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*MATHEMATICS*
*(CONVEX SETS)*

# Monadic Functors and Convexity

by

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**Summary.** The purpose of this paper is to show that the forgetful functors from the category of compact convex sets to the category of compact spaces and to the category of sets are monadic (Theorems 3 and 4). These results show that, roughly speaking, the structure of a compact convex set is (in some sense) algebraic. On the other hand, it turns out that the forgetful functor from the category of convex sets to the category of sets is not monadic (Theorem 2), hence the convex structure on a set is not an algebraic structure. Theorem 1 gives some sufficient conditions for a functor to be monadic; this result is used in the proof of Theorem 3.

**1.** Let $\mathfrak{A}$ be a category. A *monad* (=*triple*, cf. [2, 4]) $T=(T, \eta, \mu)$ on $\mathfrak{A}$ is a covariant functor $T: \mathfrak{A}\to\mathfrak{A}$ with two natural transformations $\eta: I_{\mathfrak{A}}\to T$ and $\mu: T^2\to T$ satisfying the following conditions: $\mu_A T(\mu_A)=\mu_A \mu_{TA}$ and $\mu_A \eta_{TA}=\iota_{TA}=\mu_A T(\eta_A)$ for each object $A$ in $\mathfrak{A}$. A *T-algebra* is a pair $(A, a)$ where $a: TA\to A$ is a morphism in $\mathfrak{A}$ such that $aT(a)=a\mu_A$ and $a\eta_A=\iota_A$. A *morphism* $f^{\Delta}: (A, a)\to(A', a')$ *of T-algebras* is a triple $(f, a, a')$ where $f: A\to A'$ is any $\mathfrak{A}$-morphism such that $fa=a' T(f)$. $\mathfrak{A}^T$ will denote the category of $T$-algebras and their morphisms.

Let $\Psi: \mathfrak{B}\to\mathfrak{A}$ be a functor having a left adjoint $\Phi: \mathfrak{A}\to\mathfrak{B}$ with front and back adjunctions $\eta: I_{\mathfrak{A}}\to\Psi\Phi$, $\rho: \Phi\Psi\to I_{\mathfrak{B}}$. In this situation, one obtains a monad $T= =(\Psi\Phi, \eta, \Psi\rho_{\Phi})$ on $\mathfrak{A}$ and a canonical comparison functor $\Theta: \mathfrak{B}\to\mathfrak{A}^T$ defined by $\Theta B=(\Psi B, \Psi\rho_B)$, $\Theta\beta=(\Psi\beta, \Psi\rho_B, \Psi\rho_{B'})$ for $\beta: B\to B'$ in $\mathfrak{B}$.

A functor $\Psi: \mathfrak{B}\to\mathfrak{A}$ is *monadic* (=*tripleable*) if it has a left adjoint and the corresponding comparison functor $\Theta$ is an equivalence of categories. A category $\mathfrak{B}$ is *monadic* over $\mathfrak{A}$ if there exists a monadic functor $\Psi: \mathfrak{B}\to\mathfrak{A}$.

For unexplained terms and notation we refer the reader to [2, 4, 7].

We shall use the following strengthened version of a well-known theorem of Linton [1] (Linton has proved the theorem under an additional assumption that any epimorphism in $\mathfrak{A}$ is a retraction in $\mathfrak{A}$):

THEOREM 1. *Let a category $\mathfrak{A}$ have kernel pairs of retractions, and let a category $\mathfrak{B}$ have kernel pairs and coequalizers. Let $\Psi: \mathfrak{B}\to\mathfrak{A}$ be a functor having a left adjoint. If*

(i) *for every morphism $f$ in $\mathfrak{B}$, $f$ is a coequalizer if and only if $\Psi f$ is a coequalizer,*

(ii) *for every parallel pair* $(f, g)$ *in* $\mathfrak{B}$, $(f, g)$ *is a kernel pair if (and only if)* $(\Psi f, \Psi g)$ *is a kernel pair,*

*then the comparison functor* $\Theta\colon \mathfrak{B}\to\mathfrak{A}^T$ *is an equivalence of categories.*

The proof is omitted.

**2.** By a vector space we shall always mean a real vector space. Let $E$ and $E'$ be vector spaces. Let $K\subset E$ and $K'\subset E'$ be convex sets. A map $f\colon K\to K'$ is affine if it preserves the convex combinations, i.e., $f(sx+(1-s)y)=sf(x)+(1-s)f(y)$ for $x, y$ in $K$, $0\leqslant s\leqslant 1$.

Conv is the category of convex sets and affine maps.

THEOREM 2. *The category* Conv *is not monadic over* Ens.

Proof. Let $B=\{x\in R\colon 0\leqslant x\leqslant 1\}$, and let $A=\{(x, y)\colon 0<x<1 \,\&\, 0<y<1\}\cup\cup\{(0, 0)\}\cup\{(1, 1)\}$. We shall consider the maps $\pi_1, \pi_2\colon A\to B$ defined by $\pi_1(x, y)=x$, $\pi_2(x, y)=y$ for $(x, y)$ in $A$. $\pi_1$ and $\pi_2$ are morphisms in Conv, and $(\pi_1, \pi_2)$ is an equivalence relation in Conv, i.e., for every object $K$ of Conv, $(\langle K, \pi_1\rangle_{\mathrm{Conv}}, \langle K, \pi_2\rangle_{\mathrm{Conv}})$ is a kernel pair in Ens (cf. [7], 10.2.3).

Let $g$ be a coequalizer of $(\pi_1, \pi_2)$ in Conv. It is clear that $g$ is a constant map. We shall show that $(\pi_1, \pi_2)$ is not a kernel pair of $g$ in Conv. Indeed, let $p_1, p_2\colon B\times B\to\to B$ be the canonical projections. Of course, they are morphisms in Conv. We have then $gp_1=gp_2$, but there is no morphism $h$ in Conv such that $\pi_1 h=p_1$ and $\pi_2 h=p_2$. Hence $(\pi_1, \pi_2)$ is not a kernel pair of $g$. Since $(\pi_1, \pi_2)$ is not a kernel pair of its coequalizer, it is not a kernel pair in Conv.

For every right adjoint functor $\Psi$: Conv$\to$Ens the pair $(\Psi\pi_1, \Psi\pi_2)$ is a kernel pair in Ens. Thus, by Linton's Theorem ([1], Theorem 3), Conv is not monadic over Ens.

In particular, the forgetful functor $\Box$: Conv$\to$Ens is not monadic (cf. [7], 23.5.6).

**3.** By a compact convex set we shall mean a compact convex subset of a locally convex Hausdorff space.

Compconv is the category of compact convex sets and continuous affine maps.

The forgetful functor $\Box$: Compconv$\to$Comp has a left adjoint $\mathcal{S}$: Comp$\to\to$Compconv, where $\mathcal{S}(X)$ is the set of all probability measures on the compact space $X$. Composing $\mathcal{S}$ with the Stone—Čech functor $\beta$ (restricted to discrete spaces) we get a functor $\mathcal{S}\circ\beta$: Ens$\to$ Compconv which assigns to each set $X$ the free compact convex set $\mathcal{S}(\beta(X))$ generated by $X$. $\mathcal{S}\circ\beta$ is a left adjoint of the forgetful functor $\Box$: Compconv$\to$Ens (cf. [7], 23.7.2).

LEMMA 1. *A morphism* $f$ *in* Compconv *or* Comp *is a coequalizer if and only if it is a surjection.*

LEMMA 2. *Let* $\Box$: Compconv$\to$Comp *be the forgetful functor, and let* $f_1, f_2\colon A\to B$ *be morphisms in* Compconv. *If* $(\Box f_1, \Box f_2)$ *is a kernel pair in* Comp, *then* $(f_1, f_2)$ *is a kernel pair in* Compconv.

Proof. Let $f: B \to C$ be a coequalizer of $(f_1, f_2)$ in Comp. Thus $(f_1, f_2)$ is a kernel pair of $f$ in Comp. $B$ is a compact convex subset of a locally convex Hausdorff space $(E, \tau)$. We can assume that $0 \in B$ and $E = \operatorname{span} B$.

Let $M = \{x \in E$: there exist $b_1, b_2$ in $B$ and $t$ in $R$ such that $f(b_1) = f(b_2)$ and $x = t(b_1 - b_2)\}$. It can be proved that $M$ is a linear subspace of $E$. It can be also shown that $f(b_1) = f(b_2)$ if and only if $b_1 - b_2 \in M$ (the proof of these facts is omitted). Consequently, for every $x$ in $E$ the set $(x + M) \cap B$ is empty or is an inverse image of a one-point set; hence it is compact.

The sets $B - B$ and $(B - B) \cap M$ are compact. Indeed, the map $g: B \times B \to B - B$ defined by $g(b', b'') = b' - b''$ for $(b', b'')$ in $B \times B$ is a continuous surjection. (The topologies in $B$ and $B - B$ are induced by $\tau$ and the topology in $B \times B$ is the product topology.) Hence $B - B$ is compact and $g$ is a closed map. Therefore it is enough to prove that

$$D = g^{-1}((B - B) \cap M)$$

is a closed subset of $B \times B$. Let $(b', b'')$ be an element of the closure of $D$. For every neighbourhood $U$ of zero in $E$ the set

$$W = [(b' + U) \cap B] \times [(b'' + U) \cap B]$$

is a neighbourhood of $(b', b'')$ in $B \times B$. Thus there exists $(b_1, b_2)$ in $W \cap D$. But the condition $(b_1, b_2) \in D$ is equivalent to $f(b_1) = f(b_2)$. Hence $f(b') = f(b'')$ and $(b', b'') \in D$.

It is easy to see that the set $K = B - B$ is absolutely convex, absorbing and compact in $(E, \tau)$. Let $\|\cdot\|_K$ be the Minkowski functional of $K$. Then $(E, \|\cdot\|_K)$ is a Banach space. Moreover, there exists a Banach space $(F, \|\cdot\|)$ such that $(E, \|\cdot\|_K)$ is isometrically isomorphic to the conjugate of $(F, \|\cdot\|)$. (Cf. [5], 13.6 and [8], Proposition 1). Of course $(E, \sigma(E, F))$ is a locally convex Hausdorff space. The topologies $\tau$ and $\sigma(E, F)$ coincide on $K$ (and consequently on $B$). Since $M \cap K$ is compact in $(E, \tau)$ it is compact in $(E, \sigma(E, F))$. Hence by the well-known Krein—Šmulan theorem (cf. [3], IV.6.3) $M$ is $\sigma(E, F)$-closed.

Let $\mu$ be the quotient topology on $E/M$ determined by $\sigma(E, F)$. Then $(E/M, \mu)$ is a locally convex Hausdorff space. Let $\pi: E \to E/M$ be the quotient map. Then the map

$$p: B \to \pi(B)$$

defined by $p(b) = \pi(b)$ for $b$ in $B$ is a Compconv-morphism. It follows immediately from the definition of $p$ that there exists a unique homeomorphism (Comp-isomorphism) $h: \pi(B) \to C$ such that $f = hp$. We shall show that $(f_1, f_2)$ is a kernel pair of $p$ in Compconv. Let $(g_1, g_2)$ be a parallel pair in Compconv such that $pg_1 = pg_2$, i.e., $fg_1 = fg_2$. Hence there exists a unique morphism $u$ in Comp such that $f_1 u = g_1$ and $f_2 u = g_2$. Simple verification shows that $u$ is an affine map, i.e., it is a morphism in Compconv. This completes the proof of Lemma 2.

THEOREM 3. *The forgetful functor* $\Box$: Compconv$\to$Comp *is monadic.*

The Theorem immediately follows from Lemmas 1 and 2 and Theorem 1.

THEOREM 4. *The forgetful functor* $\square$: Compconv→Ens *is monadic.*

Proof. The functor $\square$: Compconv→Ens is the composition of the forgetful functors $\square_1$: Compconv→Comp and $\square_2$: Comp→Ens. It is well known that the functor $\square_2$ is monadic (cf. [2], VI.9). Hence by the Linton theorem ([1], Theorem 3) $\square_2$ satisfies conditions (i) and (ii) of Theorem 1. Therefore the functor $\square$: Compconv→Ens satisfies these conditions and consequently, by Theorem 1, is monadic.

**4.** It is not difficult to prove that the forgetful functor $\square$: Compsaks→Compconv (cf. [6]) satisfies conditions (i) and (ii) of Theorem 1 and consequently is monadic. Similar arguments as in the proof of Theorem 4 show that the forgetful functors

$\square$: Compsaks→Comp
$\square$: Compsaks→Conv
$\square$: Compsaks→Ens
$\square$: Compconv→Conv

are also monadic.

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**Т. Сьвирщ, Монадические функторы в теории выпуклости**

**Содержание.** В настоящей работе приводится доказательство, что пренебрегающие функторы из категории компактных выпуклых множеств в категорию компактных пространств и в категорию множеств — монадические. Доказывается также, что категория выпуклых множеств не монадическая над категорией множеств.