Non-commutative geometry and matrix models

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outline

1. Part I: general aspects of NC / matrix geometry
   - motivation, some history
   - basic examples of noncommutative spaces (S^2_N, T^2_θ, $\mathbb{R}^4_\theta$ etc.)
   - quantization of Poisson / symplectic structures
   - basic noncommutative field theory
   - spectral aspects, Connes NCG
2. Part II: Matrix models and dynamical geometry
   - Yang-Mills matrix models, noncommutative gauge theory
   - general geometry in matrix models (branes, curvature)
   - nonabelian gauge fields, fermions, SUSY
   - quantization of M.M: heat kernel expansion, UV/IR mixing
   - aspects of (emergent) gravity, outlook

literature:

- review article:
  H.S., “Emergent Geometry and Gravity from Matrix Models: an Introduction”. Class.Quant.Grav. 27 (2010) 133001, arXiv:1003.4134
- brief qualitative intro:
  H. S., “On Matrix Geometry“ arXiv:1101.5003
- ... (later)

issue in quantum mechanics $\leftrightarrow$ gravity:

the cosmological constant problem

QM predicts vacuum energy (cosm.const.)

$$(E_{\rm vac})_{QM} = \pm\int d^3k \, \tfrac{1}{2}\hbar\omega(k) \sim \begin{cases} O(10^3 GeV)^4, & \text{SUSY} \\ O(10^{19} GeV)^4 & \text{no SUSY} \end{cases}$$

$$(c.c.)_{obs} = (2.10^{-12} GeV)^4$$

both described by $\int d^4x \sqrt{g}\Lambda^4 \Rightarrow$ discrepancy

$$\boxed{\frac{(c.c.)_{\rm QM}}{(c.c.)_{\Lambda\rm CDM}} \geq \left(\frac{10^3}{10^{-12}}\right)^4 = 10^{60}}$$

$\Rightarrow$ ridiculous fine-tuning

... maybe we're missing something !!?

how to adress this in quantum theory of gravity?

- string theory $\rightarrow$ vast set ($> 10^{500}$) of possible "vacua"
  $\rightarrow$ "landscape", lack of predictivity
  $\rightarrow$ "anthropic principle" (= give up ?)
- loop quantum gravity (?)

try different approach:

1. noncommutative (NC) space-time, NC geometry
2. dynamical NC space(time):

(Yang-Mills) Matrix models

pre-geometric, BG independent
natural quantization
hopefully large separation of scales $\Lambda_{\rm Planck} \leftrightarrow \Lambda_{c.c.}$

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recall:

Q.M. & G.R. $\Rightarrow$ break-down of classical space-time

measure object of size Δx

Q.M. $\Rightarrow$ energy $E \geq \hbar k \sim \frac{\hbar}{\Delta x}$

G.R. $\Rightarrow$ $\Delta x \geq R_{\text{Schwarzschild}} \sim GE \geq \frac{\hbar G}{\Delta x}$

$\Rightarrow$ $(\Delta x)^2 \geq \hbar G = L_{Pl}^2, \qquad L_{Pl} = 10^{-33} cm$

more precise version: (Doplicher Fredenhagen Roberts 1995)

$$\Delta x^0 (\sum_i \Delta x^i) \geq L_{Pl}^2, \qquad \sum_{i \neq j} \Delta x^i \Delta x^j \geq L_{Pl}^2$$

... space-time uncertainty relations, follows from

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \qquad \text{(cf. Q.M.)}$$

... noncommutative (quantum) space-time

"fuzzy", "foam-like" structure of space-time (no singularities?)

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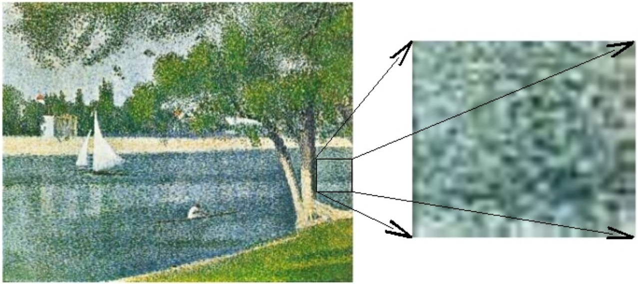
inspiration: Quantum Mechanics!

class. mechanics	quantum mechanics
phase space $\mathbb{R}^2$	"quantized phase space"
functions $f(q,p) \in \mathcal{C}(\mathbb{R}^2)$	Heisenberg algebra $f(Q,P)$
$[p,q] = 0$	$[P,Q] = i\hbar\,\mathbf{1}$



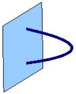
historical comments:

- old idea (Heisenberg 1938 "universal length")
- Snyder 1947: Lorentz-invariant NC space-time algebra
- Mathematik: von Neumann "pointless geometry", Connes NC (differential) geometry
- first field-theoretical models 199x
  - fuzzy sphere Madore, Grosse, ...
  - quantized spaces with quantum group symmetry Wess-Zumino, ...
- Connes-Lott: $M^4 \times \{1, -1\}$ (standard model, Higgs interpreted as connection in internal NC "2-point space")
- Matrix Models BFSS, IKKT 1996
- NCG on D-branes (string theory) 1998 (Chu Ho Douglas Hull Schomerus Seiberg Witten etc 1998 ff
- NC QFT, UV/IR mixing, solitons/instantons, new phase transitions ("striped phase"), matrix models, ...

NC spaces in other physical contexts

- 2D– systems in strong magnetic fields
  projection on lowest Landau level
  $\Rightarrow$ coordinates $\hat{x}_i$ satisfy $[\hat{x}_i, \hat{x}_j] = \frac{i}{B}\varepsilon_{ij}$
- Quantum Hall effect Bellisard 1994
- string theory (10 dimensions, zoo of objects)

  strings end on "D-branes" (=submanifolds)

  D-branes in background B–field $\Rightarrow$ strings induce
  NC field theory (NCFT) on D-branes (NC gauge theory)
  (Chu Ho Douglas Hull Schomerus Seiberg Witten etc 1998 ff)
  $\Rightarrow$ D-branes = NC space
- 3D quantum gravity $\rightarrow$ NC spaces (Freidel etal)

NC field theory = (quantum) field theory on NC spaces

studied during past $10-15$ years:

- starting point: NC space & diff. calculus
- Lorentz invariance broken by $\theta^{\mu\nu}$
- straightforward for scalar FT
  quantization $\Rightarrow$ UV/IR mixing due to $\Delta x^\mu \Delta x^\nu \geq L_{NC}^2$
  problem for renormalization dim $[\theta^{\mu\nu}] = [L^2]$
- NC gauge theory:
  - straightforward for $U(n)$
  - less clear for other gauge groups
  - NC standard model proposed (only effective, not quantiz.)
  - new processes ($Z \to 2\gamma$, ...)
- generalized (quantum group) symmetries:
  ($\mathbb{R}^4_q$, κ-Poincare,...)
  hard to reconcile with “2nd quantization“
  possibly modified dispersion relations

Gravity $\leftrightarrow$ NC spaces

many possible approaches:

1. deformation of class. GR
   - insist on diffeos (Aschieri, Wess etal, ...)
   - start with generalized local Lorentz invariance (Chamseddine, ...)
   - etc.

   problems: extra structure structure $\theta^{\mu\nu}$? dynamical? quantization?

2. start with fundamentally different model, s.t. dynamical NC space(time) “emerges”

   (Yang-Mills) Matrix Models

   + contains also gauge theory & matter
   + can be quantized (?!)
   + distinct from GR, $\rightarrow$ hope for c.c. problem
   - not clear if close enough to GR

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Poisson $\leftrightarrow$ symplectic structure

$\{.,.\}: \quad \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ ... Poisson structure if

$$\begin{aligned} \{f,g\} + \{g,f\} &= 0, && \text{anti-symmetric} \\ \{f,\{g,h\}\} + \text{cyclic} &= 0 && \text{Jacobi} \end{aligned}$$

$\leftrightarrow$ tensor field $\theta^{\mu\nu}(x)\partial_\mu \otimes \partial_\nu$ with

$$\theta^{\mu\nu} = -\theta^{\nu\mu}, \quad \theta^{\mu\mu'}\partial_{\mu'}\theta^{\nu\rho} + \text{cyclic} = 0$$

assume $\theta^{\mu\nu}$ non-degenerate

Then:

$$\begin{aligned} \omega &:= \tfrac{1}{2}\theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu && \in \Omega^2\mathcal{M} \quad \text{closed,} \\ d\omega &= 0 \end{aligned}$$

... symplectic form

Quantized Poisson (symplectic) spaces

$(\mathcal{M}, \theta^{\mu\nu}(x))$ ... $2n$-dimensional manifold with Poisson structure

Its quantization $\mathcal{M}_\theta$ is NC algebra such that

$$\begin{aligned}\mathcal{I}: \ \mathcal{C}(\mathcal{M}) &\to \mathcal{A} \subset \mathcal{L}(\mathcal{H}) \\ f(x) &\mapsto \hat{f}(X)\end{aligned}$$

such that

$$\begin{aligned}\hat{f}\,\hat{g} &= \mathcal{I}(fg) + O(\theta) \\ [\hat{f}, \hat{g}] &= \mathcal{I}(i\{f,g\}) + O(\theta^2)\end{aligned}$$

("nice") $\Phi \in Mat(\infty, \mathbb{C}) \quad \leftrightarrow \quad$ quantized function on $\mathcal{M}$

<u>furthermore:</u>

$$\begin{aligned}(2\pi)^n \mathrm{Tr}\,\mathcal{I}(\phi) &\sim \int \frac{\omega^n}{n!}\phi = \int d^{2n}x\, \rho(x)\, \phi(x) \\ \rho(x) &= \mathrm{Pfaff}\,(\theta^{-1}_{\mu\nu}) \ \ldots \ \text{symplectic volume}\end{aligned}$$

<u>note:</u> $\dim(\mathcal{H}) \sim \mathrm{Vol}(\mathcal{M})$, large!! (cf. Bohr-Sommerfeld)

Example: quantized phase space $\mathbb{R}^2_\hbar$

consider $X^\mu = \begin{pmatrix} Q \\ P \end{pmatrix}$, Heisenberg C.R.

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}\,\mathbf{1}, \qquad \mu,\nu = 1,...,2, \qquad \theta^{\mu\nu} = \hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\mathcal{A} \subset \mathcal{L}(\mathcal{H}) \cong \mathrm{Mat}(\infty, \mathbb{C})$ ... functions on $\mathbb{R}^2_\hbar$

uncertainty relations $\Delta X^\mu \Delta X^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|$

Weyl-quantization: Poisson structure $\{x^\mu, x^\nu\} = \theta^{\mu\nu}$

$$\mathcal{I}: \quad \mathcal{C}(\mathbb{R}^2) \to \mathcal{A},$$

$$\phi(x) = \int d^2k\, e^{ik_\mu x^\mu}\hat{\phi}(k) \quad \mapsto \quad \int d^2k\, e^{ik_\mu X^\mu}\hat{\phi}(k) =: \Phi(X) \in \mathcal{A}$$

$(L^2(\mathbb{R}^2) \leftrightarrow$ Hilbert-Schmidt$)$

interpretation:

$X^\mu \in \mathcal{A} \cong \mathrm{Mat}(\infty, \mathbb{C})$ ... quantiz. coord. function on $\mathbb{R}^2_\hbar$
$\Phi(X^\mu) \in \mathrm{Mat}(\infty, \mathbb{C})$ ... observables (functions) on $\mathbb{R}^2_\hbar$

star product

= pull-back of multiplication in $\mathcal{A}$:

$$f \star g := \mathcal{I}^{-1}(\mathcal{I}(f)\mathcal{I}(g))$$

Weyl quantization map $\rightarrow$ explicit formula (for $\theta^{\mu\nu} = const$):

$$(f \star g)(x) = f(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g(x)$$

proof:

$$e^{ik_\mu X^\mu} \star e^{ip_\mu X^\mu} = e^{\frac{i}{2}\theta^{\mu\nu}k_\mu p_\nu} e^{i(k_\mu + p_\mu)X^\mu}$$

note:

$$\begin{aligned} x^\mu \star x^\nu &= x^\mu x^\nu + \frac{i}{2}\theta^{\mu\nu} \\ [x^\mu, x^\nu]_\star &= i\theta^{\mu\nu} \end{aligned}$$

... CCR

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remarks:

- $\mathcal{I}$ not unique, not Lie-algebra homomorphism
- need quantization of general Poisson structure $\theta^{\mu\nu}(x)$

  such that

  $$[X^\mu, X^\nu] \sim i\{x^\mu, x^\nu\} = i\theta^{\mu\nu}(x)$$

  (always assume $\theta^{\mu\nu}(x)$ non-deg $\Rightarrow$ symplectic)

- existence, precise def. of quantization non-trivial (formal, strict, ...)
  need strict quantization (operators)
  established for Kähler (Schlichenmaier etal),
  almost-Kähler (Uribe etal)
- quantization map $\rightarrow$ map NCFT $\Longleftrightarrow$ ordinary QFT
  ("Seiberg-Witten map")

semi-classical limit:

work with commutative functions (de-quantization map),

replace commutators by Poisson brackets

i.e. replace

$$\begin{aligned} \hat{F} &\rightarrow f = \mathcal{I}^{-1}(F) \\ [\hat{F}, \hat{G}] &\rightarrow i\{f, g\} \qquad (+O(\theta^2), \quad \text{drop}) \end{aligned}$$

i.e. keep only leading order in θ

is independent of specific quantization $\mathcal{I}$

Noncommutative geometry

Gelfand-Naimark theorem:

every commutative C^* - algebra $\mathcal{A}$ with 1 is isomorphic to a C^* - algebra of continuous functions on compact Hausdorff space $\mathcal{M}$.

idea: replace $\mathcal{A}$ $\Rightarrow$ noncomm. algebra of "functions" $\mathcal{A}$

manif. $\mathcal{M}$ $\to$ functions $\mathcal{C}(\mathcal{M})$ $\to$ NC algebra $\mathcal{A}$

but: need additional (geometrical) structures

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classical geometry	noncomm. geometry
$\mathcal{C}(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{C}\}, \quad \theta^{\mu\nu}$ comm. algebra, metric structure $g_{\mu\nu}$ diff. calculus	$\mathcal{A} \subset \mathcal{L}(\mathcal{H}) \cong Mat(\infty, \mathbb{C})$ NC algebra, e.g. $[x^\mu, x^\nu] = i\theta^{\mu\nu}\mathbf{1}$ Dirac/Laplace operator $\not{D}, \ \Delta$ NC diff. calculus (A. Connes)
field theory: $\Delta\phi = \lambda\phi$ $\phi \in \mathcal{C}^\infty(\mathcal{M})$	NC field theory: $\Delta\phi = \lambda\phi,$ $\phi \in \mathcal{A}$
QFT $\int_{\mathcal{C}(\mathcal{M})} d\phi \, e^{-S(\phi)}$	NC QFT $\int_{\mathcal{A}} d\phi \, e^{-S(\phi)}$
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The fuzzy sphere

classical S^2 :

$$\left.\begin{array}{rcl} x^a : S^2 & \hookrightarrow & \mathbb{R}^3 \\ x^a x^a & = & 1 \end{array}\right\} \Rightarrow \mathcal{A} = \mathcal{C}^\infty(S^2)$$

fuzzy sphere S^2_N : (Hoppe, Madore)

let $X^a \in \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... 3 hermitian matrices

$$\begin{array}{rcl} [X^a, X^b] & = & \frac{i}{\sqrt{C_N}} \varepsilon^{abc} X^c , \qquad C_N = \frac{1}{4}(N^2 - 1) \\ X^a X^a & = & \mathbb{1}, \end{array}$$

realized as $X^a = \frac{1}{\sqrt{C_N}} J^a$ ... $N-$ dim irrep of $\mathfrak{su}(2)$ on $\mathbb{C}^N$,

generate $\mathcal{A} \cong \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... alg. of functions on S^2_N

$SO(3)$ action:

$$\begin{array}{rcl} \mathfrak{su}(2) \times \mathcal{A} & \to & \mathcal{A} \\ (J^a, \phi) & \mapsto & [X^a, \phi] \end{array}$$

decompose $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ into irreps of $SO(3)$:

$$\begin{aligned}\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C}) \cong (\mathrm{N}) \otimes (\bar{\mathrm{N}}) &= (1) \oplus (3) \oplus \ldots \oplus (2N-1) \\ &= \{\hat{Y}^0_0\} \oplus \{\hat{Y}^1_m\} \oplus \ldots \oplus \{\hat{Y}^{N-1}_m\}.\end{aligned}$$

... fuzzy spherical harmonics (polynomials in X^a); UV cutoff !

quantization map:

$$\begin{aligned}\mathcal{I}: \quad \mathcal{C}(S^2) &\to \mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C}) \\ Y^l_m &\mapsto \begin{cases} \hat{Y}^l_m, & l < N \\ 0, & l \geq N \end{cases}\end{aligned}$$

satisfies

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S^2_N ... quantization of $(S^2, N\omega)$

Coherent states (Perelomov)

class. geometry: let $p \in S^2$ ... north pole

$$\begin{aligned} SO(3) &\to S^2 \\ g &\mapsto g \triangleright p \end{aligned}$$

stabilizer group $U(1) \subset SO(3) \Rightarrow S^2 \cong SO(3)/U(1)$

fuzzy functions: $\mathcal{A} = \mathcal{L}(\mathcal{H})$, $\mathcal{H} = |m, L\rangle$, $m = -L, ..., L$ $N = 2L + 1$

consider group orbit ($|L, L\rangle$ ... highest weight state)

$$\begin{aligned} SO(3) &\to \mathcal{H} \\ g &\mapsto |\psi_g\rangle := \pi_N(g)|L, L\rangle \end{aligned}$$

note: projector $|\psi_g\rangle\langle\psi_g| \in \mathcal{A}$ is independent of $U(1) \subset SO(3)$

$$\begin{aligned} S^2 \cong SO(3)/U(1) &\to \mathcal{A} \\ p &\mapsto \Pi_p := |\psi_g\rangle\langle\psi_g| \quad =: 4\pi\delta_N^{(2)}(x - p) \end{aligned}$$

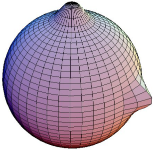
def. $|p\rangle := |\psi_{g(p)}\rangle \sim |\psi_{g(p)'}\rangle$

can show

$$\begin{aligned} \int_{S^2} |p\rangle\langle p| &= c\mathbb{1} \\ |\langle p|p'\rangle| &= \big(\cos(\vartheta/2)\big)^{N-1}, \qquad \vartheta = \measuredangle(p,p') \\ p_a X^a|p\rangle &= |p\rangle \\ \langle p|X^a|p\rangle &= \mathrm{Tr} X^a \Pi_p \; = \; p^a \in S^2, \end{aligned}$$

overcomplete

localization $p \approx p'$



$$\boxed{X^a \sim x^a : \quad S^2 \hookrightarrow \mathbb{R}^3}$$

coherent states minimize uncertainty

$$\begin{aligned} (\Delta X^1)^2 + (\Delta X^2)^2 + (\Delta X^3)^2 \quad &= \sum_a \langle p|X^a X^a|p\rangle - \langle p|X^a|p\rangle\langle p|X^a|p\rangle \\ &\geq \frac{N-1}{2C_N} \sim \frac{1}{2N} \end{aligned}$$

intuition

$S^2 \approx$ "fuzzy common spectrum" of $\quad X^a \sim x^a: \quad S^2 \hookrightarrow \mathbb{R}^3$

semi-classical limit:

- $X^a \sim x^a := \langle p|X^a|p\rangle: \;\; S^2 \hookrightarrow \mathbb{R}^3$
- or: replace

$$
\begin{aligned}
X^a &\rightarrow \mathcal{I}^{-1}(X^a) = x^a, \quad \Phi \rightarrow \mathcal{I}^{-1}(\phi) = \phi(p) \in \mathcal{C}(S^2), \\
[\phi, \Psi] &\rightarrow i\{\phi(p), \psi(p)\}
\end{aligned}
$$

error $\sim \frac{1}{N}$

local description:

near "north pole" $|L,L\rangle$: $\quad X^3 \approx 1, \qquad X^1 \approx X^1 \approx 0$

$$
\begin{aligned}
X^3 &= \sqrt{1-(X^1)^2-(X^2)^2} \\
[X^1,X^2] &= \frac{i}{\sqrt{C_N}} X^3 =: \theta^{12}(X) \ \approx \frac{2i}{N}
\end{aligned}
$$

cf. Heisenberg algebra!

quantum cell $\Delta X^1 \Delta X^2 \geq \frac{1}{N}$, area $\Delta A \sim \frac{4\pi}{N}$
inferred from Poisson structure

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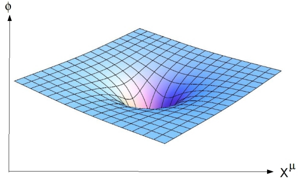
---

note: can modify $X^3 = X^3(X^1, X^2)$



2D submanifolds $\mathcal{M}^2 \subset \mathbb{R}^3$

metric structure of fuzzy sphere

$SO(3)$ symmetry $\Rightarrow$ obviously "round sphere"

---

metric encoded in NC Laplace operator

$$\Box : \ \mathcal{A} \to \mathcal{A}, \qquad \Box\phi = [X^a, [X^b, \phi]]\delta_{ab}$$

$SO(3)$ invariant: $\Box(g \triangleright \phi) = g \triangleright (\Box\phi) \qquad \Rightarrow \ \Box\hat{Y}^l_m = c_l \hat{Y}^l_m$

note: $\Box = \frac{1}{C_N} J^a J^a$ on $\quad \mathcal{A} \cong (N) \otimes (\bar{N}) \cong (1) \oplus (3) \oplus ... \oplus (2N-1)$

$\Rightarrow$
$$\boxed{\Box\hat{Y}^l_m = \frac{1}{C_N} l(l+1)\hat{Y}^l_m}$$

spectrum identical with classical case $\Delta_g\phi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi)$

$\Rightarrow$ effective metric of $\Box$ = round metric on S^2

Symplectomorphisms

$\exists$ $U(N)$ action on $\mathcal{A} = \mathrm{Mat(N, \mathbb{C})}$:

$$\phi \to U\phi U^{-1}$$

infinitesimal version: $U = e^{i\alpha H}$,

$$\phi \to \phi + i[H, \phi]$$

semi-classical version:

$$\phi \to \phi + \{H, \phi\}, \qquad H \in \mathcal{C}(\mathcal{M})$$

Hamiltonian VF $i_X\omega = dH$, infinites. symplectomorphism on (S^2, ω)

is area-preserving diffeo: $dVol = \omega, \quad \mathcal{L}_X\omega = (i_X d + d i_X)\omega = 0$

in 2D: all (local ...) APD's
in 4D: special APD's ($\cong$ action of symplectomorphism group)

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Scalar field theory on S^2_N

consider $\mathcal{H} := \mathcal{A} = \mathrm{Mat}(N,\mathbb{C})$ ... space of functions on S^2_N

Hilbert space structure:

$$\langle\phi,\phi'\rangle = \frac{4\pi}{N}\mathrm{Tr}(\phi^\dagger\phi) \sim \int_{S^2}\phi^\dagger\phi \qquad \text{cf. } \mathcal{H}\approx L^2(S^2)$$

action for free real scalar field $\phi = \phi^\dagger$:

$$\begin{aligned} S[\phi] &= \tfrac{4\pi}{N}\mathrm{Tr}(\phi\Box\phi + \mu^2\phi + \lambda\phi^4) \\ &= \tfrac{4\pi}{N}\mathrm{Tr}(-[X^a,\phi][X^a,\phi] + \mu^2\phi + \lambda\phi^4) \\ &\sim \textstyle\int_{S^2}(\phi\Delta_g\phi + \mu^2\phi + \lambda\phi^4) \end{aligned}$$

... deformation of classical FT on S^2, built-in UV cutoff

scalar QFT on S^2_N

most natural: "functional" (matrix) integral approach

$$Z = \int \mathcal{D}\phi e^{-S[\phi]}$$

$$\langle \phi_{l_1 m_1} \cdots \phi_{l_n m_n} \rangle = \frac{\int [\mathcal{D}\phi] e^{-S[\phi]} \phi_{l_1 m_1} \cdots \phi_{l_n m_n}}{\int [\mathcal{D}\phi] e^{-S[\phi]}}, \qquad [\mathcal{D}\phi] = \prod d\phi_{lm}$$

$$\phi = \sum \phi_{lm} \hat{Y}^l_m$$

... deformation & regularization of (euclid.) QFT on S^2, UV cutoff

propagator: as usual, $\langle \phi_{lm} \phi_{l'm'} \rangle = \delta^{lm}_{l'm'} \frac{1}{l(l+1)+\mu^2}$

vertices: $V = \lambda \sum \phi_{l_1 m_1} \cdots \phi_{l_n m_n} \mathrm{Tr}(\hat{Y}^{l_1}_{m_1} \ldots \hat{Y}^{l_4}_{m_4})$

perturb. expansion, Gaussian integrals $\Rightarrow$ Wick's theorem,
distinction planar $\leftrightarrow$ nonplanar diagrams

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large phase factors, oscillations for $l_1 l_2 \geq \Lambda^2_{NC}$

$\Rightarrow$ distinct from usual QFT, UV/IR mixing *Minwalla, V. Raamsdonk, Seiberg*

central feature of NC QFT, obstacle for perturb. renormalization

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central feature of NC QFT, obstacle for perturb. renormalization

Note: "functional integral" $\int \mathcal{D}\phi \exp(S[\phi])$ very natural:

- $\exists$ $U(N)$ action on $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$:

$$\phi \to U\phi U^{-1}$$

  ($\cong$ action of symplectomorphism group)

- can diagonalize $\phi = U^{-1}\mathrm{diag}(\lambda_1, ..., \lambda_N)U$, decompose

$$\begin{aligned} \int \mathcal{D}\phi &= \int dU \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda_i), \\ \Delta(\lambda_i) &= \prod_{i<j}(\lambda_i - \lambda_j) \end{aligned}$$

- action for scalar field theory NOT invariant under $U(N)$
  (cf. M.M. approach to 2D gravity)

  BUT

- action for gauge theory IS invariant under $U(N)$!!! (later)

scales

$[x, x] \approx \theta$ ... NC length scale, e.g. $\theta \sim \frac{R^2}{N}$ on S^2_N

UV cutoff: $\Lambda_{UV} = \frac{N}{R}$, IR cutoff: $\Lambda_{IR} = \frac{1}{R}$

$\Rightarrow$ 3 scales:

$$\Lambda_{IR} = \sqrt{\frac{1}{N\theta}} \quad \ll \quad \Lambda_{NC} = \sqrt{\frac{1}{\theta}} \quad \ll \quad \Lambda_{UV} = \sqrt{\frac{N}{\theta}}$$

in particular: $\Lambda_{IR}\Lambda_{UV} = \Lambda_{NC}^2 \quad (= \frac{1}{\Delta A})$

very general: for compact NC spaces: $\dim \mathcal{H} \sim Vol < \infty$

$\Rightarrow$ NC implies naturally large separation of scales!

product $\phi_k \phi_l$ semi-classical $\Leftrightarrow$ $pq \leq \Lambda_{NC}^2$ (uncertainty rel.n)

”anomalous” aspects in scalar QFT on S^2_N

- 1-loop effective action does NOT reduce to commutative result for $N \to \infty$
- interaction vertices rapidly oscillating, unless $pq \ll \Lambda_{\rm NC}^2$

  (loop effects probe area quantum $\Delta A \sim 1/N$)
- new physics!

scaling limits:

- commutative sphere limit $\quad S^2_N \to S^2$

  $X^a \to RX^a, \quad R$ = fixed, $N \to \infty$

- quantum plane limit: $\quad S^2_N \to \mathbb{R}^2_\theta$

  $R = \sqrt{N\theta}, \quad \theta$ = fixed:

$$z^a = J^a \sqrt{\frac{\theta}{N}} \quad \sim \sqrt{N}\, X^a$$

consider "north pole" of S^2_N:

$$[z^1, z^2] = i\frac{\theta}{N} J_3 \quad \stackrel{N\to\infty}{\approx} \quad i\theta$$

therefore: S^2_N can be used as regularization of $\mathbb{R}^2_\theta$

similarly:

compactification of $\mathbb{R}^{2n}_\theta$ using e.g. $\mathbb{C}P^n_N$, $\; S^2_N \times S^2_N$, etc.

additional structure on S^2_N

- embedding sequence

$$S^2_N \subset S^2_{N+1} \subset ...$$

(map $\hat{Y}^l_m \to \hat{Y}^l_m$), S^2 recovered in inductive limit

- Dirac operator:

$$\begin{aligned} \not{D}\Psi &= \sqrt{C_N}\,(\sigma_a[X^a,.]+1)\Psi, \qquad \chi = \tfrac{1}{2}\sigma_a\{X^a,.\} \\ \{\not{D},\chi\} &= 0 \end{aligned}$$

but $\chi^2 \approx 1$, $\exists$ top mode with $\chi = 0$

- Jordan-Schwinger: $X^a = a^+_\alpha(\sigma^a)^\beta_\alpha a_\beta$ on $\mathcal{F}_N = (a^+...a^+|0\rangle)_N$
- differential calculus:
  differential forms $\Omega^*(S^2_N)$, Leibnitz rule etc.

differential calculus on S^2_N

graded bimodule $\Omega^\star_N$ over $\mathcal{A} = S^2_N$ with

- $d^2 = 0$
- graded Leibnitz rule $d(\alpha\beta) = d\alpha\beta + (-1)^{|\alpha|}\alpha d\beta$

turns out: radial one-form does not decouple,

$$df = [\omega, f], \qquad \omega = -C_N X^a dX^a \qquad \text{(cf. Connes)}$$

$\Omega^\star_N = \oplus_{n=0}^3 \Omega^n_N$, need $\Omega^3_N = f_{abc}(X) dX^a dX^b dX^c$

can introduce frame:

$$\xi^a = \Omega X^a + \sqrt{C_N}\,\varepsilon^{abc} X^b dX^c, \qquad [f(X), \xi^a] = 0 \qquad \text{Madore}$$

most general one-form:

$$\begin{aligned} A &= A_a \xi^a \quad \in \Omega^1_N, \qquad A_a \in \mathcal{A} = \mathrm{Mat}(N, \mathbb{C}) \\ F &= dA + AA = (B_a B_b + i\varepsilon_{abc} B_c)\xi^a \xi^b \quad \in \Omega^2_N, \\ B &= \omega + A = (X^a + A^a)\xi^a \quad \in \Omega^1_N \end{aligned}$$

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Fuzzy torus T_N^2

def. $U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & & \\ 0 & & \dots & 0 & 1 \\ 1 & 0 & \dots & & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i \frac{1}{N}} & & & \\ & & e^{2\pi i \frac{2}{N}} & & \\ & & & \ddots & \\ & & & & e^{2\pi i \frac{N-1}{N}} \end{pmatrix}$

satisfy

$$\begin{aligned} UV &= qVU, \qquad U^N = V^N = 1, \qquad q = e^{2\pi i \frac{1}{N}} \\ [U,V] &= (q-1)VU \end{aligned}$$

generate $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... quantiz. algebra of functions on T_N^2

$\mathbb{Z}_N \times \mathbb{Z}_N$ action:

$$\begin{aligned} \mathbb{Z}_N \times \mathcal{A} &\to \mathcal{A} \\ (\omega^k, \phi) &\mapsto U^k \phi U^{-k} \end{aligned}$$

similar other $\mathbb{Z}_N$

$\mathcal{A} = \oplus_{n,m=0}^{N-1} U^n V^m$ ... harmonics quantization map:

$$\begin{aligned} \mathcal{I}: \quad \mathcal{C}(T^2) &\to \mathcal{A} = \mathrm{Mat}(\mathrm{N},\mathbb{C}) \\ e^{in\varphi} e^{im\psi} &\mapsto \begin{cases} U^n V^m, & |n|,|m| < N/2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

satisfies

$$\begin{aligned} \mathcal{I}(fg) &= \mathcal{I}(f)\mathcal{I}(g) + O(\tfrac{1}{N}), \\ \mathcal{I}(i\{f,g\}) &= [\mathcal{I}(f),\mathcal{I}(g)] + O(\tfrac{1}{N^2}) \end{aligned}$$

Poisson structure $\{e^{i\varphi}, e^{i\psi}\} = \frac{2}{N} e^{i\varphi} e^{i\psi}$ on T^2 $(\Leftrightarrow \{\varphi,\psi\} = -\frac{2}{N})$

integral: $\frac{4\pi^2}{N}\mathrm{Tr}(\mathcal{I}(f)) = \int_{T^2} \omega f, \qquad \omega = d\varphi d\psi$

T^2_N ... quantization of $(T^2, N\omega)$

quantization map:

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T^2_N ... quantization of $(T^2, N\omega)$

metric on T^2_N ? ... "obvious", need extra structure:

embedding $\quad T^2 \hookrightarrow \mathbb{R}^4$ via $\quad x^1 + ix^2 = e^{i\varphi}, \; x^3 + ix^4 = e^{i\psi}$

quantization of embedding maps $x^a \sim X^a$: 4 hermitian matrices

$$X^1 + iX^2 := U, \qquad X^3 + iX^4 := V$$

satisfy

$$\begin{aligned}
[X^1, X^2] &= 0 = [X^3, X^4] \\
(X^1)^2 + (X^2)^2 &= 1 = (X^3)^2 + (X^4)^2 \\
[U, V] &= (q-1)\, VU
\end{aligned}$$

Laplace operator:

$$\begin{aligned}
\Box\phi &= [X^a, [X^b, \phi]]\delta_{ab} \\
&= [U, [U^\dagger, \phi]] + [V, [V^\dagger, \phi]] = 2\phi - U\phi U^\dagger - U^\dagger\phi U - (\% V) \\
\Box(U^n V^m) &= -([n]_q^2 + [m]_q^2)\, U^n V^m \sim -(n^2 + m^2)\, U^n V^m
\end{aligned}$$

where

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin(n\pi/N)}{\sin(\pi/N)} \sim n \qquad \text{("q-number")}$$

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$\mathrm{spec}\,\Box \approx \mathrm{spec}\,\Delta_{T^2}$ below cutoff

therefore:

geometry of (embedded) fuzzy torus $T^2_N \hookrightarrow \mathbb{R}^4$ = flat

momentum space is compactified! $[n]_q$

<u>note:</u> could have started with

$$\begin{aligned} \tilde{U}V &= qV\tilde{U}, \qquad \tilde{U}^N = V^N = 1, \qquad \tilde{q} = e^{2\pi i \frac{k}{N}} \\ [\tilde{U}, V] &= (\tilde{q} - 1)V\tilde{U} \end{aligned}$$

related via $\tilde{U} = U^k$

generate same $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... quantiz. algebra of functions on $\tilde{T}^2_N$

different embedding:

$$\begin{aligned} \tilde{X}^1 + i\tilde{X}^2 &:= \tilde{U} = U^k = (X^1 + iX^2)^k, \\ X^3 + iX^4 &:= V \end{aligned}$$

related to previous by “winding“ map $\tilde{Z} = Z^k$

compare: noncommutative torus T^2_θ Connes

$$UV = qVU, \qquad q = e^{2\pi i\theta}$$
$$U^\dagger = U^{-1}, \quad V^\dagger = V^{-1}$$

generate $\mathcal{A}$ ... algebra of functions on T^2_θ

note: all $U^n V^m$ independent, $\mathcal{A}$ infinite-dimensional

in general non-integral (spectral) dimension, ...

for $\theta = \frac{p}{q} \in \mathbb{Q}$: ∞ -dim. center $\mathcal{C} = \langle U^{nq} V^{mq} \rangle$

fuzzy torus $T^2_N \cong T^2_\theta / \mathcal{C}, \qquad \theta = \frac{1}{N}$

center $\mathcal{C}$ ... infinite sector (“winding modes”)
NC torus T^2_θ very subtle, “wild”

fuzzy torus T^2_N “stable” under deformations

Fuzzy $\mathbb{C}P^n$

(Grosse & Strohmaier, Balachandran etal)

consider

$$\mathbb{C}P^2 = \{g^{-1}\lambda_8 g, \quad g \in SU(3)\} \subset su(3) \cong \mathbb{R}^8 \qquad \text{... (co)adjoint orbit}$$

$$\lambda_8 = \mathrm{diag}(1,1,-2)$$

fuzzy version:

$$\mathcal{A} := \mathbb{C}P^2_N := \mathcal{L}(V_N, \mathbb{C}) = \mathrm{Mat}(\mathrm{d_N}, \mathbb{C})$$

V_N ... irrep of $su(3)$ w. highest weight $(N,0)$, $\quad d_N = \dim V_N$

$$X^a = c_N\, \pi_N(T_a), \qquad c_N \approx \frac{R}{N}$$

satisfy the relations

$$[X^a, X^b] = i c_N f^{abc}\, X^c$$

$$\delta_{ab} X^a X^b = R^2, \qquad d^{abc} X^a X^b = R\, \frac{2N/3+1}{\sqrt{\frac{1}{3}N^2+N}}\, X^c.$$

again:

- $X^a \sim x^a : \mathbb{C}P^2 \hookrightarrow \mathbb{R}^8$ quantiz. embedding map
- $\exists$ $SU(3)$ action on $\mathcal{A}$ $\Rightarrow$ $\mathcal{A} \cong \oplus_{k=1}^N (k,k)$ (harmonics)
  $\Rightarrow$ quantization map $\mathcal{I} : \mathcal{C}(\mathbb{C}P^2) \to \mathcal{A}$
  ... quantiz. of $(\mathbb{C}P^2, \omega)$, ω ... Kirillov–Kostant symplectic form
  intrinsic UV cutoff
- $\Box = [X^a, [X^b, .]]\delta_{ab}$ ...same spectrum as Δ_g on $\mathbb{C}P^2$
- ... goes through for any (compact) coadjoint orbit

the Moyal-Weyl quantum plane $\mathbb{R}^2_\theta$

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \mathbb{1}, \qquad \mu,\nu = 1,...,2, \qquad \theta^{\mu\nu} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

Heisenberg alg. $\mathcal{A} = \mathcal{L}(\mathcal{H})$, coord.- functions on $\mathbb{R}^2_\theta$

uncertainty relations $\Delta X^\mu \Delta X^\nu \geq |\theta^{\mu\nu}|$

Weyl-quantization: $\mathcal{I}: \quad \mathcal{C}(\mathbb{R}^2) \to \mathcal{A}$, Poisson structure $\theta^{\mu\nu}$

$$\phi(x) = \int d^2k\, e^{ik_\mu x^\mu} \hat{\phi}(k) \leftrightarrow \int d^2k\, e^{ik_\mu X^\mu} \hat{\phi}(k) =: \Phi(X) \in \mathrm{Mat}(\infty, \mathbb{C})$$

interpretation:

$X^\mu \in \mathcal{A} \cong \mathrm{Mat}(\infty,\mathbb{C})$ ... quantiz. coord. function on $\mathbb{R}^2_\theta$
$\Phi(X^\mu) \in \mathrm{Mat}(\infty,\mathbb{C})$ ... function ("scalar field") on $\mathbb{R}^2_\theta$

$$(2\pi)\, Tr(\mathcal{I}(\phi)) \sim \int \rho\, \phi(x), \qquad \rho = \sqrt{|\theta^{-1}_{\mu\nu}|}$$

<u>note</u>:

- $\partial_\mu \phi(X) := -i\theta^{-1}_{\mu\nu}[X^\nu, \phi(X)] \sim \partial_\mu \phi(X)$
  ... inner derivations
- translations: $U_p := e^{ip_\mu X^\mu}$,

$$U\phi(X^\nu)U^{-1} = \phi(X^\nu - \theta^{\mu\nu}p_\mu)$$

  translations (symplectomorphisms!) are inner!
- Laplace operator: $\Box = [X^\mu, [X^\nu, \phi]]\delta_{\mu\nu} \sim -G^{\mu\nu}\partial_\mu\partial_\nu\phi$

$$\boxed{G^{\mu\nu} = \theta^{\mu\mu'}\theta^{\nu\nu'}\delta_{\mu'\nu'}} \quad (!!)$$

- star product:

$$f \star g \quad := \quad \mathcal{I}^{-1}(\mathcal{I}(f)\mathcal{I}(g)) \qquad \ldots \quad \text{pull-back algebra}$$

$$(f \star g)(x) \quad = \quad f(x)e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}g(x)$$

  ... can work on $\mathbb{R}^2$ (will not)
- generalizes immediately to $\mathbb{R}^{2n}_\theta$

note:

- $\partial_\mu \phi(X) := -i\theta^{-1}_{\mu\nu}[X^\nu, \phi(X)] \sim \partial_\mu \phi(X)$
  ... inner derivations
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Hopf fibration, fuzzy spheres

consider $\mathbb{R}^4_\theta$, $[y_\mu, y_\nu] = i\theta_{\mu\nu}$. nondeg $\to$ redefine

$$\begin{aligned} z_1 &= y_1 + iy_2, \qquad z_2 = y_3 + iy_4 \\ [z_\alpha, \bar{z}_\beta] &= i\delta_{\alpha\beta}\mathbb{1}, \qquad [z, z] = 0 = [\bar{z}, \bar{z}] \qquad \text{CCR} \end{aligned}$$

define

$$X^a = \frac{1}{2}\bar{z}_\alpha\sigma^a_{\alpha\beta}z_\beta, \qquad X^0 = \frac{1}{2}\bar{z}_\alpha\sigma^0 z_\alpha = \frac{1}{2}\hat{N}$$

satisfy

$$\begin{aligned} [X^a, X^b] &= i\varepsilon^{abc}X^c, \qquad [X^0, X^a] = 0 \\ X^0 &= \bar{z}_\alpha z_\alpha, \qquad X^aX^a = \tfrac{1}{4}X^0(X^0 + 1) \end{aligned}$$

rescale $\frac{1}{\sqrt{C_N}}$ $\to$ recover S^2_N:

$$X^a \in \mathrm{End}(\mathcal{H}_N) \cong \mathrm{Mat}(\mathrm{N}, \mathbb{C}) \quad \text{on} \quad \mathcal{H}_N = \{\underbrace{a^\dagger \ldots a^\dagger}_{N}|0\rangle\} \subset \mathcal{F}$$

essentially Hopf fibration $S^2 \cong \mathbb{C}P^1 \cong S^3/U(1)$

scalar field theory on $\mathbb{R}^2_\theta$

real scalar field $\phi = \phi^+ \in \mathcal{L}(\mathcal{H})$
action functional: e.g.

$$\begin{aligned} S[\phi] &= Tr\left(\tfrac{1}{2}[X^\mu,\phi][X^\nu,\phi]\delta_{\mu\nu} + \tfrac{1}{2}m^2\phi^2 + \tfrac{\lambda}{4}\phi^4\right) \\ &\sim \int \left(\tfrac{1}{2}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2}m^2\phi^2 + \tfrac{\lambda}{4}\phi^4\right) \end{aligned}$$

equations of motion: $\frac{\delta S}{\delta\phi} = 0$

$$\Delta\phi \;:=\; -[X^\mu,[X^\nu,\phi]]\delta_{\mu\nu} = m^2\phi + \lambda\,\phi^3$$

... see Grosse's lectures

lessons

- algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$ ... quantized algebra of functions on $(\mathcal{M}, \omega)$
  no geometrical information (not even dim)
  $\dim(\mathcal{H})$ = number of "quantum cells", $(2\pi)^n \mathrm{Tr}\,\mathcal{I}(f) \sim \mathrm{Vol}_\omega \mathcal{M}$
  finite-dim. $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ sufficient for local physics
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$$X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D \qquad \text{...embedding functions}$$

  contained e.g. in matrix Laplacian $\Box = [X^a, [X^b, .]]\delta_{ab}$
  Poisson/symplectic structure encoded in C.R.

  how to extract it? general geometries?

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how to extract it? general geometries?

some spectral geometry

... extract geometric info from Laplace / Dirac op

<u>recall</u>: fuzzy Laplacian $\Box = [X^a, [X^b, .]]\delta_{ab}$

<u>classical case:</u> heat kernel expansion of Δ_g on $(\mathcal{M}, g)$ (compact)

$$\begin{aligned} \mathrm{Tr} e^{-\alpha\Delta_g} &= \sum_{n\geq 0} \alpha^{(n-d)/2} \int_{\mathcal{M}} d^d x \sqrt{|g|}\, a_n(x) \\ a_0(x) &\sim 1 \\ a_2(x) &\sim -\frac{1}{6} R[g] \end{aligned}$$

$a_n(x)$ ... Seeley-de Witt coefficients (cf. Gilkey)
physically valuable information on $\mathcal{M}$, e.g. 1-loop eff. action

$$\begin{aligned} \Gamma_{1-\mathrm{loop}} &= \mathrm{Tr}\log\Delta_g = -\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha\Delta_g} \to -\mathrm{Tr}\int_{1/\Lambda^2}^\infty \frac{d\alpha}{\alpha} e^{-\alpha\Delta_g} \\ &= -\sum_{n\geq 0} \int_{1/\Lambda^2}^\infty d\alpha\, \alpha^{(n-d-2)/2} \int_{\mathcal{M}} d^d x \sqrt{|g|}\, a_n(x) \\ &= -\sum_{n\geq 0} \int_{\mathcal{M}} d^d x \sqrt{|g|}\, \left(\frac{1}{2}\Lambda^4 a_0(x) + \Lambda^2 a_2(x) + \log\Lambda^2 a_3 + \ldots\right) \end{aligned}$$

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In particular,

$$\mathcal{N}_\Delta(\Lambda) := \#\{\mu^2 \in \mathrm{spec}\Delta_g;\ \mu^2 \leq \Lambda^2\}.$$

Weyls asymptotic formula

$$\mathcal{N}_\Delta(\Lambda) \overset{\Lambda\to\infty}{\sim} c_d \mathrm{vol}\mathcal{M}\, \Lambda^d, \qquad c_d = \frac{\mathrm{vol} S^{d-1}}{d(2\pi)^d}.$$

$\rightarrow$ (spectral) dimension d of $\mathcal{M}$

However: $\mathrm{spec}\Delta_g$ does not quite determine $g_{\mu\nu}$ uniquely

works if replace $\Delta_g \rightarrow$ spectral triple $(\mathcal{A} = \mathcal{C}^\infty(\mathcal{M}), \not{D}, \mathcal{H})$ (Connes)

suitable for generalization to NC space

Connes Noncommutative Geometry

spectral triple: $\mathcal{A}$ ... $\star$ - algebra with

- $\star$- representation $\mathcal{A} \to \mathcal{L}(\mathcal{H})$ on separable Hilbert space
- unbounded selfadjoint operator $\not{D}$ on $\mathcal{A}$ with $(\not{D} - \lambda)^{-1}$ compact such that $[\not{D}, a]$ bounded $\forall a \in \mathcal{A}$

Then $(\mathcal{A}, \mathcal{H}, \not{D})$ ... "spectral triple"

$\exists$ various refinements (real spectral triple, ...)

commutative case: $\not{D}$ ... standard Dirac op on L^2 spinors

can define differential calculus using $d = [\not{D}, .]$ (over-simplified)

$\to$ Connes-Lott interpretation of S.M (Higgs $\leftrightarrow$ NC 2-point space)

spectral action: $S_{\rm eff} := {\rm Tr}(\chi(\not{D}^2/\Lambda^2))$

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symplectic structure, Λ_{NC}, $\mathrm{Tr} \leftrightarrow \int$ etc. plays no role for fuzzy spaces:

- $\exists$ intrinsic UV cutoff $\Lambda_{UV} \sim \frac{N}{R}$

  $\Rightarrow$ no asympotic limit: $\mathcal{N}_\Delta(\Lambda) \sim \Lambda^0$ but

$$\mathcal{N}_\Delta(\Lambda) \sim \mathrm{vol}\mathcal{M}\,\Lambda^d \qquad \text{for} \quad \Lambda < \Lambda_{UV}$$

  gives correct dimension $d = 2$
- $dim = 0$ in the Connes sense
- chirality operator for $D\!\!\!/ = \sigma_a[X^a, .] + \frac{1}{R}$ problematic
- heat kernel expansion problematic for NC spaces (Gayral etal.)

  ok if put finite cutoff $\Lambda = O(\Lambda_{NC})$ (Blaschke-H.S.-Wohlgenannt 2010)

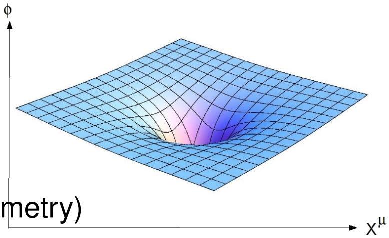
$\rightarrow$ need Matrix ("fuzzy") geometry:

abstraction & generalization:

Q: how to obtain generic matrix (fuzzy) geometries?

A: consider generic *embedded* fuzzy spaces:

$$X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$$



- inherits pull-back metric (geometry)
- (quantized) Poisson / symplectic structure via $[X^\mu, X^\nu] = i\theta^{\mu\nu}$
- easy to work with
- noncommutativity essential