Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1939. Volume XXII, № 8

MATHEMATICS

LINEAR TOPOLOGICAL SPACES AND THEIR CONNEXION WITH THE BANACH SPACES

By V. ŠMULIAN

(*Communicated by I. M. Vinogradow, Member of the Academy, 28. I. 1939*)

I. Let E_T denote a linear topological space, i. e. a linear space which is topological and in which the operations of addition and multiplication by a scalar are continuous. This definition belongs to A. Kolmogoroff (1). An other definition for a linear topological space was proposed by J. Neumann (2). J. Neumann's definition is based on the consideration of neighbourhoods which satisfy some 6 axioms. J. Neumann showed that the linear topological space in this sense is also a linear topological space in the sense of A. Kolmogoroff. V. Wehausen showed (3) that, conversely, every linear topological space in the sense of A. Kolmogoroff satisfies all 6 axioms of J. Neumann except the second.

Since, however, J. Neumann uses his second axiom only for proving the theorems 16—20 of his paper, in studying the linear topological space in the sense of A. Kolmogoroff, we can make use of the principal results of J. Neumann. Besides, Wehausen remarked (3) that locally convex linear topological spaces of Kolmogoroff coincide with convex linear topological spaces of J. Neumann.

Theorem 1. *A compact set of a linear topological space is totally bounded* (4).

This theorem, the proof of which is not difficult, contains a theorem of D. H. Hyers (5).

Theorem 2. *Let E_T denote a topologically complete* (6) *locally convex linear topological space. If S is a compact set in E_T, the smallest convex set containing S is also compact.*

To prove this theorem it is sufficient to use the preceding theorem and the theorems 14 and 11 of J. Neumann.

Definition 1. The set $S \subset E_T$ will be said to be complete, if for every fundamental [in the sense of G. Birkhoff (7)] sequence $\{x_\alpha\} \subset S$, where α runs over some directed set, there exists an element $x_0 \in E$, such that [in the sense of G. Birkhoff (7)] $x_\alpha \rightarrow x_0$.

Theorem 3. *A closed set $S \subset E_T$ is bicompact, if and only if it is totally bounded and complete.*

Proof. If S is a bicompact set, its completeness follows from the definition. By theorem 1 S is totally bounded. The second part of the theorem follows easily from the lemma of G. Birkhoff at p. 49 of his paper referred to above. We note that theorem 3 is a generalization of G. Birkhoff's theorem 12.

Definition 2. A space E_T will be said to be quasi-complete, if every its totally bounded set is complete.

From this definition follows that E_T is topologically complete, if it is quasi-complete. The following theorem shows when the converse is true.

Theorem 4. *If a topological linear space of J. Neumann is topologically complete, it is also quasi-complete.*

To prove this theorem it is necessary to use the theorem 16 of J. Neumann and the preceding theorem.

Theorem 5. *Let K be a complete, convex and closed set in a locally convex linear topological space E_T. If $U(x)$ is a continuous operation on K to its compact part, there exists an element $x \in K$ such that*

$$U(x) = x.$$

Remark. If the space is quasi-complete, the condition that K is complete can be omitted.

To prove the theorem we consider the smallest convex closed set K_1 containing $U(K)$. Since K_1 is a bicompact set and $U(K_1) \subset K_1$, it remains to use the theorem of Tychonoff on the existence of a fix point [8].

II. Let E denote a Banach space *. Then we denote by $U(f_1, \dots f_n; \varepsilon)$, where $f_i \in \overline{E}$ $(i = 1, 2, \dots, n; \varepsilon > 0)$ the set of all elements $x \in E$ for which

$$|f_1^{(x)}| < \varepsilon, \dots, |f_n^{(x)}| < \varepsilon.$$

Each set $U(f_1, \dots, f_n; \varepsilon)$ for arbitrary $f_1, \dots, f_n, \varepsilon$ we call a neighbourhood of the point θ. In this way we obtain a linear topological space (in the sense of A. Kolmogoroff) which we denote by E_T. We note that this space E_T is always locally convex.

If E is separable, E_T is a linear topological space in the sense of J. Neumann.

Lemma. *If K is a convex and closed set in E, it is also a convex and closed set in E_T.*

Theorem 6. *In order that the unit sphere of the space E should be weakly compact, it is necessary and sufficient that E_T should be topologically complete.*

Theorem 7. *In order that the unit sphere of the space E should be weakly compact, it is necessary and sufficient that it should form a compact and closed set in E_T.*

To prove these two theorems it is necessary to use our lemma, the theorem 22 of J. Neumann and the following theorem of the author**:

In order that the unit sphere E should be weakly compact, it is necessary and sufficient that for every denumerable bounded sequence of convex closed sets

$$K_1 \supset K_2 \supset \dots \supset K_n \supset \dots$$

enclosed in one another, there should exist an element x belonging to all K_n.

In proving the next theorem we shall apply a proposition which has been recently proved by H. Goldstine [10].

A) *Every $F \in \overline{\overline{E}}$ (the space conjugate to $\overline{E}$) can be represented in the following form: $F(f) = \lim f(x_\alpha)$, where $|x_\alpha| \leqslant |F|$ and α runs over some directed set.*

---

* We shall keep the denotations and the terminology of Banach [9].

** Which is to appear in the «Recueil Mathématique de Moscou».

Consequently, the regularity of E is equivalent to the following condition:

If for some $\{x_\alpha\}$ there exists $\lim f(x_\alpha)$ for $f \in \overline{E}$, where $|x_\alpha| \leqslant 1$ and α runs over some directed set, then there exists such an element $x_0 \in E$ that

$$\lim f(x_\alpha) = f(x_0) \text{ for } f \in \overline{E}.$$

It is easy to see that this criterion of H. Goldstine can be put in the following form:

The regularity of E is equivalent to the quasi-completeness of E_T.

For this it is sufficient to use the theorem 22 of J. Neumann. Now, by theorem 3 we have:

Theorem 8. *In order that the space E should be regular, it is necessary and sufficient that its unit sphere should form a bicompact set in E_T.*

III. The following theorem completes the theorem A) of H. Goldstine.

Theorem 9. *Let the unit sphere of E be weakly compact. Then every $F \in \overline{\overline{E}}$ with $|F| = 1$ can be represented in the following form:*

$$F(f) = \lim f(x_\alpha),$$

where all x_α belong to some convex set lying on the boundary of the unit sphere E and α runs over some directed set. Therefore the regularity of E in this case is equivalent to the following condition: if for some $\{x_\alpha\}$, belonging to a convex set lying on the boundary of the unit sphere E, there exists $\lim f(x_\alpha)$ for all $f \in \overline{E}$, where α runs over some directed set, then there exists an element $x_0 \in E$ such that $\lim f(x_\alpha) = f(x_0)$ for $f \in \overline{E}$.

The proof of this theorem, as well as of the theorem A) of H. Goldstine, is based upon the following fact: every $F \in \overline{\overline{E}}$ can be represented in the form

$$F(f) = \int_Q f(x)\, d\Phi(e),$$

where Q is the unit sphere of E and $\Phi(e)$ is an additive function of bounded variation on the sets $e \in Q$.

By comparison with the preceding theorem we find

Theorem 10. *In order that a space E should be regular, it is necessary and sufficient that E_T should be topologically complete, and every convex closed (by norm) set lying on the boundary of the unit sphere E should form a bicompact set in E_T.*

After this paper was sent to print, the author got information that both theorem 8 and theorem 3 (for regularly topological spaces) had recently been proved by N. Bourbaki (C. R., Paris, 206, 1701) and H. Weyl (Actualités scient. et industr., Publ. Inst. Math. Univ. de Strasbourg, № 551) respectively.

University of Odessa.

Received
1. XI. 1938.

REFERENCES

[1] A. Kolmogoroff, Stud. Math., 5. [2] J. Neumann, Trans. of the Amer Math. Soc., 37, No 1 (1935) (we shall quote only this paper of J. Neumann) [3] V. Wehausen, Duke Math. Journal, 4, No 1 (1938). [4] Loc. cit ([2]), definition 6. [5] D. H. Hyers, Bull. of the Amer. Math. Soc., 44, No 2, th. 2 (1938). [6] Loc. cit. ([2]), definition 10. [7] G. Birkhoff, Annals of Math., 38, No 1, 40, 41, 48 (1937) (we shall quote only this paper of G. Birkhoff). [8] A. Tychonoff, Math. Annalen, III, 767 (1935). [9] S. Banach, Théorie des opérations linéaires (1932). [10] H. H. Goldstine, Duke Math. Journal, 4, No 1 (1938).