10 marca 2011
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(II) Perturbative QFT in curved spacetimes

1) Free Klein-Gordon field ✓
2) Time-ordered products
3) Renormalization (Ambiguities)
4) Yang-Mills theory
    a) Classical field theory prelims.
    b) BRST method
    c) Quantization

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1) Free KG-field

(M, g) - spacetime

KG eqn: $(\Box_g - m^2)\phi(x) = 0$

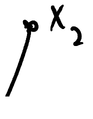
$$L_0 = \frac{1}{2}\left((\nabla\phi)^2 - m^2\phi^2 \right) d\mu$$

$\mathcal{O} = \nabla^{k_1}\phi \dots \nabla^{k_n}\phi$ arbitrary monomial in ϕ & derivatives

$$\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) = \sum_{a} \underbrace{C^{a}_{12\dots n}(x_1, \dots, x_n)}_{\text{OPE coefficients}} \mathcal{O}_a(x_n)$$

$$\phi(x_1)\phi(x_2) = H(x_1, x_2)\mathbb{1} + \phi^2(x_2) + \nabla^{\mu}_{x_2}\sigma(x_1, x_2)\,\phi\nabla_\mu\phi(x_2) + \cdots$$

$\sigma =$ signed^2 geodesic distance

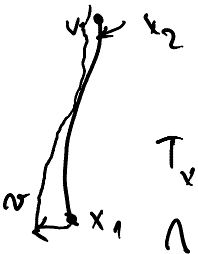


$$= \int_a^b \left(g_{\mu\nu}(\gamma(t))\, \dot\gamma^\mu(t)\, \dot\gamma^\nu(t) \right)^{1/2} dt$$

H = Hadamard form

$$= \text{const} \cdot \left(\frac{\Delta^{1/2}(x_1,x_2)}{\sigma} + \sum_{n\geq 0} v_n(x_1,x_2)\sigma^n \log \frac{\sigma}{L^2} \right) \}$$

Δ - geometric



$v \mapsto v'$

$T_{x_1}M \to T_{x_2}M$

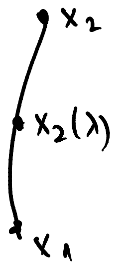
Δ determinant of this map

Δ = 0 when you have caustic

$v_n(x_1,x_2)$ determined by recursion rels.

$$v_{n+1}(x_1,x_2) = c_n \Delta^{1/2}(x_1,x_2) \int_0^1 d\lambda\, \lambda^n \underbrace{\frac{(\Box_{x_1} - m^2) v_n}{\Delta^{1/2}}}$$

taken @ argument $(x_1, x_2(\lambda))$ $\longrightarrow$



Minkowski: $\sigma = (x_1 - x_2)^2$

$\Delta = 1$

Recursion comes from imposing KG eqn. on H.

$$L \to L' \quad H \to H' = H + \underbrace{\sum v_n \sigma^n}\ \underbrace{\log \frac{L^2}{L'^2}}$$

smooth in const.

$$x_1, x_2$$

e.g. at $x_1 = x_2$ $\quad\downarrow$

$$\text{const}\cdot R + \text{const}\cdot m^2$$

Changing $L \to L'$ can compensated by changing $\phi^2 \to \phi^2 + \underbrace{(\text{const}\cdot R + \text{const}\cdot m^2)\mathbb{1}}$

$$\underbrace{\langle \phi(x_1)\,\phi(x_1)\rangle_\psi} = H(x_1, x_2)\underbrace{\langle \mathbb{1}\rangle_\psi}_{=1} + \underbrace{\langle \phi^2(x_2)\rangle_\psi}$$

$+\dots$

$\downarrow$ know this

$\Rightarrow$ approx. for LHS.

$\downarrow$ know $\langle \phi^2(x_2)\rangle_\psi = \lim_{x_1 \to x_2}\Big\{\overbrace{\langle \phi(x_1)\,\phi(x_2)\rangle_\psi}^{\to\infty} - \underbrace{H(x_1, x_2)}\Big\}$

"point splitting"

$$\phi^2(x_1)\,\phi^2(x_2) = \overbrace{H(x_1,x_2)^2}^{C^{\mathbb{1}}_{\phi^2\phi^2}}\,\mathbb{1} + \overbrace{2\,H(x_1,x_2)}^{\to\infty\ C^{\phi^2}_{\phi^2\phi^2}}\,\phi^2(x_2) + \dots + \underbrace{1}_{C^{\phi^4}_{\phi^2\phi^2}}\,\phi^4(x_2) + \dots$$

$$H^2 \sim \frac{1}{\sigma^2} \sim \frac{1}{[(x_1-x_2)^2]^2}$$

$$2H \sim \frac{2}{\sigma} \sim \frac{1}{(x_1-x_2)^2}$$

States are characterized by n-pt. fcts

$$\langle \phi^2(x_1)\, \phi^7(x_2)\, \phi(x_3) \dots \rangle_\psi$$

Simplest ones $\langle \phi(x_1) \cdots \phi(x_n) \rangle_\psi \leftarrow$

$$\underset{\uparrow}{=} \sum_{\text{pairs}}{}' \prod_{ij} \langle \phi(x_i)\phi(x_j)\rangle_\psi$$

if holds then state is called "Gaussian"

- Mink- vacuum
- Thermal states
- Not : superposition of vacuum and 2-particle state

$\langle \phi(x_1)\, \phi(x_2) \rangle_\psi$ must satisfy:

1) Because ϕ satisfies KG :

$$(\Box - m^2)_{x_1} \langle \phi(x_1)\, \phi(x_2)\rangle_\psi = 0 \qquad \text{same for } x_1 \leftrightarrow x_2$$

2) From OPE : $\langle \phi(x_1)\, \phi(x_2)\rangle_\psi = \underbrace{H}_{\text{state indep}}(x_1, x_2) + \text{smooth}_\psi$

Hadamard property

3) $\langle [\phi(x_1), \phi(x_2)] \rangle_\psi = iE(x_1, x_2)$

$$E = \underset{\uparrow}{\Delta_A} - \underset{\uparrow}{\Delta_R}$$

difference between advance & retarded fundament

4) Positivity $\int_{M\times M} \langle \phi(x_1)\,\phi(x_2)\rangle_\psi\, \overline{f(x_1)}\, f(x_2) \geq 0$

for any function $f(x)$

In practice such $\langle \phi(x_1)\,\phi(x_2)\rangle_\psi = G_\psi(x_1, x_2)$

are found by choosing +-ive frequency solutions.

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2) Time-ordered products

$$\text{"}\ \underset{\substack{\uparrow\\ \text{time ordering}}}{T}\{\underbrace{O_1(x_1)\cdots O_n(x_n)}_{O_1=\dots=O_n=\phi^4}\} = O_{\pi 1}(x_{\pi 1})\cdots O_{\pi n}(x_{\pi n})\ \text{"}$$

$x_{\pi 1} \notin J^-(x_{\pi 2}) \quad x_{\pi 2} \notin J^-(x_{\pi 3})$

...

You want this because you would like interacting fields assoc. $L = \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \underbrace{\lambda\phi^4}_{\text{self-interaction}}$

$$T(O_1(x_1)\,O_2(x_2)) = \underset{\uparrow}{\theta}(\underset{\uparrow}{T}(x_2) - T(x_1))\,O_2(x_2)\,O_1(x_1)$$

time coordinat

$$+\ \theta(T(x_1) - T(x_2))\,\underbrace{O_1(x_1)\,O_2(x_2)}_{C_{12}^{\mathbb{1}}(x_1,x_2)\,\mathbb{1}\,+\dots}$$

$$\frac{1}{[\sigma + i0T]^{(d_1+d_2)/2}}$$

d_1, d_2 dimensions of O_1, O_2

d_1, d_2 dimensions of $\mathcal{O}_1, \mathcal{O}_2$

Problem: $\mathcal{O}$ doesn't want to be multiplied by $[\sigma + i0T]^{(-d_1-d_2)/2}$

$$T(\phi^2(x_1)\,\phi^2(x_2)) = \frac{1}{\underbrace{[\sigma + i0]}_{\text{doesn't like to be }{}^2\text{'ed}}{}^2} + \dots$$

Example: Take self-enery of point charge

Self energy $\int E^2 \, d^3x = \int \underbrace{\frac{1}{r^4}}_{\text{isn't integrable at } r=0} d^3x = \infty$

Mathematically you can think of this as an extension problem: $\int \frac{1}{r^4} f(x)\, d^3x$ that is defined for all functions that vanish near $x=0$.

$= (Pf)(x)$

Introduce $f(x) \to f(x) - f(0)\psi(x) - x \cdot \partial f(0)\psi(x)$

$\psi(x)$ is an arbitrary "window fct."

I can always define $f \mapsto \int \frac{1}{r^4} Pf(x)\, d^3x$

is an extension: f if vanishes @ origin

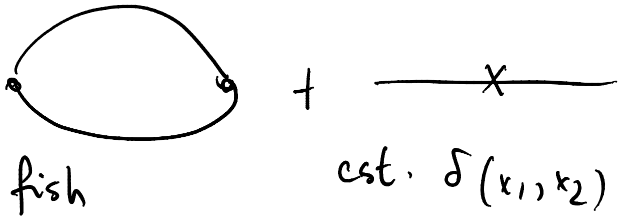
$\Rightarrow Pf = f$

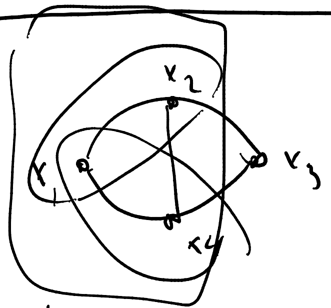
Ambiguity: how to chose ψ?

If I choose a different $\tilde{\psi} \Rightarrow$

New extension = old extension + $\delta(x) + c \cdot \partial\delta(x)$

$\Rightarrow$ $T(\phi^2(x_1)\,\phi^2(x_2))$ is defined on $M \times M$ by an extension process. Extension not unique: Different extensions differ by cst $\cdot\, \delta(x_1, x_2)$!!





contribute to $T(\phi^2(x_1)\,\phi^2(x_3) \cdot \phi^3(x_4)\,\phi^3(x_2))$

You have divergences when $x_1 = x_2$ or

$x_1 = x_4$ or

$x_1 = x_2 = x_4$

...

Need disentangle all this

60's Hepp for Minkowski space

2000's CST

There are ambiguities

Interacting fields:

Interacting fields in $\lambda\phi^4$-theory are defined by a formal power series

$$O_I(x) = O(x) + \sum_{n\geq 1} \frac{(-\lambda)^n}{n!} R\{O(x); S_I^n\}$$

$\uparrow$ free (under $O(x)$)

Haag's series

'50's

$\uparrow$ I = in the interacting theory

$$S_I = \int \phi^4(y)\, d\mu$$

$R(O_1(x_1); O_2(y_2) \dots O_n(y_n))$ = retarded product can be expressed in terms of time ordered products.

Basic idea: $O = \phi$

$$(\Box - m^2)\phi_I = \lambda \phi_I^3$$

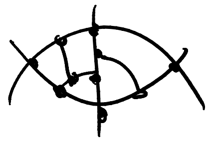
Ansatz (classically) $\phi_I = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots$

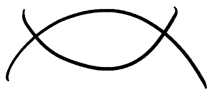
$\uparrow$ Free field

$(\Box - m^2)\phi_0 = 0$

$$(\Box - m^2)\phi_1 = \phi_0^3 \Rightarrow \phi_1(x) = \underbrace{\int \Delta_R(x,y)\, \phi_0^3(y)\, d\mu}_{R(\phi(x), S_I)}$$

$$S_I = \int \phi^4(y)\, d\mu$$





$\frac{1}{x^4}$

$d = 4 + \varepsilon$

$\phi^2(x)\ \phi^2(x$ 1 , const d^2

$$\phi^2(x_1)\,\phi^2(x_2) = \frac{1}{x^4} + \frac{\text{const}}{x^2}\phi^2$$

$$+ \ \phi^4$$