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MONADIC FUNCTORS AND CATEGORIES OF CONVEX SETS

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PREFACE

The present paper consists of two parts.

Part one is of purely categorical character. The core of it is a generalization of the well-known Linton criterion stating necessary and sufficient conditions of monadicity of a functor $\Phi : \mathfrak{B} \longrightarrow \mathfrak{A}$. In Theorem 2.1, giving sufficient conditions of monadicity, the assumption concerning the category $\mathfrak{A}$ is essentially weaker (specifically, it is not assumed that any epimorphism is a retraction), while in Theorem 2.4, giving necessary conditions, the assumptions are only slightly weaker. The proofs of these theorems are different from, and at the same time simpler than, the original proof of Linton.

Part two of the paper is devoted to the application of the above criteria to certain problems of functional analysis. They have been employed to investigate monadicity of certain forgetful functors in convexity theory. Some of the results can be expressed without using notions of the theory of categories. These are, among others, an axiomatic definition of a convex structure on a set and an axiomatic definition of a probability measure on a compact space.

Numerous properties of a convex subset of a vector space are intrinsic in the sense that they depend merely on the way of forming convex combinations of elements of the subset and they do not depend on the vector space itself. That is why the paper discusses sets with convex structures, convex sets for short, instead of convex subsets of vector spaces. It turns out that each convex structure on a set X is induced by a family of binary operations $(\textcircled{s} : X \times X \longrightarrow X)_{0<s<1}$ satisfying the following axioms

(A) $x \textcircled{s} x = x$

(B) $x \textcircled{s} y = y \textcircled{1-s} x$

(C) $(x \textcircled{s} y) \textcircled{t} z = x \textcircled{s+t-st} (y \textcircled{\frac{t}{s+t-st}} z)$

(D) $x_1 \textcircled{s} y = x_2 \textcircled{s} y \Longrightarrow x_1 = x_2$

for all x, x_1, x_2, y, z in X, $0<s<1$, $0<t<1$.

Thus a convex set can be regarded as an abstract algebra. The axioms (A)-(C) are of an equational type, whereas the axiom (D) is not. It has been proved in the paper that the category Conv of convex sets is not monadic over the category Ens of sets, and therefore there is no system of axioms of an equational type defining a convex structure on a set. The smallest category of equationally defined algebras over Ens contained the category of convex sets (treated as abstract algebras) is denoted by Sconv and its objects are called semi-convex sets. It turns out that semi-convex sets are the sets with structures satisfying the axioms (A)-(C). Therefore these axioms constitutes an "equational part" of the definition of a convex set. Technically, Sconv is the Eilenberg-Moore category of the monad $\mathbb{T}$ determined by the forgetful functor from Conv to Ens and its left adjoint $\mathcal{G}$.

Unlike the forgetful functor $\square_1$: Conv $\longrightarrow$ Ens, the forgetful functor $\square_2$ from the category Compconv of compact convex sets to the category Comp of compact spaces is monadic. This result can be expressed in the language of functional analysis as follows:

Let X be a compact space and let $\mathcal{S}(X)$ be the set of probability measures on X, convex and compact in the $*$-weak topology. Let $\gamma : \mathcal{S}(X) \longrightarrow X$ be a continuous map satisfying the following conditions:

(i) $\gamma(\delta^X_x) = x$ for each x in X,

where δ^X_x is the Dirac measure at x,

(ii) if $\lambda, \lambda_1, \lambda_2 \in \mathcal{S}(X)$ and $\gamma(\lambda_1) = \gamma(\lambda_2)$, then

$$\gamma((1-t)\lambda_1 + t\lambda) = \gamma((1-t)\lambda_2 + t\lambda) \quad \text{for } 0 \leq t \leq 1.$$

Define the convex combinations of elements of X as

$$\sum_{i=1}^{n} a_i x_i = \gamma\Big(\sum_{i=1}^{n} a_i \delta^X_{x_i}\Big) .$$

Then X becomes a compact convex set such that $\gamma(\lambda)$ is the centroid of λ for each λ in $\mathcal{S}(X)$.

Therefore the centroid of probability measure for a compact space X can be defined axiomatically as a continuous function $\gamma : \mathcal{S}(X) \longrightarrow X$ satisfying the conditions (i), (ii).

The monadicity of some thirteen forgetful functors acting between various categories of convex sets is investigated in the paper.

The existence of the monadic functor $\Psi : \mathcal{B} \longrightarrow \mathcal{O}\!\mathcal{l}$ is a measure of algebraicity of the category $\mathcal{B}$ over the category $\mathcal{O}\!\mathcal{l}$ as we encounter then a situation similar to that in case of the forgetful functor from the category of equationally defined algebras to the category of sets. As we know, such a functor is monadic. In this case the monadicity is closely linked to the fact that each set generates a free algebra and all algebras are quotient algebras of free algebras. Similarly, if $\Psi : \mathcal{L} \longrightarrow \mathfrak{A}$ is a monadic functor, then each object of the category $\mathfrak{A}$ determines a certain free object in $\mathcal{L}$, and any other object of the category $\mathcal{L}$ is a "quotient" object of some free object.

Laconically, compactness "makes the convex structure algebraic".

A simple proof of a theorem of Linton concerning the monadicity of the functor $^{*}:\mathrm{Ban}_1{}^{op} \longrightarrow \mathrm{Ban}_1$ is one of applications of the criterion 2.1. Theorem 2.1 was also employed in Semadeni's paper [11] to prove that the identical embedding of the category of spaces of continuous functions into the category Ban_1 of Banach spaces and linear contractions is monadic.

I want to thank Professors Antoni Wiweger and Zbigniew Semadeni for sugestions many of the problems discussed on the present paper, for their invaluable advices which helped me to solve the problems, as well for their assistance in the preparing the final version of the paper.

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§ 1. Preliminaries.

1.1. Let $\mathfrak{A}$ be a category. We shall denote by $\mathfrak{A}^{o}$ the class of objects of $\mathfrak{A}$. $\mathfrak{A}^{op}$ will denote the dual category of $\mathfrak{A}$.

Let A,B be elements of $\mathfrak{A}^{o}$. $\langle A,B\rangle_{\mathfrak{A}}$ or shortly $\langle A,B\rangle$ will denote the set of all morphisms from A to B in $\mathfrak{A}$. If M is a fixed object of $\mathfrak{A}$, $\langle M, ?\rangle_{\mathfrak{A}}$ is the principal covariant functor $\mathfrak{A} \longrightarrow$ Ens corresponding to M(cf. [9], 10.2.3.).

<u>A parallel pair</u> in $\mathfrak{A}$ is a pair (f,g) of morphisms of of the form: $A \overset{f}{\underset{g}{\rightrightarrows}} B$.

Let f be a morphism in $\mathfrak{A}$. A <u>kernel pair of f</u> is a pullback of the pair of equal morphisms

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[r, "f"] & A \end{tikzcd}$$

A parallel pair (f,g) is a <u>kernel pair</u> in $\mathfrak{A}$ if there exists a morphism q in $\mathfrak{A}$ such that (f,g) is a kernel pair of q.

Every kernel pair (f,g) is a <u>monic pair</u>, i.e., for any morphisms x,y in $\mathfrak{A}$, the conditions $fx = fy$ and $gx = gy$ imply $x = y$.

A morphism q is a <u>coequalizer</u> in $\mathfrak{A}$ if there exists a parallel pair (f,g) in $\mathfrak{A}$ such that q is a coequalizer of (f,g).

Let B be an object of $\mathfrak{A}$. An <u>equivalence relation</u> on B is a monic pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ such that for every M in $\mathfrak{A}^{o}$ the pair ($\langle M,f\rangle_{\mathfrak{A}}$, $\langle M,g\rangle_{\mathfrak{A}}$) is a kernel pair in Ens.

An <u>exact sequence</u> in $\mathfrak{A}$ is a diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{q} C$$

such that (f,g) is a kernel pair of q is a coequalizer of (f,g).

A morphism q is a <u>split coequalizer</u> of (f,g) if there exist morphisms f', q' in $\mathfrak{A}$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "f'"] \arrow[rr, bend left, "\iota_B"] \arrow[d, "q"'] & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "q"] \\ C \arrow[r, "q'"'] \arrow[rr, bend right, "\iota_C"'] & B \arrow[r, "q"'] & C \end{tikzcd}$$

is commutative. A split coequalizer of (f,g) is a coequalizer of (f,g). A morphism q is a split coequalizer of some parallel pair if and only if it is a retraction.

Let $\Psi : \mathfrak{B} \longrightarrow \mathfrak{A}$ be a covariant functor. Ψ <u>creates isomorphisms</u> if, given X in $\mathfrak{B}^{o}$ and an isomorphism $f:A \longrightarrow \Psi X$ in $\mathfrak{A}$, there exists a unique $\mathfrak{B}$-morphism $g:X' \longrightarrow X$ such that $\Psi g = f$ (which, of course, entails $\Psi X' = A$), and this morphism g is an isomorphism in $\mathfrak{B}$.

$\mathcal{P}$ will denote the class of all parallel pairs (f,g) in $\mathcal{L}$ such that $(\psi f, \psi g)$ has a split coequalizer in $\mathcal{O}\!\mathcal{L}$.
<u>$\mathcal{L}$ has $\mathcal{P}$ - coequalizers</u> if each pair (f,g) in $\mathcal{P}$ has a coequalizer in $\mathcal{L}$; <u>ψ reflects $\mathcal{P}$ - coequalizers</u> if for any diagram

$$(1) \qquad X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Z$$

in $\mathcal{L}$ with (f,g) in $\mathcal{P}$ and $\psi q = \mathrm{coeq}(\psi f, \psi g)$ it follows that $q = \mathrm{coeq}(f,g)$; <u>ψ preserves $\mathcal{P}$ - coequalizers</u> if, given a diagram (1) with (f,g) in $\mathcal{P}$ and $q = \mathrm{coeq}(f,g)$, it follows that $\psi q = \mathrm{coeq}(\psi f, \psi g)$.

1.2. Let $\mathcal{O}\!\mathcal{L}$ be a category. A monad (= triple, cf. [4],[6] $\mathbb{T} = (T, \eta, \mu)$ is a covariant functor

$$T : \mathcal{O}\!\mathcal{L} \longrightarrow \mathcal{O}\!\mathcal{L}$$

together with two natural transformations

$$\eta : I_{\mathcal{O}\!\mathcal{L}} \longrightarrow T, \qquad \mu : T^2 \longrightarrow T$$

satisfying the following conditions

$$\mu_A T\mu_A = \mu_A \mu_{TA} \quad , \quad \mu_A \eta_{TA} = \iota_{TA} = \mu_A T\eta_A$$

for each A in $\mathcal{O}\!\mathcal{L}^\circ$.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{O}\!\mathcal{L}$. An <u>Eilenberg-Moore algebra</u> of $\mathbb{T}$, shortly: a <u>$\mathbb{T}$ - algebra,</u> is a pair (A, α), where A is an element of $\mathcal{O}\!\mathcal{L}^\circ$ and $\alpha : TA \longrightarrow A$ is a morphism in $\mathcal{O}\!\mathcal{L}$ such that

$$\alpha T(\alpha) = \alpha\mu_A \qquad \text{and} \qquad \alpha\eta_A = \iota_A .$$

A is the <u>underlying object</u> of the $\mathbb{T}$- algebra (A, α) and α is a <u>$\mathbb{T}$ - algebra structure</u> on A.

A morphism $f^\nabla : (A, \alpha) \longrightarrow (A', \alpha')$ <u>of $\mathbb{T}$- algebras is</u> a triple (f, α, α'), where $f : A \longrightarrow A'$ is a morphism in $\mathcal{O}\!\mathcal{L}$ such that $f\alpha = \alpha' Tf$. $\mathcal{O}\!\mathcal{L}^{\mathbb{T}}$ will denote the category of $\mathbb{T}$ - algebras and their morphisms; it is the <u>Eilenberg-Moore category</u> of the monad $\mathbb{T} = (T, \eta, \mu)$.

(TA, μ_A) is a <u>free $\mathbb{T}$- algebra</u> with the underlying object TA.

1.3. Let $\psi : \mathcal{L} \longrightarrow \mathcal{O}\!\mathcal{L}$ be a functor having a left adjoint $\phi : \mathcal{O}\!\mathcal{L} \longrightarrow \mathcal{L}$ with canonical transformations

$$\eta : I_{\mathcal{O}\!\mathcal{L}} \longrightarrow \psi\phi \quad , \quad \varepsilon : \phi\psi \longrightarrow I_{\mathcal{L}}$$

Then $\mathbb{T} = (\psi\phi, \eta, \psi\varepsilon_\phi)$ is the <u>monad determined by the adjunction $(\phi, \psi, \eta, \varepsilon)$.</u>

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{O}\!\mathcal{L}$ and let

$$\phi^{\mathbb{T}} : \mathcal{O}\!\mathcal{L} \longrightarrow \mathcal{O}\!\mathcal{L}^{\mathbb{T}} \quad , \quad \psi^{\mathbb{T}} : \mathcal{O}\!\mathcal{L}^{\mathbb{T}} \longrightarrow \mathcal{O}\!\mathcal{L}$$

be functors defined by

$$\phi^{\mathbb{T}}(A) = (TA, \mu_A), \qquad \phi^{\mathbb{T}}(f) = (Tf, \mu_A, \mu_{A'})$$

for A in $\mathcal{O}\!\mathcal{L}^\circ$ and $f : A \longrightarrow A'$ in $\mathcal{O}\!\mathcal{L}$, and

$$\psi^{\mathbb{T}}(A, \alpha) = A, \qquad \psi^{\mathbb{T}}(f, \alpha, \alpha') = f$$

for (A, α) in $(\mathfrak{A}^{\mathbb{T}})^{o}$ and (f, α, α') in $\mathfrak{A}^{\mathbb{T}}$. Then the functor $\phi^{\mathbb{T}}$ is a left adjoint of $\psi^{\mathbb{T}}$ and the monad determined by this pair of adjoint functors is the given monad $\mathbb{T}$.

Furthermore, if $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ and $\psi : \mathfrak{B} \longrightarrow \mathfrak{A}$ is any pair of adjoint functors determining the monad $\mathbb{T}$, then there is a unique covariant functor $\Theta : \mathfrak{B} \longrightarrow \mathfrak{A}^{\mathbb{T}}$ such that the diagram

$$\begin{tikzcd}
& \mathfrak{B} \arrow[dr, "\psi"] \arrow[dd, "\Theta"] & \\
\mathfrak{A} \arrow[ur, "\phi"] \arrow[dr, "\phi^{\mathbb{T}}"'] & & \mathfrak{A} \\
& \mathfrak{A}^{\mathbb{T}} \arrow[ur, "\psi^{\mathbb{T}}"'] &
\end{tikzcd}$$

is commutative. The functor Θ is defined by

$$\Theta B = (\psi B, \psi \varepsilon_B) \quad , \quad \Theta\beta = (\psi\beta, \psi\varepsilon_B, \psi\varepsilon_{B'})$$

for B in $\mathfrak{B}^{o}$ and $\beta : B \longrightarrow B'$ in $\mathfrak{B}$; it is called the <u>canonical comparison functor.</u>

A functor $\psi : \mathfrak{B} \longrightarrow \mathfrak{A}$ is <u>monadic</u> [<u>quasi-monadic</u>] if it has a left adjoint $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ and the corresponding comparison functor $\Theta : \mathfrak{B} \longrightarrow \mathfrak{A}^{\mathbb{T}}$ is an isomorphism [an equivalence] of categories. A category $\mathfrak{B}$ is <u>monadic</u> [<u>quasi-monadic</u>] over $\mathfrak{A}$ if there exists a monadic [quasi-monadic] functor $\psi : \mathfrak{B} \longrightarrow \mathfrak{A}$.

We shall use the following well-known results:

1.4. <u>Lemma.</u> If $(f, \alpha, \alpha') : (A, \alpha) \longrightarrow (A', \alpha')$ is a morphism in $\mathfrak{A}^{\mathbb{T}}$, and

$$E \underset{f_2}{\overset{f_1}{\rightrightarrows}} A$$

is a kernel pair of f in $\mathfrak{A}$, then there exists a unique morphism $\gamma : TE \longrightarrow E$ in $\mathfrak{A}$ such that (E, γ) is a $\mathbb{T}$-algebra and

$$(E, \gamma) \underset{(f_2, \gamma, \alpha)}{\overset{(f_1, \gamma, \alpha)}{\rightrightarrows}} (A, \alpha)$$

are morphisms in $\mathfrak{A}^{\mathbb{T}}$. Moreover $((f_1, \gamma, \alpha), (f_2, \gamma, \alpha))$ is a kernel pair of (f, α, α') in $\mathfrak{A}^{\mathbb{T}}$.

Proof. There is a unique morphism $\gamma : TE \longrightarrow E$ such that the diagram

$$\begin{tikzcd}
& & TA \arrow[rr, "Tf"] \arrow[dr, "\alpha"] & & TA' \arrow[dd, "\alpha'"] \\
& & & A \arrow[dr, "f"] & \\
TE \arrow[uurr, "Tf_1"] \arrow[ddrr, "Tf_2"'] \arrow[r, dashed, "\gamma"] & E \arrow[urr, "f_1"] \arrow[drr, "f_2"'] & & & A' \\
& & & A \arrow[ur, "f"'] & \\
& & TA \arrow[rr, "Tf"'] \arrow[ur, "\alpha"'] & & TA' \arrow[uu, "\alpha'"']
\end{tikzcd} \qquad (1)$$

is commutative. We shall show that (E, γ) is a $\mathbb{T}$ - algebra. The diagram

$$\begin{tikzcd}
E \arrow[r, "\eta_E"] \arrow[d, "f_i"'] & TE \arrow[r, "\gamma"] \arrow[d, "Tf_i"] & E \arrow[d, "f_i"] \\
A \arrow[r, "\eta_A"'] \arrow[rr, bend right, "1_A"'] & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

is commutative for $i = 1,2$. Hence $f_i = f_i \gamma \eta_E$ for $i = 1,2$, and consequently $1_E = \gamma \eta_E$ (because (f_1, f_2) is a monic pair). The diagram

$$\begin{tikzcd}
& TE \arrow[rr, "\gamma"] \arrow[dr, "Tf_i"] & & E \arrow[dd, "f_i"] \\
& & TA \arrow[dr, "\alpha"] & \\
T^2E \arrow[uur, "\mu_E"] \arrow[r, "T^2f_i"] \arrow[ddr, "T\gamma"'] & T^2A \arrow[ur, "\mu_A"] \arrow[dr, "T\alpha"'] & & A \\
& & TA \arrow[ur, "\alpha"'] & \\
& TE \arrow[ur, "Tf_i"] \arrow[rr, "\gamma"'] & & E \arrow[uu, "f_i"']
\end{tikzcd}$$

is commutative for $i = 1,2$. Hence $f_i \gamma T(\gamma) = f_i \gamma \mu_E$ for $i = 1,2$, and consequently $\gamma T(\gamma) = \gamma \mu_E$. Thus (E, γ)

is a $\mathbb{T}$ - algebra and, by (1),

$$(E, \gamma) \underset{(f_2, \gamma, \alpha)}{\overset{(f_1, \gamma, \alpha)}{\rightrightarrows}} (A, \alpha)$$

are morphisms in $\mathfrak{A}^{\mathbb{T}}$. It is easy to see, that $((f_1, \gamma, \alpha), (f_2, \gamma, \alpha))$ is a kernel pair in of (f, α, α') in $\mathfrak{A}^{\mathbb{T}}$.

1.5. <u>Beck's Theorem</u> (cf. [3]). Let $\Psi : \mathfrak{B} \longrightarrow \mathfrak{A}$ be a functor having a left adjoint. The corresponding comparison functor $\Theta : \mathfrak{B} \longrightarrow \mathfrak{A}^{\mathbb{T}}$ is an equivalence of categories if and only if $\mathfrak{B}$ has and Ψ preserves and reflects $\mathcal{P}$ - coequalizers.

1.6. <u>Lemma</u> (cf. [3]). The canonical comparison functor Θ is an isomorphism of categories if and only if it is an equivalence of categories and Ψ creates isomorphisms.

§ 2. Strengthened versions of Linton's Theorem

The following theorem has been proved by Linton [3] under an additional assumption that any epimorphism in $\mathfrak{A}$ is a retraction in $\mathfrak{A}$. We shall show that this assumption is superfluous.

2.1. <u>Theorem.</u> Let a category $\mathfrak{A}$ have kernel pairs of retractions, and let a category $\mathfrak{B}$ have kernel pairs and coequalizers. Let $\Psi : \mathfrak{B} \longrightarrow \mathfrak{A}$ be a functor having a left adjoint. If

(i) for each morphism f in $\mathfrak{B}$, f is a coequalizer if and only if Ψf is a coequalizer,

(ii) for each parallel pair (f,g) in $\mathfrak{B}$, (f,g) is a kernel pair if (and only if) $(\Psi f, \Psi g)$ is a kernel pair,
then the canonical comparison functor

$$\Theta : \mathfrak{B} \longrightarrow \mathfrak{A}^{\mathbb{T}}$$

is an equivalence of categories.

Proof. (I) Θ is full and faithful. Indeed, for each object B in $\mathfrak{B}^\circ$, $\Psi\varepsilon_B$ is a (split) coequalizer in $\mathfrak{A}$, hence, by (i), ε_B is a coequalizer in $\mathfrak{B}$. Therefore Θ is full and faithful (cf. [6], 21.4.6).

(II) Ψ preserves exact sequences. Indeed, let

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{q} Z$$

be an exact sequence in $\mathfrak{B}$. Since Ψ is a right adjoint, $(\Psi f, \Psi g)$ is a kernel pair of Ψq. By (i), Ψq is a coequalizer in $\mathfrak{A}$, and therefore Ψq is a coequalizer of its kernel pair $(\Psi f, \Psi g)$. Thus the diagram

$$\Psi X \overset{\Psi f}{\underset{\Psi g}{\rightrightarrows}} \Psi Y \xrightarrow{\Psi q} \Psi Z$$

is an exact sequence.

(III) It is well known that, since $\mathfrak{B}$ has coequalizers,

Θ has a left adjoint

$$\Xi : \mathfrak{A}^{\mathbb{T}} \longrightarrow \mathfrak{B}$$

where

$$\Phi\Psi\Phi A \overset{\Phi\alpha}{\underset{\varepsilon_{\Phi A}}{\rightrightarrows}} \Phi A \xrightarrow{\xi(A,\alpha)} \Xi(A,\alpha)$$

is a coequalizer in $\mathfrak{B}$, for (A,α) in $(\mathfrak{A}^{\mathbb{T}})^\circ$. The unit of the adjunction

$$\eta^\nabla : I_{\mathfrak{A}^{\mathbb{T}}} \longrightarrow \Theta\Xi$$

is defined by

$$\eta^\nabla_{(A,\alpha)} = (\eta_{(A,\alpha)}, \alpha, \Psi\varepsilon_{\Xi(A,\alpha)})$$

where $\eta_{(A,\alpha)}$ is the unique morphism in $\mathfrak{A}$ such that the diagram

$$\begin{tikzcd} TA \arrow[r, "\alpha"] \arrow[dr, "\Psi\xi(A,\alpha)"'] & A \arrow[d, dashed, "\eta_{(A,\alpha)}"] \\ & \Psi\Xi(A,\alpha) \end{tikzcd}$$

is commutative (cf. [4], VI.7. Exercise 2., and [3] p. 56).

(IV) If (A,α) is a $\mathbb{T}$-algebra, $E \overset{f_1}{\underset{f_2}{\rightrightarrows}} TA$ is a kernel pair of α in $\mathfrak{A}$ and $g_i = \varepsilon_{\Phi A}\Phi f_i$ for $i = 1,2$, then $\xi(A,\alpha)$ is a coequalizer of (g_1, g_2) in $\mathfrak{B}$.

Proof of (IV). We shall show that for each morphism f in $\mathcal{B}$, $fg_1 = fg_2$ if and only if $f\,\phi\alpha = f\,\varsigma_{\phi A}$.

If $f\phi\alpha = f\varsigma_{\phi A}$, then the diagram

$$\begin{tikzcd}
 & \phi\psi\phi A \arrow[rr, "\varsigma_{\phi A}"] \arrow[dr, "\phi\alpha"] & & \phi A \arrow[dr, "f"] & \\
\phi E \arrow[ur, "\phi f_1"] \arrow[dr, "\phi f_2"'] & & \phi A \arrow[rr, "f"] & & X \\
 & \phi\psi\phi A \arrow[rr, "\varsigma_{\phi A}"'] \arrow[ur, "\phi\alpha"] & & \phi A \arrow[ur, "f"'] &
\end{tikzcd}$$

is commutative. Hence $fg_1 = fg_2$.

Assume now that $fg_1 = fg_2$. There exists, by Lemma 1.4., an $\gamma : TE \longrightarrow E$ in $\mathfrak{A}$ such that (E, γ) is a $\mathbb{T}$ - algebra, and $(E, \gamma) \underset{(f_2,\, \gamma\, ,\, \mu_A)}{\overset{(f_1,\, \gamma\, ,\, \mu_A)}{\rightrightarrows}} (TA, \mu_A)$ are morphisms of $\mathbb{T}$ - algebras.

Thus we get two commutative diagrams

$$\begin{tikzcd}
E \arrow[r, "\eta_E"] \arrow[dr, "\iota_E"'] & TE \arrow[r, "Tf_i"] \arrow[d, "\gamma"] \arrow[dr, "\psi g_i"] & T^2A \arrow[d, "\mu_A"] & \\
 & E \arrow[r, "f_i"'] & TA \arrow[r, "f"'] & \psi X
\end{tikzcd}$$

for $i = 1,2$. In particular, $\psi f\,\psi g_i = \psi f f_i$, for $i = 1,2$. It follows from $fg_1 = fg_2$ that $\psi f\,\psi g_1 = \psi f\,\psi g_2$, and consequently $\psi f f_1 = \psi f f_2$. Since α is a coequalizer of its kernel pair (f_1, f_2), there exists a unique morphism h in

$\mathfrak{A}$ such that $\psi f = h\alpha$.

$$\begin{tikzcd}
E \arrow[r, "\eta_E"] \arrow[dr, "\iota_E"'] & TE \arrow[r, shift left, "Tf_1"] \arrow[r, shift right, "Tf_2"'] \arrow[d, "\gamma"] \arrow[dr, shift left, "\psi g_1"] \arrow[dr, shift right, "\psi g_2"'] & T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] & TA \arrow[d, "\alpha"] \arrow[dd, bend left, "\psi f"] \\
 & E \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & TA \arrow[r, "\alpha"] \arrow[dr, "\psi f"'] & A \arrow[d, dashed, "h"] \\
 & & & \psi X
\end{tikzcd}$$

Hence $\psi f\,\mu_A = h\,\alpha\,\mu_A = h\,\alpha\,T(\alpha) = \psi f T\alpha$. Since ψ is faithful, we have $f\varsigma_{\phi A} = f\,\phi\alpha$.

(V) For each object (A, α) of $\mathfrak{A}^{\mathbb{T}}$ the morphism $\eta_{(A,\alpha)}$ is an isomorphism in $\mathfrak{A}$.

Proof of (V). Let $E \underset{f_2}{\overset{f_1}{\rightrightarrows}} TA$ be a kernel pair of α in $\mathfrak{A}$, and let $\gamma : TE \longrightarrow E$ be the unique morphism in $\mathfrak{A}$ such that (E, γ) is a $\mathbb{T}$ - algebra and (f_1, γ, μ_A), (f_2, γ, μ_A) are morphisms of $\mathbb{T}$ - algebras. Let $g_i = \varsigma_{\phi A}\, f_i$ for $i = 1,2$. Then the diagrams

$$\begin{tikzcd}
TE \arrow[r, "Tf_1"] \arrow[d, "\gamma"'] \arrow[dr, "\psi g_i"] & T^2A \arrow[d, "\mu_A"] \\
E \arrow[r, "f_1"'] & TA
\end{tikzcd}$$

are commutative for $i = 1,2$. By (IV), $\xi(A,\alpha)$ is a coequalizer of (g_1, g_2) in $\mathcal{B}$.

We shall consider the following morphisms in $\mathcal{B}$:

$$\begin{tikzcd}
X \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & \phi E \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "q"'] \arrow[dr, "p"] & \phi A \arrow[r, "\xi(A,\alpha)"] & \Xi(A,\alpha) \\
& Y \arrow[r, "f"'] & B \arrow[u, shift left, "p_1"] \arrow[u, shift right, "p_2"'] &
\end{tikzcd}$$

where

(p_1,p_2) is a kernel pair of $\xi(A,\alpha)$

p is the unique morphism in $\mathcal{B}$ such that $p_i p = g_i$, $i = 1,2$

(q_1,q_2) is a kernel pair of p

q is a coequalizer of (q_1,q_2)

f is the unique morphism in $\mathcal{B}$ such that $p = fq$.

It is easy to see that (q_1,q_2) is a limit of the diagram

$$\begin{tikzcd}
\phi E \arrow[dr, "g_2"] \arrow[drr, "g_1"] & & \\
& \phi A & \phi A \\
\phi E \arrow[ur, "g_2"] \arrow[urr, "g_1"'] & &
\end{tikzcd}$$

in $\mathcal{B}$. Since ψ is a right adjoint functor, $(\psi q_1, \psi q_2)$ is a limit of the diagram

$$\begin{tikzcd}
TE \arrow[dr, "\psi g_2"] \arrow[drr, "\psi g_1"] & & \\
& TA & TA \\
TE \arrow[ur, "\psi g_2"] \arrow[urr, "\psi g_1"'] & &
\end{tikzcd}$$

in $\mathcal{A}$. Since $\psi g_i = f_i$ for $i = 1, 2$, and (f_1,f_2) is a monic pair, $(\psi q_1, \psi q_2)$ is also a limit of the diagram

$$\begin{tikzcd}
TE \arrow[dr, "\gamma"] & \\
& E \\
TE \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

i.e., $(\psi q_1, \psi q_2)$ is a kernel pair of γ in $\mathcal{A}$. Since γ is a coequalizer in $\mathcal{A}$, it is a coequalizer of its kernel pair $(\psi q_1, \psi q_2)$. The diagram

$$\begin{tikzcd}
\psi X \arrow[r, shift left, "\psi q_1"] \arrow[r, shift right, "\psi q_2"'] & TE \arrow[r, "\psi q"] & \psi Y
\end{tikzcd}$$

is, by (II), an exact sequence. In particular, ψq is a coequalizer of $(\psi q_1, \psi q_2)$. Hence there exists a unique isomorphism h in $\mathcal{A}$ such that the diagram

$$\begin{tikzcd}
TE \arrow[r, "\psi q"] \arrow[dr, "\gamma"'] & \psi Y \arrow[d, dashed, "h"] \\
& E
\end{tikzcd}$$

is commutative. Thus we get the following diagram in $\mathfrak{A}$

$$\begin{tikzcd}
E \arrow[rr, shift left, "f_1"] \arrow[rr, shift right, "f_2"'] & & TA \arrow[r, "\alpha"] \arrow[dr, "\psi\xi(A,\alpha)"'] & A \arrow[d, "\eta(A,\alpha)"] \\
& & & \psi\Xi(A,\alpha) \\
TE \arrow[uu, "r"] \arrow[r, "\psi q"] \arrow[rr, bend right, "\psi p"'] & \psi X \arrow[uul, "h"'] \arrow[r, "\psi f"] & \psi B \arrow[uu, shift left, "\psi p_1"] \arrow[uu, shift right, "\psi p_2"'] &
\end{tikzcd}$$

In particular we have $f_i h = \psi p_i \, \psi f$ for $i = 1,2$, (because ψq is an epimorphism). Since h is an isomorphism and (f_1, f_2) is a kernel pair, it follows that $(f_1 h, f_2 h) = (\psi(p_1 f), \psi(p_2 f))$ is a kernel pair in $\mathfrak{A}$. Hence, by (ii), $(p_1 f, p_2 f)$ is a kernel pair in $\mathfrak{B}$. Since $\xi(A,\alpha)$ is a coequalizer of $(g_1, g_2) = (p_1 fq, p_2 fq)$, it is a coequalize of $(p_1 f, p_2 f)$, too. Thus $(p_1 f, p_2 f)$ is a kernel pair of $\xi(A,\alpha)$ and therefore the diagram

$$\begin{tikzcd}
Y \arrow[r, shift left, "p_1 f"] \arrow[r, shift right, "p_2 f"'] & \phi A \arrow[r, "\xi(A,\alpha)"] & \Xi(A,\alpha)
\end{tikzcd}$$

is an exact sequence. Consequently the diagram

$$\begin{tikzcd}
\psi Y \arrow[r, shift left, "f_1 h"] \arrow[r, shift right, "f_2 h"'] & TA \arrow[r, "\psi\xi(A,\alpha)"] & \psi\Xi(A,\alpha)
\end{tikzcd}$$

is an exact sequence (because $f_i h = \psi(p_i f)$, $i = 1,2$). In particular, $\psi\xi(A,\alpha)$ is a coequalizer of $(f_1 h, f_2 h)$, and consequently it is a coequalizer of (f_1, f_2). Since α is also a coequalizer of its kernel pair (f_1, f_2), $\eta(A,\alpha)$ is an isomorphism in $\mathfrak{A}$.

(VI) Θ is an equivalence of categories.

Proof of (VI). By (V), $\check{\eta}_{(A,\alpha)} = (\eta_{(A,\alpha)}, \alpha, \psi\xi_{\Xi(A,\alpha)})$ is an isomorphism in $\mathfrak{A}^{\mathbb{T}}$ for each $\mathbb{T}$ - algebra (A, α). For each object B of $\mathfrak{B}$ the diagram

$$\begin{tikzcd}
\Theta B \arrow[r, "\check{\eta}_{\Theta B}"] \arrow[dr, "\iota_{\Theta B}"'] & \Theta\Xi\Theta B \arrow[d, "\Theta\bar{\xi}_B"] \\
& \Theta B
\end{tikzcd}$$

is commutative, where $\bar{\xi}: \Xi\Theta \longrightarrow I_{\mathfrak{B}}$ is the counit of the adjunction. Hence $\Theta\bar{\xi}_B$ is an isomorphism in $\mathfrak{A}^{\mathbb{T}}$. Since Θ is full and faithful, $\bar{\xi}_B$ is an isomorphism in $\mathfrak{B}$. Hence $\Xi\Theta$ and $\Theta\Xi$ are naturally equivalent to identities on $\mathfrak{B}$ and $\mathfrak{A}^{\mathbb{T}}$ respectively.

This completes the proof of Theorem 2.1.

2.2. <u>Remark.</u> It is easy to see that the composition of functors satisfying the conditions 2.1.(i), (ii), also satisfies these conditions, and consequently is a monadic functor. But it is known that the composition of two monadic functors need not be monadic (cf. [6], 21.6.10.(d)). This shows that the conditions 2.1.(i),(ii), are not necessary for a functor to be monadic.

2.3. We shall now give some necessary conditions for a functor to be monadic. The following Theorem 2.4. has been proved by Linton [3] under an assumption that every epimorphism in $\mathfrak{A}$ is a retraction. We shall prove the theorem under an essentially weaker assumption that every coequalizer in $\mathfrak{A}$ is a retraction.

For example in the category Conv (cf. 3.12.) there are epimorphisms which are not retractions, but each coequalizer in Conv is a retraction.

2.4. <u>Theorem.</u> Let a category $\mathfrak{A}$ have kernel pairs and coequalizers of kernel pairs, and let every coequalizer in $\mathfrak{A}$ be a retraction. Let $\Psi : \mathcal{B} \longrightarrow \mathfrak{A}$ be a functor having a left adjoint. If the corresponding comparison functor $\Theta : \mathcal{B} \longrightarrow \mathfrak{A}^{\mathbb{T}}$ is an equivalence of categories, then

(i) $\mathcal{B}$ has kernel pairs and $\mathcal{P}$ - coequalizers,

(ii) for each q in $\mathcal{B}$, q is a coequalizer if and only if Ψq is a coequalizer,

(iii) for each parallel pair (f,g) in $\mathcal{B}$, (f,g) is a kernel pair if (and only if) $(\Psi f, \Psi g)$ is a kernel pair.

Proof of (i). By Theorem 1.5., $\mathcal{B}$ has $\mathcal{P}$ -coequalizers. We shall show that $\mathcal{B}$ has kernel pairs. Let $q : Y \longrightarrow B$ be a morphism in $\mathcal{B}$ and let $A \underset{g}{\overset{f}{\rightrightarrows}} \Psi Y$ be a kernel pair of Ψq in $\mathfrak{A}$.

Then there exists a unique $\alpha : TA \longrightarrow A$ in $\mathfrak{A}$ such that

$$(A, \alpha) \underset{(g,\alpha, \Psi \varepsilon_Y)}{\overset{(f,\alpha, \Psi \varepsilon_Y)}{\rightrightarrows}} \Theta Y$$

is a kernel pair of $\Theta q : \Theta Y \longrightarrow \Theta B$ in $\mathfrak{A}^{\mathbb{T}}$ (cf. Lemma 1.4.). Since Θ is an equivalence, there is a functor $\Xi : \mathfrak{A}^{\mathbb{T}} \longrightarrow \mathcal{B}$ which is a right and a left adjoint of Θ. Hence

$$\Xi(A, \alpha) \underset{\Xi(g,\alpha, \Psi \varepsilon_Y)}{\overset{\Xi(f,\alpha, \Psi \varepsilon_Y)}{\rightrightarrows}} \Xi\Theta Y$$

is a kernel pair of $\Xi\Theta q : \Xi\Theta Y \longrightarrow \Xi\Theta B$. The functor $\Xi\Theta$ is naturally equivalent to the functor $I_{\mathcal{B}}$. Hence there exist isomorphisms λ, μ in $\mathcal{B}$ such that the diagram

$$\begin{tikzcd}
\Xi\Theta Y \arrow[r, "\Xi\Theta q"] \arrow[d, "\lambda"'] & \Xi\Theta B \arrow[d, "\mu"] \\
Y \arrow[r, "q"'] & B
\end{tikzcd}$$

is commutative. Consequently the pair $(\lambda\Xi(f,\alpha,\psi\varsigma_Y),\lambda\Xi(g,\alpha,\psi\varsigma_Y))$ is a kernel pair of q.

Proof of (iii). Let (f,g) be a parallel pair in $\mathcal{L}$ such that $(\psi f, \psi g)$ is a kernel pair in $\mathfrak{A}$. Then $(f,g)\in\mathfrak{P}$. Indeed, there is a coequalizer p of $(\psi f, \psi g)$ in $\mathfrak{A}$ which is a retraction, and $(\psi f, \psi g)$ is a kernel pair of p. Consequently p is a split coequalizer of $(\psi f, \psi g)$ (cf. [6], 21.4.2.(b)), i.e., $(f,g)\in\mathfrak{P}$.

Thus, by Theorem 1.5., there is a coequalizer q of (f,g) and ψq is a coequalizer of $(\psi f, \psi g)$. Consequently $(\psi f, \psi g)$ is a kernel pair ψq. Hence, by Lemma 1.4., $(\Theta f, \Theta g)$ is a kernel pair of Θq in $\mathfrak{A}^{\mathbb{T}}$. Since Θ is an equivalence of categories, (f,g) is a kernel pair of q.

The converse implication is true for each right adjoint functor ψ.

Proof of (ii). Let q be a coequalizer in $\mathcal{L}$, and let (f,g) be a kernel pair of q. Since ψ is a right adjoint functor, $(\psi f, \psi g)$ is a kernel pair of ψq. Hence $(f,g)\in\mathfrak{P}$, and therefore, by 1.5., ψq is a coequalizer

of $(\psi f, \psi g)$. Consequently ψq is a coequalizer in $\mathfrak{A}$.

Let $q : A \longrightarrow B$ be a morphism in $\mathcal{L}$ such that ψq is a coequalizer in $\mathfrak{A}$. Let $E \overset{f'}{\underset{g'}{\rightrightarrows}} \psi A$ be a kernel pair of ψq. There is unique morphism $\gamma : TE \longrightarrow E$ in $\mathfrak{A}$ such that

$$(E,\gamma) \overset{(f',\gamma,\psi\varsigma_A)}{\underset{(g',\gamma,\psi\varsigma_A)}{\rightrightarrows}} (\psi A, \psi\varsigma_A) = \Theta A$$

is a kernel pair of $(\psi q, \psi\varsigma_A, \psi\varsigma_B) = \Theta q$ in $\mathfrak{A}^{\mathbb{T}}$ (cf. 1.4.).

We define $f = \varsigma_A \Phi f'$, $g = \varsigma_A \Phi g'$ (hence $f' = \psi f\, \eta_E$, $g' = \psi g\, \eta_E$):

$$\begin{tikzcd}
\Phi E \arrow[r, shift left, "\Phi f'"] \arrow[r, shift right, "\Phi g'"'] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & \Phi\psi A \arrow[d, "\varsigma_A"] \\
 & A
\end{tikzcd}$$

Let us consider the diagram:

$$\begin{tikzcd}
TE \arrow[r, shift left, "Tf'"] \arrow[r, shift right, "Tg'"'] \arrow[dr, shift left, "\psi f"] \arrow[dr, shift right, "\psi g"'] \arrow[d, "\gamma"'] & \psi\Phi\psi A \arrow[d, "\psi\varsigma_A"] & \\
E \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & \psi A \arrow[r, "\psi q"] & \psi B
\end{tikzcd}$$

It is easy to see that $\psi q\, \psi f = \psi q\, \psi g$. Since ψq is a retraction and $(f', g') = (\psi f\, \eta_E, \psi g\, \eta_E)$ is a kernel pair of ψq, there are λ, μ in $\mathfrak{A}$ such that the diagram

$$\begin{tikzcd}
\psi A \arrow[r, "\lambda"] \arrow[d, "\psi q"'] \arrow[rrr, bend left, "\iota_{\psi A}"] & E \arrow[r, "\eta_E"] & \psi\phi E \arrow[r, "\psi f"] \arrow[d, "\psi g"] & \psi A \arrow[d, "\psi q"] \\
\psi B \arrow[rr, "\mu"] \arrow[rrr, bend right, "\iota_{\psi B}"'] & & \psi A \arrow[r, "\psi q"] & \psi B
\end{tikzcd}$$

is commutative. Hence ψq is a split coequalizer of $(\psi f, \psi g)$ and $(f,g) \in \mathcal{P}$. Since ψ reflects $\mathcal{P}$ - coequalizers (cf. 1.5. q is a coequalizer of (f,g).

2.5. As an application of Theorem 2.1. we shall give a simple proof of a theorem due to F.E.J.Linton (cf. [10], 4.2).

Ban_1 is the category of Banach spaces (over $\mathbb{R}$ or $\mathbb{C}$) and linear contractions. (A linear contraction is a linear operator of norm ≤ 1.). If B is a Banach space, B^* is its conjugate space; if $\alpha : A \longrightarrow B$ is a morphism in Ban_1, then $\alpha^* : B^* \longrightarrow A^*$ defined by $\alpha^*(f) = f\alpha$ for f in B^* is a morphism in Ban_1.

The contravariant conjugate-space functor * is adjoint on the right to itself (cf. [9], 12.4.4.(a)). If B is a Banach space, $\varkappa_B$ is the canonical map from B into B^{**}.

Let $\mathfrak{A} = Ban_1$ and $\mathfrak{B} = Ban_1{}^{op}$. Then the functors

$$\phi : \mathfrak{A} \longrightarrow \mathfrak{B} \quad \text{and} \quad \psi : \mathfrak{B} \longrightarrow \mathfrak{A}$$

defined as $\phi(A) = A^*$, $\phi(\alpha) = \alpha^*$, $\psi(B) = B^*$, $\psi(\beta) = \beta^*$ are both covariant, and ϕ is a left adjoint of ψ (cf. [10], 4.1).

2.5.1. <u>Theorem</u> (Linton). The functor ψ is quasi-monadic.

Proof. A morphism $\alpha : A \longrightarrow B$ is a coequalizer in $Ban_1{}^{op}$ if and only if $\alpha : B \longrightarrow A$ is an equalizer in Ban_1, i.e., an isometrical embedding. Similary, $\alpha : A \longrightarrow B$ is a coequalizer in Ban_1 if and only if α is a quotient map (cf. [9], 11.5.5. and 11.5.7.). Since α^* is a quotient map if and only if α is an isometry, the condition 2.1.(i) is satisfied.

Let $\alpha, \beta : A \longrightarrow B$ be morphisms in Ban_1 such that (α^*, β^*) is a kernel pair in Ban_1. Let f be a coequalizer of (α^*, β^*).

Then (α^*, β^*) is a kernel pair of f:

$$\begin{tikzcd}
B^* \arrow[r, shift left, "\alpha^*"] \arrow[r, shift right, "\beta^*"'] & A^* \arrow[r, "f"] & D
\end{tikzcd}$$

Let $M = \{a^{**} \in A^{**} : \exists_{b^{**}\in B^{**}}\ \alpha^*(b^{**}) = a^{**} \ \&\ \beta^*(b^{**}) = 0\}$.
It is clear that $M = \ker f$ and consequently M is a closed subspace of A^{**}. We shall show that M is $\sigma(A^{**}, A)$ - closed.
Let $\varepsilon: M \longrightarrow A^{**}$ be the identical embedding. Then $f\varepsilon = f0 =$
Hence there is a uniqre $\lambda : M \longrightarrow B^{**}$ such that $\alpha^*\lambda = \varepsilon$ and $\beta^*\lambda = 0$.

$$\begin{tikzcd}
B^{*} \arrow[r, shift left, "\alpha^*"] \arrow[r, shift right, "\beta^*"'] & A^{**} \arrow[r, "f"] & D \\
M \arrow[u, dashed, "\lambda"] \arrow[ur, shift left, "\varepsilon"] \arrow[ur, shift right, "0"'] & &
\end{tikzcd}$$

Since $\alpha^*\lambda$ is an isometry (that is, an isometrical embedding), λ and $\alpha^*|_{\lambda(M)}$ are isometries. Moreover, $\lambda(M) = \ker\beta^*$ and hence $\alpha^*|_{\ker\beta^*} : \ker\beta^* \longrightarrow M$ is an isomoetry.

Consequently

$$\alpha^*|_{OB^*\cap\ker\beta^*} : OB^{**} \cap \ker\beta^* \longrightarrow OA^{**} \cap M$$

is an isometrical bijection. Thus, by Krein-Šmulyan Theorem (cf. [5], IV, 6.4.), M is $\sigma(A^{**}, A)$ - closed.

Let $C = \bigcap\{\ker a^{**} : a^{**} \in M\}$. Since M is $\sigma(A^{**}, A)$-closed, the polar C° is equal to M (cf. [2], 20,3,(2)). It is easy to see that $C = \{a \in A : \alpha(a) = \beta(a)\}$.

Let $\varepsilon : C \longrightarrow A$ be the canonical embedding. We shall show that (α, β) is a kernel pair of ε in Ban_1^{op}. We have the following commutative diagram

$$\begin{tikzcd}
A^{**} \arrow[r, "\varepsilon^*"] \arrow[dr, "\pi"'] & C^{**} \\
& A^{**}/M \arrow[u, "i"']
\end{tikzcd}$$

where π is the quotient map and i is an isomorphism in Ban_1. Hence ε^* is a coequalizer of (α^*, β^*) and consequently (α^*, β^*) is a kernel pair of ε^*.

Let φ_1, φ_2 be morphisms in Ban_1 such that $\varphi_1\varepsilon = \varphi_2\varepsilon$ i.e., $\varepsilon\varphi_1 = \varepsilon\varphi_2$ in Ban_1^{op}. Hence $\varepsilon^*\varphi_1^* = \varepsilon^*\varphi_2^*$. Since (α^*, β^*) is a kernel pair of ε^*, there is a unique $\vartheta : X \longrightarrow B$ such that $\varphi_1^* = \alpha^*\vartheta$, $\varphi_2^* = \beta^*\vartheta$

$$\begin{tikzcd}
B^{*} \arrow[r, shift left, "\alpha^*"] \arrow[r, shift right, "\beta^*"'] & A^{**} \arrow[r, "\varepsilon^*"] & C^{**} \\
X^{**} \arrow[u, dashed, "\vartheta"] \arrow[ur, shift left, "\varphi_1^*"] \arrow[ur, shift right, "\varphi_2^*"'] & &
\end{tikzcd}$$

It is enought to prove that there is a morphism $\varphi: B \longrightarrow X$ such that $\vartheta = \varphi^*$. Let a_1, a_2 be elements of A and let $b = \alpha(a_1) + \beta(a_2)$. Let x^{**} be an element of X^{**}. Then

$$(\varkappa_B(b)\,\vartheta)(x^*) = \vartheta(x^*)(b) = \vartheta(x^*)(\alpha(a_1) + \beta(a_2)) =$$

$$= (\alpha^*\vartheta(x^*))(a_1) + (\beta^*\vartheta(x^*))(a_2) =$$

$$= (\varphi_1^*(x^*))(a_1) + (\varphi_2^*(x^*))(a_2) = x^*(\varphi_1(a_1) + \varphi_2(a_2)) =$$

$$= (\varkappa_X(\varphi_1(a_1) + \varphi_2(a_2)))(x^*)$$

Hence $\varkappa_B(b)\,\vartheta = \varkappa_X(\varphi_1(a_1) + \varphi_2(a_2))$. It is easy to see that the set $\alpha(A) + \beta(A)$ is dense in B. Thus ϑ is $(\sigma(X^*, X), \sigma(B^*, B))$ - continuous, and consequently there is $\varphi : B \longrightarrow X$ in Ban_1 such that $\vartheta = \varphi^*$.

Therefore, by Theorem 2.1., the functor ψ is quasi-monadic.

§ 3. Categories of convex sets.

By a vector space we shall always mean a real vector space A subset K of a vector space E is called convex if $(1-t)x+ty$ i an element of K for all x, y in K, $0 \leq t \leq 1$. Let K and K' convex subsets of vector spaces E and E' respectively. A map $f : K \longrightarrow K'$ is called affine, if

$$f((1-t)x + ty) = (1-t)f(x) + tf(y)$$

for all x, y in K, $0 \leq t \leq 1$.

Intuitively, a convex structure is a way the convex combinations are defined. A convex subset K of a vector space E is provided with the obvious convex structure. The space E plays an auxiliary role here; K may be contained in many entirely different vector spaces and yet may have the same convex combinations. In this section we shall give a precise definition of the convex structure.

3.1. Let X be a set. A free vector space generated by X is the space $\mathcal{V}(X)$ of all real-valued functions f on X such that the set $\{x : f(x) \neq 0\}$ is finite. The canonical injection

$$\sigma^X : X \longrightarrow \mathcal{V}(X)$$

is $\sigma^X_x(y) = \delta_{xy}$ (Kronecker δ). The convex subset

$$G(X) = \{ f \in \mathcal{V}(X) : f \geq 0 \ \& \sum_{x \in X} f(x) = 1 \}$$

of the free vector space (intersection of the cone of nonnegative functions with the hyperplane $\sum f(x) = 1$) is called the free convex set generated by X, (cf. 4.1.1.).

3.2. Let K be a set. By a convex structure on K we shall mean a map

$$k : G(K) \longrightarrow K$$

such that there exist a vector space E and an injection $\varepsilon : K \longrightarrow E$ satisfying the following condition:

$$(1) \qquad \varepsilon(k(\sum a_i \delta^K_{x_i})) = \sum a_i \varepsilon(x_i)$$

for each $\sum a_i \delta^K_{x_i}$ in $G(K)$. The condition (1) means that $\varepsilon k : G(K) \longrightarrow E$ is affine.

The element $k(\sum a_i \delta^K_{x_i})$ of K is called a convex combination of elements x_i. We shall also write $\sum a_i x_i$ instead of $k(\sum a_i \delta^K_{x_i})$.

For every convex subset K of a vector space E there is the natural convex structure induced by the vector space structure of E. On the other hand, if k is a convex structure on a set K, then there is a vector space E' such that K is a convex subset of E' and k is induced by the vector space structure of E'. In fact, if $\varepsilon : K \longrightarrow E$ is an injectio satisfying (1), then the set $E' = K \cup (E \setminus \varepsilon(K))$ becomes a vector space in the obvious way.

Let K be a set and let k be a convex structure on K. The pair (K,k) will be called a <u>set with a convex structure</u>, or shortly if there is no danger of confusion a <u>convex set.</u>

3.3. Let (K,k) and (K', k') be sets with convex structures. A map $\varphi : K \longrightarrow K'$ is called <u>affine</u> if it preserves the convex combinations, i.e.,

$$\varphi((1-t)x + ty) = (1-t)\,\varphi(x) + t\,\varphi(y)$$

for x, y in K, $0 \leq t \leq 1$.

3.4. A structure of a <u>bounded convex set</u> on a set K is a pair (k, τ) where k is a convex structure on K and τ is a topology on K such that there exist a locally convex Hausdorff space E and an affine homeomorphism $\varepsilon : K \longrightarrow \varepsilon(K) \subset E$.

Every set with a structure of a bounded convex set can be regarded as a bounded convex subset of a locally convex Hausdorff space (e.g., $E' = K \cup (E \setminus \varepsilon(K))$.

3.5. A <u>compact convex set</u> is a bounded convex set (K, k, τ) such that τ is a compact topology.

3.6. A <u>compact Saks space</u> is a quadruple (K, k, τ, x_0), where (K, k, τ) is a compact convex set and x_0 is an element of K satisfying the following condition: for each x in K there exists an y in K such that $\frac{1}{2}x + \frac{1}{2}y = x_0$. Such a point x_0 is unique and is called the <u>center of the set K.</u>

3.7. <u>Lemma.</u> (Cf. [7], 13.6.) Let (K, k, τ) be a compact convex set, and let x_0 be an element of K. The quadruple (K, k, τ, x_0) is a compact Saks space if and only if there exists a Banach space $(X, \|\cdot\|)$ and an affine $(\mathcal{T}, \sigma(X^*, X))$ - continuous bijection

$$\varepsilon : K \longrightarrow \{ x^* \in X^* : \|x^*\| \leq 1 \} = O^* X$$

such that $\varepsilon(x_0) = 0$.

The sufficiency is obvious. If $(K, k, \mathcal{T}, x_0)$ is a compact Saks space, then X may be defined as the space of all affine real valued continuous functions on K vanishing at x_0 with the supremum norm (for a proof, see also [13], Proposition 1).

Thus each compact Saks space can be regarded as a quadru $(K, k, \mathcal{T}, x_0)$, where $K = OB = \{ y \in B : \|y\| \leq 1 \}$ and $(B, \|\cdot\|)$ is a Banach space isometrically isomorphic to the conjugate of a Banach space $(X, \|\cdot\|)$, $\mathcal{T}$ is the topol on K induced by $\sigma(B, X)$, k is induced by the vector space structure on B, and x_0 is the zero element in B. We can take $B = K \cup (X^* \setminus \varepsilon(K))$.

3.8. We shall consider the following categories and functors:

Conv is the category of convex sets and affine maps.

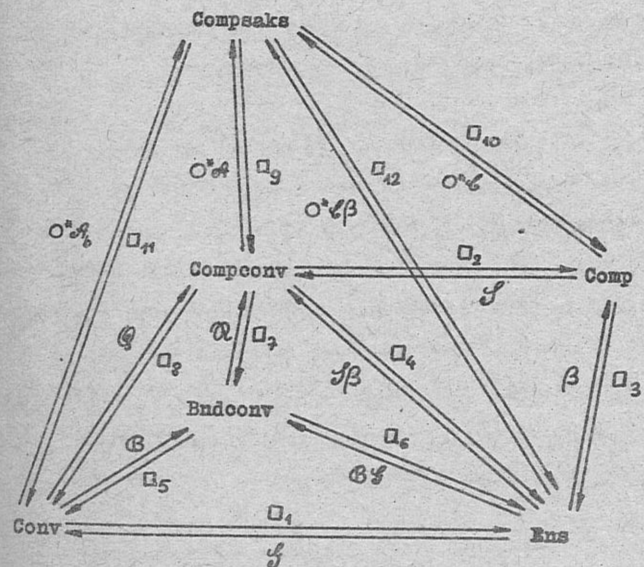
Bndconv is the category of bounded convex sets and continuous affine maps.

Compconv is the category of compact convex sets and continuous affine maps. It is a full subcategory of Bndconv.

Compsaks is the category of compact Saks spaces and centerpreserving continuous affine maps.

Comp is the category of compact (Hausdorff) topological spaces and continuous maps.

Ens is the category of sets and (all) maps.



The functors $\Box_1, \dots, \Box_{12}$ are forgetful functors. Each of them has a left adjoint. These left adjoint functors will be described in §4.

3.9. LEMMA. Each of the functors $\Box_i$, $i \neq 5, 6$, creates isomorphisms.

Proof. This is obvious for the functor $\square_7$. It is easy to see that in each of the categories Compsaks, Compconv, Comp, Conv, Ens a morphism is an isomorphism if and only if it is a bijection. Hence, if $f : A \longrightarrow \square_i(B)$ $(i \neq 5,6,7)$ is an isomorphism, then f is a bijection. Consequently the structure of B can be uniquely transfered to A in such a way that f becomes a morphism (and consequently an isomorphism) in the desired category.

3.10. LEMMA. Let $\mathfrak{A}$ be any of the categories Compsaks, Compconv, Conv, Comp. A morphism f is a coequalizer in $\mathfrak{A}$ if and only if it is a surjection.

Proof. For every morphism $f : A \longrightarrow B$ in $\mathfrak{A}$ the set $f(A) = \{f(x) : x \in A\}$ together with the induced structures is an object in $\mathfrak{A}$. Moreover the maps $f_A : A \longrightarrow f(A)$ and $\varepsilon : f(A) \longrightarrow B$ defined by $f_A(x) = f(x)$ for x in A and $\varepsilon(y) = y$ for y in $f(A)$ are morphisms in $\mathfrak{A}$ such that $\varepsilon f_A = f$.

Let f be a coequalizer of the parallel pair (f_1, f_2) in $\mathfrak{A}$. Since $f_A f_1 = f_A f_2$, there exists a unique morphism $h : B \longrightarrow f(A)$ such that $f_A = hf$. Hence $f = \varepsilon f_A = \varepsilon hf$, and consequently $\iota_B = \varepsilon h$ (because f is an epimorphism). But it is possible only if $B = \varepsilon(f(A)) = f(A)$, ie., if f is a surjection.

Let $f : A \longrightarrow B$ be a surjection, and let (f_1, f_2) be a kernel pair of f in $\mathfrak{A}$. The forgetful functor $\square_1 : \mathfrak{A} \longrightarrow$ Ens has a left adjoint, hence $(\square_1 f_1, \square_1 f_2)$ is a kernel pair of $\square_1 f$ in Ens. Since $\square_1 f$ is a surjection, it is a coequalizer of its kernel pair $(\square_1 f_1, \square_1 f_2)$ in Ens. Let $g : A \longrightarrow C$ be a morphism in $\mathfrak{A}$ such that $gf_1 = gf_2$. Then there exists a unique map $h : \square_1 B \longrightarrow \square_1 C$ such that $h\,\square_1 f = \square_1 g$. It is enough to proof that h is a morphism in $\mathfrak{A}$.

If $\mathfrak{A}$ is either Comp, Compconv or Compsaks, then h is a continuous map. Indeed, the topologies on A and on B are compact and f is a continuous surjection; hence f is a quotient map (cf. [9], 5.2.).

If $\mathfrak{A}$ is either Conv, Compconv or Compsaks, then h is an affine map. Indeed, let $b', b'' \in B$, $0 \leq t \leq 1$. There exist a', a'' in A such that $f(a') = b'$, $f(a'') = b''$. Thus

$$\begin{aligned} h(tb' + (1-t)b'') &= h(tf(a') + (1-t)f(a'')) = \\ &= hf(ta' + (1-t)a'') = g(ta' + (1-t)a'') = \\ &= tg(a') + (1-t)g(a'') = thf(a') + (1-t)hf(a'') = \\ &= th(b') + (1-t)h(b''). \end{aligned}$$

If $\mathfrak{A}$ = Compsaks, b_o is a center of an object B and c_o is a center of C, then $h(b_o) = hf(a_o) = g(a_o) = c_o$, where a_0 is a center of A.

Thus we have shown that h is a morphism in $\mathcal{O}\!\mathcal{L}$. This completes the proof.

3.11. LEMMA. There is a surjection in Bndconv which is not a coequalizer.

Proof. Let $(E, \|\cdot\|)$ be an infinitely dimensional Banach space, and let B, be the set $\{x \in E : \|x\| \leq 1\}$ with the induced convex structure and topology. Let B' be the same convex set with the topology induced by $\sigma(E,E^*)$. Suppose that there are morphisms f,g in Bndconv such that $id : B \longrightarrow B'$ defined by $id(x) = x$ for x in B, is a coequalizer of (f,g). We have also $\iota_B f = \iota_B g$, where $\iota_B : B \longrightarrow B$ is the identity on B in Bndconv, but the unique map such that the diagram

$$\begin{tikzcd} B \arrow[r, "id"] \arrow[dr, "\iota_B"'] & B' \arrow[d, dashed] \\ & B \end{tikzcd}$$

is commutative, is not continuous, i.e., it is not a morphism in Bndconv.

3.12. LEMMA. Every coequalizer in Conv is a retraction, but not every epimorphism in Conv is a retraction.

Proof. Let $f : A \longrightarrow B$ be a coequalizer in Conv. Hence, by Lemma 3.10., f is a surjection. We may assume that A,B are convex subsets of vector spaces V,W such that $V = \operatorname{span} A$, $W = \operatorname{span} B$ and $f = F|_A$, where $F : V \longrightarrow W$ is a linear map. Let $X = \ker F$, and let Y be a subspace of V such that $V = X \oplus Y$. Then $F|_Y : Y \longrightarrow W$ is a bijection. Let $G = (F|_Y)^{-1}$, and $g = G|_B$. Then $FG = \iota_W$, and consequently $fg = \iota_B$, i.e., f is a retraction.

We shall show that not every epimorphism in Conv is a retraction. Let $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $B = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ and let $\varepsilon : A \longrightarrow B$ be the inclusion map. It is easy to see that ε is an epimorphism, but it is not surjection, Consequently ε is not a retraction (cf. 2.3.).

§ 4. Monadic functors and convexity

In this section we shall discuss the adjunctions exhibited in the diagram on p. 35. Our aim is to show that the functors $\Box_i$ are monadic for $2 \leq i \leq 4$ and for $7 \leq i \leq 12$, and they are not monadic for $i = 1,5,6$.

4.1. The case $\mathrm{Conv} \underset{\mathcal{G}}{\overset{\Box_1}{\rightleftarrows}} \mathrm{Ens}$

4.1.1. Let X be a set. Then $G(X)$ is a convex subset of the free vector space $\mathcal{V}(X)$. The convex structure on $G(X)$ induced by the vector space structure of $\mathcal{V}(X)$ is called the <u>free convex structure</u> and is denoted by μ_X. The <u>set with the free convex structure</u> $(G(X), \mu_X)$ <u>generated by X</u> will be denoted by $\mathcal{G}(X)$ and will be also called the <u>free convex set generated by X</u>, if no confusion is possible.

If X' is another set and $\varphi : X \longrightarrow X'$ is a map, then

$$\mathcal{G}\varphi : \mathcal{G}(X) \longrightarrow \mathcal{G}(X')$$

defined by $\mathcal{G}\varphi(\sum a_i \sigma^X_{x_i}) = \sum a_i \sigma^{X'}_{\varphi(x_i)}$ is an affine map. Thus we get a covariant functor

$$\mathcal{G} : \mathrm{Ens} \longrightarrow \mathrm{Conv}$$

which is a left adjoint of $\Box_1$ (cf. [9], 23.5.6.).

We shall denote by G the composition $\Box_1 \mathcal{G}$.

The unit of the adjunction

$$\sigma : I_{\mathrm{Ens}} \longrightarrow \Box_1 \mathcal{G} = G$$

is defined by $\sigma = (\sigma^X)_{X \in \mathrm{Ens}^0}$, where $\sigma^X : X \longrightarrow G(X)$ is the canonical injection (cf.3.1.). The counit of the adjunction

$$\varsigma : \mathcal{G}\Box_1 \longrightarrow I_{\mathrm{Conv}}$$

is defined by $\varsigma_{(K,k)} = k$ for (K,k) in Conv^0, i.e., $\varsigma_{(K,k)}$ is the convex structure on the set K.

It is easy to see that the free convex structure μ_X on $G(X)$ is equal to $\Box_1 \varsigma_{\mathcal{G}(X)}$ for each set X, i.e., $\mathbb{T} = (G, \sigma, \mu)$, where $\mu = (\mu_X)_{X \in \mathrm{Ens}^0}$, is the monad determined by the adjunction $(\mathcal{G}, \Box_1, \sigma, \varsigma)$.

4.1.2. <u>Theorem.</u> The category Conv is not monadic over Ens.

Proof. Let $A =$

$= \{(x,y) \in \mathbb{R}\times\mathbb{R} : 0 < x < 1 \ \& \ 0 < y < 1\} \cup \{(0,0)\} \cup \{(1,1)\}$ and let $B = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. We shall consider the maps $A \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B$ defined by $\pi_1(x,y) = x$, $\pi_2(x,y) = y$ for (x,y) in A. It is easy to see that the pair (π_1, π_2) is an equivalence relation in Conv. Hence the pair $(\psi\pi_1, \psi\pi_2)$ is a kernel pair in Ens for every right adjoint functor $\psi : \mathrm{Conv} \longrightarrow \mathrm{Ens}$.

Let q be a coequalizer of (π_1, π_2) in Conv. It is clear that q is a constant map. We shall show that (π_1, π_2) is not a kernel pair of q in Conv. Indeed, let $B \times B \overset{p_1}{\underset{p_2}{\rightrightarrows}} B$ be the canonical projections. Of course, they are morphisms in Conv. We have $qp_1 = qp_2$ but there is no morphism h in Conv such that $\pi_1 h = p_1$ and $\pi_2 h = p_2$. Hence (π_1, π_2) is not a kernel pair of q. Since (π_1, π_2) is not a kernel pair of its coequalizer, it is not a kernel pair in Conv.

Thus, by Theorem 2.4., every right adjoint functor ψ : Conv $\longrightarrow$ Ens is not monadic.

In particular the forgetful functor $\Box_1$: Conv $\longrightarrow$ Ens is not monadic.

4.1.3. The Eilenberg-Moore algebras of the monad $\mathbb{T} = (G, \sigma, \mu)$ will be called the <u>semiconvex</u> sets and the morphisms of $\mathbb{T}$ - algebras will be called the <u>semiaffine</u> maps. The Eilenberg-Moore category of the monad $\mathbb{T} = (G, \sigma, \mu)$ will be denote by <u>Sconv.</u>

In other words, Sconv is the category of semiconvex sets and semiaffine maps.

Let S be a set and let $\alpha : G(S) \longrightarrow S$ be a map. The pair (S, α) is a semiconvex set if and only if the diagram

$$\begin{tikzcd}
G(G(S)) \arrow[r, "G\alpha"] \arrow[d, "\mu_S"'] & G(S) \arrow[d, "\alpha"] & S \arrow[l, "\sigma^S"'] \arrow[dl, "\iota_S"] \\
G(S) \arrow[r, "\alpha"'] & S &
\end{tikzcd}$$

is commutative, i.e., (S, α) is a semiconvex set if and only if

(1) $\qquad \alpha(\sigma^S_s) = s$ for each s in S,

(2) $$\alpha\Big(\sum_{i=1}^{n} a_i \, \sigma^S_{\alpha(\sum_{j=1}^{m_i} b_{ij} \sigma^S_{s_{ij}})}\Big) = \alpha\Big(\sum_{i=1}^{n} \sum_{j=1}^{m_i} a_i b_{ij} \, \sigma^S_{s_{ij}}\Big)$$

for all s_{ij} in S, $a_i \geqslant 0$, $b_{ij} \geqslant 0$ such that $\sum_{i=1}^{n} a_i = 1$ and $\sum_{j=1}^{m_i} b_{ij} = 1$.

4.1.4. <u>Lemma.</u> Let (K,k) be a convex set, and let $\varphi : K \longrightarrow S$ be a map. Then the following conditions are equivalent:

(1) if $\varphi(x) = \varphi(y)$, $x' \in K$ and $0 \leq t \leq 1$, then

$$\varphi((1-t)x + tx') = \varphi((1-t)y + tx')$$

(2) if $\varphi(x_i) = \varphi(y_i)$, $a_i \geqslant 0$ for $i = 1, \ldots, n$, and

$$\sum_{i=1}^{n} a_i = 1, \quad \text{then} \quad \varphi\Big(\sum_{i=1}^{n} a_i x_i\Big) = \varphi\Big(\sum_{i=1}^{n} a_i y_i\Big).$$

Proof. The implication (2) $\Rightarrow$ (1) is obvious. The converse implication will be proved by induction. Formula (2) is obviously valid if $n = 1$. Let us assume that (2) is true for some natural number n, and let $\varphi(x_i) = \varphi(y_i)$, $a_i \geqslant 0$ for $i = 1, \ldots, n+1$ and $\sum_{i=1}^{n+1} a_i = 1$. Equality $\varphi\big(\sum_{i=1}^{n+1} a_i x_i\big) = \varphi\big(\sum_{i=1}^{n+1} a_i y_i\big)$ is obvious for $a_{n+1} = 1$. If $a_{n+1} \neq 1$, then

$$\varphi\Big(\sum_{i=1}^{n+1} a_i x_i\Big) = \varphi\Big[(1 - a_{n+1}) \sum_{i=1}^{n} \frac{a_i}{1-a_{n+1}} x_i + a_{n+1} x_{n+1}\Big] =$$

$$\varphi\Big[(1 - a_{n+1}) \sum_{i=1}^{n} \frac{a_i}{1-a_{n+1}} y_i + a_{n+1} x_{n+1}\Big] =$$

$$\varphi\Big[(1 - a_{n+1}) \sum_{i=1}^{n} \frac{a_i}{1-a_{n+1}} y_i + a_{n+1} y_{n+1}\Big] = \varphi\Big(\sum_{i=1}^{n+1} a_i y_i\Big).$$

this completes the proof.

4.1.5. Theorem. Let (K,k) be a convex set and let $\varphi: K \longrightarrow S$ be a surjection satisfying 4.1.4.(1), Then there exists a unique map $\alpha: G(S) \longrightarrow S$ such that the diagram

$$\text{(1)} \qquad \begin{tikzcd} G(K) \arrow[r, "k"] \arrow[d, "G(\varphi)"'] & K \arrow[d, "\varphi"] \\ G(S) \arrow[r, "\alpha"'] & S \end{tikzcd}$$

is commutative. Moreover (S, α) is a semiconvex set (and consequently φ is a semiaffine map).

On the other hand, for every semiconvex set (S, α) there is a convex set (K,k) and a surjection $\varphi: K \longrightarrow S$ satisfying 4.1.4.(1) such that the diagram (1) is commutative.

Proof. Let $\sum_{i=1}^{n} a_i \delta^{S}_{s_i}$ be an element of $G(S)$, where $a_i > 0$ and $s_{i_1} \neq s_{i_2}$ for $i_1 \neq i_2$ (this representation is unique). Since φ is a surjection, we infer that $s_i = \varphi(x_i)$ for $i = 1, \ldots, n$, i.e.,

$$\sum_{i=1}^{n} a_i \sigma_{s_i} = \sum_{i=1}^{n} a_i \; \sigma_{\varphi(x_i)}$$

We define

$$\alpha\Big(\sum_{i=1}^{n} a_i \sigma_{s_i}\Big) = \varphi\Big(\sum_{i=1}^{n} a_i x_i\Big)$$

Let $\sum_{j=1}^{m} b_j \sigma_{y_j}$ be another element of $G(K)$ such that

$$\sum_{i=1}^{n} a_i \sigma_{s_i} = \sum_{j=1}^{m} b_j \sigma_{\varphi(y_j)} = G\varphi\Big(\sum_{j=1}^{m} b_j \sigma_{y_j}\Big)$$

Let $M_0 = \{ j : b_j = 0 \}$ and let $M_i = \{ j : \varphi(y_j) = s_i \}$ for $i = 1, \dots, n$. It clear that the sets $M_0, M_1, \dots, M_n$ are pairwise disjoint, $M_0 \cup M_1 \cup \dots \cup M_n = \{1, \dots, m\}$ and $M_i \neq \emptyset$ for $i = 1, \dots, n$. We have

$$\sum_{j=1}^{m} b_j \sigma_{\varphi(y_j)} = \sum_{i=1}^{n} \sum_{j \in M_i} b_j \sigma_{\varphi(y_j)} =$$

$$= \sum_{i=1}^{m} \sum_{j \in M_i} b_j \sigma_{s_i} = \sum_{i=1}^{n} \Big(\sum_{j \in M_i} b_j\Big) \sigma_{s_i} .$$

Hence $a_i = \sum_{j \in M_i} b_j$ for $i = 1, \dots, n$. Since $\varphi(y_j) = s_i = \varphi(x_i)$ for j in M_i, we have, by 4.1.4.(2),

$$\varphi\Big(\sum_{j=1}^{m} b_j y_j\Big) = \varphi\Big(\sum_{i=1}^{n} \sum_{j \in M_i} b_j y_j\Big) =$$

$$= \varphi\Big(\sum_{i=1}^{n} \sum_{j \in M_i} b_j x_i\Big) = \varphi\Big(\sum_{i=1}^{n} a_i x_i\Big).$$

Thus α is well-defined.

We shall show that (S, α) is a semiconvex set. By definition of α

1) $\alpha(\sigma_s) = \alpha(\sigma_{\varphi(x)}) = \varphi(x) = s$

2) $$\alpha\Big(\sum_i a_i \sigma_{\alpha(\sum_j b_{ij} \sigma_{s_{ij}})}\Big) =$$

$$= \alpha\Big(\sum_i a_i \sigma_{\alpha(\sum_j b_{ij} \sigma_{\varphi(x_{ij})})}\Big) =$$

$$= \alpha\Big(\sum_i a_i \sigma_{\varphi(\sum_j b_{ij} x_{ij})}\Big) = \varphi\Big(\sum_i a_i \sum_j b_{ij} x_{ij}\Big) =$$

$$= \alpha\Big(\sum_i a_i \sum_j b_{ij} \sigma_{\varphi(x_{ij})}\Big) = \alpha\Big(\sum_i a_i \sum_j b_{ij} \sigma_{s_{ij}}\Big),$$

i.e., by 4.1.3., (S, α) is a semiconvex set.

The converse assertion follows from the fact that if (S, α) is a semiconvex set, then the diagram

$$\begin{tikzcd}
G^2(S) \arrow[r, "\mu_S"] \arrow[d, "G\alpha"'] & G(S) \arrow[d, "\alpha"] \\
G(S) \arrow[r, "\alpha"'] & S
\end{tikzcd}$$

is commutative, and $(G(S), \mu_S)$ is a convex set. Moreover if $\alpha(\sum a_i \delta_{x_i}) = \alpha(\sum a'_j \delta_{x'_j})$, $\sum b_k \delta_{y_k}$ is an element of $G(S)$ and $0 \leq t \leq 1$, then

$$\alpha\left((1-t)\sum a_i \delta_{x_i} + t\sum b_k \delta_{y_k}\right) =$$

$$= \alpha\left((1-t)\,\delta_{\alpha(\sum a_i \delta_{x_i})} + t\,\delta_{\alpha(\sum b_k \delta_{y_k})}\right) =$$

$$= \alpha\left((1-t)\,\delta_{\alpha(\sum a'_j \delta_{x'_j})} + t\,\delta_{\alpha(\sum b_k \delta_{y_k})}\right) =$$

$$= \alpha\left((1-t)\sum a'_j \delta_{x'_j} + t\sum b_k \delta_{y_k}\right).$$

Hence we can take $(K,k) = (G(S), \mu_S)$ and $\varphi = \alpha$.

4.1.6. Corollary. Let (K,k) be a convex set and let R be an equivalence relation on K satisfying the following condition:

if xRx', then $((1-t)x + ty)R((1-t)x' + ty)$ for $0 \leq t \leq 1$, and y in K.

Let S be a subset of K and let $\varphi: K \longrightarrow S$ be a map such that $\{\varphi(x)\} = [x] \cap S$ for every x in K, where $[x]$ is the equivalence class of x. Then the pair (S, α), where $\alpha: G(S) \longrightarrow S$ is defined by

$$\alpha\left(\sum a_i \delta_{x_i}\right) = \varphi\left(\sum a_i x_i\right)$$

for x in S, is a semiconvex set.

4.1.7. Examples. (a). Let $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ and let $\alpha: G(S) \longrightarrow S$ be defined by

$$\alpha\left(\sum a_i \delta^S_{s_i}\right) = \begin{cases} (x,y) & \text{if } \sum a_i s_i = (x,y) \in S \\ (x, \sqrt{1-x^2}) & \text{if } \sum a_i s_i = (x,y) \notin S \end{cases}$$

Then, by 4.1.6., (S, α) is a semiconvex set.

(b). Let S be a set and let s_0 be an element of S. Let $\alpha: G(S) \longrightarrow S$ be defined by

$$\alpha(\sum a_i \sigma^S_{s_i}) = \begin{cases} s & \text{if} \quad \sum a_i \sigma^S_{s_i} = \sigma^S_s \\ s_0 & \text{if} \quad \sum a_i \sigma^S_{s_i} \neq \sigma^S_s . \end{cases}$$

Then, by 4.1.6., (S, α) is a semiconvex set.

4.1.8. <u>Theorem.</u> Let S be a set and let $\alpha : G(S) \longrightarrow S$ be a map. The pair (S, α) is a semiconvex set if and only if α satisfies the following conditions:

(i) $\alpha(\sigma^S_s) = s$ for each s in S

(ii) if $\alpha(f_1) = \alpha(f_2)$, $g \in G(S)$ and $0 \leq t \leq 1$, then $\alpha((1-t)f_1 + tg) = \alpha((1-t)f_2 + tg)$.

Proof. Let α satisfy the conditions (i), (ii). Then, by Lemma 4.1.4., α satisfies the condition 4.1.4.(2). Let $a_i \geq 0$, $b_{ij} \geq 0$, be such that $\sum_{i=1}^{n} a_i = 1$, $\sum_{j=1}^{m_i} b_{ij} = 1$, and let s_{ij} be elements of S. We have, by (i),

$$\alpha(\sigma^S_{\alpha(\sum_{j=1}^{m_i} b_{ij} \sigma^S_{s_{ij}})}) = \alpha(\sum_{j=1}^{m_i} b_{ij} \sigma^S_{s_{ij}})$$

for $i = 1, \dots, n$. Hence, by 4.1.2.,

$$\alpha(\sum_{i=1}^{n} a_i \sigma^S_{\alpha(\sum_{j=1}^{m_i} b_{ij} \sigma^S_{s_{ij}})}) =$$

$$= \alpha(\sum_{i=1}^{n} a_i \sum_{j=1}^{m_i} b_{ij} \sigma^S_{s_{ij}})$$

Consequently, by 4.1.3., (S, α) is a semiconvex set.

Let (S, α) be a semiconvex set. Then α satisfies the condition (I). If $\alpha(f_1) = \alpha(f_2)$, $0 \leq t \leq 1$, and g is an element of $G(S)$, then, by 4.1.3.(2),

$$\alpha((1-t)f_1 + tg) = \alpha((1-t)\sigma^S_{\alpha(f_1)} + t\sigma^S_{\alpha(g)}) =$$

$$= \alpha((1-t)\sigma^S_{\alpha(f_2)} + t\sigma^S_{\alpha(g)}) = \alpha((1-t)f_1 + tg).$$

This completes the proof of the Theorem.

4.1.9. <u>Lemma.</u> Let (K,k) be a convex set and let $\varphi : K \longrightarrow S$ be a map satisfying the following conditions:

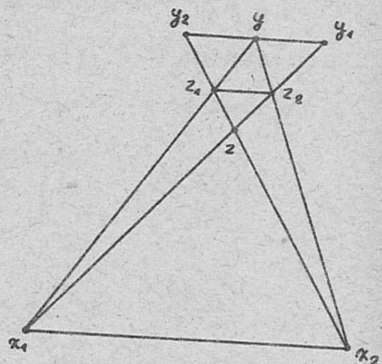
(ii) if $\varphi(x_1) = \varphi(x_2)$, $y \in K$ and $0 \leq t \leq 1$, then

$$\varphi((1-t)x_1 + ty) = \varphi((1-t)x_2 + ty),$$

(iii) if $0 < t < 1$ and $\varphi((1-t)x_1 + ty) = \varphi((1-t)x_2 + ty)$, then $\varphi(x_1) = \varphi(x_2)$.

If $0 < t < 1$, $\varphi(y_1) = \varphi(y_2)$ and $(1-t)x_1 + ty_1 = (1-t)x_2 + ty_2$, then $\varphi(x_1) = \varphi(x_2)$.

Proof. Let us denote



$$y = \tfrac{1}{2}y_1 + \tfrac{1}{2}y_2$$

$$z = (1-t)x_1 + ty_1 = (1-t)x_2 + ty_2$$

$$z_1 = \frac{1-t}{1+t}x_1 + \frac{2t}{1+t}y_1$$

$$z_2 = \frac{1-t}{1+t}x_2 + \frac{2t}{1+t}y_2$$

Then

$$\left(1 - \frac{t}{1+t}\right)z + \frac{t}{1+t}y_1 = \frac{1}{1+t}\Big[(1-t)x_1 + ty_1\Big] + \frac{t}{1+t}y_1 =$$

$$= \frac{1-t}{1+t}x_1 + \frac{2t}{1+t}y_1 = z_1 .$$

Similarly

$$\left(1 - \frac{t}{1+t}\right)z + \frac{t}{1+t}y_2 = \frac{1-t}{1+t}x_2 + \frac{2t}{1+t}y_2 = z_2$$

Hence, by (ii), $\varphi(z_1) = \varphi(z_2)$.

We have also

$$\frac{1-t}{1+t}x_1 + \frac{2t}{1+t}y = \frac{1-t}{1+t}x_1 + \frac{t}{1+t}y_1 + \frac{t}{1+t}y_2 =$$

$$= \frac{1}{1+t}\Big[(1-t)x_1 + ty_1\Big] + \frac{t}{1+t}y_2 =$$

$$= \frac{1}{1+t}\Big[(1-t)x_2 + ty_2\Big] + \frac{t}{1+t}y_2 =$$

$$= \frac{1-t}{1+t}x_2 + \frac{t}{1+t}y_2 + \frac{t}{1+t}y_2 = z_2 ,$$

i.e., $$\frac{1-t}{1+t}x_1 + \frac{2t}{1+t}y = z_2$$

and similarly

$$\frac{1-t}{1+t}x_2 + \frac{2t}{1-t}y = z_1 .$$

Since $\varphi(z_1) = \varphi(z_2)$, we obtain, by (iii), $\varphi(x_1) = \varphi(x_2)$. This completes the proof of the Lemma.

4.1.10. <u>Theorem.</u> Let X be a set and let $\varphi : G(X) \longrightarrow X$ be a map. The pair (X, φ) is a convex set if and only if φ satisfies the following conditions:

(i) $\varphi(6\frac{X}{x}) = x$ for each x in X,

(ii) if $\varphi(f_1) = \varphi(f_2)$, $g \in G(X)$ and $0 \leq t \leq 1$, then $\varphi((1-t)f_1 + tg) = \varphi((1-t)f_2 + tg)$,

(iii) if $0<t<1$ and $\varphi((1-t)f_1 + tg) = \varphi(1-t)f_2 + tg)$, then $\varphi(f_1) = \varphi(f_2)$.

In other words, (X, φ) is a convex set if and only if (X, φ) is a semiconvex set and φ satisfies the condition (iii).

Proof. Let (X, φ) be a convex set. By definition (cf.3.1), there exist a vector space E and an injection $\varepsilon : X \longrightarrow E$ such that $\varepsilon(\varphi(\sum a_i \delta^X_{x_i})) = \sum a_i\, \varepsilon(x_i)$,

$$\begin{tikzcd} G(X) \arrow[r, "\varphi"] \arrow[dr, "\varepsilon\varphi"'] & X \arrow[d, "\varepsilon"] \\ & E \end{tikzcd}$$

i.e., such that $\varepsilon\varphi$ is an affine map.

We shall verify the conditions (i) - (iii).

(i) Note that $\varepsilon(\varphi(\delta^X_x)) = \varepsilon(x)$ and ε is an injection.

(ii) Let $\varphi(f_1) = \varphi(f_2)$, $g \in G(X)$ and $0 \leq t \leq 1$. Then

$$\varepsilon\varphi((1-t)f_1 + tg) = (1-t)\,\varepsilon\varphi(f_1) + t\,\varepsilon\varphi(g) =$$
$$= (1-t)\,\varepsilon\varphi(f_2) + t\,\varepsilon\varphi(g) = \varepsilon\varphi((1-t)f_2 + tg).$$

Hence $\varphi((1-t)f + tg) = \varphi((1-t)f + tg)$.

(iii) Let $0<t<1$ and let $\varphi((1-t)f_1 + tg) = \varphi((1-t)f_2 + tg)$.

Then

$$\varepsilon\varphi((1-t)f_1 + tg) = \varepsilon\varphi((1-t)f_2 + tg)$$

and consequently

$$(1-t)\,\varepsilon\varphi(f_1) + t\,\varepsilon\varphi(g) = (1-t)\,\varepsilon\varphi(f_2) + t\,\varepsilon\varphi(g).$$

Since $0<t<1$, we have $\varepsilon\varphi(f_1) = \varepsilon\varphi(f_2)$. Hence $\varphi(f_1) = \varphi(f_2)$.

Thus for each convex set (X, φ), the convex structure satisfies the conditions (i) - (iii).

Let X be a set and let $\varphi : G(X) \longrightarrow X$ be a map satisfying the conditions (i) - (iii). $G(X)$ is a convex subset of the free vector space $\mathcal{V}(X)$ (Cf.3.1.). We define subset L of $\mathcal{V}(X)$ by

$$L = \{ t(f_1 - f_2) : t \in \mathbb{R},\ f_1, f_2 \in G(X),\ \varphi(f_1) = \varphi(f_2) \}.$$

It is clear that $0 \in L$ and $tf \in L$ for f in L and t in $\mathbb{R}$.

Let f, g be elements of L, i.e., $f = t(f_1 - f_2)$, $g = s(g_1 - g_2)$, where $s, t \in \mathbb{R}$, $\varphi(f_1) = \varphi(f_2)$ and $\varphi(g_1) = \varphi(g_2)$. We can assume that $t \geq 0$ and $s \geq 0$ (because $t(f_1 - f_2) = (-t)(f_1 - f_2)$). We shall show that $f + g \in L$. This is obvious for $t = 0$. If $t > 0$, then

$$f+g = (t+s)\left(\frac{t}{t+s} f_1 + \frac{s}{t+s} g_1\right) - \left(\frac{t}{t+s} f_2 + \frac{s}{t+s} g_2\right).$$

By (ii), $\varphi\left(\frac{t}{t+s} f_1 + \frac{s}{t+s} g_1\right) = \varphi\left(\frac{t}{t+s} f_2 + \frac{s}{t+s} g_2\right)$.

Hence $f + g$ is an element of L. Thus L is a linear subspace of $\mathcal{V}(X)$.

Let f_1, f_2 be elements of $G(X)$ such that $f_1 - f_2 \in L$. By definition of L, there are t in $\mathbb{R}$ and g_1, g_2 in $G(X)$ such that $\varphi(g_1) = \varphi(g_2)$ and $f_1 - f_2 = t(g_1 - g_2)$. Without loss of generality we can assume that $t > 0$. Hence

$$f_1 + tg_2 = f_2 + tg_1$$

and consequently

$$\frac{1}{1+t} f_1 + \frac{t}{1+t} g_2 = \frac{1}{1+t} f_2 + \frac{t}{1+t} g_1 .$$

Hence, by Lemma 4.1.9., $\varphi(f_1) = \varphi(f_2)$.

Thus $\varphi(f_1) = \varphi(f_2)$ if and only if $f_1 - f_2 \in L$, for all f_1, f_2 in $G(X)$. (The converse implication is obvious).

Let $\pi : \mathcal{V}(X) \longrightarrow \mathcal{V}(X)/L$ be the quotient map. We define $\varepsilon : X \longrightarrow \mathcal{V}(X)/L$ by $\varepsilon(x) = \pi(\delta^X_x)$ for x in X. It is clear that ε is an injection, and that the diagram

$$\begin{tikzcd} G(X) \arrow[r, "\varphi"] \arrow[d, "\cap"'] & X \arrow[d, "\varepsilon"] \\ \mathcal{V}(X) \arrow[r, "\pi"'] & \mathcal{V}(X)/L \end{tikzcd}$$

is commutative. (here $\cap$ denotes the identical embedding.) Consequently $\varepsilon\varphi = \pi|_{G(X)}$ is an affine map. Thus the pair (X, φ) is a convex set.

This completes the proof of Theorem 4.1.10.

4.1.11. <u>Theorem.</u> Conv is a full and reflective subcategory of Sconv.

Proof. It is clear that Conv is a full subcategory of Sconv.

Let (S, α) be a semiconvex set, and let L be a subset of $\mathcal{V}(S)$ defined by

$$L = \{ t(f_1 - f_2) : t \in \mathbb{R} , f_1, f_2 \in G(S), \alpha(f_1) = \alpha(f_2) \} .$$

Then L is a linear subspace of $\mathcal{V}(S)$ (cf. the proof of 4.1.10). Consequently there is a unique map $\tilde{\eta} : S \longrightarrow \mathcal{V}(S)/L$ such that the diagram

$$\begin{tikzcd} G(S) \arrow[r, "\alpha"] \arrow[d, "\cap"'] & S \arrow[d, dashed, "\tilde{\eta}"] \\ \mathcal{V}(S) \arrow[r, "\pi"'] & \mathcal{V}(S)/L \end{tikzcd}$$

is commutative. It is clear that $\tilde{\eta}(S)$ is a convex subset of $\mathcal{V}(S)/L$.

Let us denote by $\phi(S)$ the set $\tilde{\eta}(S)$, and by η_S the map $S \longrightarrow \phi(S)$ defined by $\eta_S(s) = \tilde{\eta}(s)$.

Let (K,k) be a convex set and let $\varphi : S \longrightarrow K$ be a semiaffine map. Then the diagram

$$\begin{tikzcd} G(S) \arrow[r, "G(\varphi)"] \arrow[d, "\alpha"'] & G(K) \arrow[d, "k"] \\ S \arrow[r, "\varphi"'] & K \end{tikzcd}$$

is commutative. We shall show that there is a unique affine map β such that the diagram

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "\varphi"'] & \phi(S) \arrow[d, dashed, "\beta"] \\ & K \end{tikzcd}$$

is commutative. Let

$$M = \{ t(g_1 - g_2) \in \mathcal{V}(K) : t \in \mathbb{R} , g_1, g_2 \in G(K), k(g_1) = k(g_2) \}.$$

Then M is a linear subspace of $\mathcal{V}(K)$ and $k(g_1) = k(g_2)$ if and only if $g_1 - g_2 \in M$ for all g_1, g_2 in $G(K)$ (cf. the proof of 4.1.10). Consequently $\eta_K : K \longrightarrow \phi(K)$ is an isomorphism in Conv.

Let $\mathcal{V}(\varphi) : \mathcal{V}(S) \longrightarrow \mathcal{V}(K)$ be defined by

$$\mathcal{V}(\varphi)(\sum a_i \sigma^S_{s_i}) = \sum a_i \sigma^K_{\varphi(s_i)} .$$

It is clear that $\mathcal{V}(\varphi)$ is a linear map, and the composition of $G(\varphi)$ with the identical embedding $G(K) \xrightarrow{c} \mathcal{V}(K)$ is equal to $\mathcal{V}(\varphi)|_{G(S)}$.

It is not difficult to verify that $\mathcal{V}(\varphi)(f)$ is an element of M for each f in L. Consequently there is a unique linear map u such that the diagram

$$\begin{tikzcd} \mathcal{V}(S) \arrow[r, "\mathcal{V}(\varphi)"] \arrow[d, "\pi"'] & \mathcal{V}(K) \arrow[d, "\pi"] \\ \mathcal{V}(S)/L \arrow[r, dashed, "u"'] & \mathcal{V}(K)/M \end{tikzcd}$$

is commutative. Hence there is a unique map $w : \phi(S) \longrightarrow \phi(K)$ such that the diagram

$$\begin{tikzcd} \phi(S) \arrow[r, dashed, "w"] \arrow[d, "\cap"'] & \phi(K) \arrow[d, "\cap"] \\ \mathcal{V}(S)/L \arrow[r, "u"'] & \mathcal{V}(K)/M \end{tikzcd}$$

is commutative. The map w is affine and the diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\eta_S"'] & K \arrow[d, "\eta_K"] \\ \Phi(S) \arrow[r, "w"'] & \Phi(K) \end{tikzcd}$$

is commutative. Thus $\beta = \eta_K^{-1} w$.

4.1.12. <u>Theorem.</u> Let X be a set and let $(\nu_s : X \times X \longrightarrow X)_{0<s<1}$ be a family of binary operations in X; we shall write $x \textcircled{s} y$ instead of $\nu_s(x,y)$. Thus, $(X, (\textcircled{s})_{0<s<1})$ is an abstract algebra. Let the operations $(\textcircled{s})_{0<s<1}$ satisfy the following axioms

(A) $\quad x \textcircled{s} x = x$

(B) $\quad x \textcircled{s} y = y \textcircled{1-s} x$

(C) $\quad (x \textcircled{s} y) \textcircled{t} z = x \textcircled{s+t-st} \left(y \textcircled{\frac{t}{s+t-st}} z\right)$

for x,y,z in X, $0<s,t<1$.

Then there is a unique map $\varphi : G(X) \longrightarrow X$ such that $x \textcircled{s} y = \varphi((1-s)\sigma^X_x + s\sigma^X_y)$ for x,y in X and $0<s<1$. Moreover, the pair (X, φ) is a semiconvex set.

If, in addition, the operations ν_s satisfy the axiom

(D) $\quad x_1 \textcircled{s} y = x_2 \textcircled{s} y \Longrightarrow x_1 = x_2$

for all x_1, x_2, y in X, $0<s<1$, then this unique map φ is a convex structure on the set X.

On the other hand, if (X, φ) is a semiconvex set and the binary operations $\textcircled{s} : X \times X \longrightarrow X$ are defined by

$$x \textcircled{s} y = \varphi((1-s)\sigma^X_x + s\sigma^X_y)$$

for x,y in X, $0<s<1$, then the axioms (A) – (C) are satisfied. If (X, φ) is a convex set, then the axiom (D) is also satisfied.

Proof. From (B) and (C) we obtain the identities

(1) $\quad x \textcircled{s} (y \textcircled{t} z) = \left(x \textcircled{\frac{s-st}{1-st}} y\right) \textcircled{st} z$

(2) $\quad (x \textcircled{s} y) \textcircled{t} z = \left(x \textcircled{\frac{t}{1-s+st}} z\right) \textcircled{s-st} y$

for x,y,z in X, $0<s,t<1$.

In this proof we shall write σ instead of σ^X.

Let $(\textcircled{s})_{0<s<1}$ be a family of binary operations satisfying (A) – (C). We define the function $\varphi : G(X) \longrightarrow X$ by induction with respect to the number of terms in the representation

(3) $\quad f = \sum_{i=1}^{n} a_i \sigma_{x_i}$,

where $x_i \neq x_j$ for $i \neq j$, $a_i > 0$ for $i = 1, \dots, n$ and

$\sum_{i=1}^{n} a_i = 1$.

If $n = 1$, then $f = \sigma_x$ for some x and we define:

(4) $$\varphi(\sigma_x) = x .$$

If $x_i \in X$, $a_i > 0$ for $i = 1, \dots, n+1$, $x_i \neq x_j$ for $i \neq j$ and $\sum_{i=1}^{n+1} a_i = 1$, then we define

(5) $$\varphi\Big(\sum_{i=1}^{n+1} a_i \sigma_{x_i}\Big) = \varphi\Big(\sum_{i=1}^{n} \frac{a_i}{1-a_{n+1}} \sigma_{x_i}\Big) \; a_{n+1} \; x_{n+1} .$$

It is clear that, if φ exists, it is of this form. We shall show that φ is well-defined, i.e., that the definition (5) does not depend on the way the elements of the set $\{x_1, \dots, x_{n+1}\}$ are arranged into a sequence $(x_1, \dots, x_{n+1})$. Let us assume that φ is well defined for each f of the form (3) with exactly n elements.

If $1 \leq k \leq n$, then

$$\varphi\Big(\sum_{i=1}^{n} \frac{a_i}{1-a_{n+1}} \sigma_{x_i}\Big) \; a_{n+1} \; x_{n+1} =$$

$$= \Big[\varphi\Big(\sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{a_i}{1-a_{n+1}-a_k} \sigma_{x_i}\Big) \; \frac{a_k}{1-a_{n+1}} \; x_k\Big] \; a_{n+1} \; x_{n+1} \quad \underline{(2)}$$

$$= \Big[\varphi\Big(\sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{a_i}{1-a_k-a_{n+1}} \sigma_{x_i}\Big) \; \frac{a_{n+1}}{1-a_k} \; x_{n+1}\Big] \; a_k \; x_k =$$

$$= \Big[\varphi\Big(\sum_{\substack{i=1 \\ i \neq k}}^{n+1} \frac{a_i}{1-a_k} \sigma_{x_i}\Big)\Big] \; a_k \; x_k$$

This shows that φ is well-defined. Our aim is to show that φ is a semiconvex structure on X.

We shall show that

(6) $$\varphi\Big(\sum_{i=1}^{n} a_i \sigma_{x_i}\Big) \; s \; y = \varphi\Big(\sum_{i=1}^{n} (1-s) a_i \sigma_{x_i} + s \sigma_y\Big)$$

for y, x_i in X, $x_i \neq x_j$ for $i \neq j$, $a_i > 0$, $\sum a_i = 1$.

If $y \neq x_i$ for $i = 1, \dots, n$, then the equality (6) follows from the definition of φ. If $y = x_k$ for some $1 \leq k \leq n$, then

$$\varphi\Big(\sum_{i=1}^{n} a_i \sigma_{x_i}\Big) \; s \; y = \varphi\Big(\sum_{i=1}^{n} a_i \sigma_{x_i}\Big) \; s \; x_k =$$

$$= \Big[\varphi\Big(\sum_{i \neq k} \frac{a_i}{1-a_k} \sigma_{x_i}\Big) \; a_k \; x_k\Big] \; s \; x_k =$$

$$= \varphi\Big(\sum_{i\neq k} \frac{a_i}{1-a_k}\,\sigma_{x_i}\Big)\ (a_k + s - a_k s)\ \Big(x_k\ \frac{s}{a_k + s - a_k s}\ x_k\Big)$$

$$= \varphi\Big(\sum_{i\neq k} \frac{a_i}{1 - a_k}\,\sigma_{x_i}\Big)\ (a_k(1 - s) + s)\ x_k =$$

$$= \varphi\Big(\sum_{i\neq k} \frac{a_i}{1 - a_k}(1-a_k(1-s)-s)\sigma_{x_i} + (a_k(1 - s) + s)\sigma_{x_k}\Big) =$$

$$= \varphi\Big(\sum_{i\neq k} a_i(1 - s)\,\sigma_{x_i} + a_k(1 - s)\sigma_{x_k} + s\sigma_{x_k}\Big) =$$

$$= \varphi\Big(\sum_{i=1}^{n} (1 - s)a_i\,\sigma_{x_i} + s\,\sigma_y\Big)$$

Thus, (6) is true for each y in X.

Hence, by induction, we obtain the following property of φ :

$$(7)\quad \varphi\Big(\sum_{i=1}^{n} a_i\,\sigma_{x_i}\Big)\ ⓢ\ \varphi\Big(\sum_{j=1}^{m} b_j\,\sigma_{y_j}\Big) =$$

$$= \varphi\Big(\sum_{i=1}^{n}(1 - s)a_i\,\sigma_{x_i} + \sum_{j=1}^{m} sb_j\,\sigma_{y_j}\Big).$$

The proof of (7) is similar to arguments presented above and will be omitted.

Let $\varphi\big(\sum_{i=1}^{n} a_i\,\sigma_{x_i}\big) = \varphi\big(\sum_{k=1}^{l} c_k\,\sigma_{z_k}\big)$, let $\sum_{j=1}^{m} b_j\sigma_{y_j}$ be an element of $G(X)$ and let $0 < s < 1$.

Then, by (7),

$$\varphi\Big((1-s)\sum_{i=1}^{n} a_i\,\sigma_{x_i} + s\sum_{j=1}^{m} b_j\,\sigma_{y_j}\Big) =$$

$$= \varphi\Big(\sum_{i=1}^{n} a_i\sigma_{x_i}\Big)\ ⓢ\ \varphi\Big(\sum_{j=1}^{m} b_j\sigma_{y_j}\Big) =$$

$$= \varphi\Big(\sum_{k=1}^{l} c_k\,\sigma_{z_k}\Big)\ ⓢ\ \varphi\Big(\sum_{j=1}^{m} b_j\,\sigma_{y_j}\Big) =$$

$$= \varphi\Big((1-s)\sum_{k=1}^{l} c_k\,\sigma_{z_k} + s\sum_{j=1}^{m} b_j\,\sigma_{y_j}\Big).$$

Thus, by Theorem 4.1.8., the pair (X, φ) is a semiconvex set.

Let the family $(ⓢ)_{0<s<1}$ have the properties (A) - (D), let $0 < s < 1$ and let

$$\varphi\Big((1-s)\sum_{i=1}^{n} a_i\sigma_{x_i} + s\sum_{j=1}^{m} b_j\sigma_{y_j}\Big) =$$

$$= \varphi\Big((1-s)\sum_{k=1}^{l} c_k\,\sigma_{z_k} + s\sum_{j=1}^{m} b_j\,\sigma_{y_j}\Big).$$

Hence, by (7),

$$\varphi\Big(\sum_{i=1}^{n} a_i \sigma_{x_i}\Big) \circledS \varphi\Big(\sum_{j=1}^{n} b_j \sigma_{y_j}\Big) =$$

$$= \varphi\Big(\sum_{k=1}^{l} c_k \sigma_{z_k}\Big) \circledS \varphi\Big(\sum_{j=1}^{m} b_j \sigma_{y_j}\Big)$$

and consequently, by (D), $\varphi\Big(\sum_{i=1}^{n} a_i \sigma_{x_i}\Big) = \varphi\Big(\sum_{k=1}^{l} c_k \sigma_{z_k}\Big)$.

Thus, by Theorem 4.1.10., φ is a convex structure on the set X.

The verification of the converse assertion is easy and will beomitted.

4.2. The case $\mathrm{Compconv} \underset{\mathcal{S}}{\overset{\square_2}{\rightleftarrows}} \mathrm{Comp}$

4.2.1. Let X be a compact topological space. By $\mathcal{M}(X)$ we shall denote the Banach space of Radon measures on X and by $\mathcal{C}(X)$ - the Banach space of all continuous real-valued functions on X. The set

$$\mathcal{S}(X) = \{\mu \in \mathcal{M}(X) : \mu \geq 0 \ \& \ \mu(X) = 1\}$$

(the set of probability measures on X) with the convex structure determined by the vector-space structure of $\mathcal{M}(X)$

and with the relative topology induced by $\sigma(\mathcal{M}(X), \mathcal{C}(X))$ is a compact convex set.

Let $\varphi : X \longrightarrow X'$ be a morphism in Comp. If μ is an element of $\mathcal{S}(X)$ and B is a Borel subset of X', define

$$\mathcal{S}\varphi.\mu.B = \mu(\varphi^{-1}(B))$$

Then $\mathcal{S}\varphi.\mu \in \mathcal{S}(X')$ and $\mathcal{S}\varphi : \mathcal{S}(X) \longrightarrow \mathcal{S}(X')$ is a morphism in Compconv. Moreover

$$\mathcal{S} : \mathrm{Comp} \longrightarrow \mathrm{Compconv}$$

is a covariant functor left adjoint of $\square_2$ (cf. [9], 23.7.2.).

The unit of the adjunction

$$\delta : I_{\mathrm{Comp}} \longrightarrow \square_2 \mathcal{S}$$

is defined by $\delta = (\delta^X)_{X \in \mathrm{Comp}^\circ}$, where δ^X_x is the Dirac measure for X in Comp° and x in X. The counit of the adjunction is

$$\varsigma : \mathcal{S}\square_2 \longrightarrow I_{\mathrm{Compconv}},$$

where $\varsigma_K(\mu)$ is the centroid of μ for any probability measure μ on a compact convex set K (cf. [9], 23.4.2.).

4.2.2. <u>Lemma.</u> If (f_1,f_2) is a parallel pair in Compconv such that $(\square_2 f_1, \square_2 f_2)$ is a kernel pair in Comp, then (f_1,f_2) is a kernel pair in Compconv.

Proof. Let f be a coequalizer of (f_1,f_2) in Comp. Then (f_1,f_2) is a kernel pair of f in Comp, i.e., the diagram

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \xrightarrow{f} C$$

is an exact sequence in Comp.

We can assume that B is a compact convex subset of a locally convex Hausdorff space (E,τ) such that $E = \operatorname{span} B$ and $0 \in B$.

We shall prove the Lemma in 8 steps:

(I). Let R be the relation on B defined by

$$b_1 R b_2 \iff f(b_1) = f(b_2).$$

Then R is an equivalence relation on B and the equivalence classes of R are compact. Moreover $b_1 R b_2$ if and only if there is a unique a in A such that $f_1(a) = b_1$ and $f_2(a) = b_2$.

Proof of (I). Let $X = \{x\}$ be a one-point compact convex set and let $g_i : X \longrightarrow B$ be defined by $g_i(x) = b_i$ for $i = 1,2$. Then g_1, g_2 are morphisms in Compconv and $fg_1 = fg_2$. Since (f_1,f_2) is a kernel pair of f, there is a unique morphism $h : X \longrightarrow A$ such that $f_1 h = f_2 h$. Hence $a = h(x)$.

(II). The relation R is a closed subset of $B \times B$, i.e., if b', b'' are elements of B and for every neighbourhood U of zero in E there exist b_1 in $(b' + U) \cap B$ and b_2 in $(b'' + U) \cap B$ such that $b_1 R b_2$, then $b' R b''$.

This follows immediately from the well-known Theorem of Alexandroff (cf. [1], Theorem 3.2.9 and Exercise 3.2.E).

(III). If $b_1 R b_2$, $b_3 R b_4$ and $0 \leq s \leq 1$, then $((1-s)b_1 + sb_3) R ((1-s)b_2 + sb_4)$. In particular $((1-s)b_1 + sb_2) R b_2$ for $b_1 R b_2$, $0 \leq s \leq 1$, i.e., the equivalence classes of R are convex sets.

Proof of (III). Let a, a' be elements of A such that $f_1(a) = b_1$, $f_2(a) = b_2$, $f_1(a') = b_3$, $f_2(a') = b_4$.

Then

$$f_1((1-s)a + sa') = (1-s)b_1 + sb_3$$

$$f_2((1-s)a + sa') = (1-s)b_2 + sb_4 .$$

Hence, by (I), $((1-s)b_1 + sb_3) R ((1-s)b_2 + sb_4)$.

(IV). If b_o, b_1 are elements of B and there exists $0 \leq s \leq 1$ such that $((1-s)b_o + sb_1) R b_1$, then $b_o R b_1$.

Proof of (IV). Let $t = \inf \{0 \leq s < 1 : ((1-s)b_o + sb_1) R b_1$ and let $b = (1-t)b_o + tb_1$. Since the equivalence classes of R are compact, we have $b R b_1$. Let a_o, a_1 be elements of A such that $f_1(a_0) = f_2(a_0) = b_0$, $f_1(a_1) = b_1$, $f_2(a_1) = b$, and let $a = (1-t)a_0 + ta_1$. Then

$$f_1(a) = (1-t)b_0 + tb_1 = b$$

$$f_2(a) = (1-t)b_0 + tb = (1-t^2)b_0 + t^2 b_1 .$$

Hence $((1-t^2)b_0 + t^2 b_1)Rb$ and consequently $((1-t^2)b_0 + t^2 b_1)Rb_1$. Since $t^2 < t$ for $0 < t < 1$, it follows from the definition of t that $t = 0$ and hence $b = b_0$.

(V). The set $M = \{ x \in E :$ there are b_1, b_2 in B and t in $\mathbb{R}$ such that b_1Rb_2 and $x = t(b_1-b_2)$ is a linear subspace of E.

Proof of (V). It follows immediately from the definition of M that $0 \in M$ and $sx \in M$ for x in M and s in $\mathbb{R}$.

Let $x, y,$ be elements of M, i.e., $x = t(b_1-b_2)$ and $y = s(b_3-b_4)$, where b_1Rb_2 and b_3Rb_4. We can assume that $t \geq 0$ and $s \geq 0$ (because $t(b_1-b_2) = (-t)(b_2-b_1)$). We shall show that $x + y \in M$. This is obvious if $t = 0$. If $t > 0$, then

$$x + y = (t+s)\left(\left(\frac{t}{t+s} b_1 + \frac{s}{t+s} b_3\right) - \left(\frac{t}{t+s} b_2 + \frac{s}{t+s} b_4\right)\right).$$

By (III), $\left(\frac{t}{t+s} b_1 + \frac{s}{t+s} b_3\right)R\left(\frac{t}{t+s} b_2 + \frac{s}{t+s} b_4\right)$.

Hence $x + y$ is an element of M.

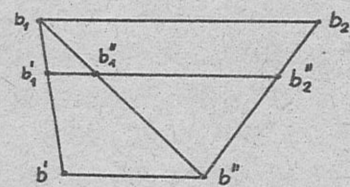
(VI). Let b_1, b_2 be elements of B. Then b_1Rb_2 if and only if $b_1 - b_2 \in M$. In other words, the equivalence class of b is the set $(b + M) \cap B$ for each b in B.

Proof of (VI). Let $b_1 - b_2 \in M$. By definition of M there are b', b'' in B such that $b_1 - b_2 = t(b' - b'')$ and $b'Rb''$. Without loss of generality we can assume that $t > 0$. Let $0 < s < 1$ and let

$$b_1' = (1-s)b_1 + sb'$$

$$b_1'' = (1-s)b_1 + sb''$$

$$b_2'' = (1-s)b_2 + sb''$$



Since $b'Rb''$, we have $b_1'Rb_1''$ (cf. (III)). By definitions

$$b_1' - b_1'' = s(b' - b'')$$

$$b_1'' - b_2'' = (1-s)(b_1 - b_2) = (1-s)t(b' - b'') = \frac{(1-s)t}{s}(b_1' - b_1'').$$

Hence

$$b_1'' = \frac{s}{s+(1-s)t} b_2'' + \frac{(1-s)t}{s+(1-s)t} b_1'$$

Since $t>0$ and $0<s<1$, we have

$$\frac{s}{s+(1-s)t} > 0, \quad \frac{(1-s)t}{s+(1-s)t} > 0, \quad \frac{s}{s+(1-s)t} + \frac{(1-s)t}{s+(1-s)t} = 1.$$

Hence, by (IV), $b_1' R b_2''$.

Let U be a neighbourhood of zero in E. Since multiplication by scalars is continuous, there exists $0<s<1$ such that $b_1' \in (b_1 + U)\cap B$ and $b_2'' \in (b_2 + U) \cap B$. Hence, by (II), we obtain $b_1 R b_2$.

The converse implication is obvious.

(VII). The sets $B - B$ and $(B - B)\cap M$ are compact.

Proof of (VII). The map $g : B\times B \longrightarrow B - B$ defined by $g(b',b'') = b' - b''$ for (b',b'') in $B\times B$ is a continuous surjection. (The topologies in B and $B - B$ are induced by τ and the topology in $B\times B$ is the product topology). Hence $B - B$ is compact and g is a closed map. Therefore it is enough to prove that the set

$$G = g^{-1}((B - B)\cap M)$$

is a closed subset of $B\times B$. Let (b',b'') be an element of the closure of G. For every neighbourhood U of zero in E the set

$$W = [(b' + U)\cap B] \times [(b'' + U)\cap B]$$

is a neighbourhood of (b',b'') in $B\times B$. Thus there exists (b_1,b_2) in $W\cap G$. But the condition $(b_1,b_2)\in G$ is equivalent to $b_1 R b_2$. Hence, by (II), $b' R b''$, and consequently $(b',b'')\in G$. This completes the proof of (VII).

(VIII). (f_1,f_2) is a kernel pair in Compconv.

Proof of (VIII). It easy to see that the set $K = B - B$ is absolutely convex, absorbing and compact in (E,τ). Let $\|\cdot\|_K$ be the Minkowski functional of K. Then $(E, \|\cdot\|_K)$ is a Banach space and there exists a Banach space $(X, \|\cdot\|)$ such that $(E, \|\cdot\|_K)$ is isometrically isomorphic to the conjugate of $(X, \|\cdot\|)$ (cf. Lemma 3.7.). Of course $(E, \sigma(E,X))$ is a locally convex Hausdorff space. The topologies τ and $\sigma(E,X)$ coincide on K (and consequently on B). Since $M\cap K$ is compact in (E,τ), it is compact in $(E, \sigma(E,X))$. Hence, by the well-known Krein-Šmulyan Theorem (cf. [5], IV.6.4), M is $\sigma(E,X)$ - closed.

Let μ be the quotient topology on E/M determined by $\sigma(E,X)$. Then $(E/M, \mu)$ is a locally convex Hausdorff space. Let

$$\pi : B \longrightarrow E/M$$

be the quotient map. Then the map

$$p : B \longrightarrow \pi(B)$$

defined by $p(b) = \pi(b)$ for b in B is a Compconv-morphism.

It follows immediately from the definition of p that there exists a unique homeomorphism (a Comp-isomorphism) $h : \pi(B) \longrightarrow C$ such that $f = hp$. We shall show that (f_1, f_2) is a kernel pair of p in Compconv. Let (g_1, g_2) be a parallel pair in Compconv such that $pg_1 = pg_2$.

$$\begin{tikzcd} A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & B \arrow[r, "f"] \arrow[dr, "p"] & C \\ D \arrow[u, dashed, "u"] \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & & \pi(B) \arrow[u, dashed, "h"'] \end{tikzcd}$$

Hence $fg_1 = fg_2$ and therefore there exists a unique morphism u in Comp such that $f_i u = g_i$ for $i = 1,2$. It is enough to prove that u is an affine map. Let d', d'' be elements of D and let $0 \leq t \leq 1$. Then

$$f_i(u(td' + (1-t)d'')) = g_i(td' + (1-t)d'') + =$$

$$= tg_i(d') + (1-t)g_i(d'') + = tf_i u(d') + (1-t)f_i u(d'') =$$

$$f_i(tu(d') + (1-t)u(d''))$$

for $i = 1,2$. Since (f_1, f_2) is a monic pair,

$$u(td' + (1-t)d'') = tu(d') + (1-t)u(d'').$$

This completes the proof of (VIII) and the proof of Lemma 4.2.2..

4.2.3. <u>Theorem.</u> The forgetful functor

$$\Box_2 : \text{Compconv} \longrightarrow \text{Comp}$$

is monadic.

Proof. From Theorem 2.1. and Lemmas 3.10., 4.2.2. it follows immediately that $\Box_2$ is quasimonadic. Hence, by 3.9. and 1.6., $\Box_2$ is monadic.

4.2.4. By Theorem 4.2.3., the canonical comparison functor

$$\circledast : \text{Compconv} \longrightarrow \text{Comp}^{\mathbb{T}}$$

is an isomorphism of categories. Therefore, if X is a compact space and

$$\gamma : \mathcal{S}(X) \longrightarrow X$$

is a continuous map such that the diagram

$$\begin{tikzcd} \mathcal{S}(\mathcal{S}(X)) \arrow[r, "\delta_{\mathcal{S}(X)}"] \arrow[d, "\mathcal{S}(\gamma)"'] & \mathcal{S}(X) \arrow[d, "\gamma"] & X \arrow[l, "\delta^X"'] \arrow[dl, "\iota_X"] \\ \mathcal{S}(X) \arrow[r, "\gamma"] & X & \end{tikzcd}$$

is commutative, then there is the unique compact convex set (K, k, τ) such that

$$(X, \gamma) = \Theta(X,k,\tau) = (\square_2(X,k,\tau), \varrho_X) = ((X,\tau), \varrho_X)$$

i.e., $X = (X,\tau)$, $\gamma = \varrho_X$. In other words, for any compact space X and for any continuous map

$$\gamma : \mathcal{S}(X) \longrightarrow X$$

satisfying the following conditions

(i) $\quad \gamma(\delta^X_x) = x \quad$ for x in X

(ii) $\quad \gamma\mathcal{S}(\gamma) = \gamma\varrho_{\mathcal{S}(X)}$

there is the unique convex structure k on X such that (X,k,τ) is a compact convex set (τ is the given topology on X), and $\gamma(\lambda)$ is the centroid of λ , for each λ in $\mathcal{S}(X)$.

It is clear that the set

$$\left\{ \sum_{i=1}^{n} a_i \delta^{\mathcal{S}(X)}_{\lambda_i} : a_i \geqslant 0, \sum_{i=1}^{n} a_i = 1, \quad \lambda_i \in \mathcal{S}(X) \right\}$$

is dense in $\mathcal{S}(\mathcal{S}(X))$. Hence the condition (ii) is equivalent to

(iii) $\quad \gamma(\sum_{i=1}^{n} a_i \delta^X_{\gamma(\lambda_i)}) = \gamma(\sum_{i=1}^{n} a_i \lambda_i)$

for λ_i in $\mathcal{S}(X)$ and $a_i \geqslant 0$ such that $\sum a_i = 1$.

Thus we get the following theorem:

4.2.5. <u>Theorem.</u> Let (X,τ) be a compact space, and let

$$\gamma : \mathcal{S}(X) \longrightarrow X$$

be a continuous map satisfying the conditions 4.2.4.(i) and (iii). Then there is the unique convex structure k on X such that (X,k,τ) is a compact convex set and $\gamma(\lambda)$ is the centroid of λ for each λ in $\mathcal{S}(X)$.

This means that the conditions 4.2.4.(i), (iii) give an axiomatic characterization of the centroid of measure on a compact space.

4.2.6. It is clear that the conditions 4.2.4.(i), (iii) are equivalent to

(i) $\quad \gamma(\delta^X_x) = x \quad$ for x in X,

(iv) $\quad$ if $\gamma(\lambda_1) = \gamma(\lambda_2)$, $\quad \lambda \in \mathcal{S}(X)$ and

$0 \leq t \leq 1$ then $\gamma((1-t)\lambda_1 + t\lambda) =$

$= \gamma((1-t)\lambda_2 + t\lambda)$ (cf.4.1.8.).

4.3. The case $\mathrm{Comp} \underset{\beta}{\overset{\square_3}{\rightleftarrows}} \mathrm{Ens}$

It is known that the forgetful functor $\square_3 : \mathrm{Comp} \longrightarrow \mathrm{Ens}$ is monadic (cf. [12]).

The Stone-Čech functor β (restricted to discrete spaces) is a left adjoint of $\square_3$. Recall that $\beta(X)$ is the Stone-Čech compactification of the set X with the discrete topology, and if $\varphi : X \longrightarrow X'$ is a map, then $\beta(\varphi) : \beta(X) \longrightarrow \beta(X')$ is the unique extension of φ to a continuous map.

The unit of the adjunction is given by the canonical injections

$$\eta_X : X \longrightarrow \square_3 \beta(X).$$

We can regard X as a dense subset of $\beta(X)$.

The counit of the adjunction is given by the canonical surjections

$$\varrho_Y : \beta \square_3 Y \longrightarrow Y$$

defined as follows: given any compact space Y, ϱ_Y is the unique extension of the identity map $\square_3 Y \longrightarrow Y$ to a continuous map.

4.4. The case: $\mathrm{Compconv} \underset{\mathcal{S}\beta}{\overset{\square_4}{\rightleftarrows}} \mathrm{Ens}$.

4.4.1. Composing $\mathcal{S}$ with the Stone-Čech functor β (restricted to discrete spaces) we get the functor

$$\mathcal{S}\beta : \mathrm{Ens} \longrightarrow \mathrm{Compconv}$$

which assigns to each set X the free compact convex set $\mathcal{S}(\beta(X))$ generated by X. $\mathcal{S}\beta$ is a left adjoint of $\square_4$. (Cf. [9] , 23.7.2.)

4.4.2. The forgetful functor

$$\square_4 : \mathrm{Compconv} \longrightarrow \mathrm{Ens}$$

is monadic.

Proof. The functor $\square_4$ is the composition of the functors

$$\square_2 : \mathrm{Compconv} \longrightarrow \mathrm{Comp}$$

and

$$\square_3 : \mathrm{Comp} \longrightarrow \mathrm{Ens}.$$

The functor $\square_3$ is monadic (cf.4.3.). Hence, by 2.4., $\square_3$ satisfies conditions 2.1.(i), (ii). Therefore $\square_4 = \square_3 \square_2$

also satisfies these conditions and consequently, by Theorem 2.1., is quasimonadic. Thus, by Lemmas 3.9. and 1.6., $\square_4$ is monadic.

4.5. The case Bndconv $\underset{\mathcal{B}}{\overset{\square_5}{\rightleftarrows}}$ Conv

4.5.1. We shall now consider the canonical injection of a convex set (K,k) into a pseudo-normed vector space.

Let (K,k) be a convex set. Let L be the subspace of the free vector space $\mathcal{V}(K)$ defined by

$$L = \{t(f_1 - f_2) : t \in \mathbb{R}, \quad f_1, f_2 \in G(K), \quad k(f_1) = k(f_2)\}.$$

Let $E(K,k) = \mathcal{V}(K)/L$ and let $\pi_L : \mathcal{V}(K) \longrightarrow E(K,k)$ be the quotient map. Then there is the unique injection $\varepsilon : K \longrightarrow E(K,k)$ such that the diagram

$$\begin{tikzcd} G(K) \arrow[r, "k"] \arrow[d, "\cap"'] & K \arrow[d, "\varepsilon"] \\ \mathcal{V}(K) \arrow[r, "\pi_L"] & E(K,k) \end{tikzcd}$$

is commutative (cf. the proof of 4.1.10.).

Let $\xi : K \longrightarrow \mathbb{R}$ be an affine map. Then $\xi k : G(K) \longrightarrow \mathbb{R}$ is affine and there exists the

unique linear functional $\tilde{\xi}$ on $\mathcal{V}(K)$ such that $\tilde{\xi}|G(K) = \xi k$. By definition, L is contained in ker $\tilde{\xi}$. Hence there is the unique linear functional $\bar{\xi}$ on $E(K,k)$ such that $\bar{\xi} = \tilde{\xi}\pi_L$. It is clear that $\xi = \bar{\xi}\varepsilon$.

Thus the map from the vector space $E^{\#}(K,k)$ of all linear functionals on $E(K,k)$ onto the vector space of all affine functionals on K defined by

(1) $$\bar{\xi} \longmapsto \xi = \bar{\xi}\varepsilon$$

is a Vect-isomorphism.

It is clear that the map

$$\|\cdot\| : \mathcal{V}(K) \longrightarrow \mathbb{R}$$

defined by $\|\sum a_i \sigma^K_{x_i}\| = \sum |a_i|$ is a norm in $\mathcal{V}(K)$. Consequently, the map

$$p : E(K,k) \longrightarrow \mathbb{R}$$

defined by $p(f+L) = \inf \{ \|f + g\| : g \in L \}$ is a pseudo-norm. It is easy to see that $p(\varepsilon(x)) = 1$ for each x in K.

We shall show that for each $\bar{\xi}$ in $E^{\#}(K,k)$

(2) $$\sup \{ \bar{\xi}(f+L) : p(f+L) \leq 1 \} = \sup \{ \bar{\xi}\varepsilon(x) : x \in K \}.$$

The inequality " $\geqslant$ " is obvious. Let $p(f+L) < 1$. Then there is an $g = \sum a_i \delta^K_{x_i}$ in $f + L$ such that $\| g \| = \sum |a_i| \leq 1$. Hence

$$|\bar{\xi}(f+L)| = |\bar{\xi}\pi(g)| = |\bar{\xi}\pi(\sum a_i \delta^K_{x_i})| = |\sum a_i \bar{\xi}\varepsilon(x_i)| \leq$$

$$\leq \sum |a_i| |\bar{\xi}\varepsilon(x_i)| \leq \sum |a_i| \sup\{ |\bar{\xi}\varepsilon(x)| : x \in K \} \leq$$

$$\leq \sup\{ |\bar{\xi}\varepsilon(x)| : x \in K \}.$$

Thus (2) is true.

Let $E^*(K,k)$ be the conjugate space of $(E(K,k),p)$ and let

$$\mathcal{A}_b(K,k) = \{ \xi : K \longrightarrow \mathbb{R} : \xi \text{ is affine and } \xi(K) \text{ is bounded} \}.$$

Then $\mathcal{A}_b(K,k)$ is a Banach space with the suppremum norm and, by (2), the map of $E(K,k)$ onto $\mathcal{A}_b(K,k)$ defined by (1) is an isomorphism in Ban_1.
It is clear that the following conditions are equivalent:

(i) L is a closed subspace of $\mathcal{V}(K)$,

(ii) p is a norm in $E(K,k)$,

(iii) $\mathcal{A}_b(K,k)$ separates K,

(iv) there exists a topology τ on K such that (K,k,τ)

is a bounded convex set.

If these conditions are satisfied, then (K,k) is called an <u>affinely-bounded</u> set.

Let $M = \{ f+L \in E(K,k) : p(f+L) = 0 \}$. Let $F(K,k) = E(K,k)/M$, and let $\pi_M : E(K,k) \longrightarrow F(K,k)$ be the quotient map. Then p determines the quotient norm $\|\cdot\|$ on $F(K,k)$. The conjugate space of $(F(K,k), \|\cdot\|)$ is isomorphic to $E^*(K,k)$, and consequently, to $\mathcal{A}_b(K,k)$.

We have canonical map

$$\varkappa_K : K \longrightarrow \mathcal{A}_b^*(K,k)$$

defined by $\varkappa_K x . \xi = \xi(x)$ for x in K and ξ in $\mathcal{A}_b(K,k)$. Let $V(K,k) = \operatorname{span}(\varkappa_K(K)) \subset \mathcal{A}_b^*(K,k)$ with the induced norm. It is clear that there is the unique linear map

$$i : V(K,k) \longrightarrow F(K,k)$$

such that the diagram

$$\begin{tikzcd} K \arrow[r, "\varkappa_K"] \arrow[d, "\varepsilon"'] & V(K,k) \arrow[d, "i"] \\ E(K,k) \arrow[r, "\pi_M"] & F(K,k) \end{tikzcd}$$

is commutative. Moreover this map i is an isomorphism in Ban_1.

It is clear that the topology on $F(K,k)$ induced by the norm $\|\cdot\|$ is the Mackey topology $\tau(F(K,k), \mathcal{A}_b(K,k))$.

4.5.2. We shall prove that the functor $\Box_5$ has a left adjoint. Let us denote by $\mathcal{B}(K,k)$ the set $\mathcal{F}_M(\mathcal{E}(K))$ with the induced convex structure and with induced topology. Let us denote by $\eta_{(K,k)}$ the map from K to $\mathcal{B}(K,k)$ defined by $\eta_{(K,k)}(x) = \mathcal{F}_M(\mathcal{E}(x))$ for x in K.

Let B be a bounded convex set and let $\varphi : (K,k) \longrightarrow \Box_5 B$ be an affine map. We shall prove that there exists a unique continuous affine map β such that the diagram

$$\begin{tikzcd}
(K,k) \arrow[r, "(K,k)"] \arrow[dr, "\varphi"'] & \Box_5\mathcal{B}(K,k) \arrow[d, "\Box_5\beta"] \\
 & \Box_5 B
\end{tikzcd}$$

is commutative.

We can assume that B is a bounded convex subset of a locally convex Hausdorff space (X, μ). There exist: a linear map $f : B(K,k) \longrightarrow X$ and a y in X such that $\varphi(x) = f(\mathcal{E}(x)) + y$ for x in K (cf. [9], 23.1.4.). Let x_1, x_2 be elements of K such

that $\varphi(x_1) \neq \varphi(x_2)$. Then there exists a continuous linear functional g on X such that $g(\varphi(x_1)) \neq g(\varphi(x_2))$. Since g is bounded on B, gf is an element of $\mathcal{A}_b(K,k)$ and $gf(x_1) \neq gf(x_2)$. Consequently $\eta_{(K,k)}(x_1) \neq \eta_{(K,k)}(x_2)$. Thus, there exists a unique map $\beta : \mathcal{B}(K,k) \longrightarrow B$ such that $\varphi = \beta\,\eta_{(K,k)}$. It is clear that β is affine. We shall show that β is continuous. Extending β to an affine map from $F(K,k)$ to X we get a linear map $h : F(K,k) \longrightarrow X$ and y in X such that $\beta(x) = h(x)+y$ for x in $\mathcal{B}(K,k)$. Let F' denote the vector space of those linear real-valued functionals on $F(K,k)$ which are bounded on $\mathcal{B}(K,k)$, and let X' be the vector space of those linear functionals on X which are bounded on B. It is clear that $gh \in F'$ for each g in X'. Hence h is continuous with respect to the topologies $\sigma(F(K,k),F')$ and $\sigma(X,X')$. Consequently, h is continuous with respect to the Mackey topologies $\tau(F(K,k),F')$ and $\tau(X,X')$ (cf. [5], IV.7.4.). Since the topology $\tau(X,X')$ is stronger than μ, h is continuous with respect to $\tau(F(K,k),F')$ and μ. Consequently, β is a morphism in Bndconv.

Thus for every morphism $\varphi : K \longrightarrow K'$ in Conv there is the unique morphism $\mathcal{B}(\varphi)$ in Bndconv such that the diagram

$$\begin{tikzcd} K \arrow[r, "\eta_{(K,k)}"] \arrow[d, "\varphi"'] & \square_5 \mathcal{B}(K,k) \arrow[d, dashed, "\square_5 \mathcal{B}(\varphi)"] \\ K' \arrow[r, "\eta_{(K',k')}"'] & \square_5 \mathcal{B}(K',k') \end{tikzcd}$$

is commutative.

We have shown that

$$\mathcal{B} : \mathrm{Conv} \longrightarrow \mathrm{Bndconv}$$

is a convariant functor and is a left adjoint of $\square_5$,

$\eta = (\eta_{(K,k)})_{(K,k) \in \mathrm{Conv}^o}$ being the unit of the adjunction.

4.5.3. <u>Theorem.</u> The forgetful functor

$$\square_5 : \mathrm{Bndconv} \longrightarrow \mathrm{Conv}$$

is not monadic.

Proof. By Lemma 3.11., there is a surjection f in Bndconv which is not a coequalizer. By Lemma 3.10., $\square_5 f$ is a coequalizer in Conv. This shows that $\square_5$ does not satisfy the condition 2.4.(ii), and consequently is not monadic.

4.5.4. Let B be a bounded convex set. Then B is a bounded convex subset of a locally convex Hausdorff space (X, μ) and $\mathcal{B}\square_5 B$ is the same convex set with the topology induced by the Mackey topology $\tau(X,X')$, where X' is the vector space of those linear functions on X which are bounded on B. Hence the counit of the adjunction

$$(\varrho_B : \mathcal{B}\square_5 B \longrightarrow B)_{B \in \mathrm{Bndconv}^o}$$

is defined by $\varrho_B(y) = y$ for y in B. Moreover $\mathcal{B}\square_5\mathcal{B} = \mathcal{B}$.

Let $\mathbb{T} = (T, \eta, \mu)$ be the monad determined by the adjunction $(\mathcal{B}, \square_5, \eta, \varrho)$. Then

$$\mu_K = \square_5 \, \varrho_{\mathcal{B}(K)} = \iota_{\square_5 \mathcal{B}(K)}$$

for each K in Conv^o

Let $((K,k), \varphi)$ be a $\mathbb{T}$- algebra. Then $\varphi\eta_{(K,k)} = \iota_{(K,k)}$ Hence $\eta_{(K,k)}$ is an injection and consequently, by definition of $\eta_{(K,k)}$, $\square_5 \mathcal{B}(K,k) = (K,k)$. Consequently, $\eta_{(K,k)} = \iota_{(K,k)} = \varphi$.

It is easy to see that $\eta_{(K,k)}$ is an injection if and only if $\mathcal{A}_b(K,k)$ separates K. Thus $((K,k), \varphi)$ is a $\mathbb{T}$ - algebra if and only if $\mathcal{A}_b(K,k)$ separates K and $\varphi = \iota_{(K,k)}$. Consequently the category $\mathrm{Conv}^{\mathbb{T}}$ may be identified with the full subcategory of Conv. We obtain the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Bndconv} \arrow[dr, "\square_5"] \arrow[dd, "\Theta=\square_5"] & \\
\mathrm{Conv} \arrow[ur, "\mathcal{B}"] \arrow[dr, "\mathcal{B}^{\mathbb{T}}=\square_5\mathcal{B}"'] & & \mathrm{Conv} \\
& \mathrm{Conv}^{\mathbb{T}} \arrow[ur, "\square_5^{\mathbb{T}} = \text{identical embedding}"'] &
\end{tikzcd}$$

4.6. The case Bndconv $\underset{\mathcal{B}\mathcal{G}}{\overset{\square_6}{\rightleftarrows}}$ Ens.

Composing $\mathcal{B}$ with $\mathcal{G}$ (cf.4.1.) we get a left adjoint functor of $\square_6$. Similar argumentsas in the proof of Theorem 4.5.2. show that:

4.6.1. <u>Theorem.</u> The forgetful functor

$$\square_6 : \mathrm{Bndconv} \longrightarrow \mathrm{Ens}$$

is not monadic.

4.7. The case Compconv $\underset{\mathcal{R}}{\overset{\square_7}{\rightleftarrows}}$ Bndconv

4.7.1. The functor $\mathcal{R}$: Bndconv $\longrightarrow$ Compconv, a left adjoint of $\square_7$, is defined as follows. If K is any bounded convex set, let $\mathcal{A}(K)$ be the Banach space of all continuous affine real-valued functions on K with the supremum norm, and

$$\mathcal{R}(K) = \left\{ f \in O^*\mathcal{A}(K) : f(1_K) = 1 \right\}$$

with the $\sigma(\mathcal{A}^*(K), \mathcal{A}(K))$ topology. If $\varphi : K \longrightarrow K'$ is a morphism in Bndconv, then $\mathcal{R}\varphi : \mathcal{R}(K) \longrightarrow \mathcal{R}(K')$ is defined by $\mathcal{R}\varphi.f.\xi = f(\xi\varphi)$ (cf. [Sa], 2.11.).

4.7.2. <u>Theorem.</u> The forgetful functor

$$\square_7 : \mathrm{Compconv} \longrightarrow \mathrm{Bndconv}$$

is monadic.

Proof. It is clear that a morphism q is a coequalizer of a parallel pair (f,g) in Compconv if and only if $\square_7 q$ is a coequalizer of ($\square_7 f$, $\square_7 g$) in Bndconv. Hence, by Theorem 1.5., $\square_7$ is quasimonadic. Consequently, by 3.9. and 1.6., $\square_7$ is monadic.

4.8. The case $\text{Compconv} \underset{\mathcal{Q}}{\overset{\Box_8}{\rightleftarrows}} \text{Conv}.$

4.8.1. Theorem. The forgetful functor

$$\Box_8 : \text{Compconv} \longrightarrow \text{Conv}$$

is monadic.

Proof. The composition $\mathcal{R}\mathcal{B}$ (cf. 4.5., 4.7.) is a left adjoint of $\Box_8$. Let (f.g) be a parallel pair in Compconv such that $(\Box_8 f, \Box_8 g)$ is a kernel pair in Conv. Since $\Box_1$ is a right adjoint, $(\Box_1 \Box_8 f, \Box_1 \Box_8 g) = (\Box_4 f, \Box_4 g)$ is a kernel pair in Ens. The functor $\Box_4$ satisfies condition 2.1.(ii) (cf. the proof of 4.4.2.), hence (f,g) is a kernel pair in Compconv. By Lemma 3.10., a morphism q is a coequalizer in Compconv if and only if $\Box_8 q$ is a coequalizer in Conv. Hence, by Theorem 2.1., the functor $\Box_8$ is quasimonadic. Consequently, by 3.9. and 1.6., $\Box_8$ is monadic.

4.8.2. A left adjoint of $\Box_8$ can be also obtained in a way similar to the construction of $\mathcal{R}$ (cf. [8a], 2.11.):

If (K,k) is any convex set, let $\mathcal{A}_b(K,k)$ be the Banach space of all bounded affine real-valued functions on K with the supremum norm. Let

$$\mathcal{Q}(K,k) = \left\{ f \in O^*\mathcal{A}_b(K,k) : f(1_K) = 1 \right\}.$$

Of course $\mathcal{Q}(K,k)$ is a convex subset of $\mathcal{A}_b^*(K,k)$ and it is compact in the $\sigma(\mathcal{A}_b^*(K,k), \mathcal{A}_b(K,k))$ topology.

If $\varphi : K \longrightarrow K'$ is an affine map, then

$$\mathcal{Q}\varphi : \mathcal{Q}(K,k) \longrightarrow \mathcal{Q}(K',k')$$

defined by $\mathcal{Q}\varphi.f.\xi = f(\xi\varphi)$ for f in $\mathcal{Q}(K,k)$ and ξ in $\mathcal{A}_b(K',k')$ is a continuous affine map. Consequently

$$\mathcal{Q} : \text{Conv} \longrightarrow \text{Compconv}$$

is a covariant functor.

The space $\mathcal{A}_b(K,k)$ is isometrically isomorphic to the Banach space $\mathcal{A}(\mathcal{B}(K,k))$ of all continuous affine real-valued functions on $\mathcal{B}(K,k)$ with the supremum norm. Hence the functor $\mathcal{Q}$ is naturally equivalent to $\mathcal{R}\mathcal{B}$, and therefore it is left adjoint of $\Box_8$. The unit of the adjunction is given by the canonical maps

$$\eta_{(K,k)} : K \longrightarrow \Box_8 \mathcal{Q}(K,k) \text{ defined by } \eta_{(K,k)} x.\xi = \xi(x)$$

for x in K and ξ in $\mathcal{A}_b(K,k)$.

4.9. The case $\text{Compsaks} \underset{O^*\mathcal{A}}{\overset{\Box_9}{\rightleftarrows}} \text{Compconv}.$

4.9.1. The forgetful functor $\Box_9$ has a left adjoint $O^*\mathcal{A}$ where $\mathcal{A}(K)$ is the Banach space of all continuous affine real-valued functions on the compact convex set K, and

$$O^*\mathcal{A}(K) = \{ f \in (\mathcal{A}(K))^* : \|f\| \leq 1 \}$$

with the $\sigma(\mathcal{A}^*(K), \mathcal{A}(K))$ topology. If $\varphi : X \longrightarrow K'$ is a Compconv-morphism, then

$$O^*\mathcal{A}\varphi : O^*\mathcal{A}(K) \longrightarrow O^*\mathcal{A}(K')$$

is defined by $O^*\mathcal{A}\varphi f.\xi = f(\xi\varphi)$ for f in $O^*\mathcal{A}(K)$ and ξ in $\mathcal{A}(K')$ (cf. [8]).

4.9.2. <u>Lemma.</u> If (f_1,f_2) is a parallel pair in Compsaks such that $(\Box_9 f_1, \Box_9 f_2)$ is a kernel pair in Compconv, then (f_1,f_2) is a kernel pair in Compsaks.

Proof. Let f be a coequalizer of (f_1,f_2) in Compconv and let (C,c,τ) be the codomain of f:

$$A \overset{f_1}{\underset{f_2}{\rightrightarrows}} B \xrightarrow{f} C$$

Then f is a surjection (cf.3.10.). Let y_o be the center of B. We shall show that $(C.c, \tau, f(y_o))$ is a compact Saks space. Indeed, let z be an element of C. There exist elements y,y' of B such that $z = f(y)$ and $\frac{1}{2}y + \frac{1}{2}y' = y_o$. Hence we have $\frac{1}{2}z + \frac{1}{2}f(y') = f(\frac{1}{2}y + \frac{1}{2}y') = f(y_o)$. Thus $(C,c,\tau, f(y_o))$ is an object in Compsaks and f is a Compsaks-

-morphism. It is easy to verify that (f_1,f_2) is a kernel pair of f in Compsaks.

4.9.3. <u>Theorem.</u> The forgetful functor

$$\Box_9 : \text{Compsaks} \longrightarrow \text{Compconv}$$

is monadic.

Proof. By Lemma 3.10., a morphism p is a coequalizer in Compsaks if and only if $\Box_9 p$ is a coequalizer in Compconv, i.e., $\Box_9$ satisfy the condition 2.1.(i). By 4.9.2., $\Box_9$ also satisfies the condition 2.1.(ii). Hence, by Theorem 2.1., 9 is quasimonadic, and consequently, by Lemmas 3.9., 1.5., $\Box_9$ is monadic.

4.10. The case: $\text{Compsaks} \overset{\Box_{10}}{\underset{O^*\mathcal{C}}{\rightleftarrows}} \text{Comp}.$

4.10.1. The forgetful functor $\Box_{10}$ has a left adjoint $O^*\mathcal{C}$, where $\mathcal{C}(X)$ is the Banach space of all continuous real-valued functions on the compact space X, and $O^*\mathcal{C}(X)$ is the unit ball in $(\mathcal{C}(X))^*$ (cf. [8]).

4.10.2. <u>Theorem.</u> The forgetful functor

$$\Box_{10} : \text{Compsaks} \longrightarrow \text{Comp}$$

is monadic.

Proof. The functor $\square_{10}$ is the composition of the functors $\square_2$ and $\square_9$. The functors $\square_2$, $\square_9$ satisfy the conditions 2.1.(i),(ii). (cf.3.10., 4.1.2., 4.9.2.). Hence $\square_{10}$ satisfies these conditions, and by 2.1. is quasi-monadic. Consequently, by 3.9. and 1.6., $\square_{10}$ is monadic.

4.11. The case: $\text{Compsaks} \underset{O^*\mathcal{A}_b}{\overset{\square_{11}}{\rightleftarrows}} \text{Conv}.$

The functor $\square_{11}$ is the composition of the functors $\square_9$ and $\square_8$, hence $O^*\mathcal{A}_b$ is a left adjoint of $\square_{11}$.

4.11.1. <u>Theorem.</u> The forgetful functor

$$\square_{11} : \text{Compsaks} \longrightarrow \text{Conv}$$

is monadic

Proof. Similar as 4.10.2..

4.12. The case: $\text{Compsaks} \underset{O^*\ell\beta}{\overset{\square_{12}}{\rightleftarrows}} \text{Ens}$

The functor $\square_{12}$ is the composition of the functors $\square_9$ and $\square_4$, hence $O^*\ell\beta$ is a left adjoint of $\square_{12}$.

4.12.1. <u>Theorem.</u> The forgetful functor

$$\square_{12} : \text{Compsaks} \longrightarrow \text{Ens}$$

is monadic.

Proof. Similar as 4.10.2..

4.13. The case: $\text{Compsaks} \underset{O^*\mathcal{AR}}{\overset{\square_{13}}{\rightleftarrows}} \text{Bndconv}.$

The functor $\square_{13}$ is the composition of the functors $\square_9$ and $\square_7$, hence $O^*\mathcal{AR}$ is a left adjoint of $\square_{13}$.

4.13.1. <u>Theorem.</u> The forgetful functor

$$\square_{13} : \text{Compsaks} \longrightarrow \text{Bndconv}$$

is monadic.

Proof. Similar as 4.10.2..

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