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MATHEMATICS

ON SOME GEOMETRICAL PROPERTIES OF THE SPHERE IN A SPACE OF THE TYPE (B)

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Let E denote throughout the following a Banach space (1). A function $\|x\|$ is said to be w e a k l y d i f f e r e n t i a b l e (2) at the point x_0, if there exists

$$\lim_{h\to 0}\frac{\|x_0+h\cdot x\|-\|x_0\|}{h}\qquad (x\in E)\qquad (1)$$

S. Mazur has shown (2) that weak differentiability of the norm $\|x\|$ at the point x_0 is equivalent to the existence of a single supporting hyperplane to the sphere $\|x\|\leqslant 1$ at the point x_0. If the convergence to the limit of the difference ratio $\frac{\|x_0+h\cdot x\|-\|x_0\|}{h}$ is uniform in the whole unit sphere $\|x\|\leqslant 1$, then $\|x\|$ is said to be s t r o n g l y d i f f e r e n t i a b l e (2) at the point x_0.

Let Q be an arbitrary set. Then by $E(Q)$ we denote an arbitrary linear normed space of bounded functions defined in Q, where $\|x\|=\sup_{q\in Q}|x(q)|$.

The sequence of points $\{q_n\}\subset Q$ is called extremal for the function $x_0(q)\in E(Q)$, if there exists

$$\lim_{n\to\infty} x_0(q_n)$$

and

$$\|x_0\|=|\lim_{n\to\infty}x_0(q_n)|.$$

L e m m a. *Let $x_0\in E(Q)$, $\|x_0\|=1$, and let $\{q_n\}\subset Q$ be an arbitrary extremal sequence of the function $x_0(q)$.*

Then for every element $x\in E(Q)$, $\|x\|\leqslant 1$, and for any number $0\neq|h|\leqslant\frac{1}{4}$ there exists such a sequence $\left\{q_n^{(h;x)}\right\}\subset Q(n=1,2,\ldots)$ that

$$\left.\begin{aligned}&\left|\frac{\|x_0+h\cdot x\|-\|x_0\|}{h}-\operatorname{Lim}_{n\to\infty}x(q_n)\ \operatorname{sign}\lim_{n\to\infty}x_0(q_n)\right|\leqslant\\ &\leqslant\left|\operatorname{Lim}_{n\to\infty}x(q_n)\ \operatorname{sign}\lim_{n\to\infty}x_0(q_n)-\lim_{n\to\infty}x(q_n^{(h;x)})\operatorname{sign}\lim_{n\to\infty}x_0(q_n^{(h;x)})\right|\end{aligned}\right\}\qquad(2)$$

Moreover,

$$\left|\,\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})\right|-1\,\right|\leqslant 2|h|.\qquad(3)$$

[Under the symbol Lim we understand here the generalized limit introduced by S. Banach (1)].

P r o o f. If $x\in E(Q)$, $\|x\|\leqslant 1$ and $0\neq|h|\leqslant\frac{1}{4}$, then there exists such a sequence $\{q_n^{(h;x)}\}\subset Q$ that

$$\|x_0+hx\|=\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})+h\cdot\lim_{n\to\infty}x(q_n^{(h;x)})\right|.$$

Then

$$\begin{aligned}0&\leqslant\left|\lim_{n\to\infty}x_0(q_n)\right|-\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})\right|\leqslant\left|\operatorname{Lim}_{n\to\infty}[x_0(q_n)+h\cdot x(q_n)]\right|+\\ &+|h|\cdot\left|\operatorname{Lim}_{n\to\infty}x(q_n)\right|-\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})\right|\leqslant\\ &\leqslant\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})+h\cdot\lim_{n\to\infty}x(q_n^{(h;x)})\right|+\|x\|\cdot|h|-\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})\right|\leqslant\\ &\leqslant|h|\cdot\|x\|+|h|\cdot\left|\lim_{n\to\infty}x(q_n^{(h;x)})\right|\leqslant 2\cdot|h|\cdot\|x\|\leqslant\cdot 2\cdot|h|.\end{aligned}$$

The inequality (3) is thus proved. We proceed now to prove the inequality (2). We have

$$\begin{aligned}&\|x_0+h\cdot x\|-\|x_0\|\geqslant\left|\lim_{n\to\infty}x_0(q_n)+h\cdot\operatorname{Lim}_{n\to\infty}x(q^n)\right|-1=\\ &=[1+h\cdot\operatorname{Lim}_{n\to\infty}x(q_n)\cdot\operatorname{sign}\lim_{n\to\infty}x_0(q_n)]-1=h\cdot\operatorname{Lim}_{n\to\infty}x(q_n)\cdot\operatorname{sign}\lim_{n\to\infty}x_0(q_n).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\|x_0+h\cdot x\|-\|x_0\|=\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})+h\cdot\lim_{n\to\infty}x(q_n^{(h;x)})\right|-1=\\ &=\left[\left|\lim_{n\to\infty}x_0(q_n^{(h;x)})\right|+h\cdot\lim_{n\to\infty}x(q_n^{(h;x)})\cdot\operatorname{sign}\lim_{n\to\infty}x_0(q_n^{(h;x)})\right]-1\leqslant\\ &\leqslant h\cdot\lim_{n\to\infty}x(q_n^{(h;x)})\cdot\operatorname{sign}\lim_{n\to\infty}x_0(q_n^{(h;x)})\end{aligned}$$

For the proof of inequality (2) it remains only to compare the last two inequalities.

T h e o r e m 1. *Let $x_0\in E(Q)$, $\|x_0\|=1$. Then for the strong differentiability of the norm $\|x\|$ in $E(Q)$ at the point x_0 it is sufficient that the following condition should be satisfied.*

For every extremal sequence $\{q_n\}\subset Q$ of the function $x_0(q)$ and every $x(q)\in(Q)$ ($\|x\|\leqslant 1$) the sequence

$$\{x(q_n)\cdot x_0(q_n)\}$$

converges uniformly in the unit sphere ($\|x\|\leqslant 1$) to a limit not depending on the choice of the extremal sequence $\{q_n\}$.

P r o o f. From the condition of the theorem follows that the right-hand side of the inequality (2) may be represented in the form

$$\left|\lim_{n\to\infty}[x(q_n)\cdot x_0(q_n)]-\lim_{n\to\infty}x(q_n^{(h;x)})\cdot\operatorname{sign}\lim_{n\to\infty}x_0(q_n^{(h;x)})\right|.\qquad(3)$$

In order to prove the theorem it is sufficient to show that the expression (3) tends to zero for $h\to 0$ uniformly with respect to $\|x\|\leqslant 1$. Asssume, on the contrary, that there exists such an $\varepsilon_0>0$ that for every $0<|h_p|\leqslant\frac{1}{2^{p+1}}$ $(p=1,2,\ldots)$ there may be found an $x_p\in E(Q)$, $\|x_p\|\leqslant 1$, for which

$$\left|\lim_{n\to\infty}[x_p(q_n)\cdot x_0(q_n)]-\lim_{n\to\infty}x_p(q_n^{(h_p;x_p)})\cdot\operatorname{sign}\lim_{n\to\infty}x_0(q_n^{(h_p;x_p)})\right|\geqslant\varepsilon_0>0.\qquad(4)$$

Take such a sequence $\{n_p\}\to\infty$ that

$$\left|x_0(q_{n_p}^{(h_p;x_p)})-\lim_{n\to\infty}x_0(q_n^{(h_p;x_p)})\right|\leqslant\frac{1}{p}\quad (p=1,2,\ldots),\tag{5}$$

$$\left|x_p(q_{n_p}^{(h_p;x_p)})-\lim_{n\to\infty}x_p(q_n^{(h_p;x_p)})\right|\leqslant\frac{1}{p}\quad (p=1,2,\ldots).\tag{6}$$

Evidently, $\lim_{p\to\infty}|x_0(q_{n_p}^{(h_p;x_p)})|=1$. We may suppose that $\lim_{p\to\infty}x_0(q_{n_p}^{(h_l;x_p)})$ exists and that, consequently, the sequence $\{q_{n_p}^{(h_l;x_p)}\}$ is extremal for the function $x_0(q)$. From the relations (4), (5) and (6) it follows that

$$\left|\lim_{n\to\infty}[x_p(q_n)\cdot x_0(q_n)]-x_p(q_{n_p}^{(h_p;x_p)})\cdot x_0(q_{n_p}^{(h_p;x_p)})\right|\geqslant\frac{\varepsilon_0}{2},\tag{7}$$

if p is sufficiently large. Since $\{q_n\}$ and $\{q_{n_p}^{(h_p;x_p)}\}$ are two extremal sequences for the function $x_0(q)$, from the condition of the theorem immediately follows the impossibility of the inequality (7) for arbitrarily large p. The theorem is thus proved.

Corollary 1. *If E is an arbitrary Banach space, then for strong differentiability of $\|f\|$ in $\bar{E}$ at the point f_0, $\|f_0\|=1$, it is sufficient that the following condition should be satisfied: from $f_0(x_n)\to\|f_0\|=1$, $\|x_n\|=1$ $(n=1,2,3,\ldots)$ follows that*

$$\|x_n-x_m\|\to 0\quad \textit{for } n, m\to\infty.$$

Corollary 2. *If E is an arbitrary Banach space, then for strong differentiability of $\|x\|$ in E at the point x_0, $\|x_0\|=1$, it is sufficient that the following condition should be satisfied: from $f_n(x_0)\to\|x_0\|=1$, $\|f_n\|=1$ $(n=1,2,3,\ldots)$ follows that*

$$\|f_n-f_m\|\to 0\quad \textit{for } n, m\to\infty.$$

Observe that Theorem 1 is a complement to my following theorem [7].

Theorem 2. *Let $x_0\in E(Q)$, $\|x_0\|=1$. Then for weak differentiability of the norm $\|x\|$ in $E(Q)$ at the point x_0 it is necessary and sufficient that the following condition should be satisfied: for every extremal sequence $\{q_n\}\subset Q$ of the function $x_0(q)$ and any $x(q)\in E(Q)$ the sequence*

$$\{x(q_n)\cdot x_0(q_n)\}$$

converges to a limit not depending on the choice of the extremal sequence $\{q_n\}$.

Corollary 1. *If E is a Banach space, then for weak differentiability of $\|f\|$ in $\bar{E}$ at the point f_0, $\|f_0\|=1$, it is necessary and sufficient that the following condition should be satisfied: if $f_0(x_n)\to 1$, $\|x_n\|=1$ $(n=1,2,\ldots)$, then the sequence $\{x_n\}$ converges weakly.*

Corollary 2. *If E is a Banach space, then for weak differentiability of $\|x\|$ in E at the point x_0, $\|x_0\|=1$, it is necessary and sufficient that the following condition should be satisfied: if $f_n(x_0)\to 1$, $\|f_n\|=1$, then*

$$\lim_{n\to\infty}f_n(x)\ \textit{exists for all}\ x\in E.$$

We shall say that the Banach space E is **semi-uniformly convex**, if from the relations

$$\|x_n\|=1,\ \lim_{n\to\infty}\left\|\frac{x_n+x_{n+k}}{2}\right\|=1\tag{8}$$

(the convergence being uniform in $k=1,2,\ldots$) follows that

$$\|x_n-x_m\|\to 0\quad\text{for } n, m\to\infty.$$

Every uniformly convex [3] space is semi-uniformly convex.

We shall say that the space E is **weakly semi-uniformly convex**, if from the relations (8) it follows that the sequence $\{x_n\}$ converges weakly to a certain element $x'\in E$. Every semi-uniformly convex space is weakly semi-uniformly convex.

Theorem 3. *If the space E is semi-uniformly convex, then $\|f\|$ is everywhere in $\bar{E}$ strongly differentiable.*

Proof. Let $\|f_0\|=1$, $f_0\in\bar{E}$. Suppose that $f_0(x_n)\to 1$, $\|x_n\|=1$. We have to show that in this case $\|x_n-x_m\|\to 0$ for $n, m\to\infty$. In fact,

$$\|x_n\|+\|x_{n+k}\|=2\leqslant f_0(x_n)+f_0(x_{n+k})+\varepsilon_n=f_0(x_n+x_{n+k})+$$
$$+\varepsilon_n\leqslant\|x_n+x_{n+k}\|+\varepsilon_n\leqslant\|x_n\|+\|x_{n+k}\|+\varepsilon_n,$$

where $\lim_{n\to\infty}\varepsilon_n=0$. Therefore the sequence $\{x_n\}$ satisfies the relations (8) and in virtue of the semi-uniform convexity of E the sequence $\{x_n\}$ converges. The theorem is proved.

Similarly we may prove the following proposition.

Theorem 4. *If the space E is weakly semi-uniformly convex, then $\|f\|$ is everywhere in $\bar{E}$ weakly differentiable.*

We have also the following

Theorem 5. *If the space E is uniformly convex, then from a) x_n converges weakly to x_0 and b) $\|x_n\|\to\|x_0\|$ follows $\|x_n-x_0\|\to 0$.*

D. Milman [4] has recently shown that a uniformly convex space is regular. From one result obtained by D. Milman in common with the author follows that a weakly semi-uniformly convex space is also regular.

S. Mazur [5] has proved the following proposition. Let: a) the unit sphere of E be weakly compact, b) the norm $\|x\|$ be everywhere strongly differentiable. Then:

1) For any bounded convex and closed set K and an arbitrary point $x_0\in E$ lying outside K there exists always a closed sphere such that it contains the set K and does not contain the point x_0.

2) A sequence $\{x_n\}$ converges weakly to x_0 then and only then when it is bounded and every closed sphere containing infinitely many elements of $\{x_n\}$ contains also the point x_0, i. e.

3) A sequence $\{x_n\}$ converges weakly to x_0 then and only then when it is bounded and

$$\lim_{n\to\infty}\|x_n-x\|\geqslant\|x_0-x\|\quad (x\in E).$$

If the unit sphere of E is weakly compact and if the norm $\|x\|$ is everywhere in E weakly differentiable, then the space E is regular [7]. Since, moreover, the unit sphere of a regular space is weakly compact [6], the spaces E possessing the properties a) and b) of S. Mazur's theorem referred to above coincide with regular spaces with strongly differentiable norm $\|x\|$.

From the preceding propositions easily follows that if the space E ($\bar{E}$) is semi-uniformly convex, then the space $\bar{E}$ (E) possesses the properties a) and b) of S. Mazur's theorem.

Theorem 6. *Let the space E satisfy the conditions a) and b) of S. Mazur's theorem and let $\{x_n\}\in E$, $\|x_n\|=1$, $(n=1,2,\ldots)$. Then, in order that the point x_0 should possess the property:*

$$\underline{\lim}_{n\to\infty}f(x_n)\leqslant f(x_0)\leqslant\overline{\lim}_{n\to\infty}f(x_n)\quad (f\in\bar{E}),$$

it is necessary and sufficient that every closed sphere containing almost all $\{x_n\}$ should contain also the point x_0, i. e. that

$$\overline{\lim_{n\to\infty}} \| x_n - x \| \geqslant \| x_0 - x \| \quad (x \in E).$$

Theorem 7. *Let E and $\bar{E}$ be both uniformly convex. Then, in order that $\| x_n - x_0 \| \longrightarrow 0$, it is necessary and sufficient that the following conditions should be satisfied:*

a) $\underline{\lim}_{n\to\infty} \| x_n - x \| \geqslant \| x_0 - x \| \quad (x \in E)$,

b) for some $x = x' \in E$
$$\lim_{n\to\infty} \| x_n - x' \| = \| x_0 - x' \|.$$

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REFERENCES

[1] S. Banach, Théorie des opérations linéaires (1932). [2] S. Mazur, Studia Math., IV, 70 (1933). [3] J. A. Clarkson, Trans. Amer. Math. Soc., 40, 396 (1936). [4] D. Milman, C. R. Acad. Sci. URSS, XX, No. 4 (1938). [5] S. Mazur, loc. cit., 128. [6] V. Gantmakher a. V. Šmulian, C. R. Acad. Sci. URSS, XVII, No. 3 (1937). [7] V. Šmulian, Rec. Math., 6 (48), 1 (1939).