L^p SPACES ASSOCIATED WITH VON NEUMANN ALGEBRAS

Notes by

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The main part of these notes is devoted to a complete and detailed exposition of the theory, due to U. Haagerup, of abstract L^p spaces associated with von Neumann algebras, with Haagerup's original proofs.

We also discuss measurable operators with respect to a trace, spatial derivatives, and spatial realizations of the L^p spaces.

These notes are part of the author's thesis for the lic.scient. degree at the University of Odense, where they provide the background material for [19] and [20]

CONTENTS.

Introduction.

The main part of these notes (Chapter II) is devoted to a complete and detailed exposition of the theory of abstract L^p spaces associated with von Neumann algebras. This theory was developed by U. Haagerup some seven years ago and outlined in a preprint (which now appears in [9]). Unfortunately, in spite of his intentions, Haagerup has not yet had the time for writing down his theory in full. This is our motivation for writing these notes.

The proofs that we give are (close to) those that Haagerup originally had in mind and which he has told us at various occasions.

Essential for the construction of the L^p spaces is the theory of measurable operators with respect to a trace on a von Neumann algebra (due to E. Nelson [13] and inspired by [15] and [16]); we treat this in Chapter I. Other prerequisites are the basic facts on crossed products of a von Neumann algebra with a modular automorphism group and some results on operator valued weights and the extended positive part of a von Neumann algebra; we have not included this in the text but we give detailed references, especially to parts of [7] and [8], at the places where it is needed.

After the appearance of Haagerup's L^p spaces, A. Connes proposed a definition of spatial L^p spaces based on the notion of spatial derivatives [1]. These spaces have been studied by M. Hilsum [10]. We include a discussion of them and show how their main properties follow easily from the corresponding properties of Haagerup's spaces (thus our presentation is complementary to Hilsum's work [10] where the objective is to develop the theory directly based on properties of spatial derivatives, avoiding as far as possible the dependence of Haagerup's construction). This is contained in Chapter IV.

Before this, we recall the main properties of spatial derivatives (Chapter III). We profit from this occasion to present a definition (due to U. Haagerup) of spatial derivatives that is slightly different from that given in [1] and to show how certain properties (such as the sum property) of spatial derivatives are almost immediate consequences of this new definition.

The reader will notice that these notes do not contain a special chapter on the - now classic - theory of L^p spaces with respect to a trace, due - in various formulations - to J. Dixmier [3] and R. A. Kunze [12] (see also [21] and [13]). Although this important particular case has been motivating for the development of the more general theory, we do not directly need it in our preliminaries. For the sake of completeness, however, we give the definition of L^p spaces with respect to a trace at the end of Chapter I, and in the following chapters, we point out how results concerning the trace case are related to the general results.

Another omission in these notes is the recent definition of L^p spaces as complex interpolation spaces. For this, we simply refer to [11] and [20].

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I

MEASURABLE OPERATORS WITH RESPECT TO A TRACE

In this chapter, we define the notion of measurability with respect to a trace τ on a von Neumann algebra M and show that the set $\tilde{M}$ of τ-measurable operators is a complete topological *-algebra. Our presentation is a modified version of that given by E. Nelson [13].

Let M be a - necessarily semifinite - von Neumann algebra acting on a Hilbert space H and let τ be a normal faithful semifinite trace on M.

For the convenience of the reader, we immediately give the definition of τ-measurability and state the main theorem about τ-measurable operators.

<u>Definition 14</u>: A closed densely defined operator a affiliated with M is called τ-<u>measurable</u> if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that

$$pH \subseteq D(a) \quad \text{and} \quad \tau(1-p) \leq \delta .$$

For a characterization of τ-measurable operators in terms of the spectral projections of their absolute value, see Proposition 21 below.

We denote by $\tilde{M}$ the set of τ-measurable closed densely defined operators.

Theorem 28. 1) $\tilde{M}$ is a *-algebra with respect to strong sum, strong product, and adjoint operation.

2) The sets

$$N(\varepsilon,\delta) = \{a \in \tilde{M} \mid \exists p \in M_{proj}: pH \subseteq D(a), \|ap\| \leq \varepsilon, \tau(1-p) \leq \delta\},$$

where $\varepsilon,\delta \in \mathbb{R}_+$, form a basis for the neighbourhoods of 0 for a topology on $\tilde{M}$ that turns $\tilde{M}$ into a topological vector space.

3) $\tilde{M}$ is a complete Hausdorff topological *-algebra and M is a dense subset of $\tilde{M}$.

Once this theorem has been proven, we can freely add and multiply operators from $\tilde{M}$, the operations being understood in the strong sense (see the definition below). Until then, we have to deal with unbounded operators in the usual careful way.

Although we are mainly interested in closed densely defined operators it will be convenient for us to work with more general kinds of unbounded operators. We therefore start by recalling some basic facts on arbitrary unbounded operators. Next, we recall some properties of the lattice M_{proj} of projections in M. After this, we go on to develop the theory of τ-measurability.

Preliminaries on unbounded operators.

Recall that for any (linear) operators a and b on H we can define the sum $a+b$ and the product ab as operators on H with domains

$$D(a+b) = D(a) \cap D(b), \quad (1)$$

$$D(ab) = \{\xi \in D(b) \mid b\xi \in D(a)\}. \quad (2)$$

These operations are associative so that $a+b+c$ and abc are well-defined operators. Furthermore, for all a, b and c we have

$$(a+b)c = ac+bc \quad \text{and} \quad c(a+b) \supseteq ca+cb \quad (3)$$

(with equality if $D(c) = H$).

We shall use the following terminology: an operator a on H is closed if its graph $G(a)$ is closed in $H \oplus H$; a is preclosed if the closure $\overline{G(a)}$ of its graph is the graph of some - necessarily closed - operator (the closure of a, denoted $[a]$); a is densely defined if $D(a)$ is dense in H.

If a, b and ab are densely defined, then

$$(ab)^* \supseteq b^*a^* \quad (4)$$

with equality if a is bounded and everywhere defined.

A closed densely defined operator a has a unique polar decomposition

$$a = u|a| \quad (5)$$

where $|a|$ is a positive self-adjoint operator and u a partial isometry with $\mathrm{supp}(a)$ as its initial projection and $r(a)$, the projection onto the closure of the range of a, as its final projection.

If the sum $a+b$ of two closed densely defined operators a and b is preclosed and densely defined, then the closure $[a+b]$ is called the strong sum of a and b. Similarly, the strong product is the closure $[ab]$ if ab is preclosed and densely defined.

We shall write

$$\|a\| = \sup\{\|a\xi\| \mid \|\xi\| \leq 1\}$$

for all everywhere defined operators a on H, bounded or not. For all such operators, the usual norm estimates hold:

$$\|a+b\| \leq \|a\| + \|b\| , \|ab\| \leq \|a\| \|b\| .$$

Denote by M' the commutant of M.

Definition 1. A linear operator a on H is said to be affiliated with M (and we write $a \eta M$) if

$$\forall y \in M': ya \subseteq ay . \tag{6}$$

Remark 2. Using (3), (4) and (5) one easily verifies that

(i) if a, $b \eta M$, then $a + b \eta M$ and $ab \eta M$;

(ii) if a is preclosed, resp. densely defined, and $a \eta M$, then $[a] \eta M$, resp. $a^* \eta M$;

(iii) if a is a closed densely defined operator with polar decomposition $a = u|a|$, then $a \eta M$ if and only if $u \in M$ and $|a| \eta M$.

Notation. We denote by $\overline{M}$ the set of closed densely defined operators affiliated with M.

Preliminaries on projections.

We denote by M_{proj} the lattice of (orthogonal) projections in

M. For a family $(p_i)_{i\in I}$ of projections in M, $\bigwedge_{i\in I} p_i$ (resp. $\bigvee_{i\in I} p_i$) is the projection onto $\bigcap_{i\in I} p_iH$ (resp. $\overline{\bigcup_{i\in I} p_iH}$).

Recall that

$$\Big(\bigwedge_{i\in I} p_i\Big)^{\perp} = \bigvee_{i\in I} p_i^{\perp} , \Big(\bigvee_{i\in I} p_i\Big)^{\perp} = \bigwedge_{i\in I} p_i^{\perp} \tag{7}$$

where $p^{\perp} = 1 - p$ is the projection orthogonal to p.

Two projections p and q are equivalent if $p = u^*u$ and $q = uu^*$ for some $u \in M$. We denote equivalence by $\sim$. Equivalent projections have the same trace.

By the polar decomposition theorem, we have

Lemma 3. Let a be a closed densely defined operator affiliated with M. Then

$$\mathrm{supp}(a) \sim r(a)$$

where $r(a)$ denotes the projection onto the closure of the range of a.

For any projections $p, q \in M$ we have

$$(p \vee q) - p \sim q - (p \wedge q) . \tag{8}$$

It follows that

$$\tau(p \vee q) \leq \tau(p) + \tau(q) . \tag{9}$$

More generally,

$$\tau\Big(\bigvee_{i\in I} p_i\Big) \leq \sum_{i\in I} \tau(p_i) \tag{10}$$

for any family $(p_i)_{i\in I}$ of projections in M (if I is finite, this follows by induction from (9); for the general case, use the normality of τ) .

Another consequence of (8) is this:

$$\forall p,q \in M_{proj}: p\wedge q = 0 \Rightarrow p \precsim 1 - q \qquad (11)$$

(where $\precsim$ means: "is equivalent to a subprojection of"). Indeed,

$$p = 1 - p^{\perp} = (p\wedge q)^{\perp} - p^{\perp} = (p^{\perp}\vee q^{\perp}) - p^{\perp} \sim q^{\perp} - (p^{\perp}\wedge q^{\perp}) \leq q^{\perp} = 1 - q .$$

The theory of τ-measurable operators.

Definition 4. Let $\varepsilon,\delta \in \mathbb{R}_+$. Then we denote by $D(\varepsilon,\delta)$ the set of all operators $a \,\eta\, M$ for which there exists a projection $p \in M$ such that

(i) $pH \subseteq D(a)$ and $\|ap\| \leq \varepsilon$

and

(ii) $\tau(1-p) \leq \delta$.

When $pH \subseteq D(a)$, the operator ap is everywhere defined. The requirement $\|ap\| \leq \varepsilon$ in particular implies that ap is bounded.

Note that we do not require a to be densely defined, closed or preclosed.

Proposition 5. Let $\varepsilon_1,\varepsilon_2,\delta_1,\delta_2 \in \mathbb{R}_+$. Then

(i) $D(\varepsilon_1,\delta_1) + D(\varepsilon_2,\delta_2) \subseteq D(\varepsilon_1+\varepsilon_2,\delta_1+\delta_2)$,

(ii) $D(\varepsilon_1,\delta_1)D(\varepsilon_2,\delta_2) \subseteq D(\varepsilon_1\varepsilon_2,\delta_1+\delta_2)$.

Proof. (i) Let $a \in D(\varepsilon_1,\delta_1)$ and $b \in D(\varepsilon_2,\delta_2)$. Then there exist projections $p,q \in M$ such that

$$pH \subseteq D(a) \ , \ \|ap\| \leq \varepsilon_1 \ , \ \text{and} \ \tau(1-p) \leq \delta_1 \ ,$$

$$qH \subseteq D(b) \ , \ \|bq\| \leq \varepsilon_2 \ , \ \text{and} \ \tau(1-q) \leq \delta_2 \ .$$

Put $r = p\wedge q$. Then

$$rH = pH \cap qH \subseteq D(a) \cap D(b) = D(a+b)$$

and

$$\|(a+b)r\| = \|ar+br\| \leq \|ar\| + \|br\| \leq \|ap\| + \|bq\| \leq \varepsilon_1 + \varepsilon_2 .$$

Furthermore,

$$\tau(1-r) = \tau((p\wedge q)^{\perp}) = \tau(p^{\perp}\vee q^{\perp}) \leq \tau(1-p) + \tau(1-q) \leq \delta_1 + \delta_2 .$$

This proves (i).

To prove (ii), let $a \in D(\varepsilon_1,\delta_1)$, $b \in D(\varepsilon_2,\delta_2)$ and take $p,q \in M_{proj}$ as above. Then bq , and hence $(1-p)bq$, is bounded. Denote by s the projection onto its null space:

$$sH = N((1-p)bq) .$$

Then $bq\xi \in pH \subseteq D(a)$ for all $\xi \in sH$, so that $sH \subseteq D(abq)$ and hence

$$(q\wedge s)H \subseteq D(ab) .$$

Also, $abqs = apbqs$ so that

$$ab(q\wedge s) = abqs(q\wedge s) = apbq(q\wedge s)$$

and thus

$$\|ab(q\wedge s)\| \leq \|ap\| \, \|bq\| \leq \varepsilon_1\varepsilon_2 .$$

On the other hand, using that

$$1-s = \mathrm{supp}((1-p)bq) \sim r((1-p)bq) \leq 1-p ,$$

we have

$$\tau(1-(q\wedge s)) = \tau((1-q)\vee(1-s)) \leq \tau(1-q) + \tau(1-s)$$

$$\leq \tau(1-q) + \tau(1-p) \leq \delta_1 + \delta_2 .$$

This completes the proof. ∎

<u>Proposition 6</u>. Let $\varepsilon,\delta \in \mathbb{R}_+$.

(i) Let a be a preclosed operator. Then

$$a \in D(\varepsilon,\delta) \Rightarrow [a] \in D(\varepsilon,\delta) .$$

(ii) Let a be a closed densely defined operator with polar decomposition $a = u|a|$. Then

$$a \in D(\varepsilon,\delta) \Leftrightarrow u \in M \text{ and } |a| \in D(\varepsilon,\delta) .$$

<u>Proof</u>. (i): trivial. (ii): trivial, since $a = u|a|$, $|a| = u^*a$, and $\|u\| \leq 1$. ∎

<u>Lemma 7</u>. Let $a \in \overline{M}$ and $\varepsilon,\delta \in \mathbb{R}_+$. Then

$$a \in D(\varepsilon,\delta) \Leftrightarrow \tau(\chi_{]\varepsilon,\infty[}(|a|)) \leq \delta$$

(where $\chi_{]\varepsilon,\infty[}(|a|)$ denotes the spectral projection of $|a|$ corresponding to the interval $]\varepsilon,\infty[$) .

<u>Proof</u>. "$\Leftarrow$": Put $p = \chi_{[0,\varepsilon]}(|a|)$. Then $pH \subseteq D(|a|)$ and $\||a|p\| \leq \varepsilon$.

"$\Rightarrow$": For some $p \in M_{proj}$, we have

$$\||a|p\| \leq \varepsilon \quad\text{and}\quad \tau(1-p) \leq \delta .$$

Let $|a| = \int_0^\infty \lambda \, de_\lambda$ be the spectral decomposition of $|a|$. Now for all $\xi \in pH$ we have

$$\||a|\xi\|^2 \leq \varepsilon^2 \|\xi\|^2 ,$$

and for all $\xi \in (1-e_\varepsilon)H\setminus\{0\}$ we have

$$\||a|\xi\|^2 > \varepsilon^2 \|\xi\|^2$$

since

$$\||a|\xi\|^2 = \int_0^\infty \lambda^2 \, d(e_\lambda\xi|\xi) = \int_{]\varepsilon,\infty[} \lambda^2 d(e_\lambda\xi|\xi) .$$

Hence $(1-e_\varepsilon)H \cap pH$ must be $\{0\}$, i.e. $(1-e_\varepsilon)\wedge p = 0$. By (11) we conclude that $1-e_\varepsilon \precsim 1-p$, whence $\tau(1-e_\varepsilon) \leq \delta$. ∎

<u>Proposition 8</u>. Let $a \in \overline{M}$ and $\varepsilon,\delta \in \mathbb{R}_+$. Then

$$a \in D(\varepsilon,\delta) \Leftrightarrow a^* \in D(\varepsilon,\delta) .$$

<u>Proof</u>. Let $a = u|a|$ be the polar decomposition of a. Then u is an isometry of $\chi_{]0,\infty[}(|a|) = \mathrm{supp}(a)$ onto $\chi_{]0,\infty[}(|a^*|) = \mathrm{supp}(a^*) = r(a)$. By uniqueness of the spectral decomposition, u induces for each $\lambda \in \mathbb{R}_+$ an isometry of $\chi_{]\lambda,\infty[}(|a|)$ onto $\chi_{]\lambda,\infty[}(|a^*|)$. The result follows by Lemma 7. ∎

<u>Definition 9</u>. A subspace E of H is called τ-<u>dense</u> if for all $\delta \in \mathbb{R}_+$, there exists a projection $p \in M$ such that

$$pH \subseteq E \quad\text{and}\quad \tau(1-p) \leq \delta .$$

<u>Proposition 10</u>. Let E be a τ-dense subspace of H. Then there exists an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of projections in M with

$$p_n \nearrow 1\ ,\ \tau(1-p_n) \to 0\ ,\ \text{and}\ \bigcup_{n=1}^{\infty} p_nH \subseteq E\ .$$

<u>Proof</u>. Take projections $q_k \in M$, $k \in \mathbb{N}$, such that

$$q_kH \subseteq E \ \text{and}\ \tau(1-q_k) \le 2^{-k}\ .$$

For each $n \in \mathbb{N}$, put

$$p_n = \bigwedge_{k=n+1}^{\infty} q_k\ .$$

Then

$$p_nH = \bigcap_{k=n+1}^{\infty} q_kH \subseteq E$$

and

$$\tau(1-p_n) = \tau\Big(\bigvee_{k=n+1}^{\infty}(1-q_k)\Big) \le \sum_{k=n+1}^{\infty}\tau(1-q_k) \le \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}\ .$$

It follows that

$$p_n \nearrow 1\ ;$$

indeed, denoting by p the supremum of the increasing sequence p_n, we have

$$\forall n \in \mathbb{N}:\ \tau(1-p) \le \tau(1-p_n) \le 2^{-n}$$

whence $\tau(1-p) = 0$ and $p = 1$.

Furthermore,

$$\bigcup_{n=1}^{\infty} p_nH \subseteq E\ . \ \blacksquare$$

<u>Corollary 11</u>. Let E be a τ-dense subspace of H. Then E is dense in H.

An important property of τ-dense subspaces is the following:

<u>Proposition 12</u>. Let $a,b \in \overline{M}$ and let E be a τ-dense subspace of H contained in $D(a) \cap D(b)$. Suppose that

$$a|_E = b|_E\ .$$

Then $a = b$.

The proof is based on the following lemma:

<u>Lemma 13</u>. 1) Let $p_0 \in M_{proj}$. Suppose that

$$\forall\delta \in \mathbb{R}_+\ \exists p \in M_{proj}:\ p_0\wedge p = 0 \ \text{and}\ \tau(1-p) \le \delta\ .$$

Then $p_0 = 0$.

2) Let $p_1, p_2 \in M_{proj}$. Suppose that

$$\forall\delta \in \mathbb{R}_+\ \exists p \in M_{proj}:\ p_1\wedge p = p_2\wedge p \ \text{and}\ \tau(1-p) \le \delta\ .$$

Then $p_1 = p_2$.

<u>Proof</u>. 1) Let $\delta \in \mathbb{R}_+$. Then $\tau(p_0) \le \delta$ (indeed, for some $p \in M_{proj}$ we have $p_0\wedge p = 0$ and $\tau(1-p) \le \delta$, whence $p_0 \precsim 1-p$ and $\tau(p_0) \le \tau(1-p) \le \delta$). Hence $\tau(p_0) = 0$ and $p_0 = 0$.

2) Put $p_0 = p_1 - (p_1\wedge p_2)$. Now $p_1\wedge p = p_2\wedge p$ implies $p_1\wedge p = (p_1\wedge p_2)\wedge p$ and hence $p_0\wedge p = 0$, so that 1) applies to p_0. Hence $p_0 = 0$, i.e. $p_1 = p_1\wedge p_2$. Similarly, $p_2 = p_1\wedge p_2$. In all, $p_1 = p_2$. $\blacksquare$

<u>Proof</u> of Proposition 12. Consider in the Hilbert space $H_2 = H \oplus H$ the von Neumann algebra $M_2 = \begin{pmatrix} M & M \\ M & M\end{pmatrix}$ equipped with the normal faithful semifinite trace τ_2 defined by

$$\tau_2\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \tau(x_{11}) + \tau(x_{22}) .$$

Denote by p_a and p_b the projections onto the graphs $G(a)$ and $G(b)$ of a and b. Since a and b are affiliated with M, $G(a)$ and $G(b)$ are invariant under all elements of $M_2' = \left\{\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \mid y \in M'\right\}$ and thus $p_a, p_b \in M_2$.

Let $\delta \in \mathbb{R}_+$. Then there exists a projection $p \in M$ with $pH \subseteq E$ and $\tau(1-p) \leq \delta/2$. Put $p_2 = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$. Then $\tau_2(1-p_2) \leq \delta$. Furthermore,

$$p_a \wedge p_2 = p_b \wedge p_2$$

since a and b agree on $pH \subseteq E$ and thus

$$G(a) \cap (pH \oplus pH) = \{(\xi, a\xi) \mid \xi \in pH,\ a\xi \in pH\}$$

$$= \{(\xi, b\xi) \mid \xi \in pH,\ b\xi \in pH\} = G(b) \cap (pH \oplus pH) .$$

By Lemma 13, we conclude that $p_a = p_b$, whence $a = b$. ∎

Definition 14. An operator $a \in \overline{M}$ is called τ-measurable if $D(a)$ is τ-dense, i.e. if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that

$$pH \subseteq D(a) \quad \text{and} \quad \tau(1-p) \leq \delta . \tag{12}$$

The set of τ-measurable operators $a \in \overline{M}$ is denoted $\widetilde{M}$.

Corollary 15. 1) Let $a, b \in \widetilde{M}$. If

$$a \subseteq b$$

then

$$a = b .$$

2) Let $a \in \widetilde{M}$. If a is symmetric (in particular, if a is positive), then a is self-adjoint.

Proof. Immediate from Definition 14 and Proposition 12 (for 2), use that $a \subseteq a^*$). ∎

Note that when a is closed and $p \in M_{proj}$ is such that $pH \subseteq D(a)$, then the everywhere defined operator ap is also closed and hence - by the closed graph theorem - automatically bounded. Therefore the following definition is a generalization of Definition 14.

Definition 16. Any operator $a \,\eta\, M$ is called τ-premeasurable if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that

$$pH \subseteq D(a), \quad \|ap\| < \infty, \quad \text{and} \quad \tau(1-p) \leq \delta . \tag{13}$$

By definition of the $D(\varepsilon, \delta)$, this may be reformulated as:

Remark 17. Let $a \,\eta\, M$. Then a is τ-premeasurable if and only if

$$\forall \delta \in \mathbb{R}_+ \ \exists \varepsilon \in \mathbb{R}_+ : a \in D(\varepsilon, \delta) .$$

Also note

Proposition 18. Let $a \,\eta\, M$. If a is τ-premeasurable, then a is densely defined.

Proof. $D(a)$ is τ-dense. ∎

<u>Proposition 19</u>. Let $a \, \eta \, M$. Suppose that a is τ-premeasurable and preclosed. Then

$$[a] \in \widetilde{M} .$$

<u>Proof</u>. Trivial. ∎

<u>Proposition 20</u>. Let $a,b \, \eta \, M$ be τ-premeasurable. Then $a+b$ and ab are also τ-premeasurable.

<u>Proof</u>. Combine Remark 17 and Proposition 5. ∎

We have the following characterization of τ-measurable operators:

<u>Proposition 21</u>. Let $a \in \widetilde{M}$ with polar decomposition $a = u|a|$. Then the following assertions are equivalent:

(i) a is τ-measurable,

(ii) $|a|$ is τ-measurable,

(iii) $\forall \delta \in \mathbb{R}_+ \ \exists \varepsilon \in \mathbb{R}_+ : \ a \in D(\varepsilon,\delta)$,

(iv) $\forall \delta \in \mathbb{R}_+ \ \exists \varepsilon \in \mathbb{R}_+ : \ \tau(\chi_{]\varepsilon,\infty[}(|a|)) \leq \delta$,

(v) $\tau(\chi_{]\lambda,\infty[}(|a|)) \to 0$ as $\lambda \to \infty$,

(vi) $\forall \lambda \in \mathbb{R}_+ : \ \tau(\chi_{]\lambda,\infty[}(|a|)) < \infty$.

<u>Proof</u>. The equivalence of (i), (ii), and (iii), follows from Lemma 7. Now note that

$$\chi_{]\lambda,\infty[}(|a|) \searrow \emptyset \quad \text{as} \quad \lambda \to \infty$$

so that, by the normality of τ ,

$$\tau(\chi_{]\lambda,\infty[}(|a|)) \searrow 0 \quad \text{as} \quad \lambda \to \infty$$

if $\tau(\chi_{]\lambda_0,\infty[}(|a|)) < \infty$ for some λ_0 . The equivalence of (iii), (iv), (v), and (vi) follows. ∎

<u>Corollary 22</u>. We have $M \subseteq \widetilde{M}$.

<u>Proof</u>. If a is bounded, then $\chi_{]\|a\|,\infty[}(|a|) = 0$. ∎

<u>Proposition 23</u>. Let $a \in \widetilde{M}$. Then also $a^* \in \widetilde{M}$.

<u>Proof</u>. Combine Proposition 8 and Proposition 21, (i) ⇔ (iii). ∎

<u>Proposition 24</u>. 1) Let $a,b \in \widetilde{M}$. Then $a+b$ and ab are densely defined and preclosed, and $[a+b] \in \widetilde{M}$, $[ab] \in \widetilde{M}$.

2) $\widetilde{M}$ is a *-algebra with respect to strong sum and strong product.

<u>Proof</u>. 1) Let $a,b \in \widetilde{M}$. Then also $a^*,b^* \in \widetilde{M}$. By Proposition 20, $a+b$ and a^*+b^* are τ-premeasurable. In particular, they are densely defined. Hence $(a^*+b^*)^*$ exists and $a+b \subseteq (a^*+b^*)^*$, whence $a+b$ is also preclosed. By Proposition 19, $[a+b] \in \widetilde{M}$.

A quite analogous reasoning gives the result on ab .

2) Let $a,b,c \in \widetilde{M}$. Then by Proposition 20 the operators

$$a+b+c \ , \ abc \ , \ ac+bc \ , \ ca+cb \ , \ a^*+b^* \ , \ b^*a^*$$

are all τ-premeasurable. Hence by Proposition 12, each of them admits at most one extension in $\widetilde{M}$. It follows that

$$[[a+b]+c] = [a+[b+c]] \ , \ [[ab]c] = [a[bc]] \ ,$$
$$[[a+b]c] = [[ac]+[bc]] \ , \ [c[a+b]] = [[ca]+[cb]] \ ,$$
$$[a+b]^* = [a^*+b^*] \qquad , \ [ab]^* = [b^*a^*] \ . \ ∎$$

Notation. From now on, we will omit the [ ] in the notation for strong sum and strong product.

Definition 25. For all $\varepsilon, \delta \in \mathbb{R}_+$, we put

$$N(\varepsilon,\delta) = \widetilde{M} \cap D(\varepsilon,\delta) ,$$

i.e. $N(\varepsilon,\delta)$ is the set of τ-measurable $a \in \overline{M}$ for which there exists a projection $p \in M$ such that

$$\|ap\| \le \varepsilon \quad \text{and} \quad \tau(1-p) \le \delta .$$

Lemma 26. For all $\varepsilon, \varepsilon_1, \varepsilon_2, \delta, \delta_1, \delta_2 \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$ we have

(i) $N(\varepsilon,\delta)^* = N(\varepsilon,\delta)$,

(ii) $N(|\lambda|\varepsilon,\delta) = \lambda N(\varepsilon,\delta)$,

(iii) $\varepsilon_1 \le \varepsilon_2 , \delta_1 \le \delta_2 \Rightarrow N(\varepsilon_1,\delta_1) \subseteq N(\varepsilon_2,\delta_2)$,

(iv) $N(\varepsilon_1,\delta_1) \cap N(\varepsilon_2,\delta_2) \supseteq N(\varepsilon_1 \wedge \varepsilon_2, \delta_1 \wedge \delta_2)$,

(v) $N(\varepsilon_1,\delta_1) + N(\varepsilon_2,\delta_2) \subseteq N(\varepsilon_1+\varepsilon_2, \delta_1+\delta_2)$,

(vi) $N(\varepsilon_1,\delta_1) N(\varepsilon_2,\delta_2) \subseteq N(\varepsilon_1\varepsilon_2, \delta_1+\delta_2)$.

Proof. (ii), (iii), (iv) are easily verified. (i) follows from Proposition 8 and (v), (vi) follow from Proposition 5 and Proposition 6, (i) ((v) and (vi) are to be understood in the strong sense). ∎

Proposition 27. The $N(\varepsilon,\delta)$, $\varepsilon,\delta \in \mathbb{R}_+$, form a basis for the neighbourhoods of 0 for a topological vector space topology on $\widetilde{M}$.

Proof. This follows from Lemma 26, (ii), (iii), (iv) and (v). ∎

Theorem 28. $\widetilde{M}$ is a complete Hausdorff topological *-algebra in which M is dense.

Proof. 1) To show that $\widetilde{M}$ is Hausdorff, we shall prove that

$$\bigcap_{\varepsilon,\delta \in \mathbb{R}_+} N(\varepsilon,\delta) = \{0\} .$$

Let $a \in \bigcap_{\varepsilon,\delta\in\mathbb{R}_+} N(\varepsilon,\delta)$. Then

$$\forall \delta \in \mathbb{R}_+ \ \forall \varepsilon \in \mathbb{R}_+ : \ \tau(\chi_{]\varepsilon,\infty[}(|a|)) \le \delta .$$

Since τ is faithful, this implies that all $\chi_{]\varepsilon,\infty[}(|a|) = 0$, whence $a = 0$.

2) Next let us prove that $\widetilde{M}$ is a topological *-algebra. By Lemma 26, (i), the adjoint operation is continuous. Now let $a_0, b_0 \in \widetilde{M}$ and let $\varepsilon, \delta \in \mathbb{R}_+$. Take $\mu, \lambda \in \mathbb{R}_+$ such that

$$a_0 \in N(\mu,\delta) , \quad b_0 \in N(\lambda,\delta) .$$

Then for all $a,b \in \widetilde{M}$ such that $a - a_0 \in N(\varepsilon,\delta)$ and $b - b_0 \in N(\varepsilon,\delta)$, we have

$$\begin{aligned} ab - a_0b_0 &= (a-a_0)(b-b_0) + a_0(b-b_0) + (a-a_0)b_0 \\ &\in N(\varepsilon,\delta)N(\varepsilon,\delta) + N(\mu,\delta)N(\varepsilon,\delta) + N(\varepsilon,\delta)N(\lambda,\delta) \\ &\subseteq N(\varepsilon^2,2\delta) + N(\mu\varepsilon,2\delta) + N(\lambda\varepsilon,2\delta) \\ &\subseteq N(\varepsilon(\varepsilon+\lambda+\mu),6\delta) . \end{aligned}$$

It follows that

$$(a,b) \mapsto ab$$

is continuous.

3) M is dense in $\tilde{M}$. Indeed, let $a \in \tilde{M}$ and take projections $p_n \in M$ such that

$$p_n \nearrow 1 \ , \ \tau(1-p_n) \to 0 \ , \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} p_n H \subseteq D(a)$$

(possible by Proposition 10). Then $ap_n \in M$ and

$$ap_n \to a \quad \text{in} \quad \tilde{M}$$

since $\|(ap_n - a)p_m\| = 0$ for all $m \geq n$ and $\tau(1-p_m) \to 0$ as $m \to \infty$.

4) Finally, we shall prove that the topological vector space $\tilde{M}$ is complete.

Since $\tilde{M}$ has a countable basis for the neighbourhoods of 0 (use e.g. the $N(1/n,1/m)$, $n,m \in \mathbb{N}$), we just have to show that every Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $\tilde{M}$ converges. So let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\tilde{M}$.

Since M is dense in $\tilde{M}$, we may assume that all $a_n \in M$ (if not, replace each a_n by $a_n' \in M$ such that $a_n - a_n' \in N(1/n,1/n)$, and observe that $(a_n')_{n\in\mathbb{N}}$ converges if and only if $(a_n)_{n\in\mathbb{N}}$ converges). Furthermore, we may assume that

$$\forall n \in \mathbb{N}: \ a_{n+1} - a_n \in N(2^{-(n+1)}, 2^{-n})$$

(since a subsequence of the given sequence has this property).

Now take projections $p_n \in M$ such that

$$\|(a_{n+1}-a_n)p_n\| \leq 2^{-(n+1)} \quad \text{and} \quad \tau(1-p_n) \leq 2^{-n} \ .$$

For each $n \in \mathbb{N}$, put

$$q_n = \bigwedge_{k=n+1}^{\infty} p_k \ .$$

Then

$$\tau(1-q_n) = \tau\Big(\bigvee_{k=n-1}^{\infty} (1-p_k)\Big) \leq \sum_{k=n+1}^{\infty} \tau(1-p_k) \leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n} \ ;$$

and

$$\forall m \geq n+1 \quad \forall \ell \in \mathbb{N}: \ \|(a_{m+\ell} - a_m)q_n\| \leq 2^{-m} \tag{14}$$

since $q_n \leq p_k$ for all $k \geq m \geq n+1$ and hence

$$\|(a_{m+\ell} - a_m)q_n\| \leq \sum_{k=m}^{m+\ell-1} \|(a_{k+1} - a_k)q_n\|$$
$$\leq \sum_{k=m}^{m+\ell-1} \|(a_{k+1} - a_k)p_k\| \leq \sum_{k=m}^{m+\ell-1} 2^{-(k+1)} \leq 2^{-m} \ .$$

Let $\xi \in \bigcup_{n\in\mathbb{N}} q_n H$. Then $\xi \in q_n H$ for some $n \in \mathbb{N}$ and hence by (14), the sequence $(a_m \xi)_{m\in\mathbb{N}}$ is Cauchy. Put

$$a\xi = \lim_{m\to\infty} a_m \xi \ .$$

We have now defined an operator a with $D(a) = \bigcup_{n\in\mathbb{N}} q_n H$ (note that $D(a)$ is a linear subspace because $(q_n)_{n\in\mathbb{N}}$ is an increasing sequence of projections).

By construction, a is τ-premeasurable: for all $n \in \mathbb{N}$, we have $q_n H \subseteq D(a)$ and $\tau(1-q_n) \leq 2^{-n}$. We claim that a is also preclosed. To see this, apply the preceding arguments to $(a_n^*)_{n\in\mathbb{N}}$. Hence there exists a τ-premeasurable operator b such that

$$b\eta = \lim_{m\to\infty} a_n^* \eta \ , \ \eta \in D(b) \ .$$

Then

$$\forall \xi \in D(a) \ \forall \eta \in D(b): \ (a\xi|\eta) = \lim(a_m\xi|\eta) = \lim(\xi|a_m^*\eta) = (\xi|b\eta) \ ,$$

whence

$$a \subseteq b^* .$$

Hence a is preclosed. By Proposition 19 we then have $[a] \in \widetilde{M}$. Write $a_0 = [a]$.

Finally we shall prove that actually

$$a_n \to a_0 \quad \text{in } \widetilde{M} . \tag{15}$$

Let $\varepsilon, \delta \in \mathbb{R}_+$. Take $n_0 \in \mathbb{N}$ such that $2^{-(n_0+1)} \leq \varepsilon$ and $2^{-n_0} \leq \delta$. Then for all $m \geq n_0+1$ we have

$$\|(a_0 - a_m)q_{n_0}\| \leq 2^{-(n_0+1)} \leq \varepsilon$$

and

$$\tau(1-q_{n_0}) \leq 2^{-n_0} \leq \delta$$

since

$$\forall \xi \in H: \ (a_0 - a_m)q_{n_0}\xi = \lim_{\ell\to\infty}(a_{m+\ell} - a_m)q_{n_0}\xi$$

and

$$\|(a_{m+\ell} - a_m)q_{n_0}\| \leq 2^{-m} \leq 2^{-(n_0+1)} \leq \varepsilon .$$

Hence

$$\forall m \geq n_0 + 1: \ a_0 - a_m \in N(\varepsilon, \delta) .$$

This proves (15). ∎

Examples. 1) If τ is finite, then $\widetilde{M} = \overline{M}$, i.e. all closed densely defined operators affiliated with M are τ-measurable (by Proposition 21, (vi)).

2) If $M = B(H)$ and τ is the usual trace Tr, then $\widetilde{M} = M$ (by Proposition 21, (iv), and the fact that $\mathrm{Tr}(x) < 1$, $x \geq 0$, implies $x = 0$).

3) If (X,μ) is a measure space, $M = L^\infty(X,\mu)$ and $\tau = \int \cdot\, d\mu$, then $\widetilde{M}$ is the closure of $L^\infty(X,\mu)$ for the topology of convergence in measure. •

L^p spaces with respect to a trace. For any positive self-adjoint operator a affiliated with M, we put

$$\tau(a) = \sup_{n\in\mathbb{N}} \tau\left(\int_0^n \lambda \, de_\lambda\right)$$

where

$$a = \int_0^\infty \lambda \, de_\lambda$$

is the spectral representation of a. Then for each $p \in [1,\infty[$, we can define

$$L^p(M,\tau) = \{a \in \overline{M} \mid \tau(|a|^p) < \infty\}$$

and

$$\|a\|_p = \tau(|a|^p)^{1/p} , \quad a \in L^p(M,\tau) .$$

The $(L^p(M, \ ), \|\cdot\|_p)$ are Banach spaces in which $I = \{x \in M \mid \tau(|x|) < \infty\}$ is dense, and they are all contained in (and even continuously embedded in) $\widetilde{M}$ (for this and further results, see [13]; see also [3], [12], [21], and Chapter IV).

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Notes and comments. The notion of measurable operators was introduced by I. E. Segal [15] and formed the basis for investigations in non-commutative integration theory, i.e. a theory of "integration" where $L^\infty(X,\mu)$ (corresponding to a measure space (X,μ)) is replaced by a more general von Neumann algebra. Among other things, this theory provided a framework for constructing L^p spaces associated with (semifinite) von Neumann algebras as concrete spaces of (closed densely defined) operators ([12], [21]) (isomorphic to J. Dixmier's "abstract" L^p spaces [3]).

In [13], E. Nelson gave a new approach - requiring less knowledge of von Neumann algebra techniques - to the theory, based on the notion of measurability with respect to a trace (inspired by the notion of convergence in measure introduced by W. F. Stinespring in [16]). Any τ-measurable operator is also measurable in the sense of [15, Definition 2.1], whereas the converse is not in general true. The set of τ-measurable operators is, however, big enough to contain the L^p spaces with respect to τ.

In our presentation, we have followed [13] with some modifications. In [13], $\tilde{M}$ is defined as the (abstract) completion of M with respect to a certain (measure) topology on M (given by the 0-neighbourhoods $N(\varepsilon,\delta) \cap M$, there simply called $N(\varepsilon,\delta)$); afterwards, $\tilde{M}$ is identified with a subset of the closed densely defined operators affiliated with M. As a tool, the completion of the Hilbert space H with respect to a certain (measure) topology is considered. - We have preferred to work with operators on H right from the beginning and to introduce the measure topology directly on the whole of $\tilde{M}$. When doing so, we do not need a new topology on H.

II

L^p SPACES ASSOCIATED WITH A VON NEUMANN ALGEBRA

In this chapter, we present Haagerup's theory of L^p spaces associated with a von Neumann algebra.

Let M be a von Neumann algebra and let φ_0 be a normal faithful semifinite weight on M.

We denote by N the crossed product $R(M,\sigma^{\varphi_0})$ of M by the modular automorphism group σ^{φ_0} associated with φ_0. Recall [18, Section 3; 8, Section 5] that if M is given on a Hilbert space H, then N is the von Neumann algebra on the Hilbert space $L^2(\mathbb{R},H)$ generated by the operators $\pi(x)$, $x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$(\pi(x)\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t) \ , \quad \xi \in L^2(\mathbb{R},H) \ , \ t \in \mathbb{R} \ , \qquad (1)$$

$$(\lambda(s)\xi)(t) = \xi(t-s) \ , \qquad \xi \in L^2(\mathbb{R},H) \ , \ t \in \mathbb{R} \ . \qquad (2)$$

We identify M with its image $\pi(M)$ in N (recall that π is a normal faithful representation of M).

We denote by θ the dual action of $\mathbb{R}$ in N. The θ_s, $s \in \mathbb{R}$, are automorphisms of N characterized by

$$\theta_s x = x \ , \qquad x \in M \qquad (3)$$

$$\theta_s \lambda(t) = e^{-ist}\lambda(t) \ , \ t \in \mathbb{R} \ . \qquad (4)$$

By (3), M is contained in the set of fixed points under θ. Actually

$$M = \{y \in N \mid \forall s \in \mathbb{R}: \theta_s y = y\} \qquad (5)$$

(see e.g. [5, Lemma 3.6]).

The θ_s, $s \in \mathbb{R}$, naturally extend to automorphisms, still denoted θ_s, of $\hat{N}_+$, the extended positive part of N [7, Section 1].

Recall [8, Lemma 5.2] that the formula

$$Tx = \int_{\mathbb{R}} \theta_s(x)ds \ , \ x \in N_+ \ , \tag{6}$$

defines a normal faithful semifinite operator valued weight T from N to M in the following sense: for each $x \in N_+$, Tx is the element of $\hat{N}_+$ characterized by

$$\langle Tx, \chi \rangle = \int_{\mathbb{R}} \langle \theta_s(x), \chi \rangle \ ds \tag{7}$$

for all $\chi \in N_*^+$. Note that

$$\forall s \in \mathbb{R}: \theta_s \circ T = T \ . \tag{8}$$

In view of (5), this formula implies that the values of T are actually in $\hat{M}_+$.

For each normal weight φ on M, we put

$$\tilde{\varphi} = \hat{\varphi} \circ T \tag{9}$$

where $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{M}_+$ as described in [7, Proposition 1.10]. Then $\tilde{\varphi}$ is a normal weight on N [7, Proposition 2.3]; $\tilde{\varphi}$ is called the dual weight of φ (see [6, Introduction + Section 1]). Note that (8) and (9) imply

$$\forall s \in \mathbb{R}: \tilde{\varphi} \circ \theta_s = \tilde{\varphi} \ . \tag{10}$$

If φ and ψ are normal faithful semifinite weights, then so are $\tilde{\varphi}$ and $\tilde{\psi}$, and we have [7, Theorem 4.7]:

$$\forall t \in \mathbb{R} \ \forall x \in M: \sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x) \ , \tag{11}$$

$$\forall t \in \mathbb{R}: \ (D\tilde{\varphi}:D\tilde{\psi})_t = (D\varphi:D\psi)_t \ . \tag{12}$$

<u>Lemma 1</u>. 1) The mapping

$$\varphi \mapsto \tilde{\varphi}$$

is a bijection of the set of all normal semifinite weights on M onto the set of normal semifinite weights ψ on N satisfying

$$\forall s \in \mathbb{R}: \psi \circ \theta_s = \psi \ . \tag{13}$$

2) For all normal weights φ and ψ on M and all $x \in M$, we have

(a) $(\varphi+\psi)^\sim = \tilde{\varphi} + \tilde{\psi}$,

(b) $(x\cdot\varphi\cdot x^*)^\sim = x \cdot \tilde{\varphi} \cdot x^*$,

(c) $\operatorname{supp} \tilde{\varphi} = \operatorname{supp} \varphi$.

<u>Proof</u>. That $\tilde{\varphi}$ is semifinite if φ is follows from the proof of [7, Proposition 2.3]. That $\varphi \mapsto \tilde{\varphi}$ is injective follows from the formula

$$\varphi(\dot{T}x) = \tilde{\varphi}(x) \ , \quad x \in m_T \ ,$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

Now let us prove 2). First observe that $(\varphi+\psi)^\wedge = \hat{\varphi} + \hat{\psi}$ since $\hat{\varphi} + \hat{\psi}: \hat{M}_+ \to [0,\infty]$ obviously satisfies the properties that characterize $(\varphi+\psi)^\wedge$ ([7, Proposition 1.10]); (a) follows trivially. Similarly, $(x\cdot\varphi\cdot x^*)^\wedge = x \cdot \hat{\varphi} \cdot x^*$, whence (b).

To prove (c), put $p_0 = 1 - \mathrm{supp}\,\varphi$. Then Mp_0 is the σ-weak closure in M of $N_\varphi = \{x \in M \mid \varphi(x^*x) = 0\}$. Similarly, the σ-weak closure in N of $N_{\tilde\varphi} = \{y \in N \mid \tilde\varphi(y^*y) = 0\}$ is Nq_0 where $q_0 = 1 - \mathrm{supp}\,\tilde\varphi$. Now

$$n_T\, N_\varphi \subseteq N_{\tilde\varphi}$$

since

$$\forall y \in n_T\ \forall x \in N_\varphi:\ \tilde\varphi(x^*y^*yx) = \varphi(T(x^*y^*yx))$$

$$= \varphi(x^*T(y^*y)x) \le \|T(y^*y)\|\ \varphi(x^*x) = 0\ .$$

As n_T is σ-weakly dense in N, it follows that

$$N_\varphi \subseteq \overline{N_{\tilde\varphi}}^{\sigma\text{-}w}$$

whence

$$p_0 \le q_0\ .$$

Note that we must have $q_0 \in M$ since $\tilde\varphi$, and hence $\mathrm{supp}\,\tilde\varphi$, is θ-invariant. Thus to conclude that $p_0 = q_0$ we need only show that $\varphi(q_0) = 0$. This follows from

$$\forall x \in m_T:\ \varphi(q_0\dot T(x)q_0) = \varphi(\dot T(q_0xq_0)) = \tilde\varphi(q_0xq_0) = 0$$

and the fact that $\dot T(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

We now return to 1). Let ψ be a normal semifinite weight on N satisfying (13). First suppose that ψ is also faithful. Then by [5, (proof of) Theorem 3.7], it follows that $\psi = \tilde\varphi$ for some normal faithful semifinite φ on M.

In the general case, put $q_0 = 1 - \mathrm{supp}\,\psi$. Then by (13) and (5), we have $q_0 \in M$. Now take any normal semifinite weight χ_0 on M such that $\mathrm{supp}\,\chi_0 = q_0$. Then $\tilde\chi_0$ is a normal faithful semifinite θ-invariant weight on N with $\mathrm{supp}\,\tilde\chi_0 = q_0$. Hence $\tilde\chi_0 + \psi$ is faithful and thus, as above,

$$\tilde\chi_0 + \psi = \tilde\varphi$$

for some normal faithful semifinite weight φ on M. Finally, using (b), we find that

$$\psi = (1-q_0)\cdot(\tilde\chi_0+\psi)\cdot(1-q_0)$$

$$= (1-q_0)\cdot\tilde\varphi\cdot(1-q_0)$$

$$= ((1-q_0)\cdot\varphi\cdot(1-q_0))^\sim\ . \quad \blacksquare$$

Denote by τ the normal faithful semifinite trace on N characterized by

$$\forall t \in \mathbb{R}:\ (D\tilde\varphi_0 : D\tau)_t = \lambda(t) \tag{14}$$

(for the existence, see [8, Lemma 5.2]); τ satisfies

$$\forall s \in \mathbb{R}:\ \tau\circ\theta_s = e^{-s}\tau\ . \tag{15}$$

With each $h \in \hat N_+$ we associate the normal weight $\tau(h\ \cdot)$ on N as in [8, remarks preceding Proposition 1.11]. When h is simply a positive self-adjoint operator affiliated with N (see [7, Example 1.2]), this definition agrees with that given in [14, Section 4].

We recall some facts about the mapping $h \mapsto \tau(h\ \cdot)$ (see [7, Theorem 1.12 (and its proof) and Proposition 1.11, (4)]):

<u>Lemma 2</u>. 1) The mapping

$$h \mapsto \tau(h\ \cdot)$$

is a bijection of $\hat{N}_+$ onto the set of normal weights on N. In particular, it is a bijection of the positive self-adjoint operators affiliated with N onto the normal semifinite weights on N.

2) For all $h,k \in \hat{N}_+$ and all $x \in N$, we have

(a) $\tau((h\dot{+}k)\ \cdot) = \tau(h\ \cdot) + \tau(k\ \cdot)$,

(b) $\tau((x\cdot h\cdot x^*)\ \cdot) = x\cdot\tau(h\ \cdot)\cdot x^*$,

(c) $\operatorname{supp}\tau(h\ \cdot) = \operatorname{supp} h$.

Here, $h \dot{+} k$ and $x\cdot h\cdot x^*$ denote the operations in $\hat{N}_+$ introduced in [7, Definition 1.3]. If h and k are positive self-adjoint operators such that $D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ is dense, then $h \dot{+} k$ is the simply the <u>form sum</u> of h and k [2, Corollary 4.13]. If h is a positive self-adjoint operator and x a bounded operator such that $D(h^{\frac{1}{2}}x^*)$ is dense, then $x\cdot h\cdot x^* = |h^{\frac{1}{2}}x^*|^2$.

<u>Definition 3</u>. For each normal weight φ on M we define h_φ as the unique element of $\hat{N}_+$ given by

$$\tilde{\varphi} = \tau(h_\varphi\ \cdot) \quad . \tag{16}$$

<u>Proposition 4</u>. 1) The mapping

$$\varphi \mapsto h_\varphi$$

is a bijection of the set of all normal semifinite weights on M onto the set of all positive self-adjoint operators h affiliated with N satisfying

$$\forall s \in \mathbb{R}: \theta_s h = e^{-s}h \ . \tag{17}$$

2) For all normal weights φ and ψ on M and all $x \in M$, we have

(a) $h_{\varphi+\psi} = h_\varphi \dot{+} h_\psi$,

(b) $h_{x\cdot\varphi\cdot x^*} = x\cdot h_\varphi\cdot x^*$,

(c) $\operatorname{supp} h_\varphi = \operatorname{supp}\varphi$.

<u>Proof</u>. This proposition is an immediate consequence of Lemma 1 and 2. We just need to prove that a positive self-adjoint operator h affiliated with N satisfies (17) if and only if the corresponding weight $\tau(h\ \cdot)$ is θ-invariant. This follows easily from (15). Indeed, for all $s \in \mathbb{R}$ we have

$$\tau(e^s\theta_s(h)\ \cdot) = e^s(\tau\circ\theta_s)(h\theta_{-s}(\cdot)) = \tau(h\theta_{-s}(\cdot)) = \tau(h\ \cdot)\circ\theta_{-s} \ ,$$

whence

$$e^s\theta_s(h) = h \iff \tau(e^s\theta_s(h)\ \cdot) = \tau(h\ \cdot)$$

$$\iff \tau(h\ \cdot) = \tau(h\ \cdot)\circ\theta_{-s} \ .$$

The equivalence of (17) and

$$\forall s \in \mathbb{R}: \tau(h\ \cdot) = \tau(h\ \cdot)\circ\theta_s$$

follows. ∎

The next lemma is essential. It will permit us apply results on τ-measurable operators.

As usual, $\chi_{]\gamma,\infty[}$ denotes the characteristic function for the interval $]\gamma,\infty[$.

<u>Lemma 5</u>. Let φ be a normal semifinite weight on M. Then for all $\gamma \in \mathbb{R}_+$, we have

$$\tau(\chi_{]\gamma,\infty[}(h_\varphi)) = \frac{1}{\gamma}\ \varphi(1)\ .$$

<u>Proof</u>. First let us prove the formula in the case $\gamma = 1$.

Let $s \in \mathbb{R}$. Then since θ_s is an automorphism and $\theta_s\, h_\varphi = e^{-s}\, h_\varphi$ we have

$$\theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi)) = e^s\, h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi)\ .$$

Now let $h_\varphi = \int \lambda\, de_\lambda$ be the spectral decomposition of h_φ. Then for any vector functional $\omega_{\xi,\xi}$, where ξ is a unit vector, we have

$$\begin{aligned}\langle\int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))ds, \omega_{\xi,\xi}\rangle &= \int_{\mathbb{R}} \langle e^s h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi), \omega_{\xi,\xi}\rangle\, ds \\ &= \int_{\mathbb{R}}\int_{]0,\infty[} e^s\lambda^{-1}\chi_{]1,\infty[}(e^{-s}\lambda)\, d(e_\lambda\xi|\xi)\, ds \\ &= \int_{]0,\infty[}\lambda^{-1}\Big(\int_{]-\infty,\log\lambda[} e^s ds\Big) d(e_\lambda\xi|\xi) \\ &= \int_{]0,\infty[}\lambda^{-1}\lambda\ d(e_\lambda\xi|\xi) \\ &= \|(\mathrm{supp}\ h_\varphi)\xi\|^2\end{aligned}$$

so that

$$\int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))ds = \mathrm{supp}\ h_\varphi = \mathrm{supp}\ \varphi\ .$$

Finally, since $\tilde{\varphi} = \tau(h_\varphi\ \cdot)$ we have

$$\begin{aligned}\tau(\chi_{]1,\infty[}(h_\varphi)) &= \tau(h_\varphi^{\frac12}(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))h_\varphi^{\frac12}) \\ &= \tilde{\varphi}(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi)) \\ &= \varphi\Big(\int \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))ds\Big) = \varphi(\mathrm{supp}\ \varphi) = \varphi(1)\ .\end{aligned}$$

This completes the proof in the case $\gamma = 1$. In the general case, write $\gamma = e^s$, $s \in \mathbb{R}$. Then by (15)

$$\begin{aligned}\tau(\chi_{]e^s,\infty[}(h_\varphi)) &= \tau(\chi_{]1,\infty[}(e^{-s}h_\varphi)) \\ &= \tau(\theta_s(\chi_{]1,\infty[}(h_\varphi))) \\ &= e^{-s}\ \tau(\chi_{]1,\infty[}(h_\varphi)) = e^{-s}\ \varphi(1)\ . \ \blacksquare\end{aligned}$$

By Chapter I, Proposition 21, we have

<u>Corollary 6</u>. Let φ be a normal semifinite weight on M. Then h_φ is τ-measurable if and only if $\varphi \in M_*$.

We denote by $\tilde{N}$ the set of all τ-measurable closed densely defined operators affiliated with N. Recall (Chapter I) that $\tilde{N}$ is a topological *-algebra with respect to strong sum and product. Sums and products of elements in $\tilde{N}$ will always be understood to be in the strong sense although we do not emphasize it in the notation.

We denote by $\tilde{N}_+$ the subset of all positive self-adjoint elements of $\tilde{N}$.

Note that the θ_s, $s \in \mathbb{R}$, extend to continuous *-automorphisms, still denoted θ_s, of $\tilde{N}$. We have

$$\forall s \in \mathbb{R}\ \forall \varepsilon, \delta \in \mathbb{R}_+ : \theta_s(N(\varepsilon,\delta)) = N(\varepsilon, e^{-s}\delta) \qquad (18)$$

since for all $a \in \widetilde{N}_+$

$$\tau(\chi_{]\varepsilon,\infty[}(\theta_s a)) = \tau(\theta_s(\chi_{]\varepsilon,\infty[}(a))) = e^{-s}\tau(\chi_{]\varepsilon,\infty[}(a))$$

(for the definition and properties of the 0-neighbourhoods $N(\varepsilon,\delta)$, we refer to Chapter I).

Theorem 7. 1) The mapping

$$\varphi \mapsto h_\varphi$$

extends to a linear bijection, still denoted $\varphi \mapsto h_\varphi$, of M_* onto the subspace

$$\{h \in \widetilde{N} \mid \forall s \in \mathbb{R}: \theta_s h = e^{-s}h\} \tag{19}$$

of N.

2) For all $\varphi \in M_*$ and $x,y \in M$, we have

$$h_{x\cdot\varphi\cdot y^*} = x\, h_\varphi\, y^* \tag{20}$$

and

$$h_{\varphi^*} = h_\varphi{}^* . \tag{21}$$

3) If $\varphi = u|\varphi|$ is the polar decomposition of φ, then $h_\varphi = uh_{|\varphi|}$ is the polar decomposition of h_φ. In particular,

$$|h_\varphi| = h_{|\varphi|} . \tag{22}$$

The proof will be based on Corollary 6, Proposition 4, and the following lemma.

Lemma 8. 1) Let h and k be positive self-adjoint operators such that $D(h^{\frac12}) \cap D(k^{\frac12})$ is dense. Then

$$h + k \subseteq h \dot{+} k .$$

If $h+k$ is essentially self-adjoint, then its unique self-adjoint extension is precisely $h \dot{+} k$.

2) Let h be a positive self-adjoint operator and x a bounded operator such that $D(h^{\frac12}x^*)$ is dense. Then

$$xhx^* \subseteq x \cdot h \cdot x^* .$$

If xhx^* is essentially self-adjoint, then its unique self-adjoint extension is precisely $x \cdot h \cdot x^*$.

Proof. 1) Recall that by definition $h \dot{+} k$ is the unique positive self-adjoint operator characterized by $D((h\dot{+}k)^{\frac12}) = D(h^{\frac12}) \cap D(k^{\frac12})$ and

$$\forall \xi \in D(h^{\frac12}) \cap D(k^{\frac12}):\ \|(h\dot{+}k)^{\frac12}\xi\|^2 = \|h^{\frac12}\xi\|^2 + \|k^{\frac12}\xi\|^2 . \tag{23}$$

By polarization, it follows that

$$\forall \xi,\eta \in D(h^{\frac12}) \cap D(k^{\frac12}):\ ((h\dot{+}k)^{\frac12}\xi \mid (h\dot{+}k)^{\frac12}\eta) = (h^{\frac12}\xi \mid h^{\frac12}\eta) + (k^{\frac12}\xi \mid k^{\frac12}\eta) .$$

Now let $\xi \in D(h+k) = D(h) \cap D(k)$ and $\eta \in D(h\dot{+}k)$. Then also $\xi \in D(h^{\frac12}) \cap D(k^{\frac12})$ and $\eta \in D((h\dot{+}k)^{\frac12}) = D(h^{\frac12}) \cap D(k^{\frac12})$ so that

$$\begin{aligned} ((h+k)\xi \mid \eta) &= (h\xi \mid \eta) + (k\xi \mid \eta) \\ &= (h^{\frac12}\xi \mid h^{\frac12}\eta) + (k^{\frac12}\xi \mid k^{\frac12}\eta) \\ &= ((h\dot{+}k)^{\frac12}\xi \mid (h\dot{+}k)^{\frac12}\eta) \\ &= (\xi \mid (h\dot{+}k)\eta) . \end{aligned}$$

It follows that

$$h + k \subseteq (h\dot{+}k)^* = h \dot{+} k .$$

Hence $h+k$ is preclosed and $[h+k] \subseteq h \dot{+} k$. If $[h+k]$ is self-adjoint, we must have $[h+k] = h \dot{+} k$.

2) Recall that $x \cdot h \cdot x^* = |h^{\frac{1}{2}}x^*|^2$. Now let $\xi \in D(xhx^*) = D(hx^*)$ and $\eta \in D(x \cdot h \cdot x^*)$. Then also $\xi \in D(h^{\frac{1}{2}}x^*)$ and $\eta \in D((x \cdot h \cdot x^*)^{\frac{1}{2}}) = D(h^{\frac{1}{2}}x^*)$ so that

$$(xhx^*\xi|\eta) = (hx^*\xi|x^*\eta) = (h^{\frac{1}{2}}x^*\xi|h^{\frac{1}{2}}x^*\eta) = (\xi|(x \cdot h \cdot x^*)\eta) .$$

It follows that

$$xhx^* \subseteq (x \cdot h \cdot x^*)^* = x \cdot h \cdot x^* .$$

Hence xhx^* is preclosed and $[xhx^*] \subseteq x \cdot h \cdot x^*$. If $[xhx^*]$ is self-adjoint, we must have $[xhx^*] = x \cdot h \cdot x^*$. ∎

<u>Proof</u> of Theorem 7. Let $\varphi, \psi \in M_*^+$. Then h_φ and h_ψ are positive self-adjoint and τ-measurable so that their strong sum exists and is again a positive self-adjoint τ-measurable operator. By Lemma 8, this sum then coincides with $h_\varphi \dot{+} h_\psi$. Then Proposition 4 yields

$$h_{\varphi+\psi} = h_\varphi + h_\psi ,$$

where the sum at the right hand side is now the sum in $\tilde{N}$.

Similarly for all $\varphi \in M_*^+$ and $x \in M$ we get

$$h_{x \cdot \varphi \cdot x^*} = xh_\varphi x^* . \qquad (24)$$

Now the additive and homogeneous mapping $\varphi \mapsto h_\varphi$ of M_*^+ onto $\{h \in \tilde{N}_+ \mid \forall s \in \mathbb{R}: \theta_s h = e^{-s}h\}$ extends by linearity to a linear mapping $\varphi \mapsto h_\varphi$ of M_* onto the subspace of $\tilde{N}$ spanned by $\{h \in \tilde{N}_+ \mid \forall s \in \mathbb{R}: \theta_s h = e^{-s}h\}$, i.e. onto the subspace (19) (evidently, (19) is stable under $h \mapsto h^*$ and $h \mapsto |h|$ and hence spanned by its positive elements).

By linearity, we must have (21) for all $\varphi \in M_*$. Also by linearity, (24) holds for all $\varphi \in M_*$ and $x \in M$; by polarization the equation (20) follows for all $\varphi \in M_*$ and $x,y \in M$.

In particular, if $\varphi = u|\varphi|$ is the polar decomposition of φ , we have

$$h_\varphi = h_{u|\varphi|} = uh_{|\varphi|} .$$

That this relation is the polar decomposition of h_φ follows from the fact that the initial projection for the partial isometry u is $\operatorname{supp} |\varphi| = \operatorname{supp} h_{|\varphi|}$.

Finally, $\varphi \mapsto h_\varphi$ is injective: if $h_\varphi = 0$, then $h_{|\varphi|} = |h_\varphi| = 0$, whence $|\varphi| = 0$ and $\varphi = 0$. ∎

Motivated by Theorem 7, we now give the following definition:

<u>Definition 9</u>. For each $p \in [1,\infty]$, we let

$$L^p(M) = \{a \in \tilde{N} \mid \forall s \in \mathbb{R}: \theta_s a = e^{-s/p}a\} .$$

Note that the $L^p(M)$ are linear subspaces of $\tilde{N}$ and that they are linearly spanned by their positive part $L^p(M)_+ = L^p(M) \cap \tilde{N}_+$.

By Theorem 7, we know that $L^1(M) \simeq M_*$. And:

<u>Proposition 10</u>. We have $L^\infty(M) = M$.

<u>Proof</u>. In view of (5), we just need to show that every $a \in L^\infty(M)$ is bounded. Let $a \in L^\infty(M)$. Then for all $s \in \mathbb{R}$ and all

$\lambda \in \mathbb{R}_+$ we have

$$\tau(\chi_{]\lambda,\infty[}(|a|)) = \tau(\chi_{]\lambda,\infty[}(\theta_s|a|))$$
$$= \tau(\theta_s(\chi_{]\lambda,\infty[}(|a|))) = e^{-s}\tau(\chi_{]\lambda,\infty[}(|a|)) .$$

Hence for all $\lambda \in \mathbb{R}_+$ we must have

$$\tau(\chi_{]\lambda,\infty[}(|a|)) = 0 \quad \text{or} \quad \tau(\chi_{]\lambda,\infty[}(|a|)) = \infty .$$

Since a is τ-measurable, we have $\tau(\chi_{]\lambda,\infty[}(|a|)) < \infty$ for some λ. Hence $\tau(\chi_{]\lambda,\infty[}(|a|)) = 0$ and thus $\chi_{]\lambda,\infty[}(|a|) = 0$ since τ is faithful. This means that a is bounded. ∎

Remark 11. We have seen that all elements of $L^\infty(M)$ are bounded. In contrast to this, all non-zero elements of $L^p(M)$, where $p < \infty$, are unbounded. To see this, let $a \in L^p(M)$ and suppose that $a \neq 0$. Then for some $\lambda \in \mathbb{R}_+$ we have $\chi_{]\lambda,\infty[}(|a|) \neq 0$ and hence $\tau(\chi_{]\lambda,\infty[}(|a|)) \neq 0$. Then for all $\mu \in \mathbb{R}_+$ we have

$$\tau(\chi_{]\mu,\infty[}(|a|)) \neq 0$$

since for all $s \in \mathbb{R}$

$$\tau(\chi_{]e^{s/p}\lambda,\infty[}(|a|)) = \tau(\chi_{]\lambda,\infty[}(e^{-s/p}|a|))$$
$$= \tau(\chi_{]\lambda,\infty[}(\theta_s|a|))$$
$$= \tau(\theta_s\, \chi_{]\lambda,\infty[}(|a|))$$
$$= e^{-s}\, \tau(\chi_{]\lambda,\infty[}(|a|)) \neq 0 .$$

It follows that $|a|$ must be unbounded.

Proposition 12. Let a be a closed densely defined operator affiliated with N with polar decomposition $a = u|a|$. Let $p \in [1,\infty[$. Then

$$a \in L^p(M)$$

if and only if

$$u \in M \quad \text{and} \quad |a|^p \in L^1(M) .$$

Proof. Recall that $a \in \tilde{N}$ if and only if $|a| \in \tilde{N}$. Furthermore, $|a| \in \tilde{N}$ if and only if $|a|^p \in \tilde{N}$ since $\tau(\chi_{]\lambda,\infty[}(|a|)) = \tau(\chi_{]\lambda^p,\infty[}(|a|^p))$ for all $\lambda \in \mathbb{R}_+$. For all such a and all $s \in \mathbb{R}$ we have

$$\theta_s a = e^{-s/p}a \Leftrightarrow \theta_s u = u \quad \text{and} \quad \theta_s|a|^p = e^{-s}|a|^p .$$

The result follows by Definition 9 and Proposition 10. ∎

A similar result holds for the right polar decomposition.

Definition 13. We define a linear functional tr on $L^1(M)$ by

$$\mathrm{tr}(h_\varphi) = \varphi(1) , \ \varphi \in M_* .$$

Note that

$$\mathrm{tr}(|h_\varphi|) = \mathrm{tr}(h_{|\varphi|}) = |\varphi|(1) = \|\varphi\| \tag{25}$$

for all $\varphi \in M_*$. This implies that

$$|\mathrm{tr}(a)| \leq \mathrm{tr}(|a|) \tag{26}$$

for all $a \in L^1(M)$ and that the mapping $a \mapsto \mathrm{tr}(|a|)$ is a norm on $L^1(M)$.

<u>Definition 14</u>. Let $p \in [1,\infty[$. Then we define $\|\cdot\|_p$ on $L^p(M)$ by

$$\|a\|_p = \mathrm{tr}(|a|^p)^{1/p} \ , \quad a \in L^p(M) \ .$$

For $p = \infty$, we put

$$\|a\|_\infty = \|a\| \ , \qquad a \in L^\infty(M) \ .$$

We shall see that for all p , $\|\cdot\|_p$ is a norm on $L^p(M)$.

By (26), we have

<u>Proposition 15</u>. The mapping

$$\varphi \mapsto h_\varphi : M_* \to L^1(M)$$

is an isometry of M_* onto $L^1(M)$.

<u>Lemma 16</u>. Let $p \in [1,\infty[$ and $\varepsilon,\delta \in \mathbb{R}_+$. Then

$$N(\varepsilon,\delta) \cap L^p(M) = \{a \in L^p(M) \mid \|a\|_p \leq \varepsilon\delta^{1/p}\} \ .$$

<u>Proof</u>. Let $a \in L^p(M)$. Then $|a|^p \in L^1(M)_+$ and hence $|a|^p = h_\varphi$ for some $\varphi \in M_*^+$. Now

$$\begin{aligned}\tau(\chi_{]\varepsilon,\infty[}(|a|)) &= \tau(\chi_{]\varepsilon^p,\infty[}(|a|^p)) \\ &= \frac{1}{\varepsilon^p}\varphi(1) \\ &= \frac{1}{\varepsilon^p}\||a|^p\|_1 = \frac{1}{\varepsilon^p}\|a\|_p^p \ .\end{aligned}$$

Using this we get

$$\begin{aligned} a \in N(\varepsilon,\delta) &\Leftrightarrow |a| \in N(\varepsilon,\delta) \\ &\Leftrightarrow \tau(\chi_{]\varepsilon,\infty[}(|a|)) \leq \delta \\ &\Leftrightarrow \frac{1}{\varepsilon^p}\|a\|_p^p \leq \delta \\ &\Leftrightarrow \|a\|_p \leq \varepsilon\delta^{1/p} \ . \ \blacksquare \end{aligned}$$

<u>Corollary 17</u>. On $L^1(M)$ the norm topology is exactly the topology induced from $\tilde{N}$.

We denote by $\mathbb{C}_+$ the closed half-plane $\{\alpha \in \mathbb{C} \mid \mathrm{Re}\,\alpha \geq 0\}$ and by $\mathbb{C}_+^o$ the corresponding open half-plane.

<u>Lemma 18</u>. Let $h \in \tilde{N}_+$. Then the mapping

$$\alpha \mapsto h^\alpha : \mathbb{C}_+^o \to \tilde{N}$$

is differentiable.

<u>Proof</u>. First note that all h^α , $\alpha \in \mathbb{C}_+^o$, are actually τ-measurable since h is τ-measurable.

1) Suppose that h is bounded, i.e. $h \in N_+$. Then the mapping

$$\alpha \mapsto h^\alpha : \mathbb{C}_+^o \to N$$

is differentiable with respect to the norm topology on N and

$$\frac{d}{d\alpha}h^\alpha = h^\alpha \log h \qquad (27)$$

(note that the expression at the right hand side is defined for any positive $h \in N$ since the function $\lambda \mapsto \lambda^\alpha \log \lambda$ is continuous on the closed half-plane $\mathbb{C}_+$) . This follows from spectral theory using the fact that for all $\alpha_0 \in \mathbb{C}_+^{\,o}$ we have

$$\frac{1}{\alpha-\alpha_0}(\lambda^\alpha-\lambda^{\alpha_0}) - \lambda^{\alpha_0}\log\lambda = \frac{1}{\alpha-\alpha_0}(e^{\alpha\log\lambda} - e^{\alpha_0\log\lambda}) - \log\lambda\, e^{\alpha_0\log\lambda}$$

$$\to 0 \text{ as } \alpha \to \alpha_0 \text{ uniformly in } \lambda \in \,]0,\|h\|] \;.$$

2) Now let h be any element of $\tilde{N}_+$. We claim that $\alpha \mapsto h^\alpha : \mathbb{C}_+^{\,o} \to \tilde{N}$ is differentiable with respect to the topology on $\tilde{N}$ and that (27) still holds (as above, $h^\alpha \log h$ is a well-defined positive self-adjoint operator and, by spectral theory, it is τ-measurable). Now let $\varepsilon,\delta \in \mathbb{R}_+$. Take $\lambda \in \mathbb{R}_+$ such that $\tau(\chi_{]\lambda,\infty[}(h)) \leq \delta$. Put $p = \chi_{[0,\lambda]}(h)$. Then hp is bounded and by the first part of the proof

$$\left\|\left(\frac{1}{\alpha-\alpha_0}(h^\alpha-h^{\alpha_0}) - h^{\alpha_0}\log h\right)p\right\|$$

$$= \left\|\frac{1}{\alpha-\alpha_0}((hp)^\alpha - (hp)^{\alpha_0}) - (hp)^\alpha \log(hp)\right\| \leq \varepsilon$$

for all $\alpha \in \mathbb{C}_+^{\,o}$ sufficiently close to α_0. Thus

$$\frac{1}{\alpha-\alpha_0}(h^\alpha-h^{\alpha_0}) - h^{\alpha_0}\log h \in N(\varepsilon,\delta)$$

for α sufficiently close to α_0. This proves the lemma. ∎

We denote by S the closed complex strip $\{\alpha \in \mathbb{C} \mid 0 \leq \operatorname{Re}\alpha \leq 1\}$ and by S^o the corresponding open strip.

<u>Lemma 19</u>. Let $h,k \in L^1(M)_+$. Then for $\alpha \in S^o$ we have

$$h^\alpha k^{1-\alpha} \in L^1(M) ,$$

and the mapping

$$\alpha \mapsto h^\alpha k^{1-\alpha} : S^o \to L^1(M) \tag{28}$$

is analytic.

<u>Proof</u>. That $h^\alpha k^{1-\alpha} \in L^1(M)$ follows from Definition 9 since

$$\forall s \in \mathbb{R}: \theta_s(h^\alpha k^{1-\alpha}) = (\theta_s h)^\alpha(\theta_s k)^{1-\alpha}$$

$$= e^{-\alpha s} h^\alpha e^{-(1-\alpha)s} k^{1-\alpha} = e^{-s} h^\alpha h^{1-\alpha} .$$

We want to prove that the mapping (28) is differentiable. In view of Corollary 17 we may as well prove that (28) is differentiable as a mapping into $\tilde{N}$. Now by the preceding lemma, the functions $f,g: S^o \to \tilde{N}$ defined by $f(\alpha) = h^\alpha$ and $g(\alpha) = k^{1-\alpha}$ are differentiable. It follows that for all $\alpha_0 \in S^o$ we have

$$\frac{1}{\alpha-\alpha_0}(f(\alpha)g(\alpha) - f(\alpha_0)g(\alpha_0))$$

$$= \frac{1}{\alpha-\alpha_0} f(\alpha)(g(\alpha)-g(\alpha_0)) + \frac{1}{\alpha-\alpha_0}(f(\alpha)-f(\alpha_0))g(\alpha_0)$$

$$\to f(\alpha_0)g'(\alpha_0) + f'(\alpha_0)g(\alpha_0) \quad \text{as } \alpha \to \alpha_0$$

so that also $f\cdot g: S^o \to \tilde{N}$ is differentiable. ∎

<u>Lemma 20</u>. Let $t \in \mathbb{R}$ and put

$$\tilde{N}_{\frac{1}{2}+it} = \{a \in \tilde{N} \mid \forall s \in \mathbb{R}: \theta_s a = e^{-(\frac{1}{2}+it)} a\} . \tag{29}$$

Let $a,b \in \tilde{N}_{\frac{1}{2}+it}$. Then b^*a, $ab^* \in L^1(M)$ and

$$\operatorname{tr}(b^*a) = \operatorname{tr}(ab^*) . \tag{30}$$

<u>Proof</u>. That b^*a, $ab^* \in L^1(M)$ follows from Definition 9 and (29).

To prove (30), suppose first that $a = b$. Then by Definition 13 and Lemma 5

$$\operatorname{tr}(a^*a) = \tau(\chi_{]1,\infty[}(a^*a)) = \tau(\chi_{]1,\infty[}(aa^*)) = \operatorname{tr}(aa^*) .$$

In the general case, note that $a+ib \in \tilde{N}_{\frac{1}{2}+it}$ and

$$b^*a = \frac{1}{4}\sum_{k=0}^{3} i^k(a+i^kb)^*(a+i^kb)$$

$$ab^* = \frac{1}{4}\sum_{k=0}^{3} i^k(a+i^kb)(a+i^kb)^* .$$

The result follows since tr is linear. ∎

Proposition 21. Let $p,q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $a \in L^p(M)$ and $b \in L^q(M)$. Then $ab, ba \in L^1(M)$ and

$$\mathrm{tr}(ab) = \mathrm{tr}(ba) .$$

Proof. If $p = 1$, we have $a = h_\varphi$ for some $\varphi \in M_*$ and the result follows by Theorem 7:

$$\mathrm{tr}(h_\varphi b) = \mathrm{tr}(h_{\varphi\cdot b}) = (\varphi\cdot b)(1) = (b\cdot\varphi)(1) = \mathrm{tr}(h_{b\cdot\varphi}) = \mathrm{tr}(bh_\varphi) .$$

Now suppose that $p,q \in]1,\infty[$. As usual, we easily see that ab and ba are in $L^1(M)$. By linearity, we may assume that $a \in L^p(M)_+$ and $b \in L^q(M)_+$. Now $a^p, b^q \in L^1(M)_+$ and by Lemma 19 the functions F and G on S^o defined by $F(\alpha) = \mathrm{tr}(a^{p\alpha}b^{q(1-\alpha)})$ and $G(\alpha) = \mathrm{tr}(b^{q(1-\alpha)}a^{p\alpha})$ are analytic. For all $t \in \mathbb{R}$, we have $a^{p(\frac{1}{2}+it)} \in \tilde{N}_{\frac{1}{2}+it}$ and $b^{q(\frac{1}{2}+it)} \in \tilde{N}_{\frac{1}{2}+it}$ so that by Lemma 20

$$F(\tfrac{1}{2}+it) = \mathrm{tr}(a^{p(\frac{1}{2}+it)}b^{q(\frac{1}{2}-it)}) = \mathrm{tr}(a^{p(\frac{1}{2}+it)}(b^{q(\frac{1}{2}+it)})^*)$$

$$= \mathrm{tr}((b^{q(\frac{1}{2}+it)})^*a^{p(\frac{1}{2}+it)}) = \mathrm{tr}(b^{q(\frac{1}{2}-it)}a^{p(\frac{1}{2}+it)}) = G(\tfrac{1}{2}+it) .$$

We conclude that $F = G$. In particular,

$$\mathrm{tr}(ab) = F(1/p) = G(1/p) = \mathrm{tr}(ba) . \ ∎$$

The proof of the next lemma is based on the 3 lines theorem for analytic functions (see e.g. [23, p. 93]). The 3 lines theorem also holds for analytic functions F with values in a Banach space (to see this, apply it to the scalar-valued functions $\alpha \mapsto v(F(\alpha))$, where v is in the dual of the given Banach space).

Lemma 22. Let $h,k \in L^1(M)_+$ and suppose that $\|h\|_1 = \|k\|_1 = 1$. Then for all $\alpha \in S^o$, we have

$$\|h^\alpha k^{1-\alpha}\|_1 \leq 1 .$$

Proof. Write $s = \mathrm{Re}\,\alpha$, $t = \mathrm{Im}\,\alpha$. Then $h^s \in L^{1/s}(M)$ with $\|h^s\|_{1/s} = 1 = s^{-s}\cdot s^s$, whence by Lemma 16

$$h^s \in N(s^{-s},s) .$$

Similarly,

$$k^{1-s} \in N((1-s)^{-(1-s)},1-s) .$$

It follows that

$$h^s k^{1-s} \in N(s^{-s},s)\cdot N((1-s)^{-(1-s)},1-s)$$

$$\subseteq N(s^{-s}(1-s)^{-(1-s)},s+(1-s))$$

whence also

$$h^\alpha k^{1-\alpha} = h^{it}h^s k^{1-s}k^{-it} \in N(s^{-s}(1-s)^{-(1-s)},1) .$$

Again by Lemma 16,

$$\|h^\alpha k^{1-\alpha}\|_1 \leq s^{-s}(1-s)^{-(1-s)} .$$

Since $s \mapsto s^{-s}(1-s)^{-(1-s)}$ is bounded, the function $\alpha \mapsto h^\alpha k^{1-\alpha}: S^o \to L^1(M)$ is bounded. It is analytic by Lemma 19. Hence we can apply the 3 lines theorem on each closed strip

$\{\alpha \in \mathbb{C} \mid \varepsilon \leq \operatorname{Re}\alpha \leq 1-\varepsilon\}$ and we obtain

$$\sup_{\varepsilon \leq \operatorname{Re}\alpha \leq 1-\varepsilon} \|h^\alpha k^{1-\alpha}\|_1 \leq \varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)} .$$

Hence for fixed $\alpha \in S^o$, the inequality

$$\|h^\alpha k^{1-\alpha}\|_1 \leq \varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)}$$

holds for all $\varepsilon \in \mathbb{R}_+$ such that $\varepsilon \leq \operatorname{Re}\alpha \leq 1-\varepsilon$. Since

$$\varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)} = e^{-\varepsilon \log \varepsilon} e^{-(1-\varepsilon)\log(1-\varepsilon)} \to 1 \quad \text{as} \quad \varepsilon \to 0 ,$$

it follows that

$$\|h^\alpha k^{1-\alpha}\|_1 \leq 1 .$$

This proves the lemma. ∎

<u>Theorem 23</u>. (Hölder's inequality). Let $p,q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^p(M)$ and $b \in L^q(M)$. Then

$$\|ab\|_1 \leq \|a\|_p \|b\|_q .$$

<u>Proof</u>. If $p = 1$, we have $a = h_\varphi$ for some $\varphi \in M_*$ and

$$\|h_\varphi b\|_1 = \|h_{\varphi \cdot b}\|_1 = \|\varphi \cdot b\| \leq \|\varphi\| \, \|b\|_\infty = \|h_\varphi\|_1 \cdot \|b\|_\infty$$

for all $b \in L^\infty(M) = M$. The case $q = 1$ is quite similar to this.

Now assume $p,q \in]1,\infty[$, and $\|a\|_p = 1$, $\|b\|_q = 1$. Let $a = u|a|$ be the (usual) polar decomposition of a and $b = |b^*|v$ the right polar decomposition of b. Then $|a|^p, |b^*|^q \in L^1(M)$ with $\||a|^p\|_1 = \||b^*|_r^q\|_1 = 1$ and Lemma 22 applies:

$$\begin{aligned} \|ab\|_1 &= \|u|a|\ |b^*|v\|_1 \leq \||a|\ |b^*|\|_1 \\ &= \||a|^{p/p}\ |b^*|^{q/q}\|_1 \leq 1 . \ ∎ \end{aligned}$$

<u>Proposition 24</u>. Let $p,q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^p(M)$. Then

$$\|a\|_p = \sup\{|\operatorname{tr}(ab)| \mid b \in L^q(M),\ \|b\|_q \leq 1\} .$$

<u>Proof</u>. If $p = 1$ or $p = \infty$, this is well-known (since $\operatorname{tr}(c\,h_\varphi) = \operatorname{tr}(h_\varphi c) = \varphi(c)$ for all $\varphi \in M_*$ and $c \in M$). Suppose that $1 < p < \infty$. We may assume that $\|a\|_p = 1$. Then putting $b = |a|^{p/q}u^*$, where $a = u|a|$ is the polar decomposition of a, we have $b \in L^q(M)$ with $\|b\|_q = \||a|^{p/q}u^*\|_q = \operatorname{tr}(|a|^p)^{1/q} = 1$ and

$$\operatorname{tr}(ab) = \operatorname{tr}(u|a|\ |a|^{p/q}u^*) = \operatorname{tr}(|a|^p) = 1 .$$

Hence

$$\|a\|_p = 1 \leq \sup\{|\operatorname{tr}(ab)| \mid b \in L^q(M),\ \|b\|_q \leq 1\} .$$

The converse inequality follows from Hölder's inequality (together with (26)). ∎

<u>Corollary 25</u>. $\|\cdot\|_p$ is a norm on $L^p(M)$.

<u>Proof</u>. The inequality

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p$$

follows immediately from Proposition 24. ∎

<u>Proposition 26</u>. On $L^p(M)$, the norm topology is exactly the topology induced from $\tilde{N}$.

<u>Proof</u>. Now that we know that $\|\cdot\|_p$ is a norm, this is a corollary of Lemma 16. ∎

Corollary 27. $(L^p(M), \|\cdot\|_p)$ is a Banach space.

Proof. From the definition of $L^p(M)$ it follows that it is a closed subspace of the complete topological vector space $\tilde{N}$. Hence it is complete for the uniform structure induced from $\tilde{N}$. By Lemma 16, this is simply the uniform structure coming from the norm. Hence $L^p(M)$ is a complete normed space. ∎

Corollary 28. $(L^2(M), \|\cdot\|_2)$ is a Hilbert space with the inner product

$$(a|b)_{L^2(M)} = tr(b^*a) \quad (= tr(ab^*)) , \quad a,b \in L^2(M) .$$

Proof. That $(a,b) \mapsto (a|b)_{L^2(M)}$ is an inner product defining the norm $\|\cdot\|_2$ is easily verified. By Corollary 27, $L^2(M)$ is complete. ∎

Remark 29. Let $t \in \mathbb{R}$. Define $\tilde{N}_{\frac{1}{2}+it}$ as in Lemma 20. Then

$$(a,b) \mapsto tr(b^*a)$$

is an inner product on $\tilde{N}_{\frac{1}{2}+it}$ and

$$a \mapsto tr(a^*a)^{\frac{1}{2}}$$

is a norm which we shall denote by $\|\cdot\|_2$ (as in the case $t = 0$ where $\tilde{N}_{\frac{1}{2}} = L^2(M)$). Note that

$$|tr(b^*a)| \leq \|a\|_2 \|b\|_2$$

and

$$\|a+b\|_2^2 + \|a-b\|_2^2 = 2\|a\|_2^2 + 2\|b\|_2^2$$

for all $a,b \in \tilde{N}_{\frac{1}{2}+it}$.

Remark 30. Let $p \in [1,\infty[$. Then we have a natural identification

$$L^p(M \oplus M) \simeq L^p(M) \times L^p(M) \tag{31}$$

such that

$$\forall (a,b) \in L^p(M) \times L^p(M) \simeq L^p(M \oplus M) : \|(a,b)\|_p = (\|a\|_p^p + \|b\|_p^p)^{1/p} . \tag{32}$$

To see this, write $M^{(2)} = M \oplus M$ and define the normal faithful semifinite weight $\varphi_0^{(2)}$ on $M^{(2)}$ by $\varphi_0^{(2)} = \varphi_0 \oplus \varphi_0$, i.e.

$$\varphi_0^{(2)}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \varphi_0(x) + \varphi_0(y) , \quad x,y \in M_+ .$$

Let us denote by $N^{(2)}$, $\tau^{(2)}$ etc. the objects associated with $(M^{(2)}, \varphi_0^{(2)})$ analogous to N, τ etc. associated with (M,φ_0). Then one easily verifies that $N^{(2)} \simeq N \oplus N$, $\tau^{(2)} \simeq \tau \oplus \tau$, $(M^{(2)})_* \simeq M_* \oplus M_*$, $h_{\varphi\oplus\psi}^{(2)} \simeq h_\varphi \oplus h_\psi$, $\theta_s^{(2)} \simeq \theta_s \oplus \theta_s$, $\tilde{N}^{(2)} \simeq \tilde{N} \oplus \tilde{N}$, and finally (31). Furthermore, $tr^{(2)} \simeq tr \oplus tr$ so that

$$\|(a,b)\|_p^p = tr^{(2)}\left(\left|\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right|^p\right) = tr^{(2)}\begin{pmatrix} |a|^p & 0 \\ 0 & |b|^p \end{pmatrix}$$

$$= tr(|a|^p) + tr(|b|^p) = \|a\|_p^p + \|b\|_p^p$$

for all $a,b \in L^p(M)$. This proves (32).

Proposition 31. (Clarkson's inequality.) Let $p \in [2,\infty[$. Then for all $a,b \in L^p(M)$ we have

$$\|a+b\|_p^p + \|a-b\|_p^p \leq 2^{p-1}(\|a\|_p^p + \|b\|_p^p) .$$

Proof. Using Remark 30 we may reformulate the inequality to be proved as

$$\|(a+b,a-b)\|_p \le 2^{1/q}\,\|(a,b)\|_p \tag{33}$$

where we have put $1/q = 1 - 1/p$.

Let $(a,b) \in L^p(M\oplus M)$ and $(c,d) \in L^q(M\oplus M)$ such that

$$\|(a,b)\|_p = 1 \quad \text{and} \quad \|(c,d)\|_q = 1 . \tag{34}$$

Let

$$a = uh^{1/p} , \quad b = vk^{1/p}$$

be the polar decompositions of a and b , and

$$c = f^{1/q}w , \quad d = g^{1/q}z$$

the right polar decompositions of c and d . Then $h,k,f,g \in L^1(M)_+$ and

$$\|(h,k)\|_1 = 1 , \quad \|(f,g)\|_1 = 1 .$$

For each $\alpha \in S^o$, put

$$F(\alpha) = tr((uh^\alpha+vk^\alpha)f^{1-\alpha}w + (uh^\alpha-vk^\alpha)g^{1-\alpha}z) .$$

Then

$$F(1/p) = tr((a+b)c + (a-b)d) .$$

For all $\alpha \in S^o$, we have

$$F(\alpha) = tr^{(2)}\left(\begin{pmatrix} u & 0 \\ 0 & -v \end{pmatrix}\begin{pmatrix} h^\alpha & 0 \\ 0 & k^\alpha \end{pmatrix}\begin{pmatrix} f^{1-\alpha} & 0 \\ 0 & g^{1-\alpha} \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}\right)$$

$$+ \, tr^{(2)}\left(\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} k^\alpha & 0 \\ 0 & h^\alpha \end{pmatrix}\begin{pmatrix} f^{1-\alpha} & 0 \\ 0 & g^{1-\alpha} \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}\right)$$

By Lemma 19 and 22 applied to $(h,k) \in L^1(M\oplus M)$ and $(f,g) \in L^1(M\oplus M)$ we conclude that F is analytic and

$$\forall \alpha \in S^o : \; |F(\alpha)| \le 2 . \tag{35}$$

We claim that

$$\forall t \in \mathbb{R} : \; |F(\tfrac12+it)| \le \sqrt{2} . \tag{36}$$

For the proof we apply first the Cauchy-Schwarz inequality in $\tilde{N}^{(2)}_{\frac12+it}$, next the parallelogram law in $\tilde{N}_{\frac12+it}$ (cf. Remark 29):

$$|F(\tfrac12+it)|^2$$

$$= \left| tr^{(2)}\left(\begin{pmatrix} uh^{\frac12+it}+vk^{\frac12+it} & 0 \\ 0 & uh^{\frac12+it}-vk^{\frac12+it} \end{pmatrix}\begin{pmatrix} f^{\frac12-it}w & 0 \\ 0 & g^{\frac12-it}z \end{pmatrix}\right)\right|^2$$

$$\le \left\|\begin{pmatrix} uh^{\frac12+it}+vk^{\frac12+it} & 0 \\ 0 & uh^{\frac12+it}-vk^{\frac12+it} \end{pmatrix}\right\|_2^2 \left\|\begin{pmatrix} f^{\frac12-it}w & 0 \\ 0 & g^{\frac12-it}z \end{pmatrix}^*\right\|_2^2$$

$$= (\|uh^{\frac12+it}+vk^{\frac12+it}\|_2^2+\|uh^{\frac12+it}-vk^{\frac12+it}\|_2^2)(\|f^{\frac12-it}w\|_2^2+\|g^{\frac12-it}z\|_2^2)$$

$$= (2\|uh^{\frac12+it}\|_2^2+2\|vk^{\frac12+it}\|_2^2)(\|f^{\frac12}\|_2^2+\|g^{\frac12}\|_2^2) = 2(\|h^{\frac12}\|_2^2+\|k^{\frac12}\|_2^2) = 2.$$

Finally, by the 3 lines theorem applied to each strip $\{\alpha \in \mathbb{C} \mid \varepsilon \le \mathrm{Re}\,\alpha \le \frac12\}$ where $0 < \varepsilon < 1/p$, (35) and (36) give

$$|tr((a+b)c+(a-b)d)| = |F(1/p)|$$

$$\le 2^{(\frac12-1/p)/(\frac12-\varepsilon)} \cdot (\sqrt{2})^{((1/p)-\varepsilon)/(\frac12-\varepsilon)}$$

$$\to 2^{1-2/p} \cdot 2^{1/p} = 2^{1/q} \quad \text{as} \quad \varepsilon \to \infty .$$

Hence

$$\left| tr^{(2)}\left(\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right)\right| \le 2^{1/q}$$

for all $(a,b) \in L^p(M\oplus M)$ and $(c,d) \in L^q(M\oplus M)$ satisfying (34).

By Proposition 24 applied to $L^p(M\oplus M)$ this implies that

$$\|(a+b,a-b)\|_p \leq 2^{1/q}$$

for all $(a,b) \in L^p(M\oplus M)$ with $\|(a,b)\|_p = 1$. (33) follows. ∎

By Clarkson's inequality, the Banach space $L^p(M)$, where $2 \leq p < \infty$, is uniformly convex. Hence it is reflexive (see e.g. [11, p. 127, Theorem 2]).

<u>Theorem 32</u>. Let $p \in [1,\infty[$ and $1/p + 1/q = 1$.

1) Let $a \in L^q(M)$. Then φ_a defined by

$$\varphi_a(b) = tr(ab) , \quad b \in L^p(M) ,$$

is a bounded linear functional on $L^p(M)$.

2) The mapping

$$a \mapsto \varphi_a$$

is an isometric isomorphism of $L^q(M)$ onto the dual Banach space of $L^p(M)$.

<u>Proof</u>. By Proposition 24, $a \mapsto \varphi_a$ is an isometry of $L^q(M)$ onto a subspace of the dual $L^p(M)^*$ of $L^p(M)$. Since $L^q(M)$ is complete, this subspace is closed. It follows from Proposition 24 that it is w*-dense (its orthogonal in $L^p(M)$ vanishes).

Now if $p \geq 2$, the space $L^p(M)$ is reflexive. Hence $L^p(M)^*$ is also reflexive and thus the w*-closure of the subspace $L^q(M)$ is equal to its norm closure. Hence $L^q(M) = L^p(M)^*$.

If $p < 2$, we have $q \geq 2$ and thus $L^p(M) \simeq L^q(M)^*$ via tr . It follows that $L^p(M)^* \simeq L^q(M)^{**} \simeq L^q(M)$ (via tr) . ∎

<u>Proposition 33</u>. Let $p,q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^q(M)$. Then $a \geq 0$ if and only if

$$\forall b \in L^p(M)_+ : tr(ab) \geq 0 . \qquad (37)$$

<u>Proof</u>. If $p,q \in \{1,\infty\}$, the result is well-known. Now assume that $p,q \in]1,\infty[$. If $a \in L^q(M)_+$, then $a^{1/2}ba^{1/2} \in L^1(M) \cap \tilde{N}_+ = L^1(M)_+$ and hence

$$tr(ab) = tr(a^{1/2}a^{1/2}b) = tr(a^{1/2}ba^{1/2}) \geq 0 .$$

Conversely, suppose that $a \in L^q(M)$ satisfies (37). Then $a = a^*$ since

$$tr(ab) = \overline{tr(ab)} = tr((ab)^*) = tr(ba^*) = tr(a^*b)$$

for all $b \in L^p(M)_+$. Put $a_+ = (a+|a|)/2$, $a_- = (a-|a|)/2 \in L^q(M)_+$. Then $a = a_+ - a_-$ and $a_+a_- = 0$. Put $b = a_-^{q/p}$. Then $b \in L^p(M)_+$ so that $tr(ab) \geq 0$. Now

$$tr(ab) = tr(a_+b) - tr(a_-b) = -tr(a_-b) = -tr(a_-^q) .$$

It follows that $tr(a_-^q) = 0$ whence $a_- = 0$ and $a = a_+ \in L^q(M)_+$. ∎

For each $p \in [1,\infty]$ we define left and right actions λ_p and ρ_p on $L^p(M)$ by

$$\lambda_p(x)a = xa , \quad a \in L^p(M) , \qquad (38)$$

$$\rho_p(x)a = ax , \quad a \in L^p(M) , \qquad (39)$$

for all $x \in M$. That $\lambda_p(x)$ and $\rho_p(x)$ map $L^p(M)$ into itself follows immediately from Definition 9. From Lemma 16 and the fact that $xN(\varepsilon,\delta) \subseteq N(\|x\|\varepsilon,\delta)$ for all $x \in M$ and $\varepsilon,\delta \in \mathbb{R}_+$, we get

$$\forall x \in M\ \forall a \in L^p(M):\ \|xa\|_p \leq \|x\|_\infty \|a\|_p \ . \tag{40}$$

Since $ax = (x^*a^*)^*$, we also have

$$\forall x \in M\ \forall a \in L^p(M):\ \|ax\|_p \leq \|x\|_\infty \|a\|_p \ . \tag{41}$$

Hence $\lambda_p(x)$ and $\rho_p(x)$ are bounded linear operators on $L^p(M)$.

<u>Proposition 34</u>. Let $p \in [1,\infty]$.

1) λ_p (resp. ρ_p) is a faithful representation (resp. anti-representation) of M on the Banach space $L^p(M)$.

2) For all $x \in M$, we have

$$J_p\, \lambda_p(x) J_p = \rho_p(x^*) ,$$

where J_p denotes the conjugate linear isometric involution $a \mapsto a^*$ of $L^p(M)$.

3) Let z be an element of the center of M. Then

$$\lambda_p(z) = \rho_p(z) .$$

<u>Proof</u>. 1) Suppose that $\lambda_p(x) = 0$. Then

$$\forall a \in L^p(M)\ \forall b \in L^q(M):\ tr(xab) = tr((\lambda_p(x)a)b) = 0 .$$

Since $L^1(M) = L^p(M) \cdot L^q(M)$, x must be 0.

2) For all $a \in L^p(M)$, we have

$$(J_p\lambda_p(x)J_p)(a) = (xa^*)^* = ax^* = \rho_p(x^*)a .$$

3) Clearly, $\lambda_\infty(z) = \rho_\infty(z)$. It follows that

$$\forall a \in L^1(M)\ \forall b \in L^\infty(M):\ tr(zab) = tr(abz) = tr(azb)$$

whence $\lambda_1(z) = \rho_1(z)$. In particular

$$\forall a \in L^1(M)_+:\ za = az ,$$

whence by spectral theory

$$\forall a \in L^1(M)_+:\ za^{1/p} = a^{1/p}z .$$

Thus $\lambda_p(z)$ and $\rho_p(z)$ coincide on $L^p(M)_+$. Hence $\lambda_p(z) = \rho_p(z)$. ∎

<u>Proposition 35</u>. For all $p \in [1,\infty]$, we have

$$\lambda_p(M) = \rho_p(M)' \quad \text{and} \quad \rho_p(M) = \lambda_p(M)' \tag{42}$$

(where $\rho_p(M)'$, resp. $\lambda_p(M)'$, denotes the set of bounded linear operators on $L^p(M)$ commuting with all $\rho_p(x)$, $x \in M$, resp. all $\lambda_p(x)$, $x \in M$).

<u>Proof</u>. Obviously

$$\lambda_p(M) \subseteq \rho_p(M)' \quad \text{and} \quad \rho_p(M) \subseteq \lambda_p(M)' .$$

To show (42) we need only prove either $\lambda_p(M) \supseteq \rho_p(M)'$ or $\rho_p(M) \supseteq \lambda_p(M)'$. Then the other one follows by Proposition 34, 2).

(i) First suppose that $p = \infty$. Let $T \in \lambda_\infty(M)'$. Then

$$\forall a \in L^\infty(M):\ T(a) = T(a1) = aT(1)$$

whence $T = \rho_\infty(T(1)) \in \rho_\infty(M)$.

(ii) Next we consider the case $p = 1$. Let $S \in \lambda_1(M)'$. Denote by $T: L^\infty(M) \to L^\infty(M)$ the transpose of S given by

$$\operatorname{tr}(T(a)b) = \operatorname{tr}(aS(b)) , \quad a \in L^\infty(M) , \; b \in L^1(M) .$$

Now

$$\forall x \in M \; \forall a \in L^\infty(M) \; \forall b \in L^1(M): \operatorname{tr}(T(ax)b) = \operatorname{tr}(axS(b)) = \operatorname{tr}(aS(xb)) = \operatorname{tr}(T(a)xb) .$$

Thus $T \in \rho_\infty(M)'$ and hence $T = \lambda_\infty(y)$ for some $y \in M$. It follows that

$$\forall a \in L^\infty(M) \; \forall b \in L^1(M): \operatorname{tr}(aS(b)) = \operatorname{tr}(T(a)b) = \operatorname{tr}(yab) = \operatorname{tr}(aby) ,$$

whence $S = \rho_1(y) \in \rho_1(M)$.

(iii) Now let $p \in]1,\infty[$. Let $T \in \lambda_p(M)'$. We want to define a linear mapping $S: L^1(M) \to L^1(M)$ by

$$S\left(\sum_{i=1}^n b_i a_i\right) = \sum_{i=1}^n b_i T(a_i) \tag{43}$$

for all $a_1,\dots,a_n \in L^p(M)$ and $b_1,\dots,b_n \in L^q(M)$. First let us show that

$$\sum_{i=1}^n b_i a_i = 0 \Rightarrow \sum_{i=1}^n b_i T(a_i) \tag{44}$$

so that S is well-defined.

Suppose that $\sum_{i=1}^n b_i a_i = 0$. Put $a = \left(\sum_{i=1}^n a_i{}^* a_i\right)^{\frac{1}{2}} \in L^p(M)_+$. Then all $a_i{}^* a_i \le a^2$. Hence there exist $x_1,\dots,x_n \in M$ such that

$$a_i = x_i a \quad \text{and} \quad \sum_{i=1}^n x_i{}^* x_i = \operatorname{supp} a .$$

Then

$$\left(\sum_{i=1}^n b_i x_i\right) a = \sum_{i=1}^n b_i a_i = 0$$

and

$$\operatorname{supp}\left(\sum_{i=1}^n b_i x_i\right) \le \operatorname{supp} a$$

whence

$$\sum_{i=1}^n b_i x_i = 0 .$$

It follows that

$$\sum_{i=1}^n b_i \, T(a_i) = \sum_{i=1}^n b_i \, T(x_i a) = \sum_{i=1}^n b_i x_i \, T(a) = \left(\sum_{i=1}^n b_i x_i\right) T(a) = 0$$

as wanted.

We have shown that $S: L^1(M) \to L^1(M)$ is a well-defined linear map. It is also bounded. Indeed, any $c \in L^1(M)$ may be written as a product $c = ba$ where $a \in L^p(M)$, $b \in L^q(M)$, and $\|c\|_1 = \|b\|_q \|a\|_p$. Then

$$\|S(c)\|_1 = \|bT(a)\|_1 \le \|b\|_q \|T(a)\|_p \le \|b\|_q \|T\| \|a\|_p = \|T\| \|c\|_1 .$$

Finally, since

$$\forall x \in M \; \forall b \in L^q(M) \; \forall a \in L^p(M): S(xba) = xb\,T(a) = x\,S(ba)$$

we have $S \in \lambda_1(M)'$. Hence $S = \rho_1(y)$ for some $y \in M$.

Now

$$b\,T(a) = S(ba) = bay = b\,\rho_p(y)a$$

for all $b \in L^q(M)$ and $a \in L^p(M)$. It follows that $T = \rho_p(y) \in \rho_p(M)$ as wanted. ∎

We shall denote λ_2 and ρ_2 simply by λ and ρ, and J_2 by J (i.e. $Ja = a^*$ for all $a \in L^2(M)$).

Theorem 36. 1) λ (resp. ρ) is a normal faithful representation (resp. anti-representation) of M on the Hilbert space $L^2(M)$.

2) The von Neumann algebras $\lambda(M)$ and $\rho(M)$ are commutants of each other, and

$$\rho(M) = J\,\lambda(M)J$$

3) $(\lambda(M), L^2(M), J, L^2(M)_+)$ is a standard form of M in the sense of [4, Definition 2.1].

Proof. For all $x \in M$ and $a,b \in L^2(M)$ we have

$$(\lambda(x)a|b)_{L^2(M)} = \mathrm{tr}(b^*xa) = \mathrm{tr}((x^*b)^*a) = (a|\lambda(x^*)b)_{L^2(M)}$$

so that λ is a *-representation.

Suppose that $x_i \nearrow x$ in M . Then for all $a \in L^2(M)$, we have

$$(\lambda(x_i)a|a)_{L^2(M)} = \mathrm{tr}(a^*x_ia) = \mathrm{tr}(x_iaa^*) = \langle x_i, aa^*\rangle$$

$$\nearrow \langle x, aa^*\rangle = \mathrm{tr}(xaa^*) = \mathrm{tr}(a^*xa) = (\lambda(x)a|a)_{L^2(M)} .$$

2) follows immediately from Proposition 35 and Proposition 34, 2).

3) That $L^2(M)_+$ is a self-dual cone follows from Proposition 33. Now

(i) $J\,\lambda(M)J = \rho(M) = \lambda(M)'$;

(ii) $J\,\lambda(z)J = \rho(x^*) = \lambda(z^*) = \lambda(z)^*$ for all z in the center of M ;

(iii) for all $a \in L^2(M)_+$, we have $a^* = a$;

(iv) for all $a \in L^2(M)_+$ and $x \in M$, we have $(\lambda(x)J\,\lambda(x)J)a = \lambda(x)\rho(x^*)a = xax^* \in L^2(M)_+$. ∎

Independence of the choice of φ_0 . The spaces $L^p(M)$ and their relations are independent of the choice of φ_0 (and hence canonically associated with M) . This is a consequence of the following theorem and its corollary when we recall that the spaces $(L^p(M), \|\cdot\|_p)$ are defined in terms of $\tilde{N}$, $(\theta_s)_{s\in\mathbb{R}}$, and τ .

Let φ_0 and φ_1 be normal faithful semifinite weights on M . We view the crossed products $N_0 = R(M, \sigma^{\varphi_0})$ and $N_1 = R(M, \sigma^{\varphi_1})$ as von Neumann algebras on $L^2(\mathbb{R}, H)$. They are generated by $\pi_0(x)$, $x \in M$, (resp. $\pi_1(x)$, $x \in M$) and $\lambda(s)$, $s \in \mathbb{R}$, where

$$(\pi_0(x)\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t) , \quad (\pi_1(x)\xi)(t) = \sigma_{-t}^{\varphi_1}\xi(t) ,$$

$$(\lambda(s)\xi)(t) = \xi(t-s)$$

for all $\xi \in L^2(\mathbb{R},H)$, $t \in \mathbb{R}$.

Denote by $s \mapsto \theta_s$ the dual action of $\mathbb{R}$ in N_0 and N_1 . Recall [18, Section 4] that each θ_s has the form

$$\theta_s(y) = \mu_s y \mu_s^{-1} \qquad (45)$$

where μ_s is the unitary on $L^2(\mathbb{R},H)$ given by

$$(\mu_s\xi)(t) = e^{-ist}\xi(t) , \quad \xi \in L^2(\mathbb{R},H) , \ t \in \mathbb{R} . \qquad (46)$$

Denote by τ_0 , resp. τ_1 , the trace on N_0 , resp. N_1 , given by (14).

Theorem 37. There exists an isomorphism

$$\kappa : N_0 \to N_1$$

such that

$$\forall s \in \mathbb{R}: \ \kappa \circ \theta_s \circ \kappa^{-1} = \theta_s \tag{47}$$

and

$$\tau_1 = \tau_0 \circ \kappa^{-1} . \tag{48}$$

<u>Proof</u> (cf. [18, Proposition 3.5]). We define a unitary u on $L^2(\mathbb{R},H)$ by

$$(u\xi)(t) = (D\varphi_1 : D\varphi_0)_{-t}\xi(t) \ , \ \xi \in L^2(\mathbb{R},H) \ , \ t \in \mathbb{R} .$$

Now

$$\forall x \in M: \ u\,\pi_0(x)u^* = \pi_1(x) \tag{49}$$

and

$$\forall s \in \mathbb{R}: \ u\,\lambda(s)u^* = \pi_1((D\varphi_1 : D\varphi_0)_s{}^*)\lambda(s) \tag{50}$$

since

$$(u\pi_0(x)u^*\xi)(t) = (D\varphi_1 : D\varphi_0)_{-t}\sigma_{-t}^{\varphi_0}(x)(D\varphi_1 : D\varphi_0)_{-t}{}^*\xi(t)$$

$$= \sigma_{-t}^{\varphi_1}(x)\xi(t) \ , \ t \in \mathbb{R} ,$$

and

$$(u\lambda(s)u^*\xi)(t) = (D\varphi_1 : D\varphi_0)_{-t}(D\varphi_1 : D\varphi_0)_{-(t-s)}{}^*\xi(t-s)$$

$$= (D\varphi_1 : D\varphi_0)_{-t}((D\varphi_1 : D\varphi_0)_{-t}\sigma_{-t}^{\varphi_0}((D\varphi_1 : D\varphi_0)_s))^*\xi(t-s)$$

$$= (D\varphi_1 : D\varphi_0)_{-t}\sigma_{-t}^{\varphi_0}((D\varphi_1 : D\varphi_0)_s{}^*)(D\varphi_1 : D\varphi_0)_{-t}{}^*\xi(t-s)$$

$$= (\sigma_{-t}^{\varphi_1}((D\varphi_1 : D\varphi_0)_s{}^*)\lambda(s)\xi)(t) \ , \quad t \in \mathbb{R} ,$$

for all $x \in M$, $s \in \mathbb{R}$, and $\xi \in L^2(\mathbb{R},H)$.

Hence $\kappa = u(\cdot)u^*$ maps N_0 into N_1 . Similarly, $u^*(\cdot)u$ maps N_1 into N_0 . In all, we have shown that

$$\kappa : N_0 \to N_1$$

in an isomorphism of N_0 onto N_1 .

Obviously, u commutes with each μ_s (see (46)) and hence κ commutes with each θ_s , i.e. (47).

Denote by T_0 , resp. T_1 , the operator valued weights introduced as in (6). Then for all $x \in (N_0)_+$, we have

$$T_1 x = \pi_1{}^{-1}(\int\theta_s(x)ds) = \pi_1{}^{-1}(\kappa(\int\theta_s(\kappa^{-1}(x))ds))$$

$$= \pi_0{}^{-1}(\int\theta_s(\kappa^{-1}(x))ds) = T_0(\kappa^{-1}(x)) ,$$

i.e.

$$T_1 = T_0 \circ \kappa^{-1} ,$$

whence

$$\tilde{\varphi}^{(1)} = \tilde{\varphi}^{(0)} \circ \kappa^{-1}$$

for all normal weights on M (here, ${}^{(0)}$, resp. ${}^{(1)}$, denotes dual weight construction w.r.t. N_0 , resp. N_1) . In particular,

$$\tilde{\varphi}_1{}^{(1)} = \tilde{\varphi}_1{}^{(0)} \circ \kappa^{-1}$$

whence

$$(D\tilde{\varphi}_1{}^{(1)} : D(\tau_0\circ\kappa^{-1}))_t = (D(\tilde{\varphi}_1{}^{(0)}\circ\kappa^{-1}) : D(\tau_0\circ\kappa^{-1}))_t$$

$$= \kappa((D\tilde{\varphi}_1{}^{(0)} : D\tau_0)_t)$$

$$= \kappa((D\tilde{\varphi}_1{}^{(0)} : D\tilde{\varphi}_0{}^{(0)})_t)\kappa((D\tilde{\varphi}_0{}^{(0)} : D\tau_0)_t)$$

$$= \kappa(\pi_0((D\varphi_1 : D\varphi_0)_t))\kappa(\lambda(t))$$

$$= \pi_1((D\varphi_1 : D\varphi_0)_t)\pi_1((D\varphi_1 : D\varphi_0)_t{}^*)\lambda(t)$$

$$= \lambda(t)$$

for all t . Hence $\tau_0 \circ \kappa^{-1} = \tau_1$. ∎

<u>Corollary 38</u>. The mapping $\kappa: N_0 \to N_1$ extends to a topological *-isomorphism

$$\tilde{\kappa}: \tilde{N}_0 \to \tilde{N}_1 .$$

For all $\varepsilon,\delta \in \mathbb{R}_+$, we have

$$\tilde{\kappa}(N^{(0)}(\varepsilon,\delta)) = N^{(1)}(\varepsilon,\delta) .$$

<u>Proof</u>. We can define $\tilde{\kappa}$ by

$$\tilde{\kappa}(y) = uyu^* , \quad y \in \tilde{N}_0 .$$

Since $\tau_1 = \tau_0 \circ \kappa^{-1}$, $\tilde{\kappa}(y)$ is τ_1-measurable when y is τ_0-measurable. ∎

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<u>The semifinite case</u>. Suppose that M is semifinite. Then there exists a normal faithful semifinite trace τ_0 on M, and for the construction of the $L^p(M)$, we may assume that $\varphi_0 = \tau_0$. In this case, we have (when identifying $L^2(\mathbb{R})$ with $L^2(\mathbb{R})$ via Fourier transformation)

$$R(M,\sigma^{\varphi_0}) \simeq M \otimes L^\infty(\mathbb{R}) ;$$

for all $x \in M$, $f \in L^\infty(\mathbb{R})$, and $s \in \mathbb{R}$ we have

$$\theta_s(x \otimes f) \simeq x \otimes \ell(s)(f) ,$$

where $\ell(s)$ denotes translation by s in $L^2(\mathbb{R})$. Finally,

$$\tau \simeq \tau_0 \otimes e^{-s}\, ds .$$

Now one can show that for all $p \in [1,\infty]$,

$$L^p(M) \simeq L^p(M,\tau_0) \otimes \exp((\cdot)/p)$$

where $L^p(M,\tau_0)$ is the L^p space with respect to the trace τ_0 as defined at the end of Chapter I.

III

SPATIAL DERIVATIVES

Spatial derivatives were introduced by A. Connes in [1]. In this chapter, we give an alternative definition (equivalent to that given in [1]) suggested to us by U. Haagerup, based on the notion of the extended positive part of a von Neumann algebra. This definition permits us to obtain very easily some elementary properties of spatial derivatives. After this, we recall their main modular properties and the characterization as (-1)-homogeneous operators.

Definition and elementary properties of spatial derivatives.

Let M be a von Neumann algebra acting on a Hilbert space H, and let ψ be a normal faithful semifinite weight on the commutant M' of M.

We shall use the following standard notation: $n_\psi = \{y \in M' \mid \psi(y^*y) < \infty\}$, H_ψ the Hilbert space completion of n_ψ with respect to the inner product $(y_1, y_2) \mapsto \psi(y_2^*y_1)$, Λ_ψ the canonical injection of n_ψ into H_ψ, π_ψ the canonical representation of M' on H_ψ.

<u>Definition 1</u>. For each $\xi \in H$, we denote by $R^\psi(\xi)$ the (densely defined) operator from H_ψ to H defined by

$$R^\psi(\xi)\Lambda_\psi(y) = y\xi \ , \quad y \in n_\psi \ . \tag{1}$$

<u>Proposition 2</u>. For all $\xi, \xi_1, \xi_2 \in H$, $x \in M$, and $y \in M'$ we have

(i) $R^{\psi}(\xi_1+\xi_2) = R^{\psi}(\xi_1) + R^{\psi}(\xi_2)$,

(ii) $R^{\psi}(x\xi) = xR^{\psi}(\xi)$,

(iii) $yR^{\psi}(\xi) \subseteq R^{\psi}(\xi)\pi_{\psi}(y)$,

and

(i)* $R^{\psi}(\xi_1)^* + R^{\psi}(\xi_2)^* \subseteq R^{\psi}(\xi_1+\xi_2)^*$,

(ii)* $R^{\psi}(x\xi)^* = R^{\psi}(\xi)^*x^*$,

(iii)* $\pi_{\psi}(y)R^{\psi}(\xi)^* \subseteq R^{\psi}(\xi)^*y$.

<u>Proof</u>. (i) and (ii) are immediate from Definition 1. (iii): For all $z \in n_{\psi}$, we have $yR^{\psi}(\xi)\Lambda_{\psi}(z) = yz\xi = R^{\psi}(\xi)\Lambda_{\psi}(yz) = R^{\psi}(\xi)\pi_{\psi}(y)\Lambda_{\psi}(z)$.

(i)*, (ii)*, and (iii)* follow from (i), (ii), and (iii) using $R^{\psi}(\xi_1)^* + R^{\psi}(\xi_2)^* \subseteq (R^{\psi}(\xi_1)+R^{\psi}(\xi_2))^*$, $(xR^{\psi}(\xi))^* = R^{\psi}(\xi)^*x^*$, and $(y^*R^{\psi}(\xi))^* = R^{\psi}(\xi)^*y^*$. ∎

<u>Definition 3</u>. A vector $\xi \in H$ is called ψ-<u>bounded</u> if the operator $R^{\psi}(\xi)$ is bounded. The set of ψ-bounded vectors is denoted $D(H,\psi)$.

<u>Notation</u>. If $\xi \in D(H,\psi)$, $R^{\psi}(\xi)$ extends to a bounded operator $H_{\psi} \to H$ which we shall also denote $R^{\psi}(\xi)$.

<u>Proposition 4</u>. The set $D(H,\psi)$ is an M-invariant dense subspace of H .

<u>Proof</u>. That $D(H,\psi)$ is an M-invariant subspace of H follows from Proposition 2, (i) and (ii). Denote by e the projection onto $\overline{D(H,\psi)}$; then $e \in M'$. Suppose that $e \neq 1$. Then $\psi(1-e) > 0$. We can write $\psi = \sum_{i \in I} \omega_{\zeta_i,\zeta_i}$ for certain $\zeta_i \in H$. Then for at least one ζ_i , we have $((1-e)\zeta_i|\zeta_i) \neq 0$ so that $(1-e)\zeta_i \neq 0$. On the other hand, we have

$$\forall y \in n_{\psi}: \|y\zeta_i\|^2 \leq \psi(y^*y) = \|\Lambda_{\psi}(y)\|^2$$

so that $\zeta_i \in D(H,\psi)$ and hence $e\zeta_i = \zeta_i$. This is a contradiction. Hence we must have $e = 1$ and $D(H,\psi)$ is dense in H . ∎

Let $\xi \in H$. By Proposition 2, (iii)*, $D(R^{\psi}(\xi)^*)$ is invariant under the action of M' . Hence the projection p onto $\overline{D(R^{\psi}(\xi)^*)}$ is in M . Considered as an operator from pH to H_{ψ} , $R^{\psi}(\xi)^*$ is closed and densely defined and hence $|R^{\psi}(\xi)^*|^2$ exists as a positive self-adjoint operator on pH which by Proposition 2, (iii)*, is affiliated with pMp . We denote by $\theta^{\psi}(\xi,\xi)$ the element of $\hat{M}_+$ (the extended positive part of M) associated with the couple $(pH, |R^{\psi}(\xi)^*|^2)$ as in [7, Example 1.2 and Lemma 1.4], i.e.

<u>Definition 5</u>. For each $\xi \in H$, we denote by

$$\theta^{\psi}(\xi,\xi)$$

the element of $\hat{M}_+$ characterized by

$$\forall \eta \in H: \langle \omega_{\eta,\eta}, \theta^{\psi}(\xi,\xi) \rangle = \begin{cases} \|R^{\psi}(\xi)^*\eta\|^2 & \text{if } \eta \in D(R^{\psi}(\xi)^*) \\ \infty & \text{otherwise} \end{cases} \quad . \qquad (2)$$

<u>Remark 6</u>. If $\xi \in D(H,\psi)$, we simply have

$$\theta^{\psi}(\xi,\xi) = R^{\psi}(\xi)R^{\psi}(\xi)^* . \tag{3}$$

<u>Proposition 7</u>. For all $\xi \in H$ and $x \in M$, we have

$$\theta^{\psi}(x\xi,x\xi) = x\cdot\theta^{\psi}(\xi,\xi)\cdot x^* .$$

<u>Proof</u>. For all $\eta \in H$, we have, using Proposition 2, (ii)*, and Definition 5

$$\begin{aligned}\langle\omega_{\eta,\eta},\theta^{\psi}(x\xi,x\xi)\rangle &= \langle\omega_{x^*\eta,x^*\eta},\theta^{\psi}(\xi,\xi)\rangle \\ &= \langle x^*\cdot\omega_{\eta,\eta}\cdot x,\theta^{\psi}(\xi,\xi)\rangle \\ &= \langle\omega_{\eta,\eta},x\cdot\theta^{\psi}(\xi,\xi)\cdot x^*\rangle\end{aligned}$$

where the last equality simply follows from the definition of the operation $m \mapsto x\cdot m\cdot x^*$ in $\hat{M}_+$. ∎

Recall that by [7, Proposition 1.10], every normal weight φ has a unique extension, also denoted φ, to a normal weight on $\hat{M}_+$.

<u>Definition 8</u>. Let φ be a normal weight on M. We define

$$q_{\varphi}: H \to [0,\infty]$$

by

$$q_{\varphi}(\xi) = \langle\varphi,\theta^{\psi}(\xi,\xi)\rangle , \quad \xi \in H . \tag{4}$$

<u>Proposition 9</u>. Let φ be a normal weight on M. Then q_{φ} is a l.s.c. quadratic form on H i.e.

(i) $\forall \xi_1,\xi_2 \in H$: $q_{\varphi}(\xi_1+\xi_2) + q_{\varphi}(\xi_1-\xi_2) = 2q_{\varphi}(\xi_1) + 2q_{\varphi}(\xi_2)$,

(ii) $\forall \xi \in H\ \forall\lambda \in \mathbb{C}$: $q_{\varphi}(\lambda\xi) = |\lambda|^2 q_{\varphi}(\xi)$,

(iii) q_{φ} is lower semi-continuous.

<u>Proof</u>. (ii) is immediate. For the proof of (i) and (iii), first suppose that $\varphi = \omega_{\eta,\eta}$ for some $\eta \in H$. Then

$$q_{\varphi}(\xi) = \langle\omega_{\eta,\eta},\theta^{\psi}(\xi,\xi)\rangle = \begin{cases}\|R^{\psi}(\xi)^*\eta\|^2 & \text{if } \eta \in D(R^{\psi}(\xi)^*) \\ \infty & \text{otherwise}\end{cases} . \tag{5}$$

Let $\xi_1,\xi_2 \in H$. We shall prove that

$$q_{\varphi}(\xi_1+\xi_2) + q_{\varphi}(\xi_1-\xi_2) \le 2q_{\varphi}(\xi_1) + 2q_{\varphi}(\xi_2) . \tag{6}$$

If either $\eta \notin D(R^{\psi}(\xi_1)^*)$ or $\eta \notin D(R^{\psi}(\xi_2)^*)$, the right hand side of (6) is $+\infty$ and hence (6) holds. Now suppose that $\eta \in D(R^{\psi}(\xi_1)^*)$ and $\eta \in D(R^{\psi}(\xi_2)^*)$. Then by Proposition 2,(i)*, also $\eta \in D(R^{\psi}(\xi_1+\xi_2)^*)$ and $\eta \in D(R^{\psi}(\xi_1-\xi_2)^*)$. Furthermore,

$$\begin{aligned}&\|R^{\psi}(\xi_1+\xi_2)^*\eta\|^2 + \|R^{\psi}(\xi_1-\xi_2)^*\eta\|^2 \\ &= \|R^{\psi}(\xi_1)^*\eta+R^{\psi}(\xi_2)^*\eta\|^2 + \|R^{\psi}(\xi_1)^*\eta-R^{\psi}(\xi_2)^*\eta\|^2 \\ &= 2\|R^{\psi}(\xi_1)^*\eta\|^2 + 2\|R^{\psi}(\xi_2)^*\eta\|^2 .\end{aligned}$$

Thus we have proved (6) in all cases.

By (6) applied to $\xi_1+\xi_2$ and $\xi_1-\xi_2$ we get

$$4(q_{\varphi}(\xi_1)+q_{\varphi}(\xi_2)) = q_{\varphi}(2\xi_1) + q_{\varphi}(2\xi_2) \le 2q_{\varphi}(\xi_1+\xi_2) + 2q_{\varphi}(\xi_1-\xi_2) .$$

In all, we have shown (i).

By (5), we have

$$<\omega_{\eta,\eta},\theta^{\psi}(\xi,\xi)> = \sup\{|(R^{\psi}(\xi)^*\eta|\zeta)|^2 \mid \zeta \in D(R^{\psi}(\xi)),\ \|\zeta\| \leq 1\}$$

$$= \sup\{|(\eta|R^{\psi}(\xi)\Lambda_{\psi}(y))|^2 \mid y \in n_{\psi},\ \|\Lambda_{\psi}(y)\| \leq 1\}$$

$$= \sup\{|(\eta|y\xi)|^2 \mid y \in n_{\psi},\ \|\Lambda_{\psi}(y)\| \leq 1\}$$

for all $\xi \in H$. Since each $\xi \mapsto |(\eta|y\xi)|^2$ is continuous, this implies (iii).

Now let φ be an arbitrary normal weight. Then we can write

$$\varphi = \sum_{i\in I} \omega_{\eta_i,\eta_i}$$

and thus (cf. the proof of [7, Proposition 1.10])

$$\forall\xi \in H:\ q_{\varphi}(\xi) = <\varphi,\theta^{\psi}(\xi,\xi)> = \sum_{i\in I} <\omega_{\eta_i,\eta_i},\theta^{\psi}(\xi,\xi)> .$$

Now (i) and (iii) follow by the first part of the proof. ∎

Remark 10. Let φ be a normal weight on M. Write

$$\mathrm{Dom}(q_{\varphi}) = \{\xi \in H \mid q_{\varphi}(\xi) < \infty\} . \tag{7}$$

Then for all $x \in n_{\varphi}$ and $\xi \in D(H,\psi)$, we have

$$x^*\xi \in \mathrm{Dom}(q_{\varphi}) . \tag{8}$$

Indeed,

$$q_{\varphi}(x^*\xi) = <\varphi,\theta^{\psi}(x^*\xi,x^*\xi)>$$

$$= <\varphi,x^*\cdot\theta^{\psi}(\xi,\xi)\cdot x>$$

$$\leq \|\theta^{\psi}(\xi,\xi)\| <\varphi,x^*x> < \infty .$$

In particular, if φ is semifinite then $\mathrm{Dom}(q_{\varphi})$ is dense in H (since n_{φ}^* is strongly dense in M).

Definition 11. For each normal weight φ on M, we define the spatial derivative $\frac{d\varphi}{d\psi}$ as the unique element of $\hat{B(H)}_+$ such that

$$\forall\xi \in H:\ <\omega_{\xi,\xi},\frac{d\varphi}{d\psi}> = <\varphi,\theta^{\psi}(\xi,\xi)> . \tag{9}$$

The existence of $\frac{d\varphi}{d\psi}$ follows from Proposition 9 and [7, proof of Lemma 1.4].

Remark 12. If φ is semifinite, $\frac{d\varphi}{d\psi}$ is simply a positive self-adjoint operator on H (since in this case, $\{\xi \in H \mid <\omega_{\xi,\xi},\frac{d\varphi}{d\psi}> < \infty\} = \mathrm{Dom}(q_{\varphi})$ is dense in H). Note that

$$\forall\xi \in H:\ q_{\varphi}(\xi) = \begin{cases} \left\|\left(\frac{d\varphi}{d\psi}\right)^{\frac{1}{2}}\xi\right\|^2 & \text{if } \xi \in D\left(\left(\frac{d\varphi}{d\psi}\right)^{\frac{1}{2}}\right) \\ \infty & \text{otherwise} \end{cases} . \tag{10}$$

We shall see below (Proposition 22) that the definition of $\frac{d\varphi}{d\psi}$ given here agrees with that given in [1]. (This is not quite obvious. Note that in [1, Lemma 6], the quadratic form q is only defined on the subspace $D(H,\psi)$, and then extended by [1, Lemma 5] to the whole of H.)

Lemma 13. Let φ_1,φ_2, $(\varphi_i)_{i\in I}$, and φ be normal weights on M and let $x \in M$. Then

(i) $\forall m \in \hat{M}_+:\ <\varphi_1+\varphi_2,m> = <\varphi_1,m> + <\varphi_2,m>$,

(ii) $\forall m \in \hat{M}_+:\ <x\cdot\varphi\cdot x^*,m> = <\varphi,x^*\cdot m\cdot x>$,

(iii) if $\varphi_i \nearrow \varphi$, then $\forall m \in \hat{M}_+:\ <\varphi_i,m> \nearrow <\varphi,m>$.

Proof. (i) and (ii) are immediate consequences of [7, Proposition 1.10] (or its proof). As for (iii), we have by the proof of [7, Proposition 1.10], using the notation from there,

$$\langle\varphi_i, m\rangle = \sup_n \langle\varphi_i, \int_0^n \lambda \, de_\lambda\rangle + \infty\cdot\varphi_i(p)$$

$$\nearrow \sup_n \langle\varphi, \int_0^n \lambda \, de_\lambda\rangle + \infty\cdot\varphi(p) = \langle\varphi, m\rangle \ . \quad \blacksquare$$

Theorem 14. For all normal weights φ_1, φ_2, and φ on M and all $x \in M$ we have

(a) $$\frac{d(\varphi_1+\varphi_2)}{d\psi} = \frac{d\varphi_1}{d\psi} + \frac{d\varphi_2}{d\psi} \ ,$$

(b) $$\frac{d(x\cdot\varphi\cdot x^*)}{d\psi} = x \cdot \frac{d\varphi}{d\psi} \cdot x^* \ .$$

Remark 15. The sums and products occurring at the right hand side of (a) and (b) are to be understood in the sense of the operations in $\hat{B(H)}_+$. In particular, if φ_1, φ_2, $\varphi_1+\varphi_2$ are semifinite, $\frac{d\varphi_1}{d\psi} + \frac{d\varphi_2}{d\psi}$ is the form sum of the positive self-adjoint operators $\frac{d\varphi_1}{d\psi}$ and $\frac{d\varphi_2}{d\psi}$. Similarly, if $x\cdot\varphi\cdot x^*$ is semifinite, $x\cdot\frac{d\varphi}{d\psi}\cdot x^*$ is the form product.

Remark 16. In [1], the sum property is simply stated without proof. It seems to be difficult to give a proof using only the methods of [1] (one only gets "$\geq$") . - The product property is stated (and proved) only for invertible $x \in M$.

Proof of Theorem 14. Let $\xi \in H$. Then, using successively Definition 11, Lemma 13, Definition 11 again, and the definition of the sum in $\hat{B(H)}_+$, we get

$$\langle\omega_{\xi,\xi}, \frac{d(\varphi_1+\varphi_2)}{d\psi}\rangle = \langle\varphi_1+\varphi_2, \theta^\psi(\xi,\xi)\rangle$$

$$= \langle\varphi_1, \theta^\psi(\xi,\xi)\rangle + \langle\varphi_2, \theta^\psi(\xi,\xi)\rangle$$

$$= \langle\omega_{\xi,\xi}, \frac{d\varphi_1}{d\psi}\rangle + \langle\omega_{\xi,\xi}, \frac{d\varphi_2}{d\psi}\rangle$$

$$= \langle\omega_{\xi,\xi}, \frac{d\varphi_1}{d\psi} + \frac{d\varphi_2}{d\psi}\rangle \ .$$

Similarly,

$$\langle\omega_{\xi,\xi}, \frac{d(x\cdot\varphi\cdot x^*)}{d\psi}\rangle = \langle x\cdot\varphi\cdot x^*, \theta^\psi(\xi,\xi)\rangle = \langle\varphi, x^*\cdot\theta^\psi(\xi,\xi)\cdot x\rangle$$

$$= \langle\varphi, \theta^\psi(x^*\xi, x^*\xi)\rangle = \langle\omega_{x^*\xi,x^*\xi}, \frac{d\varphi}{d\psi}\rangle$$

$$= \langle x^*\cdot\omega_{\xi,\xi}\cdot x, \frac{d\varphi}{d\psi}\rangle = \langle\omega_{\xi,\xi}, x\cdot\frac{d\varphi}{d\psi}\cdot x^*\rangle$$

where we have used Lemma 13 and Proposition 7. $\blacksquare$

Theorem 17. Let $(\varphi_i)_{i\in I}$ and φ be normal weights on M . Suppose that

$$\varphi_i \nearrow \varphi \ .$$

Then

$$\frac{d\varphi_i}{d\psi} \nearrow \frac{d\varphi}{d\psi} \ .$$

Remark 18. In particular, if φ is semifinite, we have $\frac{d\varphi_i}{d\psi} \nearrow \frac{d\varphi}{d\psi}$ in the usual sense of positive self-adjoint operators.

Proof of Theorem 17. For all $\xi \in H$, we have by Lemma 13

$$\langle\omega_{\xi,\xi}, \frac{d\varphi_i}{d\psi}\rangle = \langle\varphi_i, \theta^\psi(\xi,\xi)\rangle$$

$$\nearrow \langle\varphi, \theta^\psi(\xi,\xi)\rangle = \langle\omega_{\xi,\xi}, \frac{d\varphi}{d\psi}\rangle \ . \quad \blacksquare$$

<u>Lemma 19</u>. Let φ be a normal semifinite weight on M. Write $p = \operatorname{supp}\varphi$. Then for all $m \in \hat{M}_+$, we have

$$\langle\varphi,m\rangle = 0 \iff p\cdot m\cdot p = 0 .$$

<u>Proof</u>. Let $m = \int_0^\infty \lambda\, de_\lambda + \infty\cdot(1-r)$ be the spectral resolution of m. Put $x_n = \int_0^n \lambda\, de_\lambda$, $n \in \mathbb{N}$. Then

$$\begin{aligned}\langle\varphi,m\rangle = 0 &\iff \forall n \in \mathbb{N}: \langle\varphi,x_n\rangle = 0 \quad\text{and}\quad \langle\varphi,1-r\rangle = 0\\ &\iff \forall n \in \mathbb{N}: p\cdot x_n\cdot p = 0 \quad\text{and}\quad p\cdot(1-r)\cdot p = 0\\ &\iff p\cdot m\cdot p = 0 . \blacksquare\end{aligned}$$

<u>Theorem 20</u>. Let φ be a normal semifinite weight on M. Then

$$\operatorname{supp}\left(\frac{d\varphi}{d\psi}\right) = \operatorname{supp}(\varphi) . \tag{11}$$

In particular, $\frac{d\varphi}{d\psi}$ is injective if and only if φ is faithful.

<u>Proof</u>. Put $p = \operatorname{supp}\varphi \in M$. Now for all $\xi \in H$, we have, using Lemma 19 and Proposition 7:

$$\begin{aligned}\xi \in \ker\left(\frac{d\varphi}{d\psi}\right) &\iff \langle\omega_{\xi,\xi}, \frac{d\varphi}{d\psi}\rangle = 0\\ &\iff \langle\varphi,\theta^\psi(\xi,\xi)\rangle = 0\\ &\iff p\cdot\theta^\psi(\xi,\xi)\cdot p = 0\\ &\iff \theta^\psi(p\xi,p\xi) = 0\\ &\iff p\xi = 0\\ &\iff \xi \in (1-p)H .\end{aligned}$$

Since $\ker\left(\frac{d\varphi}{d\psi}\right) = \operatorname{supp}\left(\frac{d\varphi}{d\psi}\right)^\perp$, the result follows. $\blacksquare$

<u>Proposition 21</u>. Let $\xi \in H$. Then there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $D(H,\psi)$ satisfying

$$\xi_n \to \xi \quad\text{as}\quad n \to \infty$$

and such that

$$q_\varphi(\xi_n) \to q_\varphi(\xi) \quad\text{as}\quad n \to \infty \tag{12}$$

for all normal weights φ on M.

<u>Proof</u>. Let

$$\theta^\psi(\xi,\xi) = \int_0^\infty \lambda\, de_\lambda + \infty\cdot(1-p)$$

be the spectral resolution of $\theta^\psi(\xi,\xi)$. Then p is the projection onto $\overline{D(R^\psi(\xi)^*)}$. For each $n \in \mathbb{N}$, the operator $R^\psi(e_n\xi)^*$, being closed and everywhere defined (since $R^\psi(e_n\xi)^* = R^\psi(\xi)^* e_n$), must be bounded; hence $R^\psi(e_n\xi)$ is bounded and $e_n\xi \in D(H,\psi)$.

Take a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $D(H,\psi)$ such that $\zeta_n \to \xi$ (possible by Proposition 4). Then also $(1-p)\zeta_n \in D(H,\psi)$.

Now for each $n \in \mathbb{N}$, put

$$\xi_n = e_n\xi + (1-p)\zeta_n \in D(H,\psi) .$$

Then

$$\xi_n \to p\xi + (1-p)\xi = \xi \quad\text{as}\quad n \to \infty .$$

We claim that $(\xi_n)_{n\in\mathbb{N}}$ satisfies (12).

Hence, let φ be a normal weight on M. We consider two cases. If $\langle\varphi,\theta^\psi(\xi,\xi)\rangle = \infty$, (12) is trivially true; indeed, by the lower semicontinuity of q_φ, we have

$$\infty = q_\varphi(\xi) \le \liminf_{n\to\infty} q_\varphi(\xi_n) .$$

Now suppose that $\langle\varphi,\theta^{\psi}(\xi,\xi)\rangle < \infty$. We can write

$$\varphi = \sum_{i\in I} \omega_{\eta_i,\eta_i}$$

for certain $\eta_i \in H$. Then all

$$\langle\omega_{\eta_i,\eta_i},\theta^{\psi}(\xi,\xi)\rangle < \infty$$

so that $\eta_i \in D(R^{\psi}(\xi)^*) \subseteq pH$, whence

$$\omega_{\eta_i,\eta_i} = p\cdot\omega_{\eta_i,\eta_i}\cdot p \ .$$

Hence

$$\varphi = p\cdot\varphi\cdot p \ .$$

Now using

$$\begin{aligned} p\cdot\theta^{\psi}(\xi_n,\xi_n)\cdot p &= \theta^{\psi}(p\xi_n,p\xi_n) \\ &= \theta^{\psi}(e_n\xi,e_n\xi) \\ &= e_n\cdot\theta^{\psi}(\xi,\xi)\cdot e_n \\ &\nearrow p\cdot\theta^{\psi}(\xi,\xi)\cdot p \end{aligned}$$

it follows that

$$\begin{aligned} \langle\varphi,\theta^{\psi}(\xi_n,\xi_n)\rangle &= \langle\varphi,p\cdot\theta^{\psi}(\xi_n,\xi_n)\cdot p\rangle \\ &\nearrow \langle\varphi,p\cdot\theta^{\psi}(\xi,\xi)\cdot p\rangle \\ &= \langle\varphi,\theta^{\psi}(\xi,\xi)\rangle \ . \ \blacksquare \end{aligned}$$

Using Proposition 21, we can now prove that our definition of $\frac{d\varphi}{d\psi}$ agrees with Connes' [1]. Note that we also prove the existence of a biggest positive self-adjoint operator satisfying (13) below so that we do not need [1 , Lemma 5].

Proposition 22. Let φ be a normal semifinite weight on M .

1) The operator $\frac{d\varphi}{d\psi}$ is the biggest positive self-adjoint operator d satisfying

$$\forall\xi \in D(H,\psi):\ q_{\varphi}(\xi) = \begin{cases} \|d^{\frac12}\xi\|^2 & \text{if } \xi \in D(d^{\frac12}) \\ \infty & \text{otherwise} \end{cases} \ . \tag{13}$$

2) The operator $\frac{d\varphi}{d\psi}$ is the unique positive self-adjoint operator satisfying (13) and

$$d^{\frac12} = [d^{\frac12}|_{D(H,\psi)\cap D(d^{\frac12})}] \ . \tag{14}$$

Proof. 1) The operator $\frac{d\varphi}{d\psi}$ is characterized by (10). Hence, in particular, (13) holds.

Now let d be any positive self-adjoint operator satisfying (13). We shall prove that $d \le \frac{d\varphi}{d\psi}$. Let $\xi \in D((\frac{d\varphi}{d\psi})^{\frac12})$. By Proposition 21, there exist $\xi_n \in D(H,\psi)$ such that $\xi_n \to \xi$ and

$$q_{\varphi}(\xi_n) \to q_{\varphi}(\xi) \ .$$

On the other hand, the mapping $p\colon H \to [0,\infty]$ defined by

$$p(\xi) = \begin{cases} \|d^{\frac12}\xi\|^2 & \text{if } \xi \in D(d^{\frac12}) \\ \infty & \text{otherwise} \end{cases} \tag{15}$$

is lower semi-continuous (since $p(\xi) = \int_0^{\infty} \lambda\, d(e_{\lambda}\xi|\xi) = \sup \int_0^n \lambda\, d(e_{\lambda}\xi|\xi)$, where $d = \int_0^{\infty} \lambda\, de_{\lambda}$ is the spectral resolution of d) , whence

$$p(\xi) \le \liminf_{n\to\infty} q_{\varphi}(\xi_n) = q_{\varphi}(\xi) = \left\|\left(\frac{d\varphi}{d\psi}\right)^{\frac12}\xi\right\|^2 \ .$$

This shows that $d \le \frac{d\varphi}{d\psi}$.

2) First, let us show that $d = \frac{d\varphi}{d\psi}$ satisfies (14). Let $\xi \in D(d^{\frac{1}{2}})$. Take a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $D(H,\psi)$ as in Proposition 21. Since $q_\varphi(\xi_n) \to q_\varphi(\xi) = \|d^{\frac{1}{2}}\xi\|^2 < \infty$, we may assume that all $q_\varphi(\xi_n) < \infty$, i.e. all $\xi_n \in D(H,\psi) \cap D(d^{\frac{1}{2}})$. Now $\xi_n \to \xi$ and $\|d^{\frac{1}{2}}\xi_n\|^2 \to \|d^{\frac{1}{2}}\xi\|^2$. It follows that $d^{\frac{1}{2}}\xi_n \to d^{\frac{1}{2}}\xi$. Indeed,

$$0 \leq \limsup_{n\to\infty} \|d^{\frac{1}{2}}\xi - d^{\frac{1}{2}}\xi_n\|^2$$

$$= \limsup_{n\to\infty} (2\|d^{\frac{1}{2}}\xi\|^2 + 2\|d^{\frac{1}{2}}\xi_n\|^2 - \|d^{\frac{1}{2}}\xi + d^{\frac{1}{2}}\xi_n\|^2)$$

$$= 2\|d^{\frac{1}{2}}\xi\|^2 + 2\lim_{n\to\infty}\|d^{\frac{1}{2}}\xi_n\|^2 - \liminf_{n\to\infty}\|d^{\frac{1}{2}}\xi + d^{\frac{1}{2}}\xi_n\|^2 \leq 0 .$$

Next, assume that d is a positive self-adjoint operator satisfying (13) and (14). We shall prove that then d is the maximal positive self-adjoint operator satisfying (13). Define $p: H \to [0,\infty]$ as above (15). Then

$$\forall \xi \in H: q_\varphi(\xi) \leq p(\xi) . \qquad (16)$$

Indeed, if $\xi \notin D(d^{\frac{1}{2}})$, this is trivially true; if $\xi \in D(d^{\frac{1}{2}})$, take, by (14), $\xi_n \in D(H,\psi) \cap D(d^{\frac{1}{2}})$ such that

$$\xi_n \to \xi \quad \text{and} \quad d^{\frac{1}{2}}\xi_n \to d^{\frac{1}{2}}\xi .$$

Then $q_\varphi(\xi_n) = \|d^{\frac{1}{2}}\xi_n\|^2 \to \|d^{\frac{1}{2}}\xi\|^2 = p(\xi)$ so that

$$q_\varphi(\xi) \leq \liminf_{n\to\infty} q_\varphi(\xi_n) = p(\xi) .$$

Finally, (16) implies that $\frac{d\varphi}{d\psi} \leq d$, whence $\frac{d\varphi}{d\psi} = d$ by 1). ∎

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We recall [1, proof of Theorem 9]:

Example 23. Let M be the left algebra associated with a left Hilbert algebra $\mathfrak{A}$ in H. Let φ_0, resp. ψ_0, be the canonical weight on M, resp. M', associated with $\mathfrak{A}''$, resp. $\mathfrak{A}'$. Then

$$\frac{d\varphi_0}{d\psi_0} = \Delta ,$$

where Δ is the modular operator associated with φ_0.

Spatial derivatives are preserved by spatial isomorphisms:

Proposition 24. Let M_1 be a von Neumann algebra acting on a Hilbert space H_1. Suppose that

$$u: H \to H_1$$

is a unitary such that

$$u\, M\, u^* = M_1 .$$

Then for all normal semifinite weights φ on M and all normal faithful semifinite weights ψ on M', $u\cdot\varphi\cdot u^*$ and $u\cdot\psi\cdot u^*$ are weights on M_1 and M_1' respectively, and we have

$$\frac{d(u\cdot\varphi\cdot u^*)}{d(u\cdot\psi\cdot u^*)} = u \frac{d\varphi}{d\psi} u^* .$$

The proof is left to the reader.

Modular properties of spatial derivatives.

Here, we first recall, without proof, some main results from

[1] and then state some immediate corollaries. For the first theorems, recall that the spatial derivatives occurring in them are injective by Theorem 20.

Theorem 25. Let φ_1 and φ_2 be normal faithful semifinite weights on M, and let ψ be a normal faithful semifinite weight on M'. Then

$$\forall t \in \mathbb{R}: (D\varphi_1 : D\varphi_2)_t = \left(\frac{d\varphi_1}{d\psi}\right)^{it} \left(\frac{d\varphi_2}{d\psi}\right)^{-it} .$$

Theorem 26. Let φ and ψ be normal faithful semifinite weights on M and M', respectively. Then

(i) $\forall x \in M\ \forall t \in \mathbb{R}: \sigma_t^{\varphi}(x) = \left(\frac{d\varphi}{d\psi}\right)^{it} x \left(\frac{d\varphi}{d\psi}\right)^{-it}$,

(ii) $\forall y \in M'\ \forall t \in \mathbb{R}: \sigma_t^{\psi}(y) = \left(\frac{d\varphi}{d\psi}\right)^{-it} y \left(\frac{d\varphi}{d\psi}\right)^{it}$.

Corollary 27. Let φ and ψ be normal faithful semifinite weights on M and M', respectively. Then

(i) $\frac{d\varphi}{d\psi}\ \eta\ M'$ if and only if φ is a trace,

(ii) $\frac{d\varphi}{d\psi}\ \eta\ M$ if and only if ψ is a trace,

(iii) $\frac{d\varphi}{d\psi}\ \eta\ Z(M)$ if and only if both φ and ψ are traces.

Theorem 28. Let φ and ψ be normal faithful semifinite weights on M and M', respectively. Then

$$\left(\frac{d\varphi}{d\psi}\right)^{-1} = \frac{d\psi}{d\varphi} .$$

Property (ii) in Theorem 26 characterizes operators having the form $\frac{d\varphi}{d\psi}$:

Theorem 29. Let ψ be a normal faithful semifinite weight on M', and let a be a positive self-adjoint operator on H. Then the following are equivalent:

(i) $a = \frac{d\varphi}{d\psi}$ for some (necessarily unique) normal semifinite weight φ on M,

(ii) $\forall y \in M'\ \forall t \in \mathbb{R}: \sigma_{-t}^{\psi}(y) a^{it} = a^{it} y$.

Note that $\varphi \mapsto \frac{d\varphi}{d\psi}$ is injective by (9) combined with [1, Proposition 3].

Corollary 30. There is a bijective correspondence, characterized by the equation

$$\frac{d\tau}{d\tau'} = 1 , \tag{17}$$

between the sets of normal faithful semifinite traces τ and τ' on M and M', respectively. (In particular, M is semifinite if and only if M' is semifinite.)

Proof. Given τ, there exists by Theorem 26 and 29 a weight τ' on M' such that $\frac{d\tau}{d\tau'} = 1$. By Corollary 27, τ' is a trace. Conversely, given τ', by the same arguments, we can define τ. ∎

In case of algebras on standard form, this correspondence reduces to the usual correspondence given by J:

Corollary 31. Suppose that (M,H,J,P) is a standard form of M in the sense of [4, Definition 2.1]. Then for all normal faithful semifinite traces τ on M we have

$$\tau' = \tau(J \cdot J) .$$

Proof. Let u be the (unique) unitary carrying (M,H,J,P) onto (M,H_τ,J_τ,P_τ) . In H_τ we have by Example 23

$$\frac{d\tau}{d\tau(J\cdot J)} = \Delta_\tau = 1 .$$

Hence, by Proposition 24, also

$$\frac{d\tau}{d\tau(J\cdot J)} = 1$$

in H , whence $\tau' = \tau(J\cdot J)$. ∎

Corollary 32. Suppose that M is semifinite and that τ and τ' are normal faithful semifinite traces on M and M' related by (17). Then for all normal semifinite weights φ on M , we have

$$\forall t \in \mathbb{R}: \left(\frac{d\varphi}{d\tau'}\right)^{it} = (D\varphi : D\tau)_t .$$

Otherwise stated, for all positive self-adjoint operators $h \,\eta\, M$, we have

$$\frac{d\tau(h\ \cdot)}{d\tau'} = h .$$

Finally, we recall the notion of γ-homogeneity.

Definition 33. Let ψ be a normal faithful semifinite weight on M' , and let $\gamma \in \mathbb{R}$. A closed densely defined operator a on H with polar decomposition $a = u|a|$ is called γ-homogeneous with respect to ψ if

$$u \in M \text{ and } \forall y \in M' \ \forall t \in \mathbb{R}: \sigma_{\gamma t}{}^{\psi}(y)|a|^{it} = |a|^{it} y . \quad (18)$$

One can show (see [20, Proposition (1.1.6)]) that (18) is equivalent to requiring

$$ya \subseteq a\, \sigma_{i\gamma}{}^{\psi}(y) \quad (19)$$

for all $y \in M'$ analytic with respect to σ^{ψ} .

Note that the 0-homogeneous operators are precisely the operators affiliated with M , and the (-1)-homogeneous positive self-adjoint operators are precisely the $\frac{d\varphi}{d\psi}$. As a corollary of Theorem 29, we have

Corollary 34. Let ψ be a normal faithful semifinite weight on M' , and let $p \in [1,\infty]$. Let a be a closed densely defined operator on H with polar decomposition $a = u|a|$. Then the following are equivalent:

(i) $u \in M$ and $|a|^p = \frac{d\varphi}{d\psi}$ for some normal semifinite weight φ on M ,

(ii) a is $(-1/p)$-homogeneous.

IV

SPATIAL L^p SPACES

In this chapter, we describe the Connes/Hilsum construction of spatial L^p spaces.

Let M be a von Neumann algebra acting on a Hilbert space H and let ψ_0 be a normal faithful semifinite weight on the commutant M' of M.

The notation is as in Chapter II and III.

<u>Definition 1</u>. For each positive self-adjoint (-1)-homogeneous operator a we define the <u>integral with respect to</u> ψ_0 by

$$\int a\, d\psi_0 = \varphi(1) \;, \tag{1}$$

where φ is the (unique) normal semifinite weight on M such that $a = \frac{d\varphi}{d\psi_0}$.

<u>Notation</u>. For each $p \in [1,\infty]$, we denote by

$$\overline{M}_{-1/p}$$

the set of <u>closed densely defined</u> (-1/p)-<u>homogeneous operators on</u> H.

<u>Definition 2</u>. Let $p \in [1,\infty[$. We put

$$L^p(\psi_0) = L^p(M,H,\psi_0) = \{a \in \overline{M}_{-1/p} \mid \int |a|^p\, d\psi_0 < \infty\} \tag{2}$$

and

$$\|a\|_p = \left(\int |a|^p\, d\psi_0\right)^{1/p} \;, \; a \in L^p(\psi_0) \;. \tag{3}$$

For $p = \infty$, we put

$$L^\infty(\psi_0) = M \tag{4}$$

and write $\|\cdot\|_\infty$ for the usual operator norm on M.

Note that when a is (-1/p)-homogeneous, the operator $|a|^p$ is (-1)-homogeneous so that the integral occurring at the right hand side of (2) is defined.

The spaces $L^p(\psi_0)$ are called <u>spatial</u> L^p spaces (as opposed to the abstract L^p spaces of Haagerup).

We now follow the first part of [10] to describe the relationship between the $L^p(\psi_0)$ and Haagerup's $L^p(M)$.

Let φ_0 be a normal faithful semifinite weight on M . Put

$$d_0 = \frac{d\varphi_0}{d\psi_0} \;. \tag{5}$$

Then

$$\forall t \in \mathbb{R}\, \forall x \in M:\ \sigma_t^{\varphi_0}(x) = d_0^{\,it}\, x\, d_0^{\,-it} \;. \tag{6}$$

We define a unitary operator u_0 on the Hilbert space $L^2(\mathbb{R},H)$ by

$$(u_0\xi)(t) = d_0^{\,it}\xi(t) \;,\; \xi \in L^2(\mathbb{R},H) \;,\; t \in \mathbb{R} \;. \tag{7}$$

Recall that the crossed product $N = R(M,\sigma^{\varphi_0})$ is generated by the elements $\pi(x)$, $x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, as described in the beginning of Chapter II. We shall describe the action of $u_0(\cdot)u_0^*$ on these generating elements.

By $\ell(s)$, $s \in \mathbb{R}$, we denote the operator of translation by s in $L^2(\mathbb{R})$:

$$(\ell(s)f)(t) = f(t-s), \quad f \in L^2(\mathbb{R}), \quad t \in \mathbb{R}.$$

We identify $L^2(\mathbb{R},H)$ with $H \otimes L^2(\mathbb{R})$ (so that $v \otimes f$, $v \in H$, $f \in L^2(\mathbb{R})$, is identified with $\xi \in L^2(\mathbb{R},H)$ given by $\xi(t) = f(t)v$, $t \in \mathbb{R}$).

<u>Proposition 3</u>. 1) For all $x \in M$, we have

$$u_0 \, \pi(x) u_0^* = x \otimes 1 .$$

2) For all $s \in \mathbb{R}$, we have

$$u_0 \, \lambda(s) u_0^* = d_0^{is} \otimes \ell(s) .$$

<u>Proof</u>. Let $\xi \in L^2(\mathbb{R},H)$. Then

$$\begin{aligned}(u_0 \, \pi(x) u_0^* \xi)(t) &= d_0^{it} \sigma_{-t}^{\varphi_0}(x) d_0^{-it} \xi(t) \\ &= d_0^{it} d_0^{-it} \, x \, d_0^{it} d_0^{-it} \, \xi(t) \\ &= x \, \xi(t), \quad t \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(u_0 \, \lambda(s) u_0^* \xi)(t) &= d_0^{it} (u_0^* \xi)(t-s) \\ &= d_0^{it} d_0^{-i(t-s)} \xi(t-s) \\ &= d_0^{is} \xi(t-s), \quad t \in \mathbb{R}.\end{aligned}$$

This proves the result since for $\xi = v \otimes f$, $v \in H$, $f \in L^2(\mathbb{R})$, we have

$$((x \otimes 1)(v \otimes f))(t) = (xv \otimes f)(t) = f(t)xv = xf(t)v = x\,\xi(t), \quad t \in \mathbb{R},$$

and

$$\begin{aligned}((d_0^{is} \otimes \ell(s))(v \otimes f))(t) &= (d_0^{is} v \otimes \ell(s) f)(t) \\ &= (\ell(s)f)(t) d_0^{is} v \\ &= f(t-s) d_0^{is} v \\ &= d_0^{is} \, \xi(t-s), \quad t \in \mathbb{R}. \ \blacksquare\end{aligned}$$

We denote by T the unique positive self-adjoint operator in $L^2(\mathbb{R})$ characterized by

$$\forall s \in \mathbb{R}: \ T^{is} = \ell(s). \tag{8}$$

For the definition and properties of tensor products of closed operators we refer to [17, Section 9.33].

<u>Proposition 4</u>. For all normal semifinite weights φ on M we have

$$u_0 \, h_\varphi \, u_0^* = \frac{d\varphi}{d\psi_0} \otimes T . \tag{9}$$

<u>Proof</u>. First suppose that φ is faithful. Then

$$h_\varphi^{it} h_{\varphi_0}^{-it} = (D\tilde{\varphi}:D\tau)_t (D\tau:D\tilde{\varphi}_0)_t = (D\tilde{\varphi}:D\tilde{\varphi}_0)_t = \pi((D\varphi:D\varphi_0)_t)$$

and

$$(D\varphi:D\varphi_0)_t = \left(\frac{d\varphi}{d\psi_0}\right)^{it} \left(\frac{d\varphi_0}{d\psi_0}\right)^{-it}$$

for all $t \in \mathbb{R}$, so that by Proposition 3 and the fact that $h_{\varphi_0}^{it} = \lambda(t)$ for all $t \in \mathbb{R}$, we get

$$u_0\, h_\varphi^{\ it}\, u_0^* = (u_0 h_\varphi^{\ it} h_{\varphi_0}^{\ -it} u_0^*)(u_0 h_{\varphi_0}^{\ it} u_0^*)$$

$$= \left(\left(\frac{d\varphi}{d\psi_0}\right)^{it}\left(\frac{d\varphi_0}{d\psi_0}\right)^{-it} \otimes 1\right)\left(\left(\frac{d\varphi_0}{d\psi_0}\right)^{it} \otimes \ell(t)\right)$$

$$= \left(\frac{d\varphi}{d\psi_0}\right)^{it} \otimes T^{it}$$

for all $t \in \mathbb{R}$, and (9) follows.

In the general case, choose a normal semifinite weight χ with supp $\chi = 1-p$ where $p = \text{supp}\ \varphi$. Then $\varphi+\chi$ is a normal faithful semifinite weight and hence, by the first part of the proof,

$$u_0\, h_{\varphi+\chi}\, u_0^* = \frac{d(\varphi+\chi)}{d\psi_0} \otimes T .$$

Since $p = \text{supp}\ \frac{d\varphi}{d\psi}$ and $\pi(p) = \text{supp}\ h_\varphi$, this implies that

$$u_0\, h_\varphi\, u_0^* = u_0(\pi(p)\cdot h_{\varphi+\chi}\cdot\pi(p))u_0^*$$

$$= u_0\,\pi(p)u_0^*\cdot u_0\, h_{\varphi+\chi}\, u_0^*\cdot u_0\,\pi(p)u_0^*$$

$$= (p\otimes 1)\cdot\left(\frac{d(\varphi+\chi)}{d\psi_0} \otimes T\right)\cdot(p\otimes 1)$$

$$= \left(p\cdot\frac{d(\varphi+\chi)}{d\psi_0}\cdot p\right) \otimes T = \frac{d\varphi}{d\psi_0} \otimes T . \quad \blacksquare$$

<u>Corollary 5</u>. The mapping

$$a \mapsto u_0^*(a\otimes T)u_0$$

is a bijection of the set of positive self-adjoint (-1)-homogeneous operators a on H onto the set of positive self-adjoint operators h affiliated with $R(M,\sigma^{\varphi_0})$ satisfying

$$\forall s \in \mathbb{R}: \theta_s\, h = e^{-s} h . \tag{10}$$

Furthermore,

$$\int a\, d\psi_0 = tr(u_0^*(a\otimes T)u_0) \tag{11}$$

for all such a.

<u>Proof</u>. Since the mapping in question is nothing but $\frac{d\varphi}{d\psi_0} \mapsto h_\varphi$, it is a bijection by Proposition 4 in Chapter II. By definition, we have $\int \frac{d\varphi}{d\psi_0}\, d\psi_0 = \varphi(1) = tr(h_\varphi)$. $\blacksquare$

<u>Corollary 6</u>. Let $p \in [1,\infty[$. Let a be a closed densely defined operator on H. Then

1) $a \in \overline{M}_{-1/p}$ if and only if

$$u_0^*(a\otimes T^{1/p})u_0 \ \eta\ R(M,\sigma^{\varphi_0}) ,$$

2) $a \in L^p(\psi_0)$ if and only if

$$u_0^*(a\otimes T^{1/p})u_0 \in L^p(M) .$$

For all $a \in L^p(\psi_0)$, we have

$$\|a\|_p = \|u_0^*(a\otimes T^{1/p})u_0\|_p .$$

<u>Corollary 7</u>. Let $p \in [1,\infty[$. Then the mapping

$$a \mapsto u_0^*(a\otimes T^{1/p})u_0 \quad ($$

is a bijection of $\overline{M}_{-1/p}$ onto the set of closed densely defined operators h affiliated with $R(M,\sigma^{\varphi_0})$ satisfying

$$\forall s \in \mathbb{R}: \theta_s\, h = e^{-s/p}\, h . \quad ($$

Proof of Corollary 6 and 7. Let a be a closed densely defined operator on H with polar decomposition $a = u|a|$. Then

$$h = u_0^*(u\otimes 1)u_0(u_0^*(|a|\otimes T)u_0)^{1/p}$$

is the polar decomposition of $h = u_0^*(a\otimes T^{1/p})u_0$. Corollary 6, 1), and Corollary 7 now follow from Corollary 5 and Proposition 3, 1) (and the fact that $a \mapsto a\otimes T^{1/p}$ is injective). The rest of Corollary 6 follows from the equation $\int |a|^p \, d\psi_0 = \mathrm{tr}(|u_0^*(|a|\otimes T^{1/p})u_0|^p)$. ∎

Proposition 8. Let $p \in [1,\infty]$. Then for all $a \in L^p(\psi_0)$, we have $a^* \in L^p(\psi_0)$ and

$$\|a^*\|_p = \|a\|_p .$$

Proof. Let $a \in L^p(\psi_0)$. Then $a\otimes T^{1/p} \in u_0 L^p(M)u_0^*$. Hence also $a^*\otimes T^{1/p} = (a\otimes T^{1/p})^* \in u_0 L^p(M)u_0^*$. Thus $a^* \in L^p(\psi_0)$ by Corollary 6 and $\|a^*\|_p = \|u_0^*(a^*\otimes T^{1/p})u_0\|_p = \|u_0^*(a\otimes T^{1/p})u_0\|_p = \|a\|_p$. ∎

If we identify $L^2(\mathbb{R})$ with $L^2(\mathbb{R})$ via Fourier transformation, T is simply the multiplication operator in $L^2(\mathbb{R})$ given by multiplication by $t \mapsto e^t$, and similarly, for each $p \in [1,\infty[$, $T^{1/p}$ is simply multiplication by $t \mapsto e^{t/p}$. This observation will permit us to obtain information about operators a on H from information about the tensor products $a\otimes T^{1/p}$. First we have:

Lemma 9. Let a be a closed densely defined operator on H and f a Borel function on $\mathbb{R}$, and denote by m_f the corresponding multiplication operator on $L^2(\mathbb{R})$. Write

$$D = \{\xi \in L^2(\mathbb{R},H) \mid \xi(t) \in D(a) \text{ for a.a. } t \in \mathbb{R} \text{ and } \int \|f(t)a\xi(t)\|^2 dt < \infty\} .$$

Then $D(a\otimes m_f) = D$ and

$$((a\otimes m_f)\xi)(t) = f(t)a\xi(t) \ , \ \xi \in D \ , \ t \in \mathbb{R} .$$

Proof. Denote by $m(a,f)$ the operator in $L^2(\mathbb{R},H)$ given by

$$D(m(a,f)) = D$$

and

$$(m(a,f)\xi)(t) = f(t)a\xi(t) \ , \ \xi \in D \ , \ t \in \mathbb{R} .$$

Then $m(a,f)$ is a closed operator and

$$m(a^*,\overline{f}) \subseteq m(a,f)^*$$

(in fact, equality holds). Now evidently

$$a \odot m_f \subseteq m(a,f) ,$$

where $a \odot m_f$ denotes the algebraic tensor product of a and m_f, and hence

$$a \otimes m_f = [a\odot m_f] \subseteq m(a,f) .$$

Applying this to a^* and $\overline{f}$, we get

$$a^* \otimes m_{\overline{f}} \subseteq m(a^*,\overline{f}) .$$

Combining this, and using that $(A\otimes B)^* = A^*\otimes B^*$, we find that

$$m(a,f) \subseteq m(a^*,\overline{f})^* \subseteq (a^*\otimes m_{\overline{f}})^* = a\otimes m_f .$$

In all, we have shown that $a\otimes m_f = m(a,f)$. ∎

$(a\otimes m_f = m(a,f))$

Lemma 10. Let $p \in [1,\infty]$ and $a,b \in L^p(\psi_0)$.

Then $a+b$ is densely defined and preclosed, and

$$[a+b] \in L^p(\psi_0) .$$

Proof. 1) Denote by e the projection onto $\overline{D(a) \cap D(b)}$. Then

$$\begin{aligned}&(e\otimes 1)L^2(\mathbb{R},H)\\ &= \{\xi \in L^2(\mathbb{R},H) \mid \xi(t) = e\,\xi(t) \text{ for a.a. } t \in \mathbb{R}\}\\ &\supseteq \{\xi \in L^2(\mathbb{R},H) \mid \xi(t) \in D(a) \cap D(b) \text{ for a.a. } t \in \mathbb{R}\} .\end{aligned}$$

By Lemma 9, this set contains

$$D(a\otimes T^{1/p}) \cap D(b\otimes T^{1/p}) .$$

Now since $a \otimes T^{1/p}$, $b \otimes T^{1/p} \in u_0\, L^p(\psi)u_0{}^*$, their sum is densely defined. Hence $D(a\otimes T^{1/p}) \cap D(b\otimes T^{1/p})$ is dense in $L^2(\mathbb{R},H)$. It follows that $e = 1$. Hence $D(a+b) = D(a) \cap D(b)$ is dense in H.

2) Now let us show that $a+b$ is preclosed. By Proposition 8, a^* and b^* are in $L^p(\psi_0)$ and hence by the first part of proof, $a^* + b^*$ is densely defined. Since $a+b \subseteq (a^*+b^*)^*$, $a+b$ is preclosed.

3) Finally, let us show that

$$[a+b] \otimes T^{1/p} = [(a\otimes T^{1/p}) + (b\otimes T^{1/p})] . \tag{14}$$

First, by the characterization of $a \otimes T^{1/p}$ given in Lemma 9 we obviously have

$$(a\otimes T^{1/p}) + (b\otimes T^{1/p}) \subseteq [a+b] \otimes T^{1/p} ,$$

whence

$$[(a\otimes T^{1/p}) + (b\otimes T^{1/p})] \subseteq [a+b] \otimes T^{1/p} .$$

On the other hand, again by that characterization,

$$[a+b] \otimes T^{1/p} \subseteq ((a^*\otimes T^{1/p}) + (b^*\otimes T^{1/p}))^* ,$$

and finally

$$((a^*\otimes T^{1/p}) + (b^*\otimes T^{1/p}))^* = [(a\otimes T^{1/p}) + (b\otimes T^{1/p})]$$

since $*$ is an involution in $L^p(M)$ (and hence respects the strong sum). In all, we have proved (14). Now the right hand side of (14) is in $u_0\, L^p(M)u_0{}^*$. Hence by Corollary 6, $[a+b] \in L^p(\psi_0)$. ∎

Lemma 11. Let $p,p_1,p_2 \in [1,\infty]$ such that $1/p = 1/p_1 + 1/p_2$. Let $a \in L^{p_1}(\psi_0)$ and $b \in L^{p_2}(\psi_0)$. Then ab is densely defined and preclosed and

$$[ab] \in L^p(\psi_0) .$$

Proof. 1) Denote by e the projection onto $D(ab)$. Then, using Lemma 9, we have

$$\begin{aligned}&D((a\otimes T^{1/p})(b\otimes T^{1/p}))\\ &\subseteq \{\xi \in D(b\otimes T^{1/p}) \mid b\xi(t) \in D(a) \text{ for a.a. } t \in \mathbb{R}\}\\ &\subseteq \{\xi \in L^2(\mathbb{R},H) \mid \xi(t) \in D(b) \text{ for a.a. } t \in \mathbb{R}\\ &\qquad \text{and } b\xi(t) \in D(a) \text{ for a.a. } t \in \mathbb{R}\}\\ &\subseteq \{\xi \in L^2(\mathbb{R},H) \mid \xi(t) \in D(ab) \text{ for a.a. } t \in \mathbb{R}\}\\ &\subseteq \{\xi \in L^2(\mathbb{R},H) \mid \xi(t) = e\xi(t) \text{ for a.a. } t \in \mathbb{R}\}\\ &= (e\otimes 1)L^2(\mathbb{R},H) .\end{aligned}$$

Hence $e = 1$ and ab is densely defined.

2) By 1) applied to b^* and a^*, b^*a^* is densely defined. Since $ab \subseteq (b^*a^*)^*$, ab is preclosed.

3) Finally let us show that

$$[ab] \circledast T^{1/p} = [(a \circledast T^{1/p})(b \circledast T^{1/p})] .$$

First, by Lemma 9,

$$(a \circledast T^{1/p})(b \circledast T^{1/p}) \subseteq [ab] \circledast T^{1/p},$$

whence

$$[(a \circledast T^{1/p})(b \circledast T^{1/p})] \subseteq [ab] \circledast T^{1/p} .$$

On the other hand,

$$[ab] \circledast T^{1/p} \subseteq ((b^* \circledast T^{1/p})(a^* \circledast T^{1/p}))^* = [(a \circledast T^{1/p})(b \circledast T^{1/p})] .$$

The result follows as in the proof of Lemma 10. ∎

Now we are ready to transform the results on the spaces $L^p(M)$ obtained in Chapter II into results on the $L^p(\psi_0)$ (for an alternative, see [10]).

From Corollary 7, Corollary 6, 2), and Lemma 10 we now get:

<u>Theorem 12</u>. Let $p \in [1,\infty]$. Then $(L^p(\psi_0), \|\cdot\|_p)$ is a Banach space with respect to strong sum.

The mapping $a \mapsto u_0^*(a \circledast T^{1/p})u_0$ is an isometric isomorphism of $L^p(\psi_0)$ onto $L^p(M)$.

<u>Notation</u>. From now on, the strong product $[ab]$ of operators a and b will be denoted $a \cdot b$.

<u>Proposition 13</u>. Let $p,q \in [1,\infty]$ with $1/p + 1/q = 1$. Then for all $a \in L^p(\psi_0)$ and $b \in L^q(\psi_0)$ we have

$$\int a \cdot b \, d\psi_0 = \int b \cdot a \, d\psi_0 .$$

<u>Proof</u>. We have $a \cdot b \in L^1(\psi_0)$ and $b \cdot a \in L^1(\psi_0)$, and

$$\int a \cdot b \, d\psi_0 = tr(u_0^*((a \circledast T^{1/p}) \cdot (b \circledast T^{1/p}))u_0)$$

$$= tr(u_0^*(a \circledast T^{1/p})u_0 \cdot u_0^*(b \circledast T^{1/p})u_0)$$

$$= tr(u_0^*(b \circledast T^{1/p})u_0 \cdot u_0^*(a \circledast T^{1/p})u_0)$$

$$= tr(u_0^*((b \circledast T^{1/p}) \cdot (a \circledast T^{1/p}))u_0)$$

$$= \int b \cdot a \, d\psi_0 . \quad ∎$$

The following results are now immediate corollaries of the corresponding results in Chapter II.

<u>Proposition 14</u> (Hölder's inequality). Let $p,q \in [1,\infty]$ with $1/p + 1/q = 1$. Then for all $a \in L^p(\psi_0)$ and $b \in L^q(\psi_0)$ we have

$$\|a \cdot b\|_1 \leq \|a\|_p \|b\|_q .$$

<u>Theorem 15</u>. Let $p \in [1,\infty[$ and define q by $1/q = 1 - 1/p$.

1) For each $b \in L^q(\psi_0)$, the mapping φ_b defined by

$$\varphi_b(a) = \int a \cdot b \, d\psi_0 , \quad a \in L^p(\psi_0) ,$$

is a bounded linear functional on $L^q(\psi_0)$.

2) For all $b \in L^q(\psi_0)$ we have

$$\|b\|_q = \sup\{|\int a \cdot b \, d\psi_0| \mid a \in L^p(\psi_0), \|a\|_p \leq 1\} .$$

3) The mapping

$$b \mapsto \varphi_b$$

is an isometric isomorphism of $L^q(\psi_0)$ onto the dual Banach space of $L^p(\psi_0)$.

Proposition 16. $L^2(\psi_0)$ is a Hilbert space with the inner product

$$(a,b) \mapsto \int b^* \cdot a \, d\psi_0 \, .$$

We define left and right actions of M on $L^2(\psi_0)$ by

$$\lambda(x)a = x \cdot a \, , \quad a \in L^2(\psi_0) \, ,$$

$$\rho(x)a = a \cdot x \, , \quad a \in L^2(\psi_0) \, ,$$

for all $x \in M$ (as usual, "$\cdot$" means "strong product").

Proposition 17. The quadruple $(\lambda, L^2(\psi_0), *, L^2(\psi_0)_+)$ is a standard form of M in the sense of [4].

- - - - -

L^p spaces with respect to a trace.

Suppose that τ is a normal faithful semifinite trace on M . Denote by τ' the trace on M' associated with τ via $\frac{d\tau}{d\tau'} = 1$. Now for each $p \in [1,\infty]$, the $(-1/p)$-homogeneous operators with respect to τ' are precisely the operators affiliated with M . Let a be a positive self-adjoint operator affiliated with M . Then

$$\tau(a) = \int a \, d\tau' \, , \tag{15}$$

since $\tau(a) = \tau(a \, \cdot)(1)$ and (by Chapter III, Corollary 32)

$$\frac{d\tau(a \, \cdot)}{d\tau'} = a \, . \tag{16}$$

It follows that for all $p \in [1,\infty]$, we have

$$L^p(\tau') = L^p(M,\tau) \, , \tag{17}$$

where $L^p(\tau')$ is a spatial L^p space as discussed in this chapter and $L^p(M,\tau)$ is as defined at the end of Chapter I. Hence L^p spaces as defined in this chapter are generalizations of the well-known L^p spaces with respect to a trace. On the other hand, all the results on L^p spaces that we have proved in particular apply to L^p spaces with respect to a trace, so that we have reproved the well-known properties of such spaces.

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Change of weight.

Let ψ_0 and ψ_1 be two normal faithful semifinite weights on M' . Then by Theorem 12, there exists an isometric isomorphism

$$\Phi: L^p(\psi_0) \to L^p(\psi_1) \tag{18}$$

characterized by

$$\forall a \in L^p(\psi_0): \; u_1^*(\Phi(a) \otimes T^{1/p}) u_1 = u_0^*(a \otimes T^{1/p}) u_0 \, , \tag{19}$$

where u_1 is the unitary on $L^2(\mathbb{R},H)$ constructed from $d_1 = \frac{d\varphi_0}{d\psi_1}$ in analogy with (7).

For positive injective $a \in L^p(\psi_0)$, we have $\Phi(a) = b$, where b is the positive self-adjoint operator on H characterized by

$$\forall t \in \mathbb{R}: \; b^{p\,it} = d_1^{\,it} d_0^{\,-it} a^{p\,it} \, . \tag{20}$$

Indeed, if φ is the normal faithful semifinite weight given by $a^p = \frac{d\varphi}{d\psi_0}$, then for all $t \in \mathbb{R}$ we have

$$\left(\frac{d\varphi}{d\psi_1}\right)^{it} = (D\psi_1 : D\psi_0)_{-t}\left(\frac{d\varphi}{d\psi_0}\right)^{it} = \left(\frac{d\varphi_0}{d\psi_1}\right)^{it}\left(\frac{d\varphi_0}{d\psi_0}\right)^{-it} a^{p\,it} = d_1{}^{it} d_0{}^{-it} a^{p\,it}$$

and

$$u_1{}^*\left(\left(\frac{d\varphi}{d\psi_1}\right)^{1/p} \otimes T^{1/p}\right)u_1 = \left(u_1{}^*\left(\frac{d\varphi}{d\psi_1} \otimes T\right)u_1\right)^{1/p}$$
$$= h_\varphi{}^{1/p}$$
$$= \left(u_0{}^*\left(\frac{d\varphi}{d\psi_0} \otimes T\right)u_0\right)^{1/p}$$
$$= u_0{}^*(a \otimes T^{1/p})u_0 .$$

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Note. The operators h_φ from Chapter II are themselves spatial derivatives of φ with respect to a certain weight χ_0 on the commutant $u_0{}^*(M' \otimes B(L^2(\mathbb{R})))u_0$ of $\pi(M)$ (where $\pi(M) \subseteq R(M, \sigma^{\varphi_0})$ is acting on $L^2(\mathbb{R}, H)$). Indeed, by Chapter III, Theorem 29, there exists a unique normal faithful semifinite weight χ_0 on this commutant such that

$$\forall t \in \mathbb{R}: \left(\frac{d\varphi_0}{d\chi_0}\right)^{it} = \lambda(t) = h_{\varphi_0}{}^{it} . \qquad (21)$$

It follows by Chapter III, Theorem 25, that

$$\forall t \in \mathbb{R}: \left(\frac{d\varphi}{d\chi_0}\right)^{it} = h_\varphi{}^{it}$$

for all normal faithful semifinite weights φ on M , and hence

$$h_\varphi = \frac{d\varphi}{d\chi_0} \qquad (22)$$

for all such φ . By the usual methods (cf. the proofs of Chapter II, Lemma 1, and this chapter, Proposition 4), this also holds for all normal semifinite not necessarily faithful weights.

One can show that if (M,H,J,P) is a standard form of M , then

$$\chi_0 = u_0{}^* \cdot (\varphi_0(J \cdot J) \otimes \mathrm{Tr}(T^{-1} \cdot)) \cdot u_0 ,$$

where Tr is the usual trace on $B(L^2(\mathbb{R}))$.

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