is absolutely convergent; in other words, $\ell(X)$ is the space of functions integrable on X with respect to the counting measure). The Kronecker $\delta(x,y)$ regarded as a function of y (with x fixed) determines the canonical map

$$(3.21) \qquad \delta^{(X)}: X \to \ell(X) .$$

We shall write $\delta^{(X)}(x,y)$ rather than $\delta(x,y)$ if the set X is to be stressed; $\delta(x)$ will mean the function $\delta(x,?)$ of the second variable, i.e., the x-th unit vector in $\ell(X)$. Any function f in $\ell(S)$ can be uniquely represented as

$$(3.22) \qquad f = \sum_{x \in X} f(x)\delta^{(X)}(x),$$

the series being absolutely convergent in the norm (3.20). If $\varphi: X \to Y$ is any map, then

$$\ell(\varphi): \ell(X) \to \ell(Y)$$

will denote the induced linear contraction

$$\ell(\varphi)\left[\sum_{x\in X} f(x)\delta^{(X)}(x)\right] \underset{\mathrm{df}}{=} \sum_{x\in X} f(x)\,\delta^{(Y)}(\varphi(x))$$

$$= \sum_{y\in\varphi(X)}\left[\sum_{x\in\varphi^{\leftarrow}(y)} f(x)\right]\delta^{(Y)}(y)$$

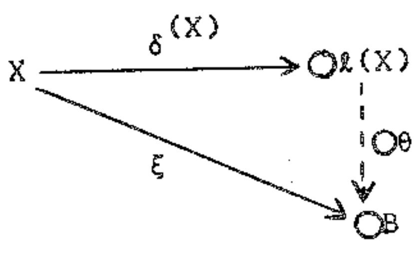
It is well known (cf. #11.4.3 and #12.1.2) that the covariant functor

(3.23) $$\ell:\mathrm{Ens}\to\mathrm{Ban}_1$$

is a left adjoint of the closed-unit-ball functor

(3.24) $$\bigcirc:\mathrm{Ban}_1\to\mathrm{Ens}$$

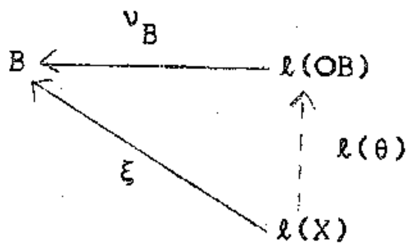
and the maps (3.21) yield the unit of this adjunction (i.e., the front adjunction, the first canonical natural transformation). In non-categorical terms the last sentence means that for any Banach space B and any map $\xi:X\to\bigcirc B$ there is a unique linear contraction $\theta:\ell(X)\to B$ such that the diagram



is commutative; of course, $\theta(\Sigma f(x)\delta(x)) = \Sigma f(x)\xi(x)$.
The counit of this adjunction is the canonical map

$$\nu_B : \ell(\mathcal{O}B) \to B \tag{3.25}$$

defined as $\nu_B(\sum_{b\in\mathcal{O}B} f(b)\delta^{(\mathcal{O}B)}(b)) = \sum_{b\in\mathcal{O}B} f(b)\,b$. It is convenient to think of the sum (3.22)as the formal sum $\Sigma^* f(x)\cdot x$. Then the map(3.25) assigns to each formal sum $\Sigma^* f(b)\cdot b$ in $\ell(\mathcal{O}B)$ the actual sum $\Sigma f(b)b$ in B. The natural transformation (3.25)is uniquely characterized by the condition: for every set X and every linear contraction $\xi:\ell(X) \to B$ there is a unique map $\theta : X \to \mathcal{O}B$ such that the diagram



is commutative; here $\theta(x) = \xi(\delta(x))$ for x in X .

The <u>free-Banach-space monad</u> is the triple

$$T = (\mathcal{O}\ell, \delta, \mu) \tag{3.26}$$

where $\mathcal{O}\ell : \mathrm{Ens} \to \mathrm{Ens}$ is the composition of (3.23)with (3.24)

δ is the natural transformation (3.21) and

$$\mu_X : O\ell O\ell(X) \to O\ell(X)$$

is the natural transformation defined as $\mu_X = O(\nu_{\ell(X)})$;

A T-algebra or an Eilenberg-Moore algebra of the monad (3.26)is any pair (X,γ) where X is a set (the underlying set of the algebra) and

(3.27) $$\gamma : O\ell(X) \to X$$

is a map (a T-algebra structure on X) such that

(3.28) $\gamma\delta^{(X)} = \iota_X$, i.e., $\gamma(\delta(x)) = x$ for x in X

and

(3.29) $$\gamma\mu_X = \gamma O\ell(\gamma) .$$

This last condition means that the diagram

$$\begin{array}{ccc} O\ell O\ell(X) & \xrightarrow{O\ell(\gamma)} & O\ell(X) \\ \Big\downarrow{\scriptstyle \mu_X} & & \Big\downarrow{\scriptstyle \gamma} \\ O\ell(X) & \xrightarrow[\gamma]{} & X \end{array}$$

is commutative.

3.4. Let $\mathcal{B}_0$ be the category in which an object means a pair (B,X), where B is a Banach space and X is a distinguished subset of B satisfying the following conditions:

$$\bigcirc^i B \subset X \subset \bigcirc B, \tag{3.30}$$

where $\bigcirc B = \{b \in B: \|b\| \leq 1\}$ and $\bigcirc^i B = \text{int} \bigcirc B = \{b \in B: \|b\| < 1\}$,

$$\forall_{x_0, x_1, \ldots \in X} \; \forall_{s_0, s_1, \ldots \in \mathbb{F}} \; \sum_{n=0}^{\infty} |s_n| \leq 1 \Rightarrow \sum_{n=0}^{\infty} s_n x_n \in X . \tag{3.31}$$

Any X satisfying (3.31) will be called <u>countably absolutely convex</u>; examples of such sets are: $\bigcirc B$, $\bigcirc^i B$, $\bigcirc B$ without some exposed points, etc.

A morphism in $\mathcal{B}_0$

$$\beta: \quad (B,X) \to (B',X') \tag{3.32}$$

means a linear operator $\beta: \; B \to B'$ satisfying $\beta(X) \subset X'$ (such an operator must be a contraction).

It is clear that if B is a Banach space and X is a subset of B satisfying (3.30) and (3.31) then the map γ which assigns to each f in $O\ell(X)$ the element

$$\gamma(f) = \sum_{x \in X} f(x)\, x \tag{3.33}$$

is a T-algebra structure on X; in virtue of (3.31) the sum in (3.33) belongs to X.

According to a general definition, a T-algebra morphism from X to a countably absolutely convex set $X' \subset B'$ is any map $\phi\colon X \to X'$ such that the diagram

$$\begin{array}{ccc} O\ell(X) & \xrightarrow{\;O\ell(\phi)\;} & O\ell(X') \\ \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \gamma'} \\ X & \xrightarrow[\;\phi\;]{} & X' \end{array}$$

is commutative, where γ' is the structure map on X'. If $\sum |s_k| \leq 1$ and $x_k \in X$, then

$$\phi(\sum s_k y_k) = \phi\gamma(\sum s_k \delta^{(X)}(x_k)) = \gamma'[O\ell\phi(\sum s_k \delta^{(X)}(x_k))]$$

$$= \gamma'[\sum s_k \delta^{(X')}(\phi(x_k))] = \sum s_k \phi(x_k) .$$

Thus there is a linear contraction $\beta\colon B \to B'$ such that $\beta(x) = \phi(x)$ for x in X and $\beta(X) \subset X'$. Conversely, any such β obviously determines a T-algebra morphism $\phi\colon X \to X'$.

3.5 Proposition. The forgetful functor

(3.34) $$U\colon \mathcal{L}_0 \to \mathrm{Ban}_1$$

which assigns to each $\mathcal{L}_0$-object (B,X) its underlying Banach space B and to each $\mathcal{L}_0$-morphism (3.32) the same β regarded as a morphism in Ban_1 has both a left adjoint and a right adjoint.

A left adjoint of (3.34) is the functor $\Phi\colon \mathrm{Ban}_1 \to \mathcal{L}_0$ which assigns to each Banach space B the pair $(B, O^1 B)$.

A right adjoint of (3.34) is the functor $\Psi\colon \mathrm{Ban}_1 \to \mathcal{L}_0$ which assigns to each B the pair (B, OB).

Proof. The proof is straightforward; for the first mentioned adjunction the unit is the identity ι_B regarded as a Ban_1-morphism from B to $U(\Phi(B)) = B$ while the counit is ι_B regarded as a $\mathcal{L}_0$-morphism from $\Phi U(B,X) = (B, O^1 B)$ to (B,X). For the second adjunction the unit is again ι_B, now regarded as a $\mathcal{L}_0$-morphism from (B,X) to $\Psi U(B,X) = (B, OB)$, while the counit is ι_B regarded as a Ban_1-morphism from $U\Psi(B) = B$ to B. ∎

3.6. Corollary. The functor (3.34) preserves products, coproducts, equalizers and coequalizers. ∎

3.7. Proposition. Let $(B_t, X_t)_{t\in T}$ be any family of $\mathcal{L}_0$-objects. Then the pair

$$\left(\prod_{t\in T}^{\infty} B_t, \mathop{\mathbb{P}}_{t\in T} X_t \right),$$

where $\prod^\infty B_t$ is the ℓ_∞-product, i.e., Ban_1-product, of $(B_t)_{t\in T}$ and $\mathbb{P}X_t$ is the Cartesian product of $(X_t)_{t\in T}$, is a $\mathcal{L}_0$-product of $(B_t, X_t)_{t\in T}$. Moreover, let $B = \coprod^1 B_t$ be the Ban_1-coproduct of $(B_t)_{t\in T}$. Let Y_t be the image of X_t under the canonical injection $B_t \to B$ and let X be the set of all b in B which are of the form

$$b = \sum_{t\in T} f(t) x_t \quad \text{with} \quad x_t \in Y_t, \; f \in O\ell(T).$$

Then (B,X) is a $\mathcal{L}_0$-coproduct of $(B_t,X_t)_{t\in T}$.

3.8. Proposition. Let α,β be any $\mathcal{L}_0$-morphisms from (B_1,X_1) to (B_2,X_2) . Let

$$B_0 = \{b \in B_1:\ \alpha(b) = \beta(b)\} \text{ and } X_0 = B_0 \cap X .$$

Then (B_0,X_0) together with the inclusion map $B_0 \to B_1$ is a $\mathcal{L}_0$-equalizer of α and β .

Furthermore, let A be the closed linear span of $\{\alpha(b)-\beta(b):\ b \in B_1\}$ in B_2 let π be the quotient map from B_2 onto $B_3 = B_2/A$ and let $X_3 = \pi(X_2)$. Then (B_3,X_3) together with the map π is a $\mathcal{L}_0$-coequalizer of α and β .

The proofs of Propositions 3.7 and 3.8 are straightforward and we omit them.

3.9. Let (B,X) and (B',X') be $\mathcal{L}_0$-objects. Define

$$(3.35)\qquad (B,X)\ \hat{\otimes}\ (B',X') = (B\ \hat{\otimes}\ B', X\ \hat{\otimes}\ X') ,$$

where $B \hat{\otimes} B'$ is the (projective) tensor product of Banach spaces B and B' (see, e.g., #20.1.10) and $Y = X \hat{\otimes} X'$ is the set of all y in $B \hat{\otimes} B'$ which can be represented as

$$y = \sum_{n=0}^{\infty} s_n(x_n \otimes x_n') \quad \text{with} \quad x_n \in X,\ x_n' \in X',\ \sum_{n=0}^{\infty} |s_n| \leq 1 .$$

Since $\|y\| \leq \sum |s_n|\ \|x_n\|\ \|x_n'\| \leq \sum |x_n| \leq 1$, Y is contained in $\bigcirc(B \hat{\otimes} B')$. On the other hand, if $a \in B \hat{\otimes} B'$ and $\|a\| < 1$, then a can be represented in the form

$$a = \sum_{n=0}^{\infty} b_n \otimes b_n'$$

with $0 \neq b_n \in B$, $0 \neq b_n' \in B'$, $\sum_{n=0}^{\infty} \|b_n\|\ \|b_n'\| < 1$.

There is an $\varepsilon > 0$ such that $\sum \|b_n\|\ \|b_n'\| < (1+\varepsilon)^{-2}$.
Denote

$$x_n = \frac{b_n}{\|b_n\|(1+\varepsilon)} , \quad x_n' = \frac{b_n'}{\|b_n'\|(1+\varepsilon)} , \quad s_n = \|b_n\|\ \|b_n'\|(1+\varepsilon)^2$$

Then $x_n \in \bigcirc^i B \subset X$, $x_n' \in \bigcirc^i B' \subset X'$ and $a = \sum s_n(x_n \otimes x_n') \in Y$. Thus Y contains $\bigcirc^i(B \hat{\otimes} B')$.
We have shown that (3.35) is a $\mathcal{L}_0$-object. A straightforward verification shows that if

$$(3.36) \qquad \beta: (B_1, X_1) \to (B_2, X_2) \quad \text{and} \quad \beta': (B_1', X_1') \to (B_2', X_2')$$

are any $\mathcal{L}_0$-morphisms, then the tensor product of operators

$$(3.37) \qquad \beta \hat{\otimes} \beta': \ B_1 \hat{\otimes} B_1' \to B_2 \hat{\otimes} B_2' .$$

maps $X_1 \hat{\otimes} X_1'$ into $X_2 \hat{\otimes} X_2'$ and yields a $\mathcal{L}_0$-morphism. It is clear that (3.35) and (3.37) yield a bifunctor

(3.38) $$\hat{\otimes} : \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0 .$$

For any $\mathcal{L}_0$-object (B,X) there are obvious canonical isomorphisms

(3.39) $$(\mathbb{F}, \bigcirc\mathbb{F}) \hat{\otimes} (B,X) \cong (B,X)$$

and

(3.40) $$(\mathbb{F}, \bigcirc^i\mathbb{F}) \hat{\otimes} (B,X) \cong (B, \bigcirc^i B) .$$

Now, the $\mathcal{L}_0$-object

(3.41) $$\mathrm{Hom}\big((B,X),(B',X')\big)$$

is defined as the Banach space $L(B,B')$ of all bounded linear operators $\beta\colon B \to B'$ with the set $L_1(X,X')$ of those β which are $\mathcal{L}_0$-morphisms, i.e., satisfy $\beta(X) \subset X'$. Any $\mathcal{L}_0$-morphism is a linear contraction; moreover, if $\|\beta\| < 1$, then

$$\beta(X) \subset \beta(\bigcirc B) \subset \bigcirc^i B' \subset X' .$$

Consequently, $\mathcal{O}^{1}L(B,B') \subset L_0(X,X') \subset \mathcal{O}L(B,B')$ and (3.41) is a $\mathcal{L}_0$-object.

If (3.36) are $\mathcal{L}_0$-morphisms, then

(3.42) $$\mathrm{Hom}(\beta,\beta'): \mathrm{Hom}((B_2,X_2),(B_1',X_1')) \to \mathrm{Hom}((B_1,X_1),(B_2',X_2'))$$

is defined as $\mathrm{Hom}(\beta,\beta')\phi = \beta'\phi\beta$ for $\phi: (B_2,X_2) \to (B_1',X_1')$. It is clear that $\mathrm{Hom}(\beta,\beta')$ is a linear contraction from $L(B_2,B_1')$ to $L(B_1,B_2')$ which maps $L_2(X_2,X_1')$ into $L_1(X_1,X_2')$, i.e., (3.42) is a $\mathcal{L}_0$-morphism. Thus

(3.43) $$\mathrm{Hom}: \mathcal{L}^* \times \mathcal{L} \to \mathcal{L}$$

is a bifunctor.

3.10. Proposition. There is a natural equivalence

(3.44) $$\mathrm{Hom}((B_1,X_1) \hat{\otimes} (B_2,X_2),(B_3,X_3)) \cong$$

$$\mathrm{Hom}((B_1,X_1),\mathrm{Hom}((B_2,X_2),(B_3,X_3)))$$

of functors from $\mathcal{L}_0^* \times \mathcal{L}_0^* \times \mathcal{L}_0$ to $\mathcal{L}_c$.

Proof. If $\phi: B_1 \hat{\otimes} B_2 \to B_3$ is a bounded linear operator, let $\hat{\phi}$ be the operator which assigns to each b_1 in B_1 the function

$$b_2 \mapsto \phi(b_1 \otimes b_2) \;;$$

then $\hat{\phi} \in L(B_1, L(B_2, B_3))$. This yields the classical natural equivalence

$$L(B_1 \hat{\otimes} B_2, B_3) \cong L(B_1, L(B_2, B_3)) \;.$$

It is clear that $\phi \in L_1(X_1 \hat{\otimes} X_2, X_3)$ if and only if

$$\bigvee_{x_1 \in X_1} \bigvee_{x_2 \in X_2} \phi(x_1 \otimes x_2) \in X_3 \;;$$

this, in turn, means that if $x_1 \in X_1$ then $\phi(x_1 \otimes ?)$ maps X_2 into X_3 , i.e., $\hat{\phi}$ belongs to the set $L_1(X_1, L_1(X_2, X_3))$. ∎

3.11. Corollary. $\mathcal{L}_o$ is a closed category.

(By a closed category we mean a symmetric, monoidal closed category in the sense of Eilenberg and Kelly [1966] ; see also MacLane [1971], p.180, and Dubuc [1970].)

3.12. We have studied the category $\mathcal{L}_0$ which may be regarded as a full subcategory of the category Ens^T of T-algebras for the free-Banach-space monad (3.26) . A natural question arises whether any T-algebra may be identified with an object of $\mathcal{L}_0$.

The following example (due to F.E.J. Linton) shows that there are T-algebras for which the structure of absolutely convex combinations are not induced by any vector-space structure.

Let X denote the three-element set $\{-1,0,1\}$. We shall consider two maps

$$\pi : O\mathbb{R} \to X \quad \text{and} \quad \gamma : O\ell(X) \to X$$

defined as follows:

$$(3.45) \qquad \pi(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 , \\ 0 & \text{if } x = 0 , \\ 1 & \text{if } 0 < x \leq 1 \end{cases}$$

and

$$(3.46) \qquad \gamma\left(\sum_{i=-1}^{1} s_i \delta_i^{(X)}\right) = \begin{cases} -1 & \text{if } s_0 = 0 \text{ and } s_{-1} - s_1 = 1 , \\ +1 & \text{if } s_0 = 0 \text{ and } s_1 - s_{-1} = 1 , \\ 0 & \text{otherwise} . \end{cases}$$

In other words, γ maps the formal linear combination $y = \sum_{i=-1}^{1}{}^{*} s_i \cdot i$ onto the ordinary sum $z = s_{-1} \cdot (-1) + s_0 \cdot 0 + s_1 \cdot 1$ if z belongs to X ; if $|z| < 1$, $\gamma(y)$ is assumed to be 0 .

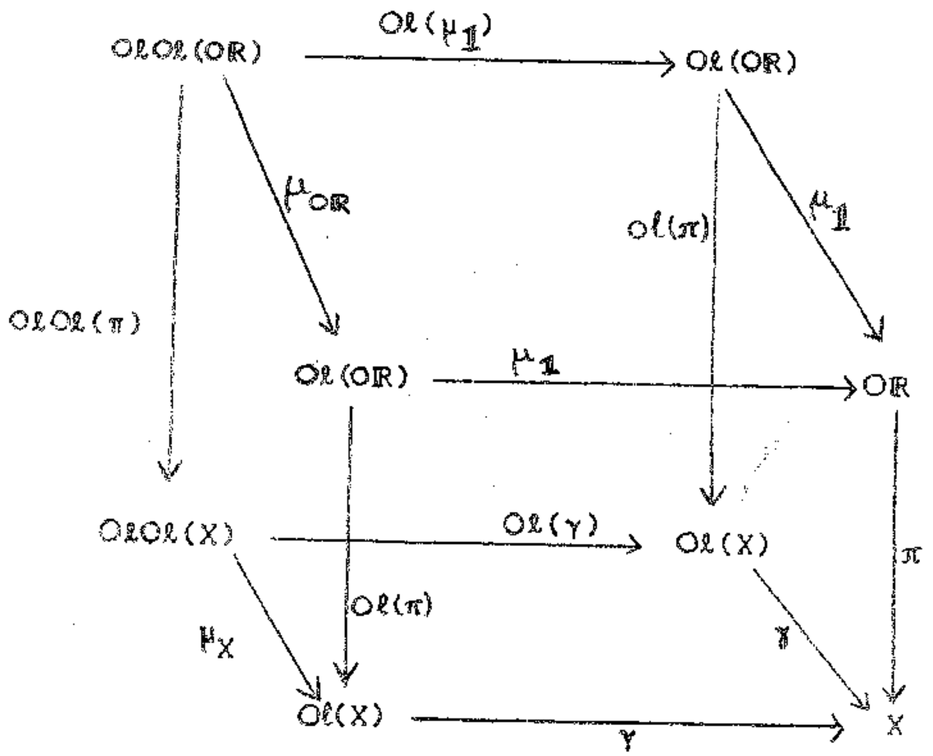
We shall show that (X, γ) is a T-algebra. Observe first that if $\mathbb{R}$ is identified with $\ell(\mathbb{1})$, where $\mathbb{1} = \{0\}$, then

$O\mathbb{R}$ is the closed interval $[-1,1]$ and

(3.47) $$\ell(O\mathbb{R}) \xrightarrow{\mu_{\mathbb{1}}} O\mathbb{R}$$

is a free T-algebra $(T(\mathbb{1}),\mu_{\mathbb{1}})$ (in the sense of Definition 2.8).

Let us consider the following diagram:



The top face of the above cube is commutative because (3.47) is a T-algebra. The four side faces are also commutative. Indeed,

(3.48) $$\pi\mu_{\mathbb{1}} = \gamma\ \ell(\pi)$$

follows by immediate computation of the value of either side at a generic element of $\mathcal{O}\ell(\mathbb{OR})$; the equality $\mathcal{O}\ell(\pi)\mathcal{O}\ell(\mu_{\mathbb{1}}) = \mathcal{O}\ell(\gamma)\mathcal{O}\ell\mathcal{O}\ell(\pi)$ follows from (3.48) by applying the functor $\mathcal{O}\ell$; moreover

$$\mu_X \mathcal{O}\ell\mathcal{O}\ell(\pi) = \mathcal{O}\ell(\pi)\mu_{\mathbb{OR}}$$

follows from naturality of (3.25) applied to the morphism $\ell(\pi)$. Thus, five sides of the cube are commutative and therefore the bottom face is commutative as well, as $\mathcal{O}\ell\mathcal{O}\ell(\pi)$ is a surjection. We have shown that the map (3.46) satisfies condition (3.29) ; condition (3.28) is immediate.

Of course, the set X cannot be made a convex subset of a real vector space. The peculiar definition (3.46) satisfies all conditions imposed on T-algebras for the monad (3.26) and yet it fails to satisfy the following one:

(3.49) if $x \in X$ & $x \neq 0$ & $s \in \mathbb{F}$ & $s \neq 0$, then $sx \neq 0$;

here 0 means the image of the zero element of $\mathcal{O}\ell(X)$ under the map γ and $sx = \gamma(s\delta_x^{(X)})$. Consequently, X cannot be embedded into any real vector space E so that γ be induced by the linear combinations in E .

3.13. We shall need some lemmas. By a <u>structure of absolutely convex combinations</u> on a set X we shall mean any function which assigns to each pair of finite sequences

$$((x_1,\dots,x_n),\ (s_1,\dots,s_n))$$

where $x_1,\dots,x_n \in X$, $s_1,\dots,s_n \in \mathbb{F}$ and $\Sigma|s_k| \le 1$, an element of X denoted by

(3.50) $$\sum_{k=1}^{n} s_k x_k \quad \text{or} \quad s_1x_1+\dots+s_nx_n ,$$

subject to (3.49) and the following conditions:

(3.51) $$s_1x_1+\dots+s_nx_n+0\cdot x = s_1x_1+\dots+s_nx_n ,$$

(3.52) $$1.x = x \quad \text{for any } x \text{ in } X,$$

(3.53) if $\pi:\{1,\dots,n\} \to \{1,\dots,n\}$ is any permutation,

then $$\sum_{k=1}^{n} s_k x_k = \sum_{k=1}^{n} s_{\pi(k)} x_{\pi(k)} ,$$

(3.54) $$\sum_k s_k\Big(\sum_m t_{km}x_{km}\Big) = \sum_{k,m} (s_k t_{km}) x_{km}$$

whenever $\sum_m |t_{km}| \le 1$ for $k = 1,\dots,n$ and $\sum_k |s_k| \le 1$.

Thus, a structure of absolutely convex combinations on X is a function

$$\bigcup_{n=1}^{\infty} [X^n \times \mathcal{O}\ell(n)] \to X$$

where $\ell(n)$ is the space of sequences $s = (s_1, \ldots, s_n)$ with $\|s\| = \Sigma |s_k|$.

3.14. By a <u>balloon</u> in a vector space B we shall mean an absolutely convex (i.e., convex balanced) absorbing subset of B. Obviously, any balloon has a natural structure of absolutely convex combinations.

By an ω-<u>balloon</u> in a normed vector space B we shall mean any countably absolutely convex balloon, i.e., a balloon X satisfying (3.31) ; if B is not complete, then (3.31) is to be understood as follows: if $\Sigma |s_n| \leq 1$, then the series $\Sigma s_n x_n$ is convergent and its sum belongs to X.

3.15. Lemma. Let X be a set with a structure of absolutely convex combinations. Then there is a vector space B with a (homogeneous) pseudo-norm $\| \;\; \|$ such that X is a balloon in B satisfying (3.30) .

This lemma is probably well known ; yet, for the reader's convenience we will outline the proof. For $n = 1$ (3.50) gives an action $(s,x) \to s \cdot x$ of the multiplicative semigroup $\mathbb{OF}$ on the set X and the condition

(3.55) $s\cdot(t\cdot x) = (st)\cdot x$ for s,t in $\mathbb{O}\mathbb{F}$, x in X

is a special case of (3.54). Let B be the set of formal products $s\cdot x$ with s in $\mathbb{F}$, x in X. Technically, B is the quotient of $X\times\mathbb{F}$ with respect to the equivalence relation $\sim$ defined as follows: if $u = \max(|s|,|t|) \neq 0$, then $(x,s)\sim(y,t)$ iff $(rs)\cdot x = (rt)\cdot y$ for some (and hence, by (3.55) for all) r such that $0 < r \leq u^{-1}$; if $s = t = 0$, the $(x,0)\sim(y,0)$. Let $\pi(x,s)$ denote the equivalence class of (x,s). The set X may be regarded as a subset of B if x is identified with $\pi(x,1)$. The action of the semigroup $\mathbb{O}\mathbb{F}$ on X is now extended to an action of the multiplicative semigroup $\mathbb{F}$ on B. Indeed, the product of t in $\mathbb{F}$ with the equivalence class $\pi(x,s)$ is defined as the equivalence class $\pi(x,ts)$; this definition does not depend on the choice of (x,s) in the equivalence class. Arbitrary linear combinations are defined as

$$\sum_{k=1}^{n} t_k\pi(x_k,s_k) = \pi\Big(\sum_{k=1}^{n} \frac{t_k s_k}{r}\, x_k, r\Big) ,$$

where $t_1,\ldots,t_n \in \mathbb{F}$, $r \geq \Sigma|t_k s_k|$. A routine inspection shows that B is a vector space over $\mathbb{F}$; moreover,

$$\text{if} \quad \sum_{k=1}^{n} |t_k| \leq 1, \quad \text{then} \quad \sum_{k=1}^{n} t_k\pi(x_k,1) = \pi\Big(\sum_{k=1}^{n} t_k x_k, 1\Big)$$

i.e., the new structure of linear combinations is an extension of the given structure of absolutely convex combinations.

Since X is a balloon in B, it determines a pseudo-norm satisfying (3.30) (see, e.g., Wilansky [1964], p.58). ∎

3.16. Lemma. If a normed vector space B contains an ω-balloon X satisfying (3.30), then B is complete.

Proof. It is enough to show that any absolutely convergent series in convergent in B (see, e.g., #3.1.2). Let $b_n \in B$, $b_n \neq 0$ $(n=0,1,\dots)$ and

$$\sum_{n=0}^{\infty} \|b_n\| < \infty$$

Denote $s_n = \|b_n\|$, $s = \Sigma s_n$, $t_n = s_n/s$, $a_n = b_n/2s_n$. Then $\|a_n\| = \frac{1}{2}$ and hence $a_n \in X$. Since $\Sigma t_n = 1$, the series $\Sigma t_n a_n = \Sigma b_n/2s$ is convergent and so is the series Σb_n. ∎

3.17. Proposition. Let (X,γ) be a T-algebra for the monad (3.26) satisfying (3.49). Then X can be embedded into an (essentially unique) Banach space so that (3.30), (3.31) and (3.33) hold.

Proof. Suppose that γ is a map (3.27) satisfying (3.28), (3.29) and (3.49). A structure of absolutely convex combinations on X can be defined as

(3.56) $$\sum_{k=1}^{n} s_k x_k \underset{df}{=} \gamma\Big(\sum_{k=1}^{n} s_k \delta^{(X)}(x_k)\Big) \quad \text{if} \quad \sum_{k=1}^{n} |s_n| \leq 1 .$$

The conditions (3.51) and (3.53) are obviously satisfied and (3.52) follows from (3.28) . We shall show that (3.54) follows from (3.29) . Assume that $\sum_m |t_{km}| \leq 1$ for $k = 1,\ldots,n$ and $\Sigma|s_k| \leq 1$ and consider the element

$$z = \sum_{k=1}^{n}{}^{*} s_k \sum_{m=1}^{p}{}^{*} t_{km} x_{km} = \sum_{k=1}^{n} s_k \delta^{(\mathcal{O}\ell X)} \Big(\sum_{m=1}^{p} t_{km} \delta^{(X)}(x_{km}) \Big)$$

of $\mathcal{O}\ell\mathcal{O}\ell(X)$. Applying μ_X to z "erases the first asterisk", i.e.

$$\mu_X(z) = \sum_k s_k \sum_m{}^{*} t_{km} x_{km} = \sum_k s_k \sum_m t_{km} \delta^{(X)}(x_{km})$$

$$= \sum_{k,m} s_k t_{km} \delta^{(X)}(x_{km})$$

and hence $\gamma\mu_X(z)$ is the right-hand side of (3.54) . On the other hand, applying $\mathcal{O}\ell(\gamma)$ to z "erases the second asterisk", i.e.,

$$\mathcal{O}\ell.\gamma.z = \sum_k{}^{*} s_k \gamma \Big(\sum_m{}^{*} t_{km} x_{km}\Big) = \sum_k{}^{*} s_k \sum_m t_{km} x_{km}$$

(3.57)

$$= \sum_k s_k \delta^{(X)} \Big(\sum_m t_{km} x_{km}\Big)$$

Consequently, $\gamma(\mathcal{O}\ell.\gamma.z)$ is the left-hand side of (3.54);

note that we cannot write (3.57) as

$$\sum_k^* \sum_m s_k t_{km} x_{km} .$$

Thus, by Lemma 3.15, X is a subset of a pseudonormed vector space B satisfying (3.30) Moreover, by (3.56) there is a linear operator

$$(3.58) \qquad \Gamma: \ \ell(X) \to B$$

such that γ is the restriction of Γ to $\bigcirc\ell(X)$. We are to show that B is actually a normed vector space, i.e., $\|x\| = 0$ implies $x = 0$; this means that X does not contain any linear subset of positive dimension. We shall need a version of (3.54) for infinite series; still, some care is needed because if $\| \ \|$ is not a norm, the convergence of a series has no meaning. Therefore we shall use the identity

$$(3.59) \qquad \gamma\Big(\sum_{k=1}^{\infty} s_k \delta\gamma(\sum_{m=1}^{\infty} t_{km}\delta(x_{km}))\Big) = \gamma(\sum_{k,m=1}^{\infty} s_k t_{km} \delta(x_{km}))$$

where $\sum_{m=1}^{\infty} |t_{km}| \leq 1$ for $k = 1,2,\ldots$ and $\sum_{k=1}^{\infty} |s_k| \leq 1$.

This identity (3.59) is, in fact, a restatement of (3.29) (note that $\gamma\delta$ is an identity but $\delta\gamma$ is not). Suppose that there is an x_0 in X and that $x_1, x_2, \ldots \in X$ where $x_n = 2^n x_0$. Denote

$$y = \gamma\left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \delta(x_n)\right)$$

In virtue of (3.59) we may write

$$y = \frac{1}{2} x_0 + \gamma\left(\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \delta(x_n)\right) = \frac{1}{2} x_0 + \gamma\left(\sum_{n=1}^{\infty} \frac{1}{2^n} \delta(x_{n-1})\right) = \frac{1}{2} x_0 + y .$$

Hence $x_0 = 0$. Thus, $\| \; \|$ is a norm. Since (3.58) a linear contraction transforming $\mathcal{O}\ell(X)$ onto X, X is an ω-balloon. Consequently, by Lemma 3.16, B is complete. ■

3.18. Any T-algebra for the monad (3.26) with $F = \mathbb{R}$ may be described as a set X with a distinguished element 0 and the following additional structures: an involution $x \to -x$, action of the multiplicative semigroup $[0,1]$, and a structure of <u>countable</u> <u>convex</u> <u>combinations</u>

(3.60) $$\sum_i s_i x_i$$

where $s_1, s_2, \ldots$ are nonnegative numbers satisfying

(3.61) $$\sum s_i = 1$$

and $x_1, x_2, \ldots$ are elements of X. One of the axioms to be satisfied is that convex combinations of convex combinations are again convex combinations according to usual formulas (for a physicist it means that the centre of gravity (3.60) can be computed by splitting the mass in any way and computing the centre of gravity for each part separately).

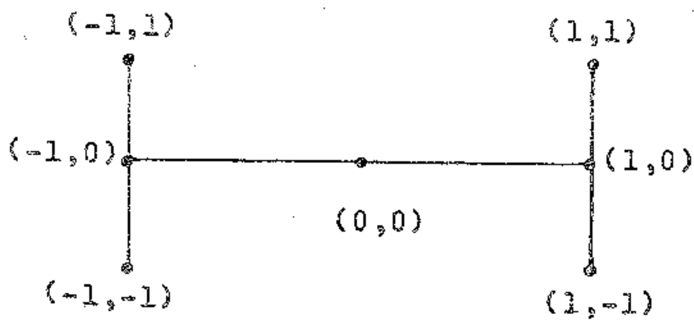
3.19. Free T-algebras for the monad (3.26) are closed unit balls of certain Banach spaces; indeed, in virtue of Definition 2.8 a free T-algebra generated by a set X is

(3.62) $$(O\ell(X), \mu_X) .$$

3.20. Exercises. (A) Show that the set

$$X = \{(s,0): -1 \leq s \leq 1\} \cup \{(-1,t): -1 \leq t \leq 1\} \cup \{(1,t): -1 \leq t \leq 1\}$$

whose picture is

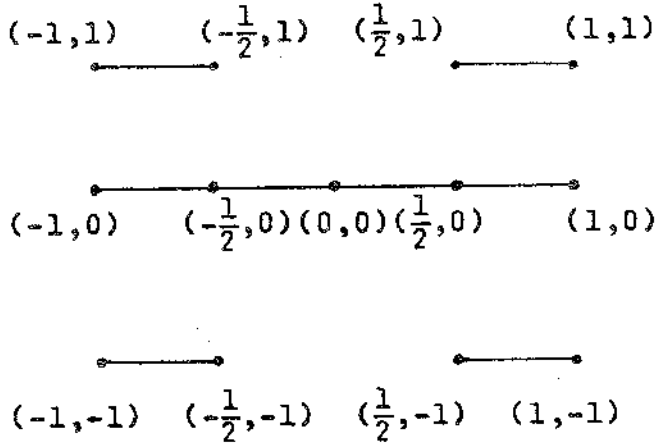


can be made a T-algebra for the monad (3.26) with

$$\gamma(\Sigma s_n \delta_{x_n}) = \begin{cases} \text{ordinary sum } z = \Sigma s_n x_n = (z_1, z_2) & \text{if } z \in X , \\ (z_1, 0) & \text{if } z \notin X . \end{cases}$$

for $x_1, x_2, \ldots$ in X , $s_1, s_2, \ldots$ in R , $\Sigma |s_n| \leqslant 1$.

(B) Show that the set X whose picture is



can be made a T-algebra if $\gamma(\Sigma s_n \delta_{x_n})$ is a suitable image of the ordinary sum $\Sigma s_n x_n$. Find x, x' in X such that

$$0 < \sup\{r \in R : r \geqslant 0 \ \& \ rx = rx'\} < 1 \ .$$

3.21. Notes. The examples in 3.12 and in Exercise (A) are due to F.E.J. Linton (unpublished).

§4. Conjugate Banach spaces. We shall consider the monad $\mathbb{T}$ obtained by composing the conjugate-space functor with itself. The problem of what is the general form of a T-algebra will be translated into the language of functional analysis; this will yield a characterization of those Banach spaces which are isometrically isomorphic to conjugate spaces. A reader not interested in categorical argument may start to read this section at 4.2 .

4.1. Let $\mathfrak{A} = Ban_1$ and $\mathfrak{B} = (Ban_1)^*$. In this way, by considering the dual category, we make the conjugate-space functor a covariant functor from $\mathfrak{A}$ to $\mathfrak{B}$, and from $\mathfrak{B}$ to $\mathfrak{A}$ as well; specifically, the functors

$$\Phi : \mathfrak{A} \to \mathfrak{B} \quad \text{and} \quad \Psi : \mathfrak{B} \to \mathfrak{A}$$

defined as $\Phi(A) = A^*$, $\Phi(\alpha) = \alpha^*$, $\Psi(B) = B^*$, $\Psi(\beta) = \beta^*$, are both covariant. Since the contravariant conjugate-space functor is adjoint on the right to itself (#12.4.4(a)), Φ is a left adjoint of Ψ ; indeed,

$$\langle \Phi A, B \rangle_{\mathfrak{B}} = \langle B, A^* \rangle_{Ban_1} \cong \langle A, B^* \rangle_{Ban_1} = \langle A, \Psi(B) \rangle_{\mathfrak{A}} .$$

Denote $\mathbb{T} = \Psi\Phi$; thus $\mathbb{T}: \mathfrak{A} \to \mathfrak{A}$ and $T(A) = A^{**}$, $T(\alpha) = \alpha^{**}$. The unit $\eta : \iota_{\mathfrak{A}} \to T$ of the adjunction of Φ and Ψ is given by the canonical maps

$$\kappa_A : A \to A^{**} .$$

The naturality of κ means that for any Banach spaces F, G and any bounded linear operator $\lambda : F \to G$ the diagram

$$\begin{array}{ccc} F & \xrightarrow{\ \lambda\ } & G \\ {\scriptstyle \kappa_F}\downarrow & & \downarrow{\scriptstyle \kappa_G} \\ F^{**} & \xrightarrow[\ \lambda^{**}\ ]{} & G^{**} \end{array}$$

is commutative (#10.6.1) , i.e.,

(4.1) $$\kappa_G \lambda = \lambda^{**} \kappa_F \quad .$$

Applying this identity to the case where $F=A$, $G=A^{**}$ and $\lambda=\kappa_A$ we get

(4.2) $$\kappa_{A^{**}} \kappa_A = (\kappa_A)^{**} \kappa_A \quad .$$

The counit $\rho : \Phi\Psi \to \iota_{\mathfrak{B}}$ of the adjunction of Φ and Ψ is also given by the canonical maps into the second conjugate spaces, this time κ_B being regarded as a $\mathfrak{B}$-morphism from B^{**} to B (!) . Each of the conditions (2.11) is now equivalent to the well-known identity

(4.3) $$(\kappa_B)^{*} \kappa_{B^*} = \iota_{B^*} \quad .$$

(#12.4.4(a)). The monad in question is $(\mathbb{T},\kappa,\mu)$, where

(4.4) $$\mu_A = (\kappa_{A^*})^* : A^{****} \to A^{**} \quad .$$

From (4.3) it follows that

(4.5) $$\mu_A \kappa_{A^{**}} = (\kappa_{A^*})^* \kappa_{A^{**}} = \iota_{A^{**}}$$

for any Banach space A ; this is the left-hand side of condition (2.4) ; the right-hand one and (2.3) can also be easily verified.

A T-algebra is now a pair (A,γ) , where A is a Banach space and $\gamma : A^{**} \to A$ is a linear contraction satisfying the conditions

(4.6) $$\gamma\kappa_A = \iota_A$$

and

(4.7) $$\gamma\gamma^{**} = \gamma\mu_A \qquad ;$$

the latter condition means that the diagram

$$\begin{array}{ccc} A^{****} & \xrightarrow{\gamma^{**}} & A^{**} \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \gamma} \\ A^{**} & \xrightarrow[\gamma]{} & A \end{array}$$

is commutative. According to (2.34) and (2.35), the comparison functor

$$\Lambda : (\mathrm{Ban}_1)^* \to (\mathrm{Ban}_1)^{\mathbb{T}}$$

is determined by

$$\Lambda(B) = (B^* , (\kappa_B)^*) \tag{4.8}$$

for any Banach space B ; if $\beta \in \langle B_1, B_2 \rangle_{Ban_1}$, i.e., if $\beta : B_2 \to B_1$ is a linear contraction, then $\Lambda(\beta)$ is the map β^* regarded as a morphism from $(B_1^* , (\kappa_{B_1})^*)$ to $(B_2^* , (\kappa_{B_2})^*)$. If we substitute $A = B^*$ and $\gamma = (\kappa_B)^*$, then (4.5) becomes the identity (4.3) while (4.7) becomes the condition

$$(\kappa_B)^*(\kappa_B)^{***} = (\kappa_B)^*(\kappa_{B^{**}})^* \tag{4.9}$$

which follows from (4.2) .

Thus, for any Banach space B the pair $(B^*, (\kappa_B)^*)$ is a T-algebra. It will be shown in 4.2 below that <u>every T-algebra (A,γ) is isomorphic to a T-algebra of the form</u> (4.8) . In other words, for any pair (A,γ) satisfying (4.5) and (4.7) there is a Banach space B and a Ban_1-isomorphism $\phi : A \to B^*$ such that the diagram

$$\begin{array}{ccc} A^{**} & \xrightarrow{\ \phi^{**}\ } & B^{***} \\ \gamma \downarrow & & \downarrow (\kappa_B)^* \\ A & \xrightarrow[\ \phi\ ]{} & B^* \end{array} \tag{4.9'}$$

is commutative (in virtue of (2.15) this means that ϕ is the underlying morphism of an isomorphism of T-algebras). Thus,

Λ is a quasi-isomorphism (equivalence) of categories and Ψ is quasi-monadic.

Before proving the above statement we shall show the uniqueness of B : if such a B exists, it is unique up to Ban_1-isomorphism (B^* is obviously determined up to isomorphism; yet, B^* need not determine B , #21.5.14) . Denote

$$(4.10) \qquad K = \text{Ker}\ \gamma = \{\alpha \in A^{**} : \gamma(\alpha) = 0\} .$$

It is clear that ϕ^{**} maps K onto the kernel of $(\kappa_B)^*$. Moreover, the annihilator of $\text{Ker}(\kappa_B)^*$ in B^{**} , i.e., the set

$$(\text{Ker}(\kappa_B)^*)^{\perp} = \{\beta \in B^{**} : \forall_{\xi \in \text{Ker}(\kappa_B)^*} \ \xi(\beta) = 0\}$$

is just the canonical image $\hat{B} = \kappa_B(B)$ of B in B^{**} ; ϕ^* maps $\hat{B}$ onto the annihilator $K^{\perp}$ of K ($K^{\perp} \subset A^*$) . Consequently, B is Ban_1-isomorphic to K and hence it is determined by A and γ uniquely up to isomorphism.

4.2. Proposition (Dixmier-Linton). Let A be a Banach space and let

$$\gamma : A^{**} \to A$$

be a linear contraction satisfying (4.6) . Let $\varepsilon : K \to A^{**}$ be the identical embedding of (4.10) into A^{**} . Then the following conditions are equivalent:

(i) There is a Banach space B and an isometric isomorphism ϕ from A onto B^{*} such that the diagram (4.10) is commutative.

(ii) Condition (4.7) holds.

(iii) $\mu_A \varepsilon^{**}(K^{**}) \subset K$, where μ_A is given by (4.4).

(iv) μ_A maps the unit ball $\mathrm{O}(\varepsilon^{**}(K^{**}))$ onto $\mathrm{O}K$.

(v) K is *weakly closed in A^{**}.

Proof. (i) ⇒ (ii). The isomorphism ϕ sends the identity (4.9) to (4.7). Indeed, consider the diagram

$$\begin{array}{ccccccc}
A^{****} & \xrightarrow{\phi^{****}} & B^{*****} & \xrightarrow{(\kappa_B)^{***}} & B^{***} & \xrightarrow{(\phi^{-1})^{**}} & A^{**} \\
\Big\downarrow{\scriptstyle (\kappa_{A^*})^{*}} & & \Big\downarrow{\scriptstyle (\kappa_{B^{**}})^{*}} & & \Big\downarrow{\scriptstyle (\kappa_B)^{*}} & & \Big\downarrow{\scriptstyle \gamma} \\
A^{**} & \xrightarrow[\phi^{**}]{} & B^{***} & \xrightarrow[(\kappa_B)^{*}]{} & B^{*} & \xrightarrow[\phi^{-1}]{} & A
\end{array}$$

The right-hand square is commutative by (i); the middle one by (4.9); the left-hand one by naturality of κ (substitute $F=B^{**}$, $G=A^{*}$, $\lambda=\phi^{*}$ in (4.1) and apply the conjugate-space functor). Consequently, the exterior rectangle is commutative and hence - in virtue of (i) - we get (ii).

(ii) ⇒ (iii). Denote $K_1 = \varepsilon^{**}(K^{**})$. Thus, $K_1 \subset A^{****}$. The definition of the map ε implies that $\gamma\varepsilon=0$. Hence $\gamma^{**}\varepsilon^{**} = 0$; this means that K_1 is contained in $\operatorname{Ker} \gamma^{**}$. On the other hand, from condition (ii) it follows that $\gamma\mu_A$ vanishes on $\operatorname{Ker} \gamma^{**}$, i.e., μ_A maps $\operatorname{Ker} \gamma^{**}$ into K. Thus, $\mu_A(K_1) \subset K$.

(iii) ⇒ (iv). From (iii) it follows that $\mu_A(\bigcirc K_1) \subset \bigcirc K$. Let $z \in \bigcirc K$.

Combining (4.4) with the naturality of κ we get

$$z = \varepsilon(z) = \mu_A \kappa_{A^{**}} \varepsilon(z) = \mu_A \varepsilon^{**} \kappa_K(z) \quad .$$

Since $\varepsilon^{**} \kappa_K(z) \in \bigcirc K_1$, z belongs to $\mu_A(\bigcirc K_1)$.

(iv) ⇒ (v) . The maps ε^{**} and μ_A are *weakly continuous, i.e., continuous with respect to the topologies

$$\sigma(K^{**},K^{*}) \ , \ \sigma(A^{****},A^{***}) \ , \ \sigma(A^{**},A^{*}) \ ,$$

respectively. Therefore the compactness of $\bigcirc^{*}(K^{*})$ (i.e., of the set $\bigcirc(K^{**})$ equipped with *weak topology) implies the compactness of $\bigcirc K = K \cap \bigcirc A^{**}$ in the *weak topology of A^{**} . Thus, by the Krein-Šmulyan theorem ([DSI],V.5.7) , K is *weakly closed in A^{**} .

(v) ⇒ (i). Denote $B = K^{\perp} = \{b \in A^{*} : \forall_{x \in K} \ x(b) = 0\}$. Since K is *weakly closed, the set

$$B^{\perp} = \{\alpha \in A^{**} : \forall_{b \in B} \ \alpha(b) = 0\}$$

coincides with K (double annihilator property).

Let us note that the condition $\gamma\kappa_A = \iota_A$ together with $\|\gamma\| \leq 1$ may be restated as

(4.11) $\quad A^{**}$ is a direct sum of $\hat{A}=\kappa_A(A)$ and K

together with

(4.12) $\quad \|\kappa_A(a) + z\| \geq \|a\|$ for a in A , z in K .

Condition (4.11) means that $\hat{A}+K = \{\kappa_A(a)+z : a\in A \ \& \ z\in K\}$ fills up all of A^{**} and $\hat{A} \cap K = \{0\}$; put differently, any α in A^{**} can be uniquely written as $\alpha = \kappa_A(a)+z$ with a in A , z in K (of course, $a=\gamma(\alpha)$ and $z=\alpha-\kappa_A(a)$). If (4.11) is satisfied, then (4.12) means that $\|a\| \leq \|\alpha\|$, i.e., $\|\gamma\| \leq 1$.

The desired isomorphism $\phi: A \to B^*$ is defined as $\phi(a) = \text{rest}_B\, \kappa_A(a)$; in other words, $\phi a.b = b(a)$ for b in B . Obviously, ϕ is linear and $\|\phi\| \leq 1$.

We shall prove that ϕ is one-to-one. Suppose that $\phi(a) = 0$ for some a in A . This means that $b(a) = 0$ for b in B . Consequently, $\kappa_A(a) \in K \cap \hat{A}$ and, by (4.11), $a=0$.

We shall now prove that ϕ maps $\bigcirc A$ onto $\bigcirc(B^*)$. The inclusion $\phi(\bigcirc A) \subset \bigcirc B^*$ follows from $\|\phi\| \leq 1$. Let $\beta \in \bigcirc B^*$. By the Hahn-Banach theorem there exists an α in A^{**} such that $\text{rest}_B\, \alpha = \beta$ and $\|\alpha\| = \|\beta\| \leq 1$. By (4.11), α can be written as $\alpha = \kappa_A(a) + z$ with $a \in A$, $z \in K$. By (4.12) , $\|a\| \leq \|\alpha\| \leq 1$, i.e., $a \in \bigcirc A$. If $b \in B$, then

$$\beta(b) = \alpha(b) = \kappa_A a.b + z(b) = b(a) \quad .$$

Thus, $\beta = \phi(a)$. We have shown that $\phi : A \to B^*$ is one-to-one and maps $\bigcirc A$ onto $\bigcirc B^*$; hence it is a linear isometrical bijection.

It remains to be shown that diagram (4.9′) is commutative.
Since $\phi = \tau^* \kappa_A$, where $\tau : B \to A^*$ is the identical embedding, the naturality of κ and (4.3) imply that

$$(\kappa_B)^* \phi^{**} = ((\kappa_A)^* \tau^{**} \kappa_B)^* = ((\kappa_A)^* \kappa_{A*} \tau)^* = \tau^* \quad .$$

We shall show that $\tau^* = \phi\gamma$. Let $\alpha \in A^{**}$. By (4.11) , $\alpha = \kappa_A(a) + z$ for some a in A , z in K . Consequently, by (4.6) and (4.10),

$$\begin{aligned} \phi\gamma(\alpha) &= \phi\gamma\kappa_A(a) + \phi\gamma(z) = \phi(a) = \tau^* \kappa_A(a) \\ &= \tau^* \kappa_A(a) + \tau^*(z) = \tau^*(\alpha) \quad . \end{aligned}$$

Thus, $(\kappa_B)^* \phi^{**} = \phi\gamma$. ■

4.3. EXERCISES

(A) Prove a statement analogous to the Proposition above assuming that $\gamma^{**} : A^{**} \to A$ is any bounded linear operator and inserting $\gamma\kappa_A = \iota_A$ and $\|\gamma\| \leq 1$ into those of the conditions (i)-(v) where $\gamma\kappa_A = \iota_A$ and $\|\gamma\| \leq 1$ are essential; use equivalent conditions (4.11) and (4.12) whenever appropriate.

(B) Show that the subspace $B = K^\perp$ of A^* in the proof of (v) ⇒ (i) above coincides with

$$\text{(4.13)} \qquad \{b \in A^* : \alpha \in A^{**} \Rightarrow \alpha(b) = b\gamma(\alpha)\} \quad .$$

(C) Show that the space (4.13) is a Ban_1-equalizer of κ_{A^*} and γ^* .

(D) Show that a Banach space A is topologically isomorphic (i.e., Ban_∞-isomorphic) to some B^* if and only if there is a *weakly closed subspace K of A^* satisfying (4.11) .

(E) Show that the space $A = C_0(N_0)$ of sequences convergent to 0 is not topologically isomorphic to any B^* . (Hint: apply (D) and Phillips's theorem or non-existence of projections $\ell_\infty(N_0) \to C_0(N_0)$, #17.7.4) .

4.4. Notes. The equivalence (i) $\Leftrightarrow$ (v) is due to Dixmier [1948] who stated it in the following form : A Banach space A is isometrically isomorphic to some B^* if and only if there is a *weakly closed subspace K of A^{**} satisfying (4.11) and (4.12) . He also proved (D) and several related results; let us quote some of them. Suppose that B is a linear subset of A^* and consider the real numbers r,s,t defined as follows:

$$r = \sup\{u : c\ell^* \bigcirc B \supset \bigcirc_u A^*\} ,$$

where $\bigcirc_u A^* = \{\xi \in A^* : \|\xi\| \leq r\}$ and $c\ell^*$ stands for the *weak closure in A^*,

$$s = \inf\{ \|a\|_B : a \in A \ \& \ \|a\| = 1\} ,$$

where $\|a\|_B = \sup\{|b(a)| : b \in \bigcirc B\}$,

$$t = \inf_{\substack{a\in A \\ \|a\|=1}} \inf_{z\in K} \| \kappa_A(a)+z \| = \rho(K,S) ,$$

where $S = \{\kappa_A(a) : \|a\| = 1\}$. Note that condition (4.12) means that $t = 1$. Dixmier proved that $r=s=t$ and r may be any number in $[0,1]$. He also proved that if B is norm-closed and separates A then $K \cap \hat{A} = \{0\}$ always holds, but condition (4.11) is satisfied if and only if B is minimal in the sense that no proper norm-closed subspace of B separates A .

The $\mathbb{T}$-algebras over the monad (T,κ,μ) were considered by F.E.J. Linton [1971] (unpublished); he proved the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) in the setting of Exercise (A) and observed that a theorem of Beck [1967] yields (C) .

See also the notes after §7 .

§5. Compact spaces. We begin with a discussion of the case of general topological spaces.

5.1. Let us consider the monad generated by the pair

$$\text{Ens} \underset{U}{\overset{D}{\rightleftarrows}} \text{Top} \; , \tag{5.1}$$

where U is the underlying-set functor ("forgetting" the topology) and D assigns to each set X the same set with the discrete topology. It is obvious that D is a left adjoint of U with the unit $\iota_X\colon X \to UD(X)$ and the counit $\rho_Y\colon DU(Y) \to Y$, where ρ_Y is the identity regarded as a continuous map from $DU(Y)$ to Y. In this case an algebra for the monad in question is a pair (X,γ), where X is a set of $\gamma\colon UD(X) \to X$ is a map satisfying (2.12) and (2.13). Condition (2.13) means that $\gamma\iota_X = \iota_X$, i.e., $\gamma = \iota_X$. Consequently, (X,γ) must be of the form (X,ι_X) and no non-trivial topology can be reconstructed from γ. The functor U in (5.1) is therefore not quasi-monadic. The Eilenberg-Moore comparison functor Λ sends some non-isomorphic objects of Top (specifically, non-homeomorphic spaces with the same cardinal number) to isomorphic algebras; yet, it maps Top onto the corresponding Eilenberg-Moore category.

Much the same argument is valid for many important full subcategories of Top (e.g., for categories of Hausdorff spaces, of metrizable spaces, etc.)

5.2. The rest of this subsection will be devoted to the monad (T,η,μ) generated by the pair

$$\text{(5.2)} \qquad \text{Ens} \underset{U}{\overset{\beta}{\rightleftarrows}} \text{Comp}$$

where β is the Stone-Čech functor, a left adjoint of the forgetful functor U. In order to simplify the notation we shall not distinguish between a set X and the same set with the discrete topology. The symbol βX will denote the Stone-Čech compactification of X ; $U\beta X$ will denote either the underlying set of βX or βX with the discrete topology. The unit of the adjunction (5.2) is given by the canonical injections

$$\text{(5.3)} \qquad \eta_X\colon \ X \to U\beta X$$

while the counit is given by the canonical surjections

$$\text{(5.4)} \qquad \rho_Y\colon \ \beta UY \to Y$$

defined as follows: given any compact space Y, we consider the identity map $UY \to Y$ and extended it to a unique continuous map from βUY onto Y . Consequently,

(5.5) $$\mu_X : U\beta U\beta X \to U\beta X$$

is the map $\mu_X = U(\rho_{\beta X})$ and $\rho_{\beta X}$ is the unique continuous extension of the identity map from $U\beta X$ to βX . Thus,

(5.6) $$\mu_X \eta_{U\beta X} = \iota_{U\beta X} .$$

If $\phi : X \to X'$ is any map, then $\beta(\phi) : \beta X \to \beta X'$ is defined as the unique continuous map rendering commutative the diagram

(5.7) $$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X' \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_{X'}} \\ \beta X & \xrightarrow[\ \beta(\phi)\ ]{} & \beta X' \end{array}$$

A T-algebra is now a pair (X,γ) , where X is a set and

(5.8) $$\gamma : U\beta X \to X$$

is a map satisfying the conditions

(5.9) $$\gamma\eta_X = 1_X$$

and

(5.10) $$\gamma\mu_X = \gamma U\beta(\gamma) \ ,$$

The last equality means that the diagram

$$\begin{array}{ccc}
U\beta U\beta X & \xrightarrow{U\beta(\gamma)} & U\beta X \\
\big\downarrow{\scriptstyle \mu_X} & & \big\downarrow{\scriptstyle \gamma} \\
U\beta X & \xrightarrow[\gamma]{} & X
\end{array}$$

is commutative. We recall that $\beta(\gamma)$ is the unique continuous map from $\beta U\beta X$ to βX such that the diagram

(5.11) $$\begin{array}{ccc}
U\beta X & \xrightarrow{\gamma} & X \\
\big\downarrow{\scriptstyle \eta_{U\beta X}} & & \big\downarrow{\scriptstyle \eta_X} \\
U\beta U\beta X & \xrightarrow[U\beta(\gamma)]{} & U\beta X
\end{array}$$

is commutative. In virtue of (2.34), the comparison functor

$$(5.12) \qquad \Lambda\colon \ \mathrm{Comp} \to \mathrm{Ens}^{T}$$

assigns to each compact space Y the pair (UY,γ) where the structure map γ is obtained from (5.4) by

$$(5.13) \qquad \gamma = U(\rho_Y) .$$

5.3. Theorem (Linton). <u>The functor</u> (5.12) <u>is an isomorphism of categories</u>. In other words, suppose that X is a set and a map (5.8) satisfies (5.9) and (5.10) Then there is a unique compact topology on X such that γ satisfies (5.13).

Proof. Condition (5.9) implies that γ is a surjection. We shall show that for any set A contained in βX the following crucial identity holds:

$$(5.14) \qquad \gamma(c\ell_{\beta X}\eta_X\gamma(A)) = \gamma(c\ell_{\beta X}A) ,$$

where $c\ell_{\beta X}$ stands for the closure operation in βX. Indeed,

$$\gamma[c\ell_{\beta X}\eta_X\gamma(A)] = \gamma[c\ell_{\beta X}(\beta(\gamma)\eta_{U\beta X}(A))]$$

$$= \gamma\beta(\gamma)[c\ell_{\beta U\beta X}\eta_{U\beta X}(A)] = \gamma\mu_X[c\ell_{\beta U\beta X}\eta_{U\beta X}(A)]$$

$$= \gamma[c\ell_{\beta X}\mu_X\eta_{U\beta X}(A)] = \gamma(c\ell_{\beta X}A) \ .$$

The above equalities follow from the commutativity of (5.11), from the continuity and the closedness (#7.1.14) of $\beta(\gamma)$, from (5.10), from the continuity and the closedness of $\rho_{\beta X}$ (which, regarded merely as a map, is the same as μ_X), and from (5.6), respectively.

If $B \subset X$, define

(5.15) $$c\ell_X B = \gamma(c\ell_{\beta X}\eta_X(B)) \ .$$

It is obvious that $c\ell_X 0 = 0$, $c\ell_X(B \cup B') = c\ell_X B \cup c\ell_X B'$, and $B \subset c\ell_X B$. Substituting $A = c\ell_{\beta X}\eta_X(B)$ in (5.14) we get

$$c\ell_X c\ell_X B = \gamma[c\ell_{\beta X}\eta_X\gamma(c\ell_{\beta X}\eta_X(B))] = \gamma[c\ell_{\beta X}c\ell_{\beta X}\eta_X(B)] = c\ell_X B \ .$$

Thus, $c\ell_X c\ell_X B = c\ell_X B$. We have shown that (5.15) determines a closure operation for a topology on X . Making use of (5.14) again, we infer that

$$\gamma(c\ell_{\beta X}A) = [c\ell_{\beta X}\eta_X\gamma(A)] = c\ell_X\gamma(A) .$$

This means that the map γ is both closed and continuous. Consequently, the topology of X is the quotient topology (determined by βX and γ) ; X is compact (#5.2.4, #7.5.4) , and the topology of X is uniquely determined by the assumptions of the Theorem. This concludes the proof. ∎

5.4. Exercises. (A) In the above theorem, condition (5.10) is necessary and sufficient in order that the quotient topology of X determined by the map (5.8) satisfying (5.9) be a Hausdorff topology.

(B) Assuming (5.8) prove that (5.10) is equivalent to (5.14)

(C) Let X, X' be compact spaces. Show that a map $\phi: X \to X'$ is continuous if and only if it is a T-algebra morphism, i.e., in virtue of (2.15) the diagram

$$\begin{array}{ccc} \beta UX & \xrightarrow{\beta U(\phi)} & \beta UX' \\ \gamma \downarrow & & \downarrow \gamma' \\ X & \xrightarrow[\phi]{} & X' \end{array}$$

is commutative, where γ and γ' are the T-algebra structures on X and X', respectively.

(D) Let Comp_0 denote the category of compact 0-dimensional spaces and continuous maps. Show that the forgetful functor $U\colon \mathrm{Comp}_0 \to \mathrm{Ens}$ has a left adjoint $\beta\colon \mathrm{Ens} \to \mathrm{Comp}_0$ but is not monadic.

(E) What is the Kleisli comparison functor in the cases considered in 5.1 and 5.2 ?

5.5 Notes. Theorem 5.3 is due to Linton [1965] who proved it using the criterion of monadicity stated in §6. E.G. Manes [1969] gave a direct topological proof. Paré [1971] (see also MacLane [1972], p. 153) found a simple proof applying a criterion based on the concept of "split forks" and "absolute coequalizers". The above proof (taken from Semadeni [1972]) is in fact, Paré's proof applied to the split fork determined by γ ; this made the proof still simpler, purely topological, and not making use of any specific construction of βX .

Linton's theorem may be laconically stated as "compact spaces are algebras" (for more details in this line see Linton [1965] and Schubert [1972]). Herrlich and Strecker [1972] proved that under some additional assumptions the converse statement holds as well; roughly speaking, among Hausdorff spaces the only "reasonable" algebras are compact spaces.

§6. Linton's criterion. One of the basic questions in the theory of monads is whether the category $\mathfrak{B}$ can be reconstructed from the monad T generated by a pair of adjoint functors

$$(6.1) \qquad \mathfrak{A} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathfrak{B} .$$

Technically, the question is whether the Eilenberg-Moore comparison functor (2.22) is an isomorphism or a quasi-isomorphism of categories. Many necessary and sufficient conditions for monadicity or quasi-monadicity of Ψ have been proved; one of the most significant and most useful for applications in other theories is Linton's criterion which is the subject of this section.

6.1. Definitions. Let $\mathfrak{B}$ be any fixed category. A parallel pair in $\mathfrak{B}$ is a pair of morphisms with common domain and common codomain, i.e., a pair of the form

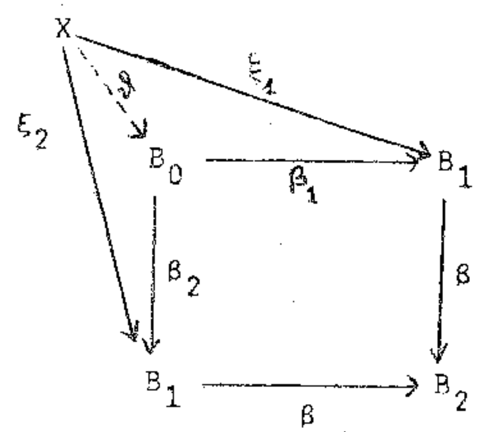
$$(6.2) \qquad \beta_1 : \mathbb{B}_0 \to \mathbb{B}_1 , \quad \beta_2 : \mathbb{B}_0 \to \mathbb{B}_1 .$$

A kernel pair of a morphism $\beta : \mathbb{B}_1 \to \mathbb{B}_2$ is a pullback of the pair β, β (if it exists), i.e., a pair (6.2) such that the square

$$(6.3) \qquad \begin{array}{ccc} \mathbb{B}_0 & \xrightarrow{\beta_1} & \mathbb{B}_1 \\ {\scriptstyle \beta_2}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathbb{B}_1 & \xrightarrow[\beta]{} & \mathbb{B}_2 \end{array}$$

is commutative and for any object X and any pair $\xi_1 \colon X \to B_1$, $\xi_2 \colon X \to B_1$ such that $\beta\xi_1 = \beta\xi_2$ there is a unique morphism $\vartheta \colon X \to B_0$ making commutative the diagram

(6.4)



If a kernel pair exists, it is unique up to commuting isomorphism (#11.6.2) .

Suppose that $\mathcal{K}$ is one of the categories Ens, Comp, Vect, Ban_1 and let $\beta \colon B_1 \to B_2$ be any morphism in $\mathcal{K}$. Denote

(6.5) $$B_0 = \{(b,b') \colon b \in B_1 \ \& \ b' \in B_1 \ \& \ \beta(b_1) = \beta(b_2)\} .$$

It is clear that B_0 is a [closed] subset [subspace] of $B_1 \times B_1$ (in the respective category) and the maps

(6.6) $$\beta_1(b,b') = b \quad , \quad \beta_2(b,b') = b'$$

(restrictions of coordinate projections) form a kernel pair of β . Note that [the underlying set of] B_0 is the equivalence relation on B_1 determined by β .

6.2. Lemma. Let β_1, β_2 be a kernel pair of $\beta: B_1 \to B_2$. Then β is a monomorphism if and only if $\beta_1 = \beta_2$; if this is the case, then ι_{B_1}, ι_{B_1} is also a kernel pair of β.

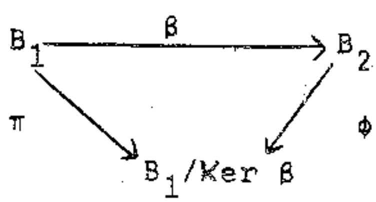
Proof. Suppose that β is a monomorphism. Denote $\beta_1 = \beta_2 = \iota_{B_1}$. Then (6.3) is commutative. Suppose that $\beta\xi_1 = \beta\xi_2$. Since β is left-cancellable, $\xi_1 = \xi_2$. Denote $\vartheta = \xi_1$. Then (6.4) is commutative; moreover, ϑ is unique.

Assume now that $\beta_1 = \beta_2$. We have to show that β is a monomorphism. Let ξ_1, ξ_2 be any morphisms such that $\beta\xi_1 = \beta\xi_2$. Then there exists a ϑ such that $\xi_1 = \beta_1\vartheta$ and $\xi_2 = \beta_2\vartheta = \beta_1\vartheta$. Thus, $\xi_1 = \xi_2$. ∎

6.3. Definition. A morphism $\beta: B_1 \to B_2$ is said to be a <u>coequalizer</u> iff there exists a parallel pair β_1, β_2 such that β is a coequalizer of β_1 and β_2.

We recall (#11.5.7, #7.5.1) that the coequalizers in Ens, Comp and Vect are characterized as surjections in the respective categories. The coequalizers in Ban_1 are the same as linear contractions $\beta: B_1 \to B_2$ such that the open unit ball of B_1 is mapped onto the open unit ball of B_2; this is equivalent to saying that in the canonical factorization

(6.7)

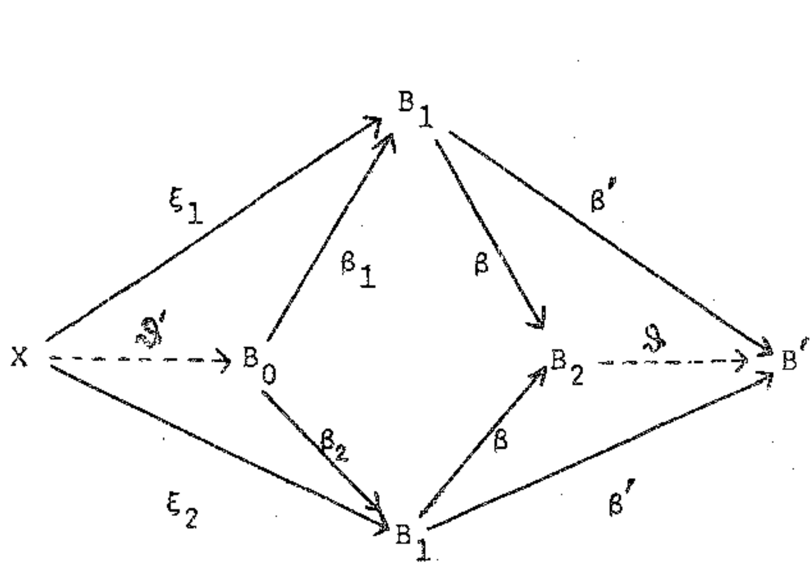


(where π is the canonical quotient map) ϕ is an isometrical bijection.

6.4. Definition. A parallel pair β_1, β_2 is said to be a <u>kernel pair</u> iff there exists a morphism β such that β_1, β_2 is a kernel pair of β.

6.5. Proposition. If $\beta: B_1 \to B_2$ is a coequalizer of the parallel pair (6.2) and β_1, β_2 is a kernel pair, then β_1, β_2 is a kernel pair of β.

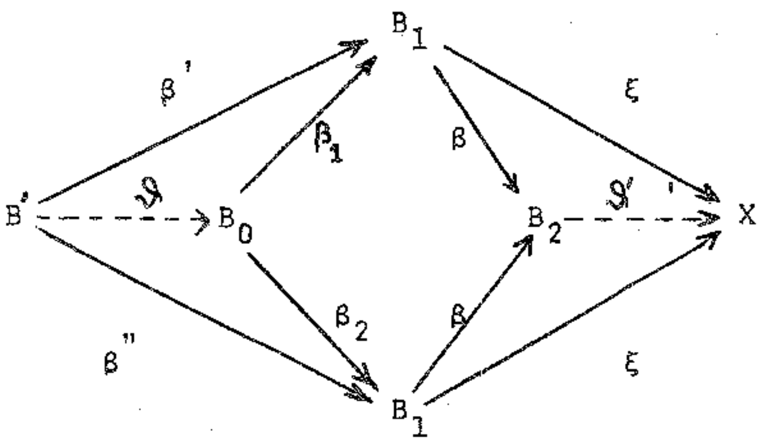
Proof. By assumption, β_1, β_2 is a kernel pair of a morphism $\beta': B_1 \to B'$. We have to show that it is a kernel pair of its coequalizer. Since β is a coequalizer of β_1, β_2, the diagram (6.3) is commutative. Moreover, $\beta'\beta_1 = \beta'\beta_2$ implies the existence of a $\vartheta: B_2 \to B'$ such that $\vartheta\beta = \beta'$. Suppose that $\beta\xi_1 = \beta\xi_2$. Then $\vartheta\beta\xi_1 = \vartheta\beta\xi_2$, i.e., $\beta'\xi_1 = \beta'\xi_2$. Since β_1, β_2 is a kernel pair of β', there exists a unique ϑ' such that the diagram



is commutative. We have shown that β_1, β_2 is a kernel pair of β . ∎

6.6. Proposition. If (6.2) is a kernel pair of $\beta: B_1 \to B_2$ and β is a coequalizer, then β is a coequalizer of β_1, β_2 .

Proof. Let β be a coequalizer of a parallel pair $\beta', \beta'': B' \to B_1$. We have to show that it is a coequalizer of its kernel pair β_1, β_2 . Since β_1, β_2 is a kernel pair of β , the diagram (6.3) is commutative and, in virtue of $\beta\beta' = \beta\beta''$, there exists a unique $\vartheta: B' \to B_0$ such that $\beta_1\vartheta = \beta'$, $\beta_2\vartheta = \beta''$. Suppose that $\xi\beta_1 = \xi\beta_2$ for some morphism $\xi: B_1 \to X$. Then $\xi\beta_1\vartheta = \xi\beta_2\vartheta$, i.e., $\xi\beta' = \xi\beta''$. Since β is a coequalizer of β', β'' , there is a unique ϑ' making commutative the diagram



∎

We shall now formulate Świrszcz's generalization of Linton's theorem; the proof will be omitted.

6.7. Theorem. Let T be the monad determined by functors (6.1), where Φ is a left adjoint of Ψ. Assume that any retraction in $\mathfrak{A}$ has a kernel pair, any parallel pair in $\mathfrak{B}$ has a coequalizer, and any morphism in $\mathfrak{B}$ has a kernel pair.

If

(i) $\forall_{\beta \in \mathfrak{B}}$ [β is a coequalizer in $\mathfrak{B}$] $\Leftrightarrow$ [$\Psi(\beta)$ is a coequalizer in $\mathfrak{A}$]

(ii) for every parallel pair β_1, β_2 in $\mathfrak{B}$

[β_1, β_2 is a kernel pair] $\Leftrightarrow$ [$\Psi(\beta_1), \Psi(\beta_2)$ is a kernel pair],

then the comparison functor $\Lambda : \mathfrak{B} \to \mathfrak{A}^T$ is a quasi-isomorphism (i.e., an equivalence of categories).

A converse theorem holds under stronger assumptions.

6.8. Theorem. Let T be as above. Assume that both in $\mathfrak{A}$ and $\mathfrak{B}$ all morphisms have kernel pairs and all parallel pairs have coequalizers. Assume also that

(6.8) any epimorphism in $\mathfrak{A}$ is a retraction in $\mathfrak{A}$.

If the comparison functor $\Lambda : \mathfrak{B} \to \mathfrak{A}^T$ is a quasi-isomorphism, then conditions (i) and (ii) hold.

6.9. Remarks. Condition (i) does not mean that Ψ is coequalizer-preserving in the sense that

$$[\beta \text{ is a coequalizer of } \beta_1,\beta_2] \tag{6.9}$$
$$\Longrightarrow [\Psi\beta \text{ is a coequalizer of } \Psi\beta_1,\Psi\beta_2] \ .$$

In fact (6.9) is not satisfied in typical cases where (i) works. E.g. in the case considered in 5.2 Ψ satisfies (i) because β is a surjection iff $\Psi\beta$ is a surjection but coequalizers in Comp are constructed in a way different from that for coequalizers in Ens (#11.5.7) .

The functor Ψ , a right adjoint of Φ , is pullback-preserving (#12.5.2) ; therefore it is the implication $\Leftarrow$ which really matters in condition (ii) .

The assumptions concerning $\mathfrak{A}$ and $\mathfrak{b}$ in 6.7 and 6.8 are mild except of condition (6.8) which is so strong that among categories discussed in this booklet only Ens and Vect satisfy it (note that Ens and Vect have the property that all objects are free). E.g., the category of Abelian group does not satisfy (6.8) .

6.10. We shall now apply theorem 6.7 to the case

$$\text{Ens} \underset{\bigcirc}{\overset{\ell}{\rightleftarrows}} \text{Ban}_1$$

considered in §3 . Condition (i) obviously fails because it is not true that if a linear contraction $\beta\colon B_1 \to B_2$ is a coequalizer in Ban_1 then $\bigcirc\beta$ maps $\bigcirc B_1$ onto $\bigcirc B_2$. We can only claim that the interior of $\bigcirc B_1$ is mapped onto the interior of $\bigcirc B_2$ (and hence B_1 is mapped onto B_2) .

Let us consider the following example. Let ξ be a linear functional on a Banach space B which does not attain its norm on $\bigcirc B$, i.e., such that

$$(6.10) \qquad \|\xi\| = 1 \text{ and } |\xi(b)| < 1 \text{ for each } b \text{ in } \bigcirc B ,$$

e.g., B may be $C([0,1],\mathbb{R})$ and

$$\xi(f) = \int_0^{1/2} f(x)dx - \int_{1/2}^{1} f(x)dx$$

(#19.6.7(A), #24.5.4(b)) . Denote $H = \{b \in B \,\vdots\, \xi(b) = 0\}$. Then the canonical surjection

$$\pi : B \to B/H$$

is a coequalizer in Ban_1 and yet $\bigcirc\pi$ does not map $\bigcirc B$ onto $\bigcirc(B/H)$. Indeed, the norm in B/H is given by

$$\|b+H\| = \inf\{\|b'\| \,\vdots\, b'-b \in H\} = \inf\{\|b'\| \,\vdots\, b' \in b+H\}$$

i.e., it is the distance between the set $b+H$ and 0 . Let b_0 be any element of B such that $\xi(b_0) = 1$. Then

$$b+H = \{b' \vdots \xi(b') = \xi(b_0)\} = \{b' \vdots \xi(b') = 1\} .$$

The quotient space B/H is Ban_1-isomorphic to $\mathbb{R}$ and the unit sphere of B/H consists of two elements: b_0+H and $-b_0+H$, and none of them belongs to

$$\pi(\bigcirc B) = \{b'+H \vdots\, b' \in B\}$$

because, in virtue of (6.10), b_0+H is disjoint with $\bigcirc B$.

In §3 we have shown that replacing the category Ban_1 by the category of ω-balloons can save condition (i) of 6.7 . Yet, it cannot save condition (ii) .

6.11. Notes. Linton [1966], [1969b] proved 6.7 and 6.8 under the assumption (6.8); in fact, the theorem was mostly used in the case where $\mathfrak{A} =$ Ens . For a discussion of this theorem, see Duskin [1969] .

T. Świrszcz [1973a], [1973b] found another proof of 6.7 and observed that in this case (6.8) was superfluous. This enabled him to apply 6.7 to 7.3 below.

J. Wick-Negrepontis [1971] considered the monad generated by the pair

$$\text{(6.11)} \qquad \mathrm{Ens} \underset{\bigcirc}{\overset{C\bigcirc^{*}\ell}{\rightleftarrows}} \mathrm{Bcf}_{\mathbb{C}}$$

here $\mathrm{Bcf}_{\mathbb{C}}$ may be described as the category of commutative C^*-algebras with units and unit-preserving C^*-algebra homomorphisms. $\bigcirc$ is the closed-unit-ball functor and $C\bigcirc^*\ell$ is its left adjoint

$$C(\bigcirc^*\ell(X,\mathbb{C})) \cong C(\bigcirc\ell_\infty(X,\mathbb{C})) = C(D^X) \quad ,$$

where X is any set,

$$D = \bigcirc\mathbb{C} = \{z\in\mathbb{C} : |z|\leq 1\} \quad ;$$

D^X is the product of copies of D indexed by X , with the product topology, i.e., D^X is the unit ball $\bigcirc\ell_\infty(X,\mathbb{C})$ with the *weak topology, naturally equivalent to $\bigcirc^*\ell(X,\mathbb{C})$.

Using Linton's criterion J. Wick-Negrepontis showed that the functor $\bigcirc$ in (6.11) is quasi-monadic. She also used the theorem (proved by Duskin [1969], Section 5) that the covariant functor

$$\text{(6.12)} \qquad \text{Comp}^* \xrightarrow[\langle ?,D\rangle]{} \text{Ens}$$

obtained from the principal contravariant functor on Comp, has a left adjoint determined by $X \mapsto D^X$ and is also quasi-monadic. It turns out that the monads determined by (6.11) and (6.12) are identical; therefore their categories of algebras are identical and this yields the quasi-isomorphism of Comp* and Bcf_C , which is just the Gelfand duality between compact spaces and commutative C*-algebras with units. In this way J.Wick-Negrepontis found a new proof of the Gelfand duality theorem; yet, the proof is technically tedious and even the purely functional-analytic part of it takes more time than the classical proofs of the Gelfand duality.

A similar proof of the Kakutani duality between compact spaces and MI-spaces was found by John Gray and one of his students (unpublished).

Linton [1970a] outlined such a proof for the Stone duality between compact 0-dimensional spaces and Boolean algebras; in this case the proof is technically simpler because Boolean algebras are equationably definable algebras and one may apply general theorems about monadicity of such a category.

§7. <u>Compact</u> <u>convex</u> <u>sets</u>. We shall first discuss convex subsets of real vector spaces and then compact convex subsets of locally convex Hansdorff spaces.

7.1. Let us consider the monad determined by

$$(7.1) \qquad \text{Ens} \underset{U}{\overset{G}{\rightleftarrows}} \text{Conv}$$

where $G(X)$ is the set of formal convex combinations of elements of X , i.e., the set

$$G(X) = \{ \sum_{i=1}^{n}{}^{*} s_i x_i : s_i \geq 0 \ \text{ for } \ i=1,\ldots,n \ \text{ and } \ \sum_{i=1}^{n} s_i = 1\} \quad .$$

It is clear that G is a left adjoint of the forgetful functor U . Applying Linton's criterion 6.8 , T. Świrszcz [1973b] proved that the functor U in (7.1) is <u>not</u> monadic.

This can also be proved directly: the set X in 3.12 is an algebra for the monad generated by (7.1) and yet X is not a convex set.

7.2. We shall now consider the monad determined by

$$(7.2) \qquad \text{Comp} \underset{U}{\overset{S}{\rightleftarrows}} \text{Compconv} \quad .$$

An object of Compconv is a compact convex subset K of a locally convex Hausdorff space (understood as a set K with a compact topology and a structure of convex combinations such that K can be embedded into a locally convex Hausdorff topological vector space E ; the elements of $E \setminus K$ do not matter). A morphism in Compconv is any continuous affine map. If X is a compact space, then

$$S(X) = \{\mu \in M(X) : \mu \geq 0 \ \& \ \mu(1_X) = 1\}$$

(the set of regular Borel probability measures on X with the *weak topology) is the Choquet simplex whose extreme boundary $\partial_e S(X)$ is (homeomorphic to) the given space X (#23.7.1) . If $\phi: X \to X'$ is a morphism in Compconv, then

$$S(\phi) : S(X) \longrightarrow S(X')$$

is the induced transformation of measures. The unit of the adjunction (7.2) is

(7.3) $$\delta^{(X)}: X \longrightarrow US(X)$$

(the Dirac measures). From #12.2.1, #4.5.4 and #23.7.2 it follows that the counit

(7.4) $$\varrho_K : SU(K) \to K$$

assigns to each regular Borel probability measure μ on UK (i.e., on K) its centroid (center of gravity). Indeed,

(7.5) $$\rho_K \circ \delta^{(UK)} = 1_{UK} ,$$

i.e.

(7.6) $$\rho_K(\delta_x) = x \quad \text{for} \quad x \quad \text{in} \quad K .$$

Since (7.4) is affine,

$$\rho_K(s_1\delta_{x_1}+\ldots+s_n\delta_{x_n}) = s_1x_1+\ldots+s_nx_n ,$$

whenever $s_1,\ldots,s_n \in [0,1]$, $s_1+\ldots+s_n = 1$, and $x_1,\ldots,x_n \in K$. Thus, for each measure of the form

(7.7) $$\mu = s_1\delta_{x_1}+\ldots+s_n\delta_{x_n}$$

$\rho_K(\mu)$ is the centroid of μ. By the Krein-Milman theorem measures (7.7) are dense in $S(K)$ and ρ_K is continuous; therefore $\rho_K(\mu)$ is a centroid of μ for each μ in $S(K) = S(UK)$.

The monad in question is $(T,\delta, U(\rho_S))$, where

$$T\colon \mathrm{Comp} \to \mathrm{Comp}$$

is given by $T(X) = US(X)$, δ is given by (7.3) and

$$U(\varrho_{S(X)}) : USUS(X) \to US(X)$$

is the map assigning to each probability measure on $US(X)$ its centroid in $US(X)$. In the sequel we shall often simplify the notation omitting the letter U , i.e., using the same symbol for a compact convex set and its underlying compact space.

7.3. Proposition (Świrszcz). <u>The functor</u> U <u>in</u> (7.2) <u>is monadic</u>.

In other words, let X be any compact space and let

$$\gamma : S(X) \to X$$

be a continuous map satisfying the following conditions:

(7.8) $$\gamma(\delta_x) = x \quad \text{for} \quad x \quad \text{in} \quad X \; ,$$

and

(7.9) $$\gamma S(\gamma) = \gamma \varrho_{S(X)}$$

i.e., the diagram

$$\begin{array}{ccc} SS(X) & \xrightarrow{S(\gamma)} & S(X) \\ \downarrow{\scriptstyle \varrho_{S(X)}} & & \downarrow{\scriptstyle \gamma} \\ S(X) & \xrightarrow[\gamma]{} & X \end{array}$$

is commutative. Then there is a homeomorphism ϕ from X onto a compact convex set K such that γ becomes the centroid operation, i.e., the following diagram is commutative:

(7.10)
$$\begin{array}{ccc} S(X) & \xrightarrow{S(\phi)} & S(K) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\rho_K} \\ X & \xrightarrow[\phi]{} & K \end{array}$$

Proof. (P. Taylor). Step(I). Define convex combinations of elements of X as

(7.11)
$$sx + (1-s)y = \gamma(s\delta_x+(1-s)\delta\ )$$

where $x , y \in X$, $0 \leq s \leq 1$. From (7.8) it follows that

$$1 \cdot x + 0 \cdot y = x \quad \text{and} \quad sx + (1-s)x = x \quad ,$$

but we still have to show that (7.11) is really a structure of convex combinations induced by a vector-space structure.

Step (II). Define

$$A = \{f \in C(X,R) : \forall_{\mu \in S(X)} \ \mu(f) = f(\gamma(\mu))\} \quad .$$

Then A is a closed linear subspace of $C(X,R)$ and $1_X \in A$.

Step (III). The elements of A are affine, i.e.,

$$f(sx+(1-s)y) = sf(x) + (1-s)f(y)$$

for x,y in X , $0 \leq s \leq 1$, and f in A . Indeed, by (7.11),

$$f(sx+(1-s)y) = f(\gamma(s\delta_x+(1-s)\delta_y))$$

$$= (s\delta_x+(1-s)\delta_y)(f) = sf(x)+(1-s)f(y) \quad .$$

Step(IV). Condition (7.9) implies that γ is affine, i.e.,

$$(7.12) \qquad (s\mu_1+(1-s)\mu_2) = s\gamma(\mu_1) + (1-s)\gamma(\mu_2)$$

for μ_1, μ_2 in $S(X)$, $0 \leq s \leq 1$. Indeed, denote

$$\vartheta = s\delta_{\mu_1}^{(S(X))} + (1-s)\delta_{\mu_2}^{(S(X))} \quad .$$

Then $\vartheta \in SS(X)$ and

$$\begin{array}{ccc}
\vartheta & \xmapsto{\;S(\gamma)\;} & s\delta_{\gamma(\mu_1)}^{(X)} + (1-s)\delta_{\gamma(\mu_2)}^{(X)} \\
\Big\downarrow \rho_{S(X)} & & \Big\downarrow \gamma \\
 & & s\gamma(\mu_1)+(1-s)\gamma(\mu_2) \\
 & & \| \\
s\mu_1+(1-s)\mu_2 & \xmapsto[\;\gamma\;]{} & \gamma(s\mu_1+(1-s)\mu_2)
\end{array} \quad .$$

Thus, (7.9) yields (7.14) .

Step (V). The set

$$P = \{\mu-\nu : \mu,\nu \in S(X) \ \& \ \gamma(\mu) = \gamma(\nu)\}$$

is convex and symmetric in $M(X,\mathbb{R})$.

Step (VI). P is *weakly compact in $M(X,\mathbb{R})$. Indeed denote

$$Q = \{(\mu,\nu) : \mu,\nu \in S(X) \ \& \ \gamma(\mu) = \gamma(\nu)\} .$$

Then Q is a closed subset of $S(X) \times S(X)$ and hence it is compact; the function $(\mu,\nu) \mapsto \mu-\nu$ maps Q onto P and is continuous.

Step (VII). The set

$$L = \bigcup_{n=1}^{\infty} nP = \bigcup_{s \geq 0} sP$$

is a linear subset of $M(X,\mathbb{R})$.

Step (VIII). Let $\lambda \in L$ and $\|\lambda\| = 2$. Then $\lambda \in P$. Indeed, from (7.16) it follows that there exist μ' and ν' in $S(X)$ and $c > 0$ such that

$$\lambda = c(\mu'-\nu') \text{ and } \gamma(\mu') = \gamma(\nu') .$$

Since $\|\mu'\| = \|\nu'\| = 1$ and

$$2 = \|c(\mu'-\nu')\| \leq 2c$$

we infer that $c \geq 1$. Denote $\mu = \lambda^{+}$ and $\nu = \lambda^{-}$. Then $\mu \geq 0$, $\nu \geq 0$, $\mu \wedge \nu = 0$, $\lambda = \mu - \nu$ and

$$\|\mu\| + \|\nu\| = \|\lambda\| = 2 \quad .$$

By formula (12) in #3.5.2 ,

$$\mu = (c\mu' - c\nu')^{+} = c\mu' - (c\mu' \wedge c\nu')$$

and hence $\|\mu\| = c - c\,\|\mu' \wedge \nu'\|$. Similarly,

$$\|\nu\| = \|c\nu' - (c\mu' \wedge c\nu')\| = c - c\|\mu' \wedge \nu'\| \quad .$$

Hence $\|\mu\| = \|\nu\| = 1$ and $\mu, \nu \in S(X)$. If $c=1$, then $\mu' \wedge \nu' = 0$ and $\mu = \mu'$, $\nu = \nu'$. Thus, we may assume that $c > 1$. Denote

$$\omega = \frac{c}{c-1}(\mu' \wedge \nu') \text{ and } s = c^{-1} \quad .$$

Then $\mu' = s\mu + (1-s)$, $\nu' = s\nu + (1-s)\omega$, $\omega \in S(X)$ and $0<s<1$. Consequently,

$$s\gamma(\mu) + (1-s)\gamma(\omega) = \gamma(\mu') = \gamma(\nu') = s\gamma(\nu) + (1-s)\gamma(\omega) \quad .$$

Hence $\gamma(\mu) = \gamma(\nu)$. We have shown that $\lambda \in P$.

Step (IX). Denote $P' = \{\lambda \in L : \|\lambda\| \leq 2\}$. Then $P = P'$. Indeed, the inclusion $P \subset P'$ is obvious. Let $\lambda' \in P'$.

Denote

$$\lambda = \frac{2}{\|\lambda'\|}\lambda' = \frac{1}{t}\lambda' \quad .$$

Then $\lambda \in L$ and $\|\lambda\| = 2$. By (VIII) , $\lambda \in P$. Since $\lambda' = t\lambda$ and $0 \leq t \leq 1$, λ' belongs to P as well.

Step(X). L is *weakly closed in $M(X,\mathbb{R})$. This follows from (VIII) and the Krein-Šmulyan theorem ([DSI], V.5.7) .

Step (XI). If $x,y \in X$ and $x \neq y$, then $\delta_x - \delta_y \notin L$. Indeed, suppose that $\delta_x - \delta_y \in L$. Since $\|\delta_x - \delta_y\| = 2$, by (VIII) we get $\delta_x - \delta_y \in P$ and $\gamma(\delta_x) = \gamma(\delta_y)$; hence, by (7.8) , $x = y$.

Step (XII). Denote $L^{\perp} = \{f \in C(X,\mathbb{R}) : \forall_{\nu \in L}\ \nu(f) = 0\}$. Then $L^{\perp} \subset A$. Indeed, let $f \in L^{\perp}$ and $\mu \in S(X)$. We are to show that $\mu(f) = f(\gamma(\mu))$. Denote $x = \gamma(\mu)$ and $\nu = \mu - \delta_x$. Then, by (7.8), $\gamma(\mu) = \gamma(\delta_x)$ and $\nu \in P$. Consequently, $\nu(f) = 0$, i.e., $\mu(f) = f(x)$.

Step (XIII). Denote $A^{\perp} = \{\nu \in M(X,\mathbb{R}) : \forall_{f \in A}\ \nu(f) = 0\}$. Then $A^{\perp} = L$. Indeed, from (XII) it follows that $L^{\perp\perp} \supset A^{\perp}$. Since L is *weakly closed, $L = L^{\perp\perp}$. Hence $L \supset A^{\perp}$. On the other hand, if $\lambda \in P$, then $\lambda = \mu - \nu$ with $\mu, \nu \in S(X)$ and $\gamma(\mu) = \gamma(\nu)$; hence, for each f in A ,

$$\lambda(f) = \mu(f) - \nu(f) = f(\gamma(\mu)) - f(\gamma(\nu)) = 0 \quad ,$$

i.e., $\lambda \in A^{\perp}$. Hence $L \subset A^{\perp}$ and $L = A^{\perp}$.

Step (XIV). A separates X , i.e., $x \neq y$ implies the existence of an f in A such that $f(x) = f(y)$. Indeed, otherwise $\delta_x - \delta_y$ would belong to $A^{\perp}$ in spite of (X) and (XII) .

Step (XV). Denote $\phi(x) = \text{rest}_A\, \delta_x$ and $K = \phi(X)$. By (XIII) , ϕ is one-to-one. By (III), ϕ is affine and K is convex. Since X is compact, ϕ is a homeomorphic injection from X into $\mathcal{O}^*A$ and K is *weakly compact.

Step (XVI). The diagram (7.10) is commutative. Indeed, let $\mu \in S(X)$. Then for each f in A

$$\phi(\gamma(\mu).f = f(\gamma(\mu))$$

whereas, by #18.3.3 ,

$$\rho_K(S\phi.\mu).f = \kappa_f(\rho_K\nu) = \int_{S(K)} \kappa_f d\nu = \int_{S(X)} (\kappa_f \circ \phi) d\mu = \mu(f) \quad ,$$

where $\kappa_f(\xi) = \xi(f)$ for ξ in A^* and $\nu = S\phi.\mu$. ∎

7.4. Exercises. (A) What is the counit of the monad considered in 7.1 ? How is it related to the map (7.4) ?

(B) Show that $\gamma(\mu)$ need not coincide with $\gamma(\mu')$ in

(VII) . Hint: Find two probability measures μ, ν on the interval $I = [0,1]$ such that

$$\rho_I(\mu) = \rho_I(\nu)$$

and yet $\rho_I((\mu-\nu)^+) = \rho_I((\mu-\nu)^-)$.

7.5. Notes. The original proof of 7.3, due to T. Świrszcz [1973b] , used the generalized Linton Theorem 6.7 . In August 1973 Peter Taylor was challenged to find a direct proof of 7.3 not using theorems of category theory and produced the proof presented above. Both proofs are of approximately the same length and both use the Krein-Šmulyan theorem.

In Taylor's proof, (7.9) was used only to show that γ is affine; note that (7.12) appears much weaker than (7.9) .

Comparing 7.1 with 7.3 one may say that "convexity is not an algebraic property but compactness make it algebraic". Of course, the continuity of γ in 7.3 is crucial.

Świrszcz also proved that the forgetful functors

$$\text{Compconv} \to \text{Ens} \quad \text{and} \quad \text{Compsaks} \to \text{Ens}$$

(cf. Semadeni [1970]) are quasi-monadic. The last statement means that $(\text{Ban}_1)^*$ is quasi-monadic over Ens, a result which should be compared with 4.2 .

7.3 may be interpreted as a statement that "wherever we have a reasonable concept of the centroid of a probability measure on a compact space X , the space X must be not only convex but also locally convex". Here "reasonable" refers to (7.9) which is a property of iterated integrals related to the anologous properties for the monads for the categories Vect, Ban_1 and Conv.

Using the notation

$$(7.13) \qquad \int_K x d\mu(x)$$

for the centroid of a measure μ in $S(K)$, one may write (7.9) in the form

$$(7.14) \qquad \int_X x d\Big(\int_{S(X)} v d\lambda(v)\Big)(x) = \int_{S(X)} \Big[\int_X x dv(x)\Big] d\lambda(v)$$

for any λ in $SS(X)$. A formula of the form (7.14) can be deduced from a more general formula in Bourbaki [1956] , §3 , N°2 ; in Bourbaki's book such a formula, however, has a somewhat different meaning: the definition of the integral (7.13) uses any continuous affine function on K rather than any continuous function or K . This is perhaps the reason why a formula of the type (7.14) may imply the local convexity.

Index of Categories

Symbol	Object	Morphism
Ban_1	Banach space	linear contraction, i.e., linear operator of norm ≤ 1
$Bcf_{\mathbb{C}}$	commutative C*-algebra with unit	C*-algebra homomorphism, unit-preserving
Comp	compact (Hausdorff) space	continuous map
Conv	convex subset of a real vector space	affine map (i.e., preserving convex combinations)
Compconv	compact convex subset of a real locally convex Hausdorff vector space	continuous affine map
Ens	set	any map
Top	topological space	continuous map
Vect	vector space over $\mathbb{F}$ (where either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$)	linear map

Bibliography

Beck, J.H. [1967]: Triples, Algebras, and cohomology (thesis), Columbia U., N.Y., 1967.

Bourbaki, N. [1956]: Eléments de Mathématiques, Livre VI. Chapitre 5 : Intégration des mesures, Paris 1956.

Dixmier, J. [1948]: Sur un théorème de Banach, Duke Math.J., 15(1948), pp. 1057-1071.

Dubuc E.J. [1970]: Kan extensions in enriched category theory, Lecture Notes in Math. 145(1970).

Dunford, N., Schwartz, J.T. [1958]: Linear operators. I. General theory, New York 1958.

Duskin, J. [1969]: Variations on Beck's tripleability criterion, in: Reports of the Midwest Category Seminar III, Lecture Notes in Math. 106(1969), pp. 74-129.

Eilenberg, S., Kelly, S. [1966]: Closed categories, in: Proc. of the conference on Categorical Algebra (La Jolla 1965), New York 1966, pp. 421-562.

Eilenberg, S., Moore, J.C. [1965]: Adjoint functors and triples, Illinois J. Math. 9(1965), pp. 381-398.

Godement, R. [1958]: Théorie des faisceaux, Paris 1958.

Herrlich, H., Strecker, G.E. [1971]: Algebra topology = compactness, General Topology and its Applications 1(1971), pp. 283-287.

Huber, P.J. [1961]: Homotopy theory in general categories, Math. Annalen 144(1961), pp. 361-385.

Kleisli, H. [1965]: Every standard construction is induced by a pair of adjoint functions, Proc. Amer. Math. Soc. 16(1965), pp. 544-546.
[1973]: Resolutions in additive and non-additive categories, Queen's Papers in Pure and Applied Math. No. 32(1973).

Linton, F.E.J. [1966]: Some aspects of equational categories, in: Proc. Conference Categorical Algebra (La Jolla 1965), New York 1966, pp. 84-94.
[1969a]: An outline of functorial semantics, in: Seminar on Triples and categorical Homology Theory, Lecture Notes in Math. 80(1969), pp. 7-52.
[1969b]: Applied functorial semantics, II, in: Seminar on Triples and categorical Homology Theory, Lecture Notes in Math. 80(1969), pp. 53-74.
[1970a): Applied functorial semantics, I, in: Edgar Raymond Lorch Sixtieth Anniversary Volume, Bologna 1970, pp. 1-13.

[1970b]: Preliminary report on a pullback lemma for Banach Spaces and the functorial semantics of double dualization, April 1970; with Addendum (mimeographed, unpublished).

MacLane, S. [1971]: Categories for the working mathematician, New York 1971.

Manes, E.G. [1969]: A triple theoretic construction of compact algebras, in: Seminar on Triples and Categorical Homology Theore, Lecture Notes in Math. 80(1969), pp. 91-118.

Paré, R. [1971]: On absolute colimits, J. Algebra 19(1971), pp. 80-95.

Schubert, H. [1972]: Categories, New York 1972.

Semadeni, Z. [1970]: The Banach-Mazur functor and related functors, Prace Mat. 14(1970), pp. 171-182.
[1971]: Banach spaces of continuous functions, Vol. I, Warsaw 1971.
[1972]: A simple topological proof that the underlying set functor for compact spaces is monadic, Preprint 36, Queen's University, Kingston, Ontario, 1972.

Świrszcz, T. [1973a]: Monadic functors and categories of convex sets, preprint, Institute of Math. of the Polish Acad. of Sci., Warsaw 1973, to appear.
[1973b]: Monadic functors in convexity, to appear in Bull. Acad. Polon. Sci.

Wick-Negrepontis, J. [1971]: Duality in analysis from the point of view of triples, J. Algebra 19(1971), pp. 228-253.

Wilansky, A. [1964]: Functional analysis, New York 1964.