(IV) Additional lecture

1) Renormalization
2) Gauge theories

1) Minkowski $\quad L = \left(\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\xi R\phi^2 + \frac{\lambda}{4!}\phi^4\right) d\mu$

absent in Mink.

$$S = \sum_{n\geq 0}' \frac{(i\lambda)^n}{n!} \int T(\phi^4(x_1)\dots\phi^4(x_n))\, \underbrace{d\mu(x_1)}_{d^4x_1}\dots d\mu(x_n)$$

$\langle p_1,\dots,p_m | S | q_1,\dots,q_{m'}\rangle \longrightarrow$ amplitudes

1) The series doesn't converge (?)
2) The integrand doesn't exist as distribution $\leftarrow$ renormalization
3) IR-divergence
(e.g. for $m^2=0$) S-matrix does no exist in the simple minded sense

Talk about 2)

In CST we are interested typically in correlation fct. of interacting fields

$$\langle \phi_I(x_1)\dots\phi_I(x_n)\rangle_\Omega = ?$$

$$\phi_I(y) = \text{given by formal series in } \lambda$$

$$= \sum_{n,m} c_{n,m}\, \lambda^{n+m} \int T\big(\phi^4(x_1)\cdots\phi^4(x_m)\big)^* \cdot$$

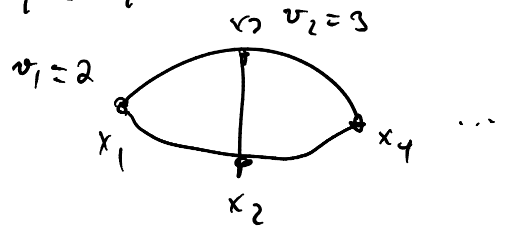
$$\cdot\, T\big(\phi(y)\,\phi^4(z_1)\cdots\phi^4(z_n)\big)$$

integration over $x_1, \dots, z_1, \dots$

$$T\big(\phi^{k_1}(x_1)\cdots\phi^{k_n}(x_n)\big) = \sum_G \overbrace{t_G(x_1,\dots,x_n)}^{\text{distributional}}$$

$$\cdot \underbrace{:\phi^{k_1-v_1}(x_1)\cdots\phi^{k_n-v_n}(x_n):_H}_{\text{smooth in } x_1,\dots,x_n}$$

G - Feynman graph w/ n vertices of valence $v_i \le k_i$



$:\cdots:_H$ like "normal ordering" in Mink space.

Mink: $:\phi(x_1)\phi(x_2): \;=\; \phi(x_1)\phi(x_2) - \langle\phi(x_1)\phi(x_2)\rangle_0 \,\mathbb{1}$

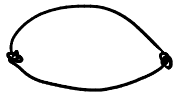
Curved: $:\phi(x_1)\phi(x_2):_H \;=\; - \underbrace{H(x_1,x_2)}_{\text{Hadamard form}} \,\mathbb{1}$

$$H = \frac{\Delta^{1/2}}{\sigma} + \sum v_n \sigma^n \ln\sigma$$

σ = $\square$'ed signed geodesic distance etc.

$$t_G(x_1,\dots,x_n) = c_G \prod_{\text{edges } e\in G} H_F\big(x_{s(e)}, x_{t(e)}\big)$$

$$H_F = \frac{\Delta^{1/2}}{\sigma + i0} + \sum v_n \sigma^n \ln(\sigma + i0)$$

G :  $t_G = \left[H_F(x_1, x_2) \right]^2$

$$\sim \underbrace{\left[\frac{1}{\sigma + i0} \right]^2}_{} + \dots$$

$\sigma \sim (x_1 - x_2)^2$ most singular part

$$\infty = \int t_G(x_1, x_2)\, f(x_1, x_2)\, d\mu(x_1)\, d\mu(x_2)$$

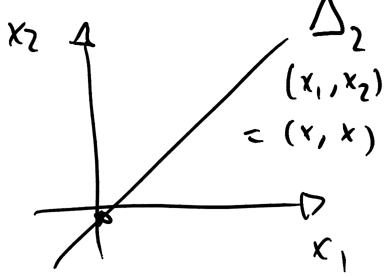
Summarize : $t_G(x_1, \dots, x_n)$ is a distribution only on the set $\{ (x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \ \forall i \neq j \}$

$$= M^n \setminus \bigcup_{I \subset \{1, \dots, n\}} \Delta_I$$

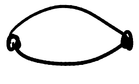
$$\Delta_I = \{ (x_1, \dots, x_n) \mid x_i = x_j \ \forall i, j \in I \}$$

$I = \{1, \dots, n\}$

$\Rightarrow \Delta_I = \{ (x, \dots, x) \}$



Task extend t_G to all M^n, e.g.

G :  : define $\int \underbrace{\left[H_F(x_1, x_2) \right]^2}_{\left[\frac{1}{\sigma + i0}\right]^2} f(x_1, x_2)$ for <u>any</u> f

Trick: Try to define

$$u_z(f) := \int \left[\frac{1}{\sigma + i0} \right]^{z} f(x_1, x_2) \qquad z \in \mathbb{C}$$

is well defined for $\mathrm{Re}(z) < 2$

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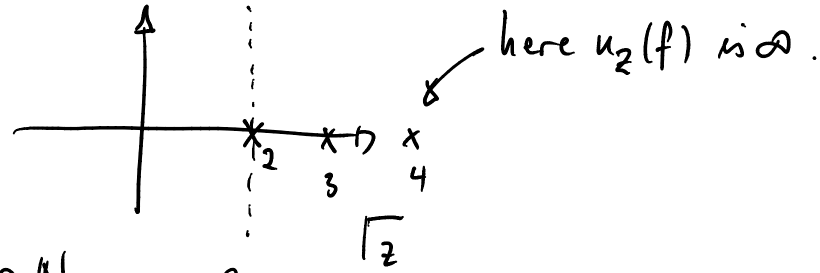
Analytically continue in the complex



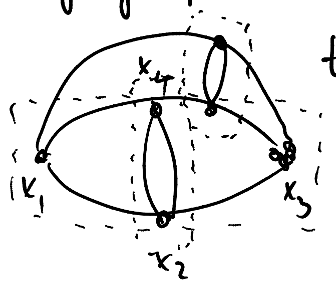
near $z = n \in \mathbb{N}$, $n \geq 2$

you have $u_z(f) = \frac{cst.}{n-z} \left(\nabla^{\frac{n-2}{2}} \delta\right)(f)$ + regular

$$\delta(f) = \int f(x,x)\, d\mu(x)$$

$$u_z(x_1, x_2) = \underbrace{\frac{cst.}{n-z} \nabla^{\frac{n-2}{2}} \delta(x_1,x_2)}_{\text{renormalize = subtract off}} + \text{regular}$$

Arbitrary graph G



t_G has "many divergences".

t_G is singular near each diagonal

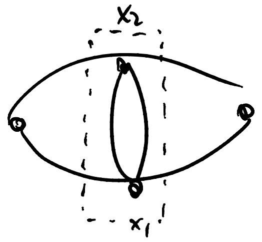
Have remove poles iteratively

For each propagator $H_F(x_i, x_j)$ introduce a complex power z_{ij}

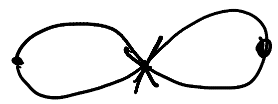
Define $u_{\vec{z}}(f) = \int \prod_{e \text{ edge of } G} \left[\frac{1}{\sigma_e + i0}\right]^{z_e} f(x_1, \dots, x_n)$

remove poles in $z_e \in \mathbb{N}$ or $\sum_{\text{subset}} z_e \in \mathbb{N}$ etc.

of e's



gives rise $\propto$ $\delta(x_1, x_2)$



Local conterterm

In example of interest, we need

$$T\left(\phi^4(x_1) \cdots \phi^4(x_n)\right)$$

two different renormalization prescription T, $\tilde{T}$

$$T\left(e^{\lambda \int \phi^4(x)\, j(x)\, d\mu(x)}\right)$$

$$= \tilde{T}\Big(e^{\lambda \int \phi^4(x)\, j(x) + Z_1(\lambda)\phi^4(x)\, j(x) + Z_2(\lambda)\, m^2 \phi^2(x)\, j(x) + Z_3(\lambda)\, R\, \xi\, \phi^2(x)\, j(x) + Z_4(\lambda)\, (\nabla\phi)^2(x)\, j(x) \cdots}\Big)$$

(incl. derivatives of j)

Z_i = formal power series.

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2) Gauge theories

$$L_{ym} = {}^*F_I \wedge F^I \qquad I\text{- indices of } \mathfrak{g}$$

$$F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$$

$$\mathcal{D}_\mu = \nabla_\mu + i\lambda A_\mu$$

$$\mathcal{D}_\mu = \underbrace{\nabla_\mu}_{\substack{\text{non-dyn.}\\ \text{backgr.}}} + i\lambda \underbrace{A_\mu}_{\text{dynamical}}$$

Free YM-field ($\lambda = 0$) does not have a propagator.

$$L = L_{ym} + L_{gf} + L_{gh} = L_0 + \lambda L_1 + \lambda^2 L_2$$

Additional fields $B^I, C^I, \bar{C}^I$

$$L_{gf} = * B^I \left(i g_I + \tfrac{1}{2} B_I \right)$$

$$L_{gh} = \mathcal{D} C^J \wedge \frac{\delta(g_I \bar{C}^I)}{\delta A^J}$$

$$g_I = (\nabla \cdot A)_I \quad \longleftarrow \text{Lorentz gauge}$$

$$\quad = t^\mu A^I_\mu \quad \longleftarrow \text{axial gauge}$$

L_0 has an "H_F"

L has an invariance under BRST-trafo:

$$sA^I = dC^I + i\lambda f^I{}_{JK} A^J C^K$$

$$sC^I = -\frac{i\lambda}{2} f^I{}_{JK} C^J C^K$$

$$\ldots$$

$$\boxed{s^2 = 0}, \qquad s \int L = 0$$

$$\int L = \int L_{ym} + s \int \psi$$

$$s\{O_1, O_2\}_{P.B.} = \underbrace{\{sO_1, O_2\}_{P.B.} + \{O_1, sO_2\}_{P.B.}}$$

$$\text{Observables} = \frac{\ker s}{\operatorname{Im} s} \quad \leftarrow \text{this still has a } \{\cdot,\cdot\}_{P.B}$$

$\Downarrow$

Poisson-algebra of YM

$$\frac{\ker s}{\operatorname{Im} s} = \Big\{ O = \prod_j \operatorname{Tr}(g, \dots, \nabla^k \mathrm{Riem}) \prod p_i(C)$$

$$\times \prod_k \Theta(F, \mathcal{D}F, \dots, \mathcal{D}^l F)$$

$\Theta \leftarrow$ invariant polyn. of $\mathfrak{g}$

$\times$ Chern-Simons type

QFT $\quad O_I(x) =$ defined by same type of formula

s implemented by $\{Q_{BRST}, \cdot\}$

$$Q_{BRST} = \int_\Sigma {*J_{BRST}} \quad \leftarrow \text{Noether current of } s$$

Problems: 1) $\quad d{*J_{BRST}} = 0$ as operator equ.

2) $\quad Q_{BST}^2 = 0$ as operator equ.

This works for an appropriate choice of scheme. ($\to$ Ward identities)