Quantum information geometric approach to foundations of quantum theory

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Approaches to quantum foundations

1. **Quantum logics**
   - main contributions: von Neumann'32, Birkhoff–von Neumann'36, Mackey'57, Gleason'57, Piron'64, Kochen–Specker'65, ..., Randall–Foulis'81, Aerts'81, ...
   - main structure: orthomodular orthocomplemented posets or lattices
   - main advantage: generalisation of structure of Kolmogorov's probability theory
   - main disadvantage: no-go for tensor products

2. **Operator algebras**
   - main contributions: Jordan'32'33, Jordan–von Neumann–Wigner'34, von Neumann–Murray'36+, Gel'fand–Naĭmark'43, Segal'47, Haag–Kastler'64, ...
   - main structure: Jordan–Banach, C^*, or W^* algebras
   - main advantage: generalisation of the notion of observable from quantum mechanics
   - main disadvantage: lack of conceptual justification of the basic axioms

3. **Convex sets**
   - main contributions: Ludwig'64+, Mielnik'68+, Davies–Lewis'70, ..., Alfsen–Shultz'78+,...
   - main structure: ordered real vector equipped with a distinguished positive cone and a linear functional
   - main advantage: operational probabilistic semantics
   - main disadvantage: lack of direct relationship with "physical" model construction

Quantum information geometric approach

1. **Key insights:**
   - Main idea:Consider state spaces as more important structurally and conceptually than algebras of observables and consider interpretation of quantum states as integrals to be more fundamental than interpretation of quantum states as measures.
     $\rightarrow$ Expectation values instead of eigenvalues.
   - Structural consequence: Investigate geometric structures on state spaces for the purpose of axiomatisation and model construction.
     $\rightarrow$ Quantum information geometry instead of spectral theory.
   - Conceptual consequence: Interpret framework of quantum states as an environment for information processing.
     $\rightarrow$ Information theoretic/quantum bayesian interpretation instead of ontic interpretation.

2. **In what follows:**
   - Nonlinear geometries of spaces of quantum states (density matrices) will be investigated.
   - Quantum dynamics will be defined in terms of these geometries.
   - New approach to measurement theory will be proposed.

1. Geometry of quantum states and nonlinear generalisation of quantum dynamics

- Geometric structures on spaces of quantum states:
  relative entropies & Poisson brackets
- Lüders' rules $\rightarrow$ constrained relative entropy maximisations
- Unitary evolution $\rightarrow$ nonlinear hamiltonian flows

Probability theory:

- Underlying structure: measure space $(\mathcal{X}, \mu)$
- Main spaces: **Probabilistic models:**

$$\mathcal{M}(\mathcal{X}, \mu) \subseteq L_1(\mathcal{X}, \mu)^+ := \{p : \mathcal{X} \to \mathbb{R} \mid \int_{\mathcal{X}} \mu|p| < \infty, p \geq 0\}$$

- e.g. Gaussian models: $\{p(\chi, (m, s)) = \frac{1}{\sqrt{2\pi}s} \mathrm{e}^{-\frac{(\chi - m)^2}{2s^2}} \mid (m, s) \in \Theta \subseteq \mathbb{R} \times \mathbb{R}^+\}$.
- Observables (estimators): functions $f : \mathcal{X} \to \mathbb{R}$

Quantum mechanics:

- Underlying structure: Hilbert space $\mathcal{H}$
- Main spaces: **Spaces of density matrices:**

$$\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+ := \{\rho \in \mathfrak{B}(\mathcal{H}) \mid \mathrm{tr}_{\mathcal{H}}(|\rho|) < \infty, \rho \geq 0\}$$

- e.g. Gibbs states: $\{\mathrm{e}^{-\beta H} \mid \beta \in]0, \infty[\}$, for a fixed self-adjoint H.
- Observables: self-adjoint operators $x : \mathcal{H} \to \mathcal{H}$

Quantum information models and quantum information distances

trace class operators: $\mathcal{T}(\mathcal{H}) := \{\rho \in \mathfrak{B}(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}_{\mathcal{H}}|\rho| < \infty\}$
we will consider arbitrary sets of denormalised quantum states: $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$

Quantum information distances $D : \mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}) \to [0, \infty]$s.t. $D(\rho, \sigma) = 0 \iff \rho = \sigma$.

- E.g.
  - $D_1(\rho, \sigma) := \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$ [Umegaki'62]
  - $D_{1/2}(\rho, \sigma) := 2\left\|\sqrt{\rho} - \sqrt{\sigma}\right\|^2_{\mathfrak{G}_2(\mathcal{H})} = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$ (Hilbert–Schmidt norm^2)
  - $D_{L_1(\mathcal{N})}(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_{\mathcal{T}(\mathcal{H})} = \frac{1}{2}\mathrm{tr}_{\mathcal{H}}|\rho - \sigma|$ (L_1/predual norm)
  - $D_\gamma(\rho, \sigma) := \frac{1}{\gamma(1-\gamma)}\mathrm{tr}_{\mathcal{H}}(\gamma\rho + (1-\gamma)\sigma - \rho^\gamma\sigma^{1-\gamma})$; $\gamma \in \mathbb{R} \setminus \{0, 1\}$ [Hasegawa'93]
  - $D_{\alpha,z}(\rho, \sigma) := \frac{1}{1-\alpha} \log \mathrm{tr}_{\mathcal{H}}(\rho^{\alpha/z}\sigma^{(1-\alpha)/z})^z$; $\alpha, z \in \mathbb{R}$ [Audenauert–Datta'14]

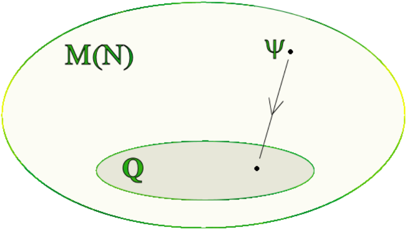
  for $\mathrm{ran}(\rho) \subseteq \mathrm{ran}(\sigma)$, and with all $D(\rho, \sigma) := +\infty$ otherwise.
- Various "quantum geometries" will arise from different additional conditions imposed on pairs $(\mathcal{M}(\mathcal{H}), D)$:
  - Different choices of $\mathcal{M}(\mathcal{H})$ reflect different assumptions on the available possible knowledge (description of experimental situation).
  - Different choices of D reflect different assumptions regarding the convention of "best/optimal" estimation/inference.
  - Both choices are case-to-case-dependent and should be operationally justified.

Quantum entropic projections

Let $\mathcal{Q} \subseteq \mathcal{T}(\mathcal{H})^+$ be such that
for each $\psi \in \mathcal{M}(\mathcal{H})$
there exists a unique solution

$$\boxed{\mathfrak{P}^D_{\mathcal{Q}}(\psi) := \arg\inf_{\rho \in \mathcal{Q}} \{D(\rho, \psi)\}\,.}$$

It will be called an entropic projection.



E.g.

- for $D_{1/2}(\rho, \sigma) = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$,
  consider the entropic projections $\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}$
  where $\mathcal{Q}$ are images of closed convex subspaces $\tilde{\mathcal{Q}} \subseteq \mathcal{K}^+ := \mathfrak{G}_2(\mathcal{H})^+$
  under the mapping $\tilde{\mathcal{Q}} \ni \sqrt{\rho} \mapsto \rho \in \mathcal{Q}$.
  They coincide with the ordinary projection operators in $\mathfrak{B}(\mathcal{K}) \cong \mathfrak{B}(\mathcal{H} \otimes \mathcal{H}^\star)$.
- for $D_1(\rho, \sigma) = \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$
  and $\mathcal{M}(\mathcal{H}) = \mathcal{T}(\mathcal{H})^+_1$, $\psi \in \mathcal{T}(\mathcal{H})^+_1$, $h \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$, then [Araki'77, Donald'90]

$$\exists!\ \psi^h := \arg\inf_{\rho \in \mathcal{T}(\mathcal{H})^+_1} \{D_1(\rho, \psi) + \mathrm{tr}_{\mathcal{H}}(\rho h)\}\,.$$

Bayes–Laplace rule and maximum relative entropy

- Fundamental principle of statistical inference in the bayesian statistics:

$$\text{the Bayes–Laplace rule:} \qquad p(x) \mapsto p_{\mathrm{new}}(x) := \frac{p(x)p(b|x)}{p(b)}.$$

- Williams'80, Warmuth'05, Caticha&Giffin'06: the Bayes–Laplace rule is a special case of

$$p(\chi) \mapsto p_{\mathrm{new}}(\chi) := \arg\inf_{q\in\mathcal{Q}} \{D_1(q,p)\},$$

where D_1 is the Kullback–Leibler distance

$$D_1(q,p) := \int_{\mathcal{X}} \mu(\chi)q(\chi)\log\left(\frac{q(\chi)}{p(\chi)}\right).$$

- Douven&Romeijn'12: the Bayes–Laplace rule is also a special case of

$$p \mapsto \arg\inf_{q\in\mathcal{Q}} \{D_1(p,q)\} = \mathfrak{P}^{D_0}_{\mathcal{Q}}(p),$$

where $D_0(p,q) = D_1(q,p)$.

Lüders' rules

- Lüders' rules provide the basic paradigm for the description of quantum state change due to measurement of an observable $x = \sum_i \lambda_i P_i$:

$$\rho \mapsto \rho_{\mathrm{new}} := \sum_i P_i \rho P_i \ \ (\text{‘weak’} = \text{‘nonselective’}),$$

$$\rho \mapsto \rho_{\mathrm{new}} := \frac{P \rho P}{\mathrm{tr}_{\mathcal{H}}(P\rho)} \ \ (\text{‘strong’} = \text{‘selective’})$$

- Bub'77'79, Caves–Fuchs–Schack'01, Fuchs'02, Jacobs'02: Lüders' rules should be considered as rules of inference (conditioning) that are quantum analogues of the Bayes–Laplace rule.
- Yet, no mathematically exact relationship was provided.

Quantum bayesian inference from quantum entropic projections

- RPK'13'14, F.Hellmann–W.Kamiński–RPK'14:
  1. weak Lüders' rule is a special case of
  $$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
  with
  $$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0 \;\forall i\}$$
  2. strong Lüders' rule derived from
  $$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
  with
  $$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0,\ \mathrm{tr}_{\mathcal{H}}(\sigma P_i) = p_i \;\forall i\}$$
  under the limit $p_2,\ldots,p_n \to 0$.
  3. hence, weak and strong Lüders' rules are special cases of quantum entropic projection $\mathfrak{P}^{D_0}_{\mathcal{Q}}$ based on relative entropy $D_0(\sigma,\rho) = D_1(\rho,\sigma)$.

Meaning: the rule of maximisation of relative entropy (entropic projection on the subspace of constraints) can be considered as a nonlinear generalisation of the dynamics describing "quantum measurement". [RPK'10'11]

Quantum Poisson structure

- Consider the space of self-adjoint trace-class operators: $\mathcal{T}(\mathcal{H})^{\mathrm{sa}} := \mathcal{T}(\mathcal{H}) \cap \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$.
- It can be equipped with a following real Banach smooth manifold structure:
  - tangent spaces: $\mathbf{T}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong \mathcal{T}(\mathcal{H})^{\mathrm{sa}}$
  - cotangent spaces: $\mathbf{T}^{\circledast}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong (\mathcal{T}(\mathcal{H})^{\mathrm{sa}})^\star \cong \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$
- Bóna'91,'00: a Poisson manifold structure on $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$ is defined by a commutator of an algebra:

$$\{h, f\}(\rho) := \mathrm{tr}_{\mathcal{H}}\left(\rho\, \mathrm{i}[\mathbf{d}h(\rho), \mathbf{d}f(\rho)]\right) \forall f, h \in \mathrm{C}^\infty(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}; \mathbb{R})\ \forall \rho \in \mathcal{T}(\mathcal{H})^{\mathrm{sa}}.$$

- So, if $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ is a smooth submanifold of $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$, then every $f \in \mathrm{C}^\infty(\mathcal{M}(\mathcal{H}); \mathbb{R})$ determines a hamiltonian vector field:

$$\mathfrak{X}_f(\rho) = -\{\cdot, f\}(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho\, \mathrm{i}[\mathbf{d}(\cdot), \mathbf{d}f(\rho)]).$$

- More generally, we can choose arbitrary real Banach Lie subalgebra $\mathcal{A}$ of $\mathfrak{B}(\mathcal{H})$ such that: (i) it has a unique Banach predual $\mathcal{A}_\star$ in $\mathcal{T}(\mathcal{H})$; (ii) there exists at least one $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ which is a smooth submanifold of $\mathcal{A}_\star$.

Nonlinear quantum hamiltonian dynamics

For each hamiltonian vector field, the corresponding Hamilton equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\rho(t)) = \{h, f\}(\rho(t)) = \mathrm{i}\,\mathrm{tr}_{\mathcal{H}}\left([\rho(t), \mathbf{d}h(\rho(t))]\mathbf{d}f(\rho(t))\right).$$

The above equation is equivalent to the Bóna equation ['91'00]

$$\boxed{\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [\mathbf{d}h(\rho(t)), \rho(t)].}$$

Hence,

The Poisson structure $\{\cdot,\cdot\}$ induced by a commutator of $\mathfrak{B}(\mathcal{H})$ allows to introduce various nonlinear hamiltonian evolutions on spaces $\mathcal{M}(\mathcal{H})$ of quantum states, generated by arbitrary real-valued smooth functions on $\mathcal{M}(\mathcal{H})$.

The solutions of Bóna equation are state-dependent unitary operators $U(\rho, t)$.
They do not form a group, but satisfy a cocycle relationship:

$$U(\rho, t+s) = U((\mathrm{Ad}(U(\rho, t)))(\rho), s)U(\rho, t) \;\; \forall t, s \in \mathbb{R}.$$

In a special case, when $h(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho H)$ for $H \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$,
the Bóna equation turns to the von Neumann equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H, \rho(t)].$$

Geometry $\Rightarrow$ dynamics

- **Two elementary geometric structures:**
  - $D(\cdot,\cdot)$ represents the convention of "best estimation/inference"
  - $\{h,\cdot\}$ represents a convention of causality ("internal dynamics")
- **Two elementary forms of quantum dynamics:**
  - entropic projections $\mathfrak{P}^D_Q$ generated by quantum distances $D(\cdot,\cdot)$
  - hamiltonian flows w^h_t generated by nonlinear hamiltonian vector fields $\{h,\cdot\}$
- **They allow for two main descriptions of total information dynamics**:
  a) **Sequential processing:** entropic projections composed with hamiltonian flows: $\phi \mapsto \mathfrak{P}^D_{\mathcal{Q}}(\eta)\circ w^h_t(\phi)$
     - ⋆ nonlinear and nonmarkovian
     - ⋆ allows for arbitrary correlations between subsystems
     - ⋆ from the bayesian perspective, $w^h_t(\phi)$ is a prior for $\mathfrak{P}^D_{\mathcal{Q}}(\eta)$-updating
  b) **Parallel processing:** infinitesimal hamiltonian flows perturbed by dissipative dynamics given by free falls along geodesics determined by entropic projections

Quantum causal inferences by entropic-hamiltonian dynamics

A 'sequential' (global) form of quantum dynamics is defined as a causal inference $\mathfrak{P}^D_{\mathcal{Q}} \circ w^h_t$.

- It generalises unitary evolution followed by a "projective measurement".
- Postulate: consider the setting of causal inferences $\mathfrak{P}^D_{\mathcal{Q}} \circ w^h_t$ as an alternative to the paradigm of semigroups of CPTP maps.
- Basic idea: every CPTP instrument [Davies–Lewis'70] can be decomposed into:
  (1) tensor product of initial state with uncorrelated environment,
  (2) unitary evolution,
  (3) projective measurement,
  (4) partial trace.
- It remains to prove that (4) and (3+4) are entropic projections.
  - M.Munk-Nielsen'15: (4) is entropic projection at least for strictly positive states.
  - *Work in progress by RPK+MMN'16:* proving that (3+4) for all states and (4) for nonfaithful ones.

Now let us consider the 'parallel' (local) quantum dynamics (information processing). For this purpose we need first to investigate the local (differential-geometric) structures induced by relative entropy.

Smooth quantum information geometries

Under some conditions, D induces a generalisation of smooth riemannian geometry on $\mathcal{M}(\mathcal{N})$.

- $\mathcal{M}(\mathcal{H}) := \{\rho(\theta) \in \mathcal{T}(\mathcal{H}) \mid \rho(\theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n \text{ open}, \theta \mapsto \rho(\theta) \text{ smooth}\}$ is a C-manifold
- Jenčová'05: a general construction of smooth manifold structure on the space of all strictly positive states over arbitrary W^*-algebra, with tangent spaces given by noncommutative Orlicz spaces.
- Eguchi'83/Ingarden et al'82/Lesniewski–Ruskai'99/Jenčová'04:
  Every smooth distance D with positive definite hessian determines
  a riemannian metric $\mathbf{g}^D$ and a pair $(\nabla^D, \nabla^{D^\dagger})$ of torsion-free affine connections:
  $$\begin{aligned}\mathbf{g}_\phi(u,v) &:= -\partial_{u|\phi}\partial_{v|\omega} D(\phi,\omega)|_{\omega=\phi},\\ \mathbf{g}_\phi((\nabla_u)_\phi v, w) &:= -\partial_{u|\phi}\partial_{v|\phi}\partial_{w|\omega} D(\phi,\omega)|_{\omega=\phi},\\ \mathbf{g}_\phi(v, (\nabla^\dagger_u)_\phi w) &:= -\partial_{u|\omega}\partial_{w|\omega}\partial_{v|\phi} D(\phi,\omega)|_{\omega=\phi},\end{aligned}$$
  which satisfy the characteristic equation of the Norden['37]–Sen['44] geometry,
  $$\mathbf{g}^D(u,v) = \mathbf{g}^D(\mathbf{t}_c^{\nabla^D}(u), \mathbf{t}_c^{\nabla^{D^\dagger}}(v)) \;\; \forall u,v \in \mathbf{T}\mathcal{M}(\mathcal{N}).$$
- A riemannian geometry $(\mathcal{M}(\mathcal{N}), \mathbf{g}^D)$ has Levi-Civita connection $\bar{\nabla} = (\nabla^D + \nabla^{D^\dagger})/2$.

Example

- $\mathcal{M}(\mathcal{N}) = \mathcal{T}(\mathcal{H}) \cap \{\rho > 0, \mathrm{tr}_{\mathcal{H}}(\rho) = 1\}$
  and $D_1(\rho, \sigma) = \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$
  give the Mori['55]–Kubo['56]–Bogolyubov['62] metric $\mathbf{g}^{D_1}$
  and Nagaoka['94]–Hasegawa['95] connections $(\nabla^{D_1}, \nabla^{D_1 \dagger})$:

$$\mathbf{g}^{D_1}_{\rho}(x, y) = \mathrm{tr}_{\mathcal{H}} \left(\int_0^{\infty} \mathrm{d}\lambda x \frac{1}{\lambda \mathbb{I} + \rho} y \frac{1}{\lambda \mathbb{I} + \rho} \right),$$

$$\mathbf{t}^{\nabla^{D_1}}_{\rho,\omega}(x) = x - \mathrm{tr}_{\mathcal{H}}(\omega x), \;\; \mathbf{t}^{\nabla^{D_1 \dagger}}_{\rho,\omega}(x) = x.$$

Hessian geometries = dually flat Norden–Sen geometries

If $(\mathcal{M}, \mathbf{g}, \nabla, \nabla^\dagger)$ is a Norden–Sen geometry with flat ∇ and $\nabla^\dagger$, then:

1. there exists a unique pair of functions $\Phi : \mathcal{M} \to \mathbb{R}$, $\Phi^{\mathbf{L}} : \mathcal{M} \to \mathbb{R}$ such that $\mathbf{g}$ is their hessian metric,

$$\mathbf{g}_{ij}(\rho) = \frac{\partial^2 \Phi(\rho(\theta))}{\partial\theta^i \partial\theta^j} \mathrm{d}\theta^i \otimes \mathrm{d}\theta^j, \quad \mathbf{g}_{ij}(\rho) = \frac{\partial^2 \Phi^{\mathbf{L}}(\rho(\eta))}{\partial\eta^i \partial\eta^j} \mathrm{d}\eta^i \otimes \mathrm{d}\eta^j,$$

   where: $\{\theta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla}_{ijk}(\rho(\theta)) = 0 \; \forall\rho \in \mathcal{M}$, $\{\eta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla^\dagger} ijk(\rho(\eta)) = 0 \; \forall\rho \in \mathcal{M}$.

2. the Eguchi equations applied to the Brègman distance

$$D_\Phi(\rho, \sigma) := \Phi(\rho) + \Phi^{\mathbf{L}}(\sigma) - \sum_i \theta^i(\rho)\eta^i(\sigma)$$

   yield $(\mathbf{g}, \nabla, \nabla^\dagger)$ above.

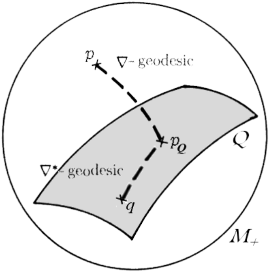
Smooth generalised pythagorean theorem

Let $(\mathcal{M}, \mathbf{g}, \nabla, \nabla^\dagger)$ be a hessian geometry. Then for any $\mathcal{Q} \subseteq \mathcal{M}$ which is:

- $\nabla^\dagger$-autoparallel $:= \nabla^\dagger_u v \in \mathbf{T}\mathcal{Q}\ \forall u, v \in \mathbf{T}\mathcal{Q}$;
- $\nabla^\dagger$-convex $:= \forall \rho_1, \rho_2 \in \mathcal{Q}\ \exists!\ \nabla^\dagger$-geodesics in $\mathcal{Q}$ connecting ρ_1 and ρ_2;

there exists a unique projection

$$\mathcal{M} \ni \rho \mapsto \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho) := \arg\inf_{\sigma \in \mathcal{Q}} \{D_\Phi(\sigma, \rho)\} \in \mathcal{Q}.$$



- it is equal to a unique projection of ρ onto $\mathcal{Q}$ along a ∇-geodesic that is $\mathbf{g}$-orthogonal at $\mathcal{Q}$.
- it satisfies a generalised pythagorean equation

$$D_\Phi(\omega, \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho)) + D_\Phi(\mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho), \rho) = D_\Phi(\omega, \rho) \quad \forall (\omega, \rho) \in \mathcal{Q} \times \mathcal{M}.$$

Hence, for Brègman distances D_Φ the local entropic projections are equivalent with geodesic projections.

Local effective dynamics = parallel quantum information processing

- One can combine locally the entropic projections with hamiltonian flows, by passing to the derived geodesic projections, and combining both in a single formula for effective dynamics.
- Given a hamiltonian observable h and a relative entropy D, the 1-form $dh(\phi) - d_{\nabla^D}(\phi)$ represents a local perturbation of causal dynamics by the information flow along entropic geodesics.
- In particular, $D_{1/2} = 2\left\|\sqrt{\rho} - \sqrt{\sigma}\right\|^2_{\mathcal{H}}$ gives Wigner–Yanase metric $\mathbf{g}^{1/2}$, with $d_{\mathbf{g}^{1/2}}(\rho,\sigma) = 2\arccos(\mathrm{tr}_{\mathcal{H}}(\sqrt{\rho}\sqrt{\sigma}))$. The free fall along the geodesics of Levi-Civita connection $\nabla^{1/2}$ encodes the continuous process of projective measurement.
- The resulting effective dynamics can be given mathematically exact form in terms of a continuous-time regularised path-integral

$$\lim_{\varepsilon\to+0}\int\mathcal{D}\phi(\cdot)\mathrm{e}^{\mathrm{i}\int_\gamma \mathrm{d}t\left\langle\Omega_{\phi(t)},\mathbf{d}_{\nabla^{1/2}}(\phi(t)\Omega_{\phi(t)}\right\rangle_{\mathcal{H}_{\phi(t)}}}\cdot$$
$$\cdot\mathrm{e}^{-\mathrm{i}\int_\gamma \mathrm{d}t\left\langle\Omega_{\phi(t)},\pi_{\phi(t)}(\mathbf{d}h(\phi(t)))\Omega_{\phi(t)}\right\rangle}\mathrm{e}^{-\frac{\varepsilon}{2}\int_\gamma \mathrm{d}t\mathbf{g}^{1/2}_{ab}(\phi(t))\dot{\phi}^a\dot{\phi}^b},$$

- If evaluated only on boundary pure states, and for $h(\phi) = \phi(\mathcal{H})$, it is known (Daubechies–Klauder'85, Anastopoulos–Savvidou'03) to be equal to a propagator $\left\langle\Omega(t=s), e^{-\mathrm{i}Hs}\Omega(t=0)\right\rangle_{\mathcal{H}}$.

3. Quantum information geometric approach to foundations of quantum theory beyond quantum mechanics

- States on W^*-algebras as noncommutative integrals
- Information theoretic/bayesian measurement theory
- Towards purely geometric reconstruction

Towards new foundations

Idea:

- consider spaces $\mathcal{M}(\mathcal{H})$ as fundamental
- allow any nonlinear functions $\mathcal{M}(\mathcal{H}) \to \mathbb{R}$ as observables
- define geometry of $\mathcal{M}(\mathcal{H})$ by means of $D(\cdot,\cdot)$ and $\{\cdot,\cdot\}$
- define dynamics of $\mathcal{M}(\mathcal{H})$ by means of $\mathfrak{P}^{D}_{\mathcal{Q}}(\cdot,\cdot)$ and $w_t^{\{h,\cdot\}}$

Questions:

- what's up with Hilbert spaces? (are they necessary? if not, then what?)
- what's up with spectral theory, probability, Born rule, etc?

Answers:

- replace Hilbert spaces by W^*-algebras
- replace density matrices by positive integrals on W^*-algebras
- this setting is an exact generalisation of Kolmogorov's measure theoretic setting for probability theory
- build up all remaining semantics for quantum theory
  in the analogy to semantics of probability theory and statistical inference
  (hence: no Born rule, no probabilities, no spectral theory)

W^*-algebras and integration

- A W^*-algebra $\mathcal{N}$:
  - an algebra over $\mathbb{R}$ or $\mathbb{C}$ with unit $\mathbb{I}$,
  - with * operation s.t. $(xy)^* = y^*x^*$, $(x+y)^* = x^* + y^*$, $(x^*)^* = x$, $(\lambda x)^* = \lambda^* x^*$,
  - that is also a Banach space,
  - with $\cdot$, $+$, * continuous in the norm topology (implied by the condition $\|x^*x\| = \|x\|^2$),
  - such that there exists a Banach space $\mathcal{N}_\star$ satisfying the Banach space duality: $(\mathcal{N}_\star)^\star \cong \mathcal{N}$,
- Special cases:
  - if $\mathcal{N}$ is commutative
    then $\exists$ a measure space $(\mathcal{X}, \mu)$ s.t. $\mathcal{N} \cong L_\infty(\mathcal{X}, \mu)$ and $\mathcal{N}_\star \cong L_1(\mathcal{X}, \mu)$
  - if $\mathcal{N}$ is "type I factor"
    then $\exists$ a Hilbert space $\mathcal{H}$ s.t. $\mathcal{N} \cong \mathfrak{B}(\mathcal{H})$ and $\mathcal{N}_\star \cong \mathcal{T}(\mathcal{H})$.
- Hence, the element $\phi \in (\mathcal{N}_\star)^+$ provides a joint generalisation of probability density and of density operator. By means of embedding of $\mathcal{N}_\star$ into $\mathcal{N}^\star$, it is also an integral on $\mathcal{N}$.
- Key fact: The above setting allows to develop full-fledged integration theory on noncommutative W^*-algebras, which generalises integration theory on measure spaces (with partial integration, conditional expectations, $L_p(\mathcal{N})$ spaces,...).

New kinematics: quantum models and observables

General quantum information models:

For any W^*-algebra $\mathcal{N}$, $\mathcal{M}(\mathcal{N})$ will be defined as an arbitrary subset of a positive part of a Banach predual space of $\mathcal{N}$, $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$.

Special cases:

- $\mathcal{N}$ is commutative $\Rightarrow \mathcal{M}(\mathcal{N}) = \mathcal{M}(\mathcal{X}, \mu)$
- $\mathcal{N}$ is type I factor $\Rightarrow \mathcal{M}(\mathcal{N}) = \mathcal{M}(\mathcal{H})$.

We do not assume that:

- $\mathcal{M}(\mathcal{N})$ is convex ($\iff$ probabilistic mixing)
- $\mathcal{M}(\mathcal{N})$ is smooth ($\iff$ asymptotic estimation)
- $\mathcal{M}(\mathcal{N})$ is normalised ($\iff$ frequentist interpretation)

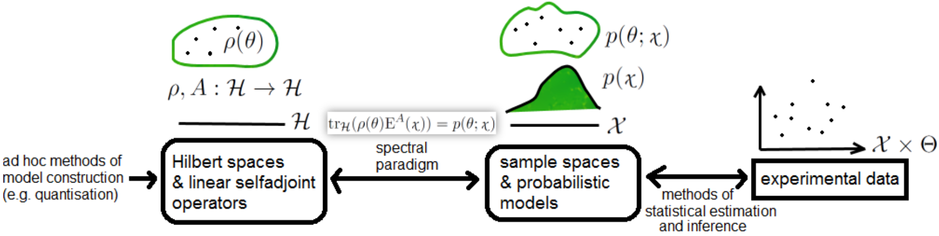
Observables:

Observables are defined as arbitrary functions $f : \mathcal{M}(\mathcal{N}) \to \mathbb{R}$.

Hence: smooth observables define hamiltonian vector fields.

Each "observable in the old sense" $x \in \mathcal{N}^{\mathrm{sa}}$ determines a corresponding "observable in the new sense" by $f_x(\phi) := \phi(x)$.

Orthodox quantum mechanical paradigm (von Neumann, 1926-1932):

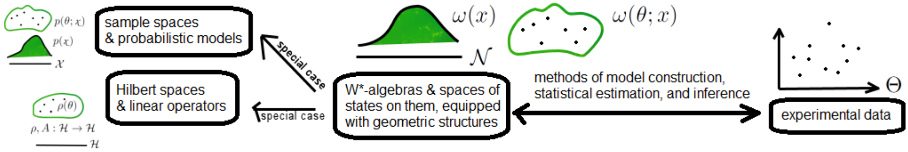


- a solution of a particular problem (solid mathematical framework providing unifying foundations for 'wave mechanics' and 'matrix mechanics')
- von Neumann'1935: "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space anymore."

Some key observations:

- Probability theory is just a special case of integration theory on W^*-algebras.
- From the perspective of this theory, quantum states are **just** integrals, so there is no a priori reason why "general" quantum theory (beyond QM) should depend on probabilities.
- Quantum states (and structures over them) can be associated directly with the epistemic data by generalising the methods of associating epistemic data with probabilities (and with structures over them).

New paradigm:



Basic object of interest: spaces $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$ of states over W^*-algebras $\mathcal{N}$.

- Quantum theoretic kinematics generalises and replaces probability theory.
- Quantum theoretic dynamics generalises and replaces causal statistical inference.
- Nonlinear information geometry of spaces of quantum states replaces the role of (linear) spectral theory of quantum mechanics.
- Replace the use of eigenvalues and expectations of self-adjoint operators on $\mathcal{H}$ (or in $\mathcal{N}$) by observables $f : \mathcal{M}(\mathcal{N}) \to \mathbb{R}$. Given any model construction rule $\mathbb{R}^n \supset \Theta \ni \theta \mapsto \rho(\theta) \in \mathcal{M}(\mathcal{N})$, and the set of experimental functions $f_\Theta : \Theta \to \mathbb{R}$ the set of observables relevant to the problem is given by $\{f : \mathcal{M}(\mathcal{N}) \to \mathbb{R} \mid f_\Theta = f \circ \theta\}$

Locally quantum mechanical quantum information theory

1. **local kinematics (only in tangent space)**:
   - states: vectors of $\mathbf{T}_\phi\mathcal{M}(\mathcal{N})$ (configurations: $\phi(\theta)\to\theta\to\frac{\partial}{\partial\theta}$)
   - effects: vectors of $\mathbf{T}^{\circledast}_\phi\mathcal{M}(\mathcal{N})$ (observables: $f\to\mathbf{d}f(\phi)$)
2. **local dynamics (only in tangent space)**:
   - causality: hamiltonian causality is local
   - inference: arbitrary entropic projections are nonlocal, but the Norden–Sen geometries derived from relative entropies allow to localise entropic projections
   - causality+inference: as presented few slides ago
3. **main insight**: Quantum mechanics holds locally, but does not have to hold globally. The degree to which it does not hold is measured by the differential geometric structure of the state space.
4. **reconstruction of W^*-algebras**: Can we start from *arbitrary* sets $\mathcal{M}$, equipped with geometric structures $\{\cdot,\cdot\}$ and $D(\cdot,\cdot)$, without knowing that they are over W^*-algebras, and reconstruct $\mathcal{M}=\mathcal{M}(\mathcal{N})$ from some conditions? $\to$ **work in progress!**
5. Basic idea of a proof: W^*-algebras $=$ LJBW*-algebras $=$ BLP submanifolds extendible to convex hull, with observables having Jordan structure $=$ BLP submanifolds (=Poisson spaces) $\mathcal{M}$ with riemannian structure induced from relative entropy and Kähler compatibility condition on the convex hull of $\mathcal{M}$ $\leftarrow$ **main conjecture**

References

Main papers:

- RPK, 2016, Local quantum information dynamics, arXiv:1605.02063.
- RPK, 2016, Towards quantum information geometric foundations, soon on arXiv :).

Earlier results and insights:

- M.I.Munk-Nielsen, 2015, Quantum measurements from entropic projections, MSc Thesis, Perimeter Institute, Waterloo, Canada.
- RPK, 2014, Lüders' and quantum Jeffrey's rules as entropic projections, arXiv:1408.3502.
- F. Hellmann, W. Kamiński, RPK, 2014, Quantum collapse rules from maximum relative entropy principle, New J. Phys. **18** (2016), 013022. arXiv:1407.7766.
- RPK, 2013, W^*-algebras and noncommutative integration, arXiv:1307.4818.
- RPK, 2013, Information geometric foundations of quantum theory, Ph.D. thesis, Institute for Theoretical Physics, University of Warsaw.
- RPK, 2011, Information dynamics and new geometric foundations for quantum theory, Växjö 2011 Proceedings, AIP Conf. Proc. **1424**, 200. arXiv:1110.4492.
- RPK, 2011, On principles of inductive inference, MAXENT 2011 Proceedings, AIP Conf. Proc. **1443**, 22. arXiv:1109.3142.
- RPK, 2011, The general form of γ-family of quantum relative entropies, Open Sys. Inf. Dyn. **18**, 191. arXiv:1106.2225.
- RPK, 2010, Quantum theory as inductive inference, MAXENT 2010 Proceedings, AIP Conf. Proc. **1305**, 28. arXiv:1009.2423.

∞-dim dynamics: Banach–Lie–Poisson spaces

- A real Banach–Lie–Poisson space is defined as a pair $(X, \{\cdot,\cdot\})$ s.t.
  1) X is a Banach space over $\mathbb{R}$
  2) $(X, \{\cdot,\cdot\})$ is a Banach Poisson manifold:
     - ⋆ $(\mathrm{C}^\infty_{\mathrm{F}}(X;\mathbb{R}), \{\cdot,\cdot\})$ is a Lie algebra,
     - ⋆ $\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2\{f_1, f_3\}\ \forall f_1, f_2, f_3 \in \mathrm{C}^\infty_{\mathrm{F}}(X;\mathbb{R})$,
     - ⋆ $\{\cdot, f\}$ is a vector on X,
  3) $X^\star := \mathrm{C}(X;\mathbb{R}) \subseteq \mathrm{C}^\infty_{\mathrm{F}}(X;\mathbb{R})$ is a Banach Lie algebra w.r.t. $[\cdot,\cdot]$ related to $\{\cdot,\cdot\}$ by

$$\{f,k\}(z) = \left([\mathfrak{D}^{\mathrm{F}}_z f, \mathfrak{D}^{\mathrm{F}}_z k]\right)(z)\quad \forall f,k \in \mathrm{C}^\infty_{\mathrm{F}}(X;\mathbb{R})\ \forall z \in X.$$

- Under these conditions, the hamiltonian vector associated to *any* $k \in \mathrm{C}^\infty_{\mathrm{F}}(X;\mathbb{R})$ reads

$$\mathfrak{X}_k(z) = -\{\cdot, k\}(z) = ([k,\cdot])(z),$$

  and the Hamilton equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(w^h_t(x)) = \{h, f(w^h_t(x))\}\ \ \forall f \in \mathrm{C}^\infty_{\mathrm{F}}(X;\mathbb{R})$$

  determines a unique local map $w^h_t : X \to X$ called a hamiltonian flow.
- Odzijewicz–Ratiu'03: for arbitrary W^*-algebra $\mathcal{N}$: $X = \mathcal{N}^{\mathrm{sa}}_\star$, $X^\star = \mathcal{N}^{\mathrm{sa}}$.
- Symplectic leaves: spaces of states with finite and constant rank.