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ON UNIFORMLY CONVEX FUNCTIONS AND UNIFORMLY SMOOTH FUNCTIONS

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ABSTRACT. In this paper, we study uniformly convex functions and uniformly smooth functions in the framework of the nonstandard analysis. We show that in a Banach space, a proper, lower semicontinuous and convex function f is uniformly convex on the whole space if and only if its conjugate function is uniformly Fréchet differentiable on $R(\partial f)$. Let $\varphi : [0,\infty) \to (-\infty,\infty]$ be a function. We characterize the uniform convexity and the uniform smoothness of the function $x \mapsto \varphi(\|x\|)$ on bounded balls in a normed linear space. We also show sufficient conditions which ensure the uniform convexity and the uniform smoothness of the function $x \mapsto \varphi(\|x\|)$ on a whole normed linear space.

1. Introduction. In 1983, Zălinescu [10] studied the uniformly convex functions with some characterizations and examples of such functions. He showed that if a proper, lower semicontinuous and convex function defined on a reflexive Banach space is uniformly convex on the whole space then its conjugate function is uniformly Fréchet differentiable on the interior of the domain of the conjugate function and that the converse is true under some conditions. Let $\psi : [0,\infty) \to [0,\infty]$ be a function. He also characterized the uniform convexity of the function $x \mapsto \int_0^{\|x\|} \psi(t)\,dt$ defined on bounded balls in a Banach space. On the other hand, it is well known that in a Hilbert space H,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in H$ and $0 \le \lambda \le 1$. Lim [5], Prus and Smarzewski [6], Smarzewski [7] and Xu [8, 9] have studied inequalities that are analogous to (1.1) in a Banach space. These inequalities are related to the uniform convexity and the uniform Fréchet differentiability of the functional $x \mapsto \|x\|^p$.

In this paper, we study uniformly convex functions and uniformly smooth functions in the framework of the nonstandard analysis [3]. Let E be a real normed linear space, let $f : E \to (-\infty,\infty]$ be a function and let Y be a subset of E such that there exists $\varepsilon > 0$ with $Y + \{x \in E : \|x\| \le \varepsilon\} \subset \operatorname{dom} f$. We mean that f is uniformly smooth on Y if

$$\lim_{t\to 0} \frac{f(y+tu) - f(y)}{t}$$

exists uniformly for $y \in Y$ and $u \in E$ with $\|u\| = 1$. If f is convex and for each $y \in Y$, $\sup_{\|u\|=1} \overline{\lim_{t\to 0}} \left| \frac{f(y+tu)-f(y)}{t} \right| < \infty$, then f is uniformly smooth on Y if and only if f is uniformly Fréchet differentiable on Y. We show that in a Banach space, a proper, lower semicontinuous and convex function f is uniformly convex on the whole space if and only if its conjugate function is uniformly Fréchet differentiable on $R(\partial f)$. Let $\varphi : [0,\infty) \to$

$(-\infty, \infty]$ be a function. We characterize the uniform convexity and the uniform smoothness of the function $x \mapsto \varphi(\|x\|)$ on bounded balls in a normed linear space. We also show sufficient conditions which ensure the uniform convexity and the uniform smoothness of the function $x \mapsto \varphi(\|x\|)$ on a whole normed linear space.

2. Nonstandard analysis. We adopt the notational conventions and the framework for the nonstandard analysis described in [3]. For convenience, we state some definitions. We denote the set of all real numbers and the set of all positive real numbers by $\mathbb{R}$ and $\mathbb{R}_+$ respectively. Let $a, b \in {}^*\mathbb{R}$. We define symbols $\simeq, \gtrsim, \lesssim, \gnsim$ and $\lnsim$ as follows:

$a \simeq b$ if for any $\varepsilon \in \mathbb{R}_+$, $|a-b| < \varepsilon$;
$a \gtrsim b$ if $a > b$ or $a \simeq b$;
$a \lesssim b$ if $a < b$ or $a \simeq b$;
$a \gnsim b$ if $a > b$ and $a \not\simeq b$;
$a \lnsim b$ if $a < b$ and $a \not\simeq b$.

We recall that a is finite if there exists a standard positive real number c with $|a| \le c$ and that a is infinite if a is not finite. Let E be a normed linear space and let $x, y \in {}^*E$. We write $x \simeq y$ if $\|x-y\| \simeq 0$, and we denote by $\mu_E(x)$ the set $\{z \in {}^*E : z \simeq x\}$.

3. Preliminaries. Throughout this paper, all vector spaces are real, o denotes the origin of a vector space and if E is a normed linear space then $E^\#$ denotes its topological dual. Let E be a normed linear space. We write $\langle x^\#, x\rangle$ in place of $x^\#(x)$ for $x \in E$ and $x^\# \in E^\#$. Let C and D be subsets of E. $C+D$ denotes the set $\{x+y \in E : x \in C, y \in D\}$ and $\operatorname{Int} C$ denotes the set of all interior points of C. C is said to be convex if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $0 \le \lambda \le 1$. For a positive real number a, $S_E(a)$ and $B_E(a)$ denote $\{x \in E : \|x\| = a\}$ and $\{x \in E : \|x\| \le a\}$ respectively. E is said to be uniformly convex if for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $x, y \in S_E(1)$,

$$\|x-y\| \ge \varepsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \le 1-\delta.$$

E is said to be uniformly smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists uniformly for $x, y \in S_E(1)$. The modulus of convexity of E and the modulus of smoothness of E are defined respectively by

$$\delta(\varepsilon) = \inf\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \ge \varepsilon\}, \quad 0 \le \varepsilon \le 2,$$

and

$$\rho(\tau) = \sup\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\}, \quad \tau > 0.$$

It is easy to see that E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$ and that E is uniformly smooth if and only if $\lim_{\tau \downarrow 0} \frac{\rho(\tau)}{\tau} = 0$. In the nonstandard representation, E is uniformly convex if and only if for any $x, y \in {}^*E$ such that $\|x\|, \|y\|$ are finite and $\|x\| \simeq \|y\|$,

$$x \not\simeq y \text{ implies } \left\|\frac{x+y}{2}\right\| \lnsim \frac{\|x\| + \|y\|}{2},$$

and E is uniformly smooth if and only if

$$\frac{\|x+u\| - \|x\|}{\|u\|} \simeq \frac{\|x-u\| - \|x\|}{-\|u\|}$$

for any $x \in {}^*E$ such that $\|x\|$ is finite and $\|x\| \not\simeq 0$ and for any $u \in {}^*E \setminus \{o\}$ with $u \simeq o$. Let $f : E \to (-\infty, \infty]$ be a function. $\operatorname{dom} f$ denotes the set $\{x \in E : f(x) < \infty\}$. f is said to be proper if $\operatorname{dom} f \ne \emptyset$. Let X be a convex subset of E. f is said to be convex on X if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in \operatorname{dom} f \cap X$ and for any $\lambda \in [0,1]$. f is said to be strictly convex on X if the above inequality is strict for $x \ne y$ and $\lambda \in (0,1)$. Let $g : E \to (-\infty, \infty]$ be a proper and convex function. $g^\# : E^\# \to (-\infty, \infty]$ denotes the conjugate function of g which is defined by

$$g^\#(x^\#) = \sup\{\langle x^\#, x\rangle - g(x) : x \in E\}, \quad x^\# \in E^\#,$$

and $g^{\#\#} : E \to (-\infty, \infty]$ denotes the second conjugate function of g which is defined by

$$g^{\#\#}(x) = \sup\{\langle x^\#, x\rangle - g^\#(x^\#) : x^\# \in E^\#\}, \quad x \in E.$$

It is well known (cf. [1]) that $g = g^{\#\#}$ if and only if g is lower semicontinuous. The subdifferential of g at $x \in E$ is the set

$$(\partial g)(x) = \{x^\# \in E^\# : g(y) \ge g(x) + \langle x^\#, y-x\rangle \text{ for all } y \in E\}.$$

By ∂g, we mean the set $\{(x, x^\#) \in E \times E^\# : x^\# \in (\partial g)(x)\}$ and by $R(\partial g)$, we mean the set $\bigcup\{(\partial g)(x) : x \in E\}$. It is well known (cf. [1]) that $(x, x^\#) \in \partial g$ if and only if $\langle x^\#, x\rangle = g(x) + g^\#(x^\#)$. Let φ be a real valued convex function defined on an open interval I of $\mathbb{R}$. It is also well known (cf. [4]) that φ is continuous on I and that if φ is differentiable on I then its derivative φ' is continuous on I.

4. Uniformly convex functions and uniformly smooth functions. We start this section by some definitions. Let $g : E \to (-\infty, \infty]$ be a function and let X be a convex subset of E. g is said to be uniformly convex on X if for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that

$$\|x-y\| \ge \varepsilon \text{ implies } g\Big(\frac{x+y}{2}\Big) \le \frac{g(x)+g(y)}{2} - \delta$$

for any $x, y \in \operatorname{dom} g \cap X$. Let $h : E \to (-\infty, \infty]$ be a function and let Y be a subset of E such that there exists $\varepsilon \in \mathbb{R}_+$ with $Y + B_E(\varepsilon) \subset \operatorname{dom} h$. We say that h is uniformly smooth on Y if

$$\lim_{t\to 0} \frac{h(y+tu) - h(y)}{t}$$

exists uniformly for $y \in Y$ and $u \in S_E(1)$. We recall that h is uniformly Fréchet differentiable on Y if for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $y \in Y$, there exists $y^\# \in E^\#$ such that

$$0 < |t| \le \delta \text{ and } u \in S_E(1) \text{ implies } \left|\frac{h(y+tu) - h(y)}{t} - \langle y^\#, u\rangle\right| \le \varepsilon.$$

If h is convex and for each $y \in Y$, $\sup_{\|u\|=1} \varlimsup_{t\to 0} \left|\frac{h(y+tu)-h(y)}{t}\right| < \infty$, then h is uniformly smooth on Y if and only if h is uniformly Fréchet differentiable on Y. In the nonstandard representation, h is uniformly smooth on Y if and only if

(4.1) $$t \simeq 0 \text{ and } s \simeq 0 \text{ implies } \frac{h(y+tu) - h(y)}{t} \simeq \frac{h(y+su) - h(y)}{s}$$

for all $y \in {}^*Y$, $u \in {}^*S_E(1)$ and $t, s \in {}^*\mathbb{R} \setminus \{0\}$. If h is convex, then h is uniformly smooth on Y if and only if

$$u \simeq o \text{ implies } \frac{h(y+u)-h(y)}{\|u\|} \simeq \frac{h(y-u)-h(y)}{-\|u\|}$$

for all $y \in {}^*Y$ and $u \in {}^*E \setminus \{o\}$. Concerning uniform convexity and uniform smoothness, we have the following propositions. The first one is Remark 2.6 in [10].

Proposition 4.1 (Zălinescu). *Let E be a normed linear space and let X be a convex subset of E. Let $f : E \to (-\infty, \infty]$ be a proper and convex function. Then the following are equivalent;*

(i) *f is uniformly convex on X, i.e., for any $x, y \in {}^*(\text{dom}\, f \cap X)$,*

$$x \not\simeq y \text{ implies } f\Big(\frac{x+y}{2}\Big) \lnsim \frac{f(x)+f(y)}{2},$$

(ii) *for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $x, y \in \text{dom}\, f \cap X$ and $0 \le \lambda \le 1$,*

$$\|x-y\| \ge \varepsilon \text{ implies } f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) - \lambda(1-\lambda)\delta,$$

i.e., for any $x, y \in {}^*(\text{dom}\, f \cap X)$ and for any $\lambda \in {}^*(0,1)$,

$$x \not\simeq y \text{ implies } \frac{f(\lambda x + (1-\lambda)y)}{\lambda(1-\lambda)} \lnsim \frac{\lambda f(x) + (1-\lambda) f(y)}{\lambda(1-\lambda)}.$$

Proposition 4.2. *Let E be a normed linear space and let $f : E \to (-\infty, \infty]$ be a function. Let Y be a subset of E such that there exists $\varepsilon \in \mathbb{R}_+$ with $Y + B_E(\varepsilon) \subset \text{dom}\, f$. Assume that f is uniformly smooth on Y. Then for any $y, z \in {}^*Y$ with $y \ne z$ and for any $\lambda \in {}^*(0,1)$,*

$$y \simeq z \text{ implies } \frac{f(\lambda y + (1-\lambda)z)}{\lambda(1-\lambda)\|y-z\|} \simeq \frac{\lambda f(y) + (1-\lambda) f(z)}{\lambda(1-\lambda)\|y-z\|},$$

i.e., for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $y, z \in Y$ and $0 \le \lambda \le 1$,

$$\|y-z\| \le \delta \text{ implies } \Big|\lambda f(y) + (1-\lambda) f(z) - f(\lambda y + (1-\lambda) z)\Big| \le \lambda(1-\lambda)\varepsilon\|y-z\|.$$

PROOF. Let $y, z \in {}^*Y$ such that $y \ne z$ and $y \simeq z$, and let $\lambda \in {}^*(0,1)$. We may assume $\lambda \in {}^*(0, \frac{1}{2}]$. From (4.1), we get

$$\begin{aligned}\frac{f(y)-f(z)}{\|y-z\|} &= \frac{f\Big(z + \|y-z\| \cdot \dfrac{y-z}{\|y-z\|}\Big) - f(z)}{\|y-z\|} \\ &\simeq \frac{f\Big(z + \lambda\|y-z\| \cdot \dfrac{y-z}{\|y-z\|}\Big) - f(z)}{\lambda\|y-z\|} \\ &= \frac{f(\lambda y + (1-\lambda)z) - f(z)}{\lambda\|y-z\|},\end{aligned}$$

and hence we have

$$\frac{f(\lambda y + (1-\lambda)z)}{\lambda\|y-z\|} \simeq \frac{\lambda f(y) + (1-\lambda) f(z)}{\lambda\|y-z\|}.$$

Since $\lambda \not\simeq 1$, we obtain

$$\frac{f(\lambda y + (1-\lambda)z)}{\lambda(1-\lambda)\|y-z\|} \simeq \frac{\lambda f(y) + (1-\lambda) f(z)}{\lambda(1-\lambda)\|y-z\|}.$$

By the transfer principle, we obtain the standard representation. □

Let φ be a real valued convex function defined on an open interval I of $\mathbb{R}$. It is well known that if φ is strictly convex on I then for any bounded and closed interval $J(\subset I)$, φ is uniformly convex on J and that if φ is differentiable on I then for any bounded and closed interval $J(\subset I)$, φ is uniformly smooth on J. Let $\psi : \mathbb{R} \to \mathbb{R}$ be uniformly smooth on $\mathbb{R}$. It is easy to see that if $t, s \in {}^*\mathbb{R} \setminus \{0\}$, $t \ne s$ and $t \simeq s$ then $\psi'(t) \simeq \frac{\psi(s)-\psi(t)}{s-t} \simeq \psi'(s)$.

Next, we show relation between a proper, lower semicontinuous and convex function defined on a Banach space and its conjugate function. The following was partly obtained by Zălinescu [10]. In the following, the proof of (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) is essentially same as the proof by Zălinescu.

Theorem 4.1. *Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Then the following conditions are equivalent;*

(i) *f is uniformly convex on E,*

(ii) *for any $(x, x^\#) \in {}^*(\partial f)$ and for any $y \in {}^*E$,*

$$y \not\simeq x \text{ implies } f(y) \gnsim f(x) + \langle x^\#, y-x\rangle,$$

(iii) *for any $(x, x^\#) \in {}^*(\partial f)$ and for any $y \in {}^*E$,*

$$y \not\simeq x \text{ implies } \frac{f(y)-f(x)}{\|y-x\|} \gnsim \frac{\langle x^\#, y-x\rangle}{\|y-x\|},$$

(iv) *for any $(x, x^\#) \in {}^*(\partial f)$, for any $u^\# \in {}^*E^\# \setminus \{o\}$ with $u^\# \simeq o$ and for any $y \in {}^*E$,*

$$\langle x^\# + u^\#, y-x\rangle + f(x) \ge f(y) \text{ implies } y \simeq x,$$

(v) *${}^*(R(\partial f)) + \mu_{E^\#}(o) \subset {}^*(\text{dom}\, f^\#)$, and for any $(x, x^\#) \in {}^*(\partial f)$ and for any $u^\# \in {}^*E^\# \setminus \{o\}$ with $u^\# \simeq o$,*

$$\frac{f^\#(x^\# + u^\#) - f^\#(x^\#)}{\|u^\#\|} \simeq \Big\langle \frac{u^\#}{\|u^\#\|}, x \Big\rangle,$$

i.e., there exists $\varepsilon \in \mathbb{R}_+$ such that $R(\partial f) + B_{E^\#}(\varepsilon) \subset \text{dom}\, f^\#$, and $f^\#$ is uniformly Fréchet differentiable on $R(\partial f)$.

PROOF. (i) $\Rightarrow$ (ii). Let $(x, x^\#) \in {}^*(\partial f)$ and let $y \in {}^*E$ with $y \not\simeq x$. We may assume $y \in {}^*(\text{dom}\, f)$. Since $y \not\simeq x$, we have $f(\frac{x+y}{2}) \lnsim \frac{f(x)+f(y)}{2}$. Hence, by $(x, x^\#) \in {}^*(\partial f)$, we get

$$\begin{aligned} f(y) &\gnsim 2f\Big(\frac{x+y}{2}\Big) - f(x) \\ &= f(x) + 2\Big(f\Big(\frac{x+y}{2}\Big) - f(x)\Big) \\ &\ge f(x) + 2\Big\langle x^\#, \frac{x+y}{2} - x\Big\rangle \\ &= f(x) + \langle x^\#, y-x\rangle. \end{aligned}$$

Therefore (ii) is valid.

(ii) $\Rightarrow$ (iii). Let $(x, x^\#) \in {}^*(\partial f)$ and let $y \in {}^*E$ with $y \not\simeq x$. We may assume $y \in {}^*(\text{dom}\, f)$. If $\|y-x\|$ is finite, it is clear that (iii) is valid. Assume that $\|y-x\|$ is infinite. Put $u = x + \frac{y-x}{\|y-x\|}$. Then we have $\|u-x\| = 1$ and, by the convexity of f,

$$\frac{f(y)-f(x)}{\|y-x\|} \ge f(u) - f(x).$$

Hence, by (ii), we get

$$\frac{f(y)-f(x)}{\|y-x\|} \geq f(u)-f(x)$$
$$\gtrsim \langle x^{\#}, u-x\rangle$$
$$= \frac{\langle x^{\#}, y-x\rangle}{\|y-x\|}.$$

Therefore (iii) holds.

(iii) $\Rightarrow$ (iv). Let $(x,x^{\#}) \in {}^*(\partial f)$, let $u^{\#} \in {}^*E^{\#} \setminus \{o\}$ with $u^{\#} \simeq o$, and let $y \in {}^*E$ such that $\langle x^{\#}+u^{\#}, y-x\rangle + f(x) \geq f(y)$. Suppose $y \not\simeq x$. Then we get

$$\frac{\langle x^{\#}+u^{\#}, y-x\rangle}{\|y-x\|} \geq \frac{f(y)-f(x)}{\|y-x\|}$$
$$\gtrsim \frac{\langle x^{\#}, y-x\rangle}{\|y-x\|}.$$

So we have $\|u^{\#}\| \gtrsim 0$, which contradicts $u^{\#} \simeq o$. Therefore $y \simeq x$.

(iv) $\Rightarrow$ (v). Let $(x,x^{\#}) \in {}^*(\partial f)$ and let $u^{\#} \in {}^*E^{\#} \setminus \{o\}$ with $u^{\#} \simeq o$. First, we shall prove $x^{\#}+u^{\#} \in {}^*(\operatorname{dom} f)$. By the definition of $f^{\#}$, we have

$$f^{\#}(x^{\#}+u^{\#}) = \sup\{\langle x^{\#}+u^{\#}, y\rangle - f(y) : y \in {}^*E,$$
$$\langle x^{\#}+u^{\#}, y\rangle - f(y) \geq \langle x^{\#}+u^{\#}, x\rangle - f(x)\}.$$

Take $y \in {}^*E$ such that $\langle x^{\#}+u^{\#}, y\rangle - f(y) \geq \langle x^{\#}+u^{\#}, x\rangle - f(x)$. Then, by (iv), we get $y \simeq x$. Hence we obtain

$$\langle x^{\#}+u^{\#}, y\rangle - f(y) = (\langle x^{\#}, y\rangle - f(y)) + \langle u^{\#}, y\rangle$$
$$\leq f^{\#}(x^{\#}) + (\langle u^{\#}, y-x\rangle + \langle u^{\#}, x\rangle)$$
$$\lesssim f^{\#}(x^{\#}) + \langle u^{\#}, x\rangle.$$

So we have $x^{\#}+u^{\#} \in {}^*(\operatorname{dom} f)$. Next, we shall prove that the Fréchet derivative of f at $x^{\#}$ is x. By the definition of $f^{\#}$, we can choose $z \in {}^*E$ which satisfies

$$\frac{f^{\#}(x^{\#}+u^{\#}) - (\langle x^{\#}+u^{\#}, z\rangle - f(z))}{\|u^{\#}\|} \simeq 0$$

and

$$\langle x^{\#}+u^{\#}, z\rangle - f(z) \geq \langle x^{\#}+u^{\#}, x\rangle - f(x).$$

We have $z \simeq x$ by (iv). Since $(x,x^{\#}) \in {}^*(\partial f)$, we get

$$\langle x^{\#}+u^{\#}, z-x\rangle + f(x) - f(z)$$
$$\leq \langle x^{\#}+u^{\#}, z-x\rangle + f(x) - (\langle x^{\#}, z-x\rangle + f(x))$$
$$\leq \|u^{\#}\|\|z-x\|.$$

Hence we obtain

$$\frac{f^{\#}(x^{\#}+u^{\#}) - f^{\#}(x^{\#})}{\|u^{\#}\|}$$
$$\simeq \frac{\langle x^{\#}+u^{\#}, z\rangle - f(z) - f^{\#}(x^{\#})}{\|u^{\#}\|}$$
$$= \frac{\langle x^{\#}+u^{\#}, z\rangle - f(z) - (\langle x^{\#}, x\rangle - f(x))}{\|u^{\#}\|}$$
$$= \frac{\big(\langle x^{\#}+u^{\#}, z-x\rangle + f(x) - f(z)\big) + \langle u^{\#}, x\rangle}{\|u^{\#}\|}$$
$$\simeq \Big\langle \frac{u^{\#}}{\|u^{\#}\|}, x\Big\rangle.$$

Therefore $f^{\#}$ is uniformly Fréchet differentiable on $R(\partial f)$.

(v) $\Rightarrow$ (ii). Let $(x,x^{\#}) \in {}^*(\partial f)$ and let $y \in {}^*E$ with $y \not\simeq x$. Since $y \not\simeq x$, there exists a standard positive real number ε such that $\|y-x\| \geq 2\varepsilon$. By the transfer principle, there exists a standard positive real number δ such that for any $u^{\#} \in {}^*E^{\#}$,

$$0 < \|u^{\#}\| \leq \delta \text{ implies } \frac{f^{\#}(x^{\#}+u^{\#}) - f^{\#}(x^{\#}) - \langle u^{\#}, x\rangle}{\|u^{\#}\|} \leq \varepsilon.$$

Hence we get

$$f(y) = f^{\#\#}(y)$$
$$= \sup\{\langle y^{\#}, y\rangle - f^{\#}(y^{\#}) : y^{\#} \in {}^*E^{\#}\}$$
$$\geq \sup\{\langle x^{\#}+u^{\#}, y\rangle - f^{\#}(x^{\#}+u^{\#}) : u^{\#} \in {}^*E^{\#}, 0 < \|u^{\#}\| \leq \delta\}$$
$$\geq \sup\{\langle x^{\#}+u^{\#}, y\rangle - \Big(f^{\#}(x^{\#}) + \langle u^{\#}, x\rangle + \varepsilon\|u^{\#}\|\Big) : u^{\#} \in {}^*E^{\#}, 0 < \|u^{\#}\| \leq \delta\}$$
$$= \sup\{\langle u^{\#}, y-x\rangle - \varepsilon\|u^{\#}\| : u^{\#} \in {}^*E^{\#}, 0 < \|u^{\#}\| \leq \delta\} + \langle x^{\#}, y\rangle - f^{\#}(x^{\#})$$
$$\geq \delta\|y-x\| - \varepsilon\delta + \langle x^{\#}, y\rangle - \Big(\langle x^{\#}, x\rangle - f(x)\Big)$$
$$\gtrsim f(x) + \langle x^{\#}, y-x\rangle.$$

Therefore (ii) is valid.

(ii) $\Rightarrow$ (i). Let $y, z \in {}^*(\operatorname{dom} f)$ such that $y \not\simeq z$. By Theorem 2 in [2], there exists $(x,x^{\#}) \in {}^*(\partial f)$ such that

$$f\Big(\frac{y+z}{2}\Big) \simeq f(x) + \Big\langle x^{\#}, \frac{y+z}{2} - x\Big\rangle.$$

By (ii), we have $\frac{y+z}{2} \simeq x$, and hence $y \not\simeq x$ and $z \not\simeq x$. So we have $f(y) \gtrsim f(x) + \langle x^{\#}, y-x\rangle$ and $f(z) \gtrsim f(x) + \langle x^{\#}, z-x\rangle$. Hence we get

$$f\Big(\frac{y+z}{2}\Big) \simeq f(x) + \Big\langle x^{\#}, \frac{y+z}{2} - x\Big\rangle$$
$$= \frac{f(x) + \langle x^{\#}, y-x\rangle + f(x) + \langle x^{\#}, z-x\rangle}{2}$$
$$\lesssim \frac{f(y)+f(z)}{2}.$$

Therefore f is uniformly convex on E. □

Remark. If a Banach space is reflexive, Zălinescu [10] showed that if a proper and lower semicontinuous function $f : E \to (-\infty, \infty]$ is uniformly convex on E, then f is uniformly Fréchet differentiable on $\mathrm{Int}(\mathrm{dom}\, f^{\#})$. In the case, $R(\partial f)$ is open, and hence $R(\partial f) = \mathrm{Int}(\mathrm{dom}\, f^{\#})$.

5. Characterization of uniform convexity and uniform smoothness on bounded balls. In this section, we characterize the uniform convexity and the uniform smoothness of $\varphi(\|\cdot\|)$ on bounded balls in a normed linear space. The following is essentially same as Theorem 4.1.(ii) in [10]. Compare these statements.

Theorem 5.1 (Zălinescu). *Let E be a normed linear space and let $\varphi : [0,\infty) \to [0,\infty]$ be an increasing function. Let $M = \sup(\mathrm{dom}\,\varphi) > 0$. Then $\varphi(\|\cdot\|)$ is uniformly convex on $B_E(a)$ for any $a \in (0, M)$ if and only if φ is strictly convex on $\mathrm{dom}\,\varphi$ and E is uniformly convex.*

Proof. Suppose that φ is strictly convex on $\mathrm{dom}\,\varphi$ and that E is uniformly convex. Let $a \in (0, M)$ be a standard real number. We remark that φ is uniformly convex on $[0, a]$. Let $x, y \in {}^*B_E(a)$ such that $x \not\simeq y$. If $\|x\| \not\simeq \|y\|$, the uniform convexity of φ yields

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \le \varphi\Big(\frac{\|x\|+\|y\|}{2}\Big) \lesssim\!\!\!\!\!/\; \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2}.$$

So we may assume $\|x\| \simeq \|y\|$. Since $x \not\simeq y$, the uniform convexity of E yields

$$\Big\|\frac{x+y}{2}\Big\| \lesssim\!\!\!\!\!/\; \frac{\|x\|+\|y\|}{2}.$$

So we have

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \lesssim\!\!\!\!\!/\; \varphi\Big(\frac{\|x\|+\|y\|}{2}\Big) \le \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2}.$$

We shall show the necessity. Suppose that $\varphi(\|\cdot\|)$ is uniformly convex on $B_E(a)$ for any $a \in (0, M)$. Fix an element $x_0 \in E$ such that $\|x_0\| = 1$. Let r, s be standard real numbers such that $r, s \in [0, M]$ and $r \neq s$. Since r and s are different, we have $r \not\simeq s$. Assume $r, s < M$. In virtue of the uniform convexity of $\varphi(\|\cdot\|)$ on $B_E(\max\{|t|, |s|\})$, we get

$$\begin{aligned}\varphi\Big(\frac{r+s}{2}\Big) &= \varphi\Big(\Big\|\frac{rx_0+sx_0}{2}\Big\|\Big)\\ &\lesssim\!\!\!\!\!/\; \frac{\varphi(\|rx_0\|)+\varphi(\|sx_0\|)}{2}\\ &= \frac{\varphi(r)+\varphi(s)}{2}.\end{aligned}$$

Hence we obtain $\varphi\big(\frac{r+s}{2}\big) < \frac{\varphi(r)+\varphi(s)}{2}$. If $M \neq \infty$, $M \in \mathrm{dom}\,\varphi$ and r or s is equal to M, then we can also show $\varphi(\frac{r+s}{2}) < \frac{\varphi(r)+\varphi(s)}{2}$ from the convexity of φ. Next we shall show that E is uniformly convex. Suppose not. Let $b \in (0, M)$ be a standard real number. Then there are $x, y \in {}^*S_E(b)$ such that $x \not\simeq y$ and $\|\frac{x+y}{2}\| \simeq b$. The uniform convexity of $\varphi(\|\cdot\|)$ on $B_E(b)$ yields

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \lesssim\!\!\!\!\!/\; \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2} = \varphi(b).$$

But, by the continuity of φ on $\mathrm{Int}(\mathrm{dom}\,\varphi)$, we have

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \simeq \varphi(b),$$

which is a contradiction. Therefore E is uniformly convex. □

Using our theorem and Proposition 4.1, we have the following. Compare this with Theorem 2 in [9].

Theorem 5.2. *Let E be a normed linear space and let a be a positive real number. Let $\varphi : [0,\infty) \to [0,\infty)$ be a function such that $\varphi(0) = 0$ and it is strictly convex on $[0,\infty)$. Then E is uniformly convex if and only if there exists an increasing function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$, $g(t) > 0$ for all $t > 0$ and*

$$\varphi(\|\lambda x + (1-\lambda)y\|) \le \lambda\varphi(\|x\|) + (1-\lambda)\varphi(\|y\|) - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_E(a)$ and $0 \le \lambda \le 1$.

The following is the dual version of Theorem 5.1, which characterizes the uniform Fréchet differentiability of $\varphi(\|\cdot\|)$ on bounded balls.

Theorem 5.3. *Let E be a normed linear space and let $\varphi : [0,\infty) \to [0,\infty]$ be a strictly increasing and convex function such that $\varphi(0) = 0$ and $\varphi'(0) = 0$. Let $M = \sup(\mathrm{dom}\,\varphi) > 0$. Then $\varphi(\|\cdot\|)$ is uniformly Fréchet differentiable on $B_E(a)$ for any $a \in (0, M)$ if and only if φ is differentiable on $(0, M)$ and E is uniformly smooth.*

Proof. Suppose that φ is differentiable on $(0, M)$ and that E is uniformly smooth. Let $a \in (0, M)$ be a standard real number and let $x, u \in {}^*E$ such that $\|x\| \le a$, $u \simeq o$ and $u \neq o$. We remark that φ is uniformly smooth on $[0, a]$. If $x \simeq o$, we get

$$\begin{aligned}\frac{\varphi(\|x+u\|)-\varphi(\|x\|)}{\|u\|} &\simeq \frac{\varphi(\|x+u\|)-\varphi(\|x\|)}{\|x+u\|-\|x\|}\,\frac{\|x+u\|-\|x\|}{\|u\|}\\ &\simeq \varphi'(\|x\|)\frac{\|x+u\|-\|x\|}{\|u\|}\\ &\simeq \varphi'(0)\frac{\|x+u\|-\|x\|}{\|u\|}\\ &= 0\end{aligned}$$

and similarly,

$$\frac{\varphi(\|x-u\|)-\varphi(\|x\|)}{-\|u\|} \simeq 0.$$

If $x \not\simeq o$, we get

$$\begin{aligned}\frac{\varphi(\|x+u\|)-\varphi(\|x\|)}{\|u\|} &= \frac{\varphi(\|x+u\|)-\varphi(\|x\|)}{\|x+u\|-\|x\|}\,\frac{\|x+u\|-\|x\|}{\|u\|}\\ &\simeq \frac{\varphi(\|x-u\|)-\varphi(\|x\|)}{\|x-u\|-\|x\|}\,\frac{\|x-u\|-\|x\|}{-\|u\|}\\ &= \frac{\varphi(\|x-u\|)-\varphi(\|x\|)}{-\|u\|}.\end{aligned}$$

Hence we have

$$\frac{\varphi(\|x+u\|)-\varphi(\|x\|)}{\|u\|} \simeq \frac{\varphi(\|x-u\|)-\varphi(\|x\|)}{-\|u\|}.$$

On the other hand, it is easy to see that $x \in B_E(a)$, $u \in {}^*E \setminus \{o\}$ and $u \simeq o$ implies

$$\Big|\frac{\varphi(\|x+u\|)-\varphi(\|x\|)}{\|u\|}\Big| \lesssim \varphi'(\|x\|).$$

Therefore $\varphi(\|\cdot\|)$ is uniformly Fréchet differentiable on $B_E(a)$. Next we shall show the necessity. Suppose that $\varphi(\|\cdot\|)$ is uniformly Fréchet differentiable on $B_E(a)$ for any $a \in$

$(0, M)$. Fix an element $x_0 \in E$ such that $\|x_0\| = 1$. Let $r \in (0, M)$ be a standard real number and let $s \in {}^*\mathbb{R} \setminus \{0\}$ with $s \simeq 0$. Then we have

$$\begin{aligned}\frac{\varphi(r+s) - \varphi(r)}{s} &= \frac{\varphi(\|rx_0 + sx_0\|) - \varphi(\|rx_0\|)}{\|sx_0\|} \\ &\simeq \frac{\varphi(\|rx_0 - sx_0\|) - \varphi(\|rx_0\|)}{-\|sx_0\|} \\ &= \frac{\varphi(r-s) - \varphi(r)}{-s},\end{aligned}$$

which shows that φ is differentiable on $(0, M)$. Next we shall prove that E is uniformly smooth. Let $b \in (0, M)$ be a standard real number. Let $x \in {}^*S_E(b)$ and let $u \in {}^*E \setminus \{o\}$ with $u \simeq o$. Then we get

$$\begin{aligned}\frac{\|x+u\| - \|x\|}{\|u\|} &= \frac{\|x+u\| - \|x\|}{\varphi(\|x+u\|) - \varphi(\|x\|)} \frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|u\|} \\ &\simeq \frac{\|x-u\| - \|x\|}{\varphi(\|x-u\|) - \varphi(\|x\|)} \frac{\varphi(\|x-u\|) - \varphi(\|x\|)}{-\|u\|} \\ &\simeq \frac{\|x-u\| - \|x\|}{-\|u\|}.\end{aligned}$$

Therefore E is uniformly smooth. □

By the same argument, we have the following.

Theorem 5.4. *Let E be a normed linear space and let a be a positive real number. Let $\varphi : [0, \infty) \to \mathbb{R}$ be a function such that $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi \not\equiv 0$ on $[0, a]$. Then $\varphi(\|\cdot\|)$ is uniformly smooth on $B_E(a)$ if and only if φ is uniformly smooth on $[0, a]$ and E is uniformly smooth.*

Using Theorem 5.3 and Proposition 4.2, we have the following. Compare this with Theorem 2′ in [9].

Theorem 5.5. *Let E be a normed linear space and let a be a positive real number. Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing and convex function such that $\varphi(0) = 0$ and $\varphi'(0) = 0$. Then E is uniformly smooth if and only if there exists an increasing function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$, $g'(0) = 0$ and*

$$\varphi(\|\lambda x + (1-\lambda) y\|) \ge \lambda\varphi(\|x\|) + (1-\lambda)\varphi(\|y\|) - \lambda(1-\lambda) g(\|x - y\|)$$

for all $x, y \in B_E(a)$ and $0 \le \lambda \le 1$.

6. On uniform convexity and uniform smoothness on whole space. In this section, we show sufficient conditions which guarantee the uniform convexity and the uniform smoothness of the function $\varphi(\|\cdot\|)$ on a whole normed linear space. We begin with the uniformly convex case. We need the following lemma. We omit its proof.

Lemma 6.1. *Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing and convex function. If R, M, δ and ε are nonnegative real numbers such that $R \le M$ and $\delta \le \varepsilon$, then*

$$\varphi\Big(R + \frac{\delta}{2}\Big) - \varphi(R) \le \frac{\varphi(M+\varepsilon) - \varphi(M)}{2}.$$

Theorem 6.1. *Let E be a normed linear space and let $\varphi : [0, \infty) \to [0, \infty)$ be a function such that it is uniformly convex on $[0, \infty)$ and $\varphi(0) = 0$. If for some positive real number c, the modulus of convexity δ satisfies $\delta(\varepsilon) \ge c\varphi(\varepsilon)$ for any $\varepsilon \in [0, 2]$ and*

(6.1) $$\varliminf_{\substack{s \to \infty \\ t \downarrow 0}} \Big(\varphi(t)\varphi(s) - \varphi(ts)\Big) \ge 0,$$

then $\varphi(\|\cdot\|)$ is uniformly convex on E.

Proof. Let c be a positive real number such that $\delta(\varepsilon) \ge c\varphi(\varepsilon)$ for any $\varepsilon \in [0, 2]$ and let (6.1) be satisfied. Let $x, y \in {}^*E$ such that $x \not\simeq y$. We shall prove

(6.2) $$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \underset{\not\simeq}{\le} \frac{\varphi(\|x\|) + \varphi(\|y\|)}{2}.$$

If $\|x\| \not\simeq \|y\|$, the uniform convexity of φ yields

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \le \varphi\Big(\frac{\|x\| + \|y\|}{2}\Big) \underset{\not\simeq}{\le} \frac{\varphi(\|x\|) + \varphi(\|y\|)}{2}.$$

So we may assume $\|x\| \simeq \|y\|$. If $\|x\|, \|y\|$ are finite, by the uniform convexity of E, we have $\|\frac{x+y}{2}\| \underset{\not\simeq}{\le} \frac{\|x\|+\|y\|}{2}$. Hence we obtain

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \underset{\not\simeq}{\le} \varphi\Big(\frac{\|x\| + \|y\|}{2}\Big) \le \frac{\varphi(\|x\|) + \varphi(\|y\|)}{2}.$$

Assume that $\|x\|, \|y\|$ are infinite. Without loss of generality we may assume $\|x\| \le \|y\|$. Put

$$M = \|x\|, M + \varepsilon = \|y\|, R = \Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\| \text{ and } R + \frac{\delta}{2} = \Big\|\frac{x+y}{2}\Big\|.$$

Since $\delta \le \varepsilon$ and $R \le M$, we have

(6.3) $$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) - \varphi\Big(\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|\Big) \le \frac{\varphi(\|y\|) - \varphi(\|x\|)}{2}$$

by Lemma 6.1. If we prove

(6.4) $$\varphi\Big(\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|\Big) \underset{\not\simeq}{\le} \varphi(\|x\|),$$

then this inequality and (6.3) yield (6.2). Suppose (6.4) is false. Then the convexity of φ, $\varphi(0) = 0$ and $\varphi(t) \ge 0$ for all $t \ge 0$ yield

$$\varphi\Big(\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|\Big) \le \frac{\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|}{\|x\|}\varphi(\|x\|) \le \varphi(\|x\|),$$

and hence

$$\frac{\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|}{\|x\|}\varphi(\|x\|) \simeq \varphi(\|x\|).$$

On the other hand, $\delta(\varepsilon) \ge c\varphi(\varepsilon)$ for any $\varepsilon \in [0, 2]$ yields

$$1 - \Big\|\frac{\frac{x}{\|x\|} + \frac{y}{\|y\|}}{2}\Big\| \ge c\varphi\Big(\Big\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\Big\|\Big).$$

So we have

$$\varphi(\|x\|) \geq \frac{\left\|\frac{x+\frac{\|x\|}{\|y\|}y}{2}\right\|}{\|x\|}\varphi(\|x\|) + c\varphi\left(\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right)\varphi(\|x\|)$$
$$\simeq \varphi(\|x\|) + c\varphi\left(\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right)\varphi(\|x\|).$$

If $\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| \gnsim 0$, it is clear that $\varphi(\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|)\varphi(\|x\|) \gnsim 0$. If $\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| \simeq 0$, then by (6.1), we get

$$\varphi\left(\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right)\varphi(\|x\|) \gtrsim \varphi\left(\left\|x - \frac{\|x\|}{\|y\|}y\right\|\right) \gnsim 0.$$

Hence we have $\varphi(\|x\|) \gnsim \varphi(\|x\|)$, which is a contradiction. This completes the proof. □

The following is due to Xu [9]. In his paper, he wrote $p > 1$, but if $1 < p < 2$, there exists no normed linear space such that $\|\cdot\|^p$ is uniformly convex on the whole space.

Theorem 6.2 (Xu). *Let $p \geq 2$ be a fixed real number. Let E be a normed linear space. Then the following are equivalent;*

(i) *there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$,*
(ii) *the functional $\|\cdot\|^p$ is uniformly convex on E,*
(iii) *there exists a constant $d > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p + \lambda(1-\lambda)d\|x-y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p$$

for all $x, y \in E$ and $0 \leq \lambda \leq 1$.

PROOF. (i) ⇒ (ii). Put $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t^p$ for $t \geq 0$. It is easy to see that φ is uniformly convex on $[0,\infty)$, $\varphi(0) = 0$, and $\delta(\varepsilon) \geq c\varphi(\varepsilon)$ for all $0 \leq \varepsilon \leq 2$. The definition of φ implies $\varphi(t)\varphi(s) - \varphi(ts) = 0$ for all $t, s \geq 0$. Hence, by our theorem, $\|\cdot\|^p$ is uniformly convex on E.

(ii) ⇒ (iii). Let $x, y \in {}^*E$ such that $x \neq y$, and let $\lambda \in {}^*(0,1)$. Since $\|\cdot\|^p$ is uniformly convex, by Proposition 4.1,

$$\frac{\left\|\lambda\frac{x}{\|x-y\|} + (1-\lambda)\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)} \lnsim \frac{\lambda\left\|\frac{x}{\|x-y\|}\right\|^p + (1-\lambda)\left\|\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)}.$$

By the transfer principle, there exists a standard positive real number d such that

$$\frac{\left\|\lambda\frac{x}{\|x-y\|} + (1-\lambda)\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)} + d \leq \frac{\lambda\left\|\frac{x}{\|x-y\|}\right\|^p + (1-\lambda)\left\|\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)}$$

for all $x, y \in E$ and $\lambda \in (0,1)$, i.e.,

$$\|\lambda x + (1-\lambda)y\|^p + \lambda(1-\lambda)d\|x-y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p$$

for all $x, y \in E$ and $0 \leq \lambda \leq 1$.

(iii) ⇒ (i). Let $x, y \in {}^*S_E(1)$ with $x \neq y$. By (iii), we have

$$\left\|\frac{x+y}{2}\right\|^p + \frac{1}{4}d\|x-y\|^p \leq 1,$$

and hence

(6.5) $$\frac{1}{\|x-y\|^p} - \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^p \geq \frac{1}{4}d.$$

We claim that there exists a standard positive real number c such that

$$\frac{1}{\|u-v\|^p} - \frac{1}{\|u-v\|^p}\left\|\frac{u+v}{2}\right\| \geq c$$

for all $u, v \in S_E(1)$. Suppose not, i.e., there exist $x, y \in {}^*S_E(1)$ such that

$$\frac{1}{\|x-y\|^p} \simeq \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|.$$

Then for any standard natural number n, we have

$$\frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^n \simeq \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^{n+1}.$$

Hence we obtain

$$\frac{1}{\|x-y\|^p} \simeq \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^p,$$

which contradicts (6.5). Therefore (i) is valid. □

The dual version of Theorem 6.1 is the following.

Theorem 6.3. *Let E be a normed linear space and let $\varphi : [0,\infty) \to [0,\infty)$ be a function such that it is uniformly smooth on $[0,\infty)$, $\varphi(0) = 0$ and $\varphi'(0) = 0$. If for some positive real number c, the modulus of smoothness ρ satisfies $\rho(\tau) \leq c\varphi(\tau)$ for all $\tau > 0$ and*

(6.6) $$\lim_{\substack{a\downarrow 0\\ A\to\infty}} \left|\varphi'(A)\frac{\varphi(\frac{a}{A})}{\frac{a}{A}}\right| = 0,$$

then $\varphi(\|\cdot\|)$ is uniformly smooth on E. Moreover, if φ is convex then $\varphi(\|\cdot\|)$ is uniformly Fréchet differentiable on E.

PROOF. Let c be a positive real number such that $\rho(\tau) \leq c\varphi(\tau)$ for all $\tau > 0$, and let (6.6) be satisfied. Let $x \in {}^*E$, let $u \in {}^*S_E(1)$ and let $t, s \in {}^*\mathbb{R} \setminus \{0\}$ with $t \simeq 0$ and $s \simeq 0$. We shall prove

(6.7) $$\frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{t} \simeq \frac{\varphi(\|x+su\|) - \varphi(\|x\|)}{s}.$$

First we assume $x = o$. Then we get

$$\frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{t} = \frac{\varphi(t) - \varphi(0)}{t}$$
$$\simeq \varphi'(0)$$
$$= 0.$$

Hence we obtain (6.7). Next we assume $x \neq o$. Then we get

$$\frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{t} - \frac{\varphi(\|x+su\|) - \varphi(\|x\|)}{s}$$
$$= \frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{\|x+tu\| - \|x\|}\frac{\|x+tu\| - \|x\|}{t} - \frac{\varphi(\|x+su\|) - \varphi(\|x\|)}{\|x+su\| - \|x\|}\frac{\|x+su\| - \|x\|}{s}$$
$$\simeq \varphi'(\|x\|)\left(\frac{\|x+tu\| - \|x\|}{t} - \frac{\|x+su\| - \|x\|}{s}\right).$$

If $\|x\|$ is finite then $\varphi'(\|x\|)$ is finite, and hence we can derive (6.7) from the uniform smoothness of E. So we may assume that $\|x\|$ is infinite. Let $\alpha = \max\{|t|, |s|\}$. Since $\rho(\tau) \leq c\varphi(\tau)$ for all $\tau > 0$, we have

$$\frac{\|\frac{x+\alpha u}{\|x\|}\| + \|\frac{x-\alpha u}{\|x\|}\|}{2} - 1 \leq c\varphi\Big(\frac{\alpha}{\|x\|}\Big),$$

i.e.,

$$\frac{\|x+\alpha u\| + \|x-\alpha u\| - 2\|x\|}{\alpha} \leq 2c\frac{\varphi(\frac{\alpha}{\|x\|})}{\frac{\alpha}{\|x\|}}.$$

So (6.6) yields

$$\begin{aligned}\Big|\varphi'(\|x\|)\Big(\frac{\|x+tu\|-\|x\|}{t} - \frac{\|x+su\|-\|x\|}{s}\Big)\Big| &\leq \Big|\varphi'(\|x\|)\frac{\|x+\alpha u\|+\|x-\alpha u\|-2\|x\|}{\alpha}\Big| \\ &\leq 2c\Big|\varphi'(\|x\|)\frac{\varphi(\frac{\alpha}{\|x\|})}{\frac{\alpha}{\|x\|}}\Big| \\ &\simeq 0.\end{aligned}$$

Therefore we obtain (6.7). This completes the proof. □

The following is also due to Xu [9]. In his paper, he wrote $q > 1$, but if $q > 2$, there exists no normed linear space such that $\|\cdot\|^q$ is uniformly Fréchet differentiable on the whole space.

Theorem 6.4 (Xu). *Let q be a fixed real number with $1 < q \leq 2$. Let E be a normed linear space. Then the following are equivalent;*

(i) *there exists a constant $c > 0$ such that $\rho(\tau) \leq c\cdot\tau^q$ for all $\tau > 0$,*
(ii) *the functional $\|\cdot\|^q$ is uniformly Fréchet differentiable on E,*
(iii) *there exists a constant $d > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^q + \lambda(1-\lambda)d\|x-y\|^q \geq \lambda\|x\|^q + (1-\lambda)\|y\|^q$$

for all $x, y \in E$ and $0 \leq \lambda \leq 1$.

PROOF. (i) ⇒ (ii). Put $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t^q$ for $t \geq 0$. It is easy to see that φ is uniformly smooth on $[0,\infty)$, $\varphi(0) = 0$ and $\varphi'(0) = 0$. The inequality $\rho(\tau) \leq c\cdot\tau^q$ for all $\tau > 0$ implies that $\rho(\tau) \leq c\varphi(\tau)$ for all $\tau > 0$. By the definition of φ, we have

$$\lim_{\substack{a\downarrow 0\\ A\to\infty}} \Big|\varphi'(A)\frac{\varphi(\frac{a}{A})}{\frac{a}{A}}\Big| = \lim_{a\downarrow 0} qa^{q-1} = 0.$$

So, by our theorem, $\|\cdot\|^q$ is uniformly Fréchet differentiable on E.

(ii) ⇒ (iii). Let M be any infinite element of ${}^*\mathbb{R}_+$. Let $x, y \in {}^*E$ with $x \neq y$. Then $\frac{x}{M\|x-y\|} \simeq \frac{y}{M\|x-y\|}$. Let $\lambda \in {}^*(0,1)$. Since $\|\cdot\|^q$ is uniformly Fréchet differentiable on E, by Proposition 4.2, we have

$$\frac{\|\frac{\lambda x}{M\|x-y\|} + \frac{(1-\lambda)y}{M\|x-y\|}\|^q}{\lambda(1-\lambda)} \simeq \frac{\lambda\|\frac{x}{M\|x-y\|}\|^q + (1-\lambda)\|\frac{y}{M\|x-y\|}\|^q}{\lambda(1-\lambda)}.$$

Hence we obtain

$$\frac{\lambda\|x\|^q + (1-\lambda)\|y\|^q - \|\lambda x + (1-\lambda)y\|^q}{\lambda(1-\lambda)}\frac{1}{M^q\|x-y\|^q} \simeq 0.$$

So we have

$$\frac{\lambda\|x\|^q + (1-\lambda)\|y\|^q - \|\lambda x + (1-\lambda)y\|^q}{\lambda(1-\lambda)}\frac{1}{M^q\|x-y\|^q} \leq 1,$$

i.e.,

$$\lambda\|x\|^q + (1-\lambda)\|y\|^q - \|\lambda x + (1-\lambda)y\|^q \leq M^q\lambda(1-\lambda)\|x-y\|^q.$$

Therefore, by the transfer principle, (iii) is valid.

(iii) ⇒ (i). Let $x \in S_E(1)$ and $u \in E\setminus\{o\}$. Since $\|x\| = 1$ and $\|x+u\| + \|x-u\| \geq 2$, we have $\|x+u\| + \|x-u\| \leq \|x+u\|^q + \|x-u\|^q$. From (iii), we can derive

$$\begin{aligned}\frac{\|x+u\| + \|x-u\|}{2} &\leq \frac{\|x+u\|^q + \|x-u\|^q}{2} \\ &\leq \Big\|\frac{(x+u)+(x-u)}{2}\Big\|^q + \frac{1}{2}\cdot\frac{1}{2}\cdot d\|(x+u)-(x-u)\|^q \\ &= \|x\|^q + 2^{q-2}d\|u\|^q.\end{aligned}$$

Hence we obtain

$$\frac{\|x+u\| + \|x-u\|}{2} - 1 \leq 2^{q-2}d\|u\|^q,$$

which implies that $\rho(\tau) \leq 2^{q-2}d\cdot\tau^q$ for all $\tau > 0$. □

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