Information geometric quantum foundations: new results & open problems

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«Unlike the Riemannian manifolds the quantum mechanical unit spheres do not differ one from another: they are all isomorphic. The worlds of the present-day quantum mechanics thus present a picture of structural monotony: they are all 'painted' on the same standard ideally symmetric surface. The formalism of the quantum theory of infinite systems and quantum field theory is not very different from that. (...) the basic structural framework of the theory is conserved at the cost of quantitative multiplication: when meeting a new level of physical reality the quantum theory responds by simply producing infinite tensor products of its basic structure. (...) It may be that present day quantum theory still represents a relatively primitive stage of development and lacks some essential evolutionary steps leading towards structural flexibility. If this were so, further development would involve a programme opposite to the 'quantization of gravity': instead of modifying general relativity to fit quantum mechanics one should rather modify quantum mechanics to fit general relativity.»

Bogdan Mielnik, 1976, *Quantum logic: is it necessarily orthocomplemented?*

$$\frac{\textbf{SR}}{\textbf{GR}} = \frac{\textbf{QM}}{\textbf{General quantum theory}} \ ?$$

$$\frac{\text{Ptolemaic epicycles}}{\text{Newtonian theory}} = \frac{\text{Feynman diagrams}}{\text{General quantum theory}} \ ?$$

Plan

1. **Global kinematics**: sets of information states equipped with relative entropies & Lie–Poisson structures
2. **Global dynamics**: entropic projections/instruments & hamiltonian flows
3. **Local kinematics**: spaces of local configurations/effects as tangent/cotangent spaces
4. **Local dynamics**:
   - *hamiltonian approach:* generalised von Neumann equation with a free fall along entropic geodesics
   - *lagrangean approach:* geometric path integral with generalised hamiltonian and entropic connection terms, weighted by the curvature-dependent measure
5. **Open problems**

1. Global theory

- Geometric structures on spaces $\mathcal{M}$ of quantum states: relative entropies & Poisson brackets
- Linear operators on Hilbert spaces $\rightarrow$ real-valued functions on $\mathcal{M}$
- Lüders' rules $\rightarrow$ constrained relative entropy maximisations on $\mathcal{M}$
- Unitary evolution $\rightarrow$ nonlinear hamiltonian flows on $\mathcal{M}$

Quantum information models and quantum information distances

trace class operators: $\mathcal{T}(\mathcal{H}) := \{\rho \in \mathfrak{B}(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}_{\mathcal{H}}|\rho| < \infty\}$
we will consider arbitrary sets of denormalised quantum states: $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$

Quantum information distances $D : \mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}) \to [0, \infty]$ s.t. $D(\rho, \sigma) = 0 \iff \rho = \sigma$.

- E.g.
  - $D_1(\rho, \sigma) := \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$ [Umegaki'62]
  - $D_{1/2}(\rho, \sigma) := 2\|\sqrt{\rho} - \sqrt{\sigma}\|^2_{\mathfrak{G}_2(\mathcal{H})} = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$ (Hilbert–Schmidt norm^2)
  - $D_{L_1(\mathcal{N})}(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_{\mathcal{T}(\mathcal{H})} = \frac{1}{2}\mathrm{tr}_{\mathcal{H}}|\rho - \sigma|$ (L_1/predual norm)
  - $D_\gamma(\rho, \sigma) := \frac{1}{\gamma(1-\gamma)}\mathrm{tr}_{\mathcal{H}}(\gamma\rho + (1-\gamma)\sigma - \rho^\gamma\sigma^{1-\gamma})$; $\gamma \in \mathbb{R} \setminus \{0, 1\}$ [Hasegawa'93]
  - $D_{\alpha,z}(\rho, \sigma) := \frac{1}{1-\alpha}\log \mathrm{tr}_{\mathcal{H}}(\rho^{\alpha/z}\sigma^{(1-\alpha)/z})^z$; $\alpha, z \in \mathbb{R}$ [Audenauert–Datta'14]
  - $D_{\mathfrak{f}}(\rho, \sigma) := \mathrm{tr}_{\mathcal{H}}(\sqrt{\rho}\,\mathfrak{f}(\mathfrak{L}_\rho\mathfrak{R}_\sigma^{-1})\sqrt{\rho})$; $\mathfrak{f}$ operator convex, $\mathfrak{f}(1) = 0$ [Kosaki'82,Petz'85]

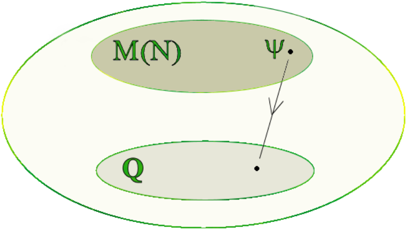
  for $\mathrm{ran}(\rho) \subseteq \mathrm{ran}(\sigma)$, and with all $D(\rho, \sigma) := +\infty$ otherwise.
- Various "quantum geometries" will arise from different additional conditions imposed on pairs $(\mathcal{M}(\mathcal{H}), D)$:
  - Different choices of $\mathcal{M}(\mathcal{H})$ reflect different assumptions on the available possible knowledge (description of experimental situation).
  - Different choices of D reflect different assumptions regarding the convention of "best/optimal" estimation/inference.
  - Both choices are case-to-case-dependent and should be operationally justified.

Quantum entropic projections

Let $\mathcal{Q} \subseteq \mathcal{T}(\mathcal{H})^+$ be such that
for each $\psi \in \mathcal{M}(\mathcal{H})$
there exists a unique solution

$$\mathfrak{P}^D_{\mathcal{Q}}(\psi) := \arg\inf_{\rho\in\mathcal{Q}} \{D(\rho,\psi)\}\,.$$

It will be called an entropic projection.



E.g.

- for $D_{1/2}(\rho,\sigma) = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$,
  consider the entropic projections $\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}$
  where $\mathcal{Q}$ are images of closed convex subspaces $\tilde{\mathcal{Q}} \subseteq \mathcal{K}^+ := \mathfrak{G}_2(\mathcal{H})^+$
  under the mapping $\tilde{\mathcal{Q}} \ni \sqrt{\rho} \mapsto \rho \in \mathcal{Q}$.
  They coincide with the ordinary projection operators in $\mathfrak{B}(\mathcal{K}) \cong \mathfrak{B}(\mathcal{H}\otimes\mathcal{H}^\star)$.
- for $D_1(\rho,\sigma) = \mathrm{tr}_{\mathcal{H}}(\rho\log\rho - \rho\log\sigma)$
  and $\mathcal{M}(\mathcal{H}) = \mathcal{T}(\mathcal{H})^+_1$, $\psi \in \mathcal{T}(\mathcal{H})^+_1$, $h \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$, then [Araki'77, Donald'90]

$$\exists!\ \psi^h := \arg\inf_{\rho\in\mathcal{T}(\mathcal{H})^+_1} \{D_1(\rho,\psi) + \mathrm{tr}_{\mathcal{H}}(\rho h)\}\,.$$

Quantum measurement, bayesianity, and maximum relative entropy

- Lüders' rules:

$$\rho \mapsto \rho_{\mathrm{new}} := \sum_i P_i \rho P_i \quad (\text{'weak'})$$

$$\rho \mapsto \rho_{\mathrm{new}} := \frac{P\rho P}{\mathrm{tr}_{\mathcal{H}}(P\rho)} \quad (\text{'strong'})$$

- Bub'77'79, Caves–Fuchs–Schack'01, Fuchs'02, Jacobs'02: Lüders' rules should be considered as rules of inference (conditioning) that are quantum analogues of

the Bayes–Laplace rule: $$p(x) \mapsto p_{\mathrm{new}}(x) := \frac{p(x)p(b|x)}{p(b)}.$$

- Williams'80, Warmuth'05, Caticha&Giffin'06: the Bayes–Laplace rule is a special case of

$$p(\chi) \mapsto p_{\mathrm{new}}(\chi) := \arg\inf_{q \in \mathcal{Q}} \{D_1(q,p)\}; \quad D_1(q,p) := \int_{\mathcal{X}} \mu(\chi) q(\chi) \log\left(\frac{q(\chi)}{p(\chi)}\right).$$

- Douven&Romeijn'12: the Bayes–Laplace rule is also a special case of

$$p \mapsto \arg\inf_{q \in \mathcal{Q}} \{D_1(p,q)\} = \mathfrak{P}^{D_0}_{\mathcal{Q}}(p),$$

where $D_0(p,q) = D_0(q,p)$.

Quantum bayesian inference from quantum entropic projections

- RPK'13'14, F.Hellmann–W.Kamiński–RPK'14:

1. weak Lüders' rule is a special case of
$$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
with
$$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0\ \forall i\}$$
2. strong Lüders' rule derived from
$$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
with
$$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0,\ \mathrm{tr}_{\mathcal{H}}(\sigma P_i) = p_i\ \forall i\}$$
under the limit $p_2,\ldots,p_n \to 0$.
3. hence, weak and strong Lüders' rules are special cases of quantum entropic projection $\mathfrak{P}^{D_0}_{\mathcal{Q}}$ based on relative entropy $D_0(\sigma,\rho) = D_1(\rho,\sigma)$.

Bayes–Laplace and Lüders' conditionings are special cases of entropic projections $\Rightarrow$ "quantum bayesianism $\subseteq$ quantum relative entropism".

Meaning: the rule of maximisation of relative entropy (entropic projection on the subspace of constraints) can be considered as a nonlinear generalisation of the dynamics describing "quantum measurement". [RPK'10'11]

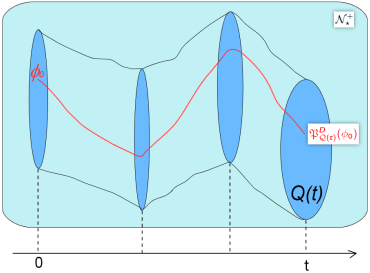
Global entropic dynamics: Quantum entropic instruments (RPK'13)

Given a set Ξ of *registration parameters*, we define a quantum entropic instrument as a map

$$\Xi \ni \eta \mapsto \mathfrak{P}^D_{\mathcal{Q}(\eta)} \in \mathrm{Hom}_{\mathbf{Set}}(\mathcal{M}_1(\mathcal{N}), \mathcal{M}_2(\mathcal{N})).$$

We will deal only with such $\mathcal{Q}$ that are convex.

- More generally, the set Ξ can be time dependent, so the entropic instrument can describe the time dependent nonlinear dynamics of quantum states 'driven' by the changes of registration parameters in time.
- E.g.: If $\{\mathcal{Q}(t) \subseteq \mathcal{N}^+_\star \mid t \in \mathbb{R}\}$ is a family of convex closed sets with $\mathfrak{P}^D_{\mathcal{Q}(0)}(\phi_0) = \phi_0$, and the map $t \mapsto \mathcal{Q}(t)$ is continuous, then $t \mapsto \mathfrak{P}^D_{\mathcal{Q}(t)}(\phi_0)$ becomes a continuous trajectory in the tube of convex sets of quantum states.



- If $\mathcal{Q}(\eta)$ is compact, then it is possible to define the corresponding quantum entropic effect as a maximal central measure on $\mathcal{Q}(\eta)$ with a barycenter $\mathfrak{P}^D_{\mathcal{Q}(\eta)}(\phi_0)$.
  (It is unique by virtue of Wils' theorem.)
- If $\mathcal{Q}(\eta)$ is equal to space of all states on some C^*-algebra $\mathcal{C}$, then every quantum entropic effect determines a unique corresponding POVM acting from Borel subsets of $\mathcal{Q}(\eta)$ to the positive part of the commutant of the GNS representation of $\mathcal{C}$ in the state $\mathfrak{P}^D_{\mathcal{Q}(\eta)}(\phi_0)$.
  (This follows from the Tomita–Ruelle theorem [cf. Halvorson'04].)

D_Ψ

- The choice of the set $\mathcal{Q}$ for which the entropic projection $\mathfrak{P}^D_{\mathcal{Q}}$ exists and is unique depends very strongly on the structure of D: the choice of principle of inference (D) determines the accepted data types ($\mathcal{Q}$).
- This leads to a question of general conditions on D that would be sufficient to guarantee existence and uniqueness, as well as good composition properties of subsequent projections (to have a category of entropic instruments).
- It turns out that this can be provided by a class of Brègman distances:

$$D_\Psi(\phi,\omega) := \widetilde{D}_\Psi(\ell(\phi),\ell(\omega))$$
$$\widetilde{D}_\Psi(x,y) := \Psi(x) - \Psi(y) - \mathfrak{D}^{\mathrm{G}}_{+}\Psi(y;x-y)$$
$$\ell : U \to X, \quad U \subseteq \mathcal{M}$$

  where X is a topological vector space, and $\Psi : X \to]-\infty,+\infty]$ is convex and lower semi-continuous.
- The sets $\mathcal{Q}$ are required to be convex closed subsets of the spaces X under the embedding ℓ. One can think of ℓ as a coordinate system on $U \subseteq \mathcal{M}$, and X as the linear parameter space used for specification of the data required for the entropic projection.

D_Ψ: Quantum distances satisfying generalised pythagorean equation

A property that is of high importance from information geometric point of view, and is also crucial geometrically, is a generalised (nonsymmetric, nonlinear) pythagorean equation.

- we say that D satisfies a generalised pythagorean equation at $\mathcal{Q}$ iff [Chencov'68]

$$D(\phi,\psi) = D(\phi,\mathfrak{P}^D_{\mathcal{Q}}(\psi)) + D(\mathfrak{P}^D_{\mathcal{Q}}(\psi),\psi) \;\; \forall(\phi,\psi)\in\mathcal{Q}\times\mathcal{M}.$$

- Thus, information distance decomposes additively under a projection onto a suitable subspace, hence we have a nonlinear **data = signal + noise** decomposition (!)
- It turns out that all Brègman distances satisfy generalised pythagorean theorem for sets that are affine under ℓ-embeddings.
- Example 1: If $\mathcal{Q}$ forms an affine subset of $\mathfrak{G}_2(\mathcal{H})^+$ under $\rho\mapsto\sqrt{\rho}$, then:

$$\left\|x-\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}(z)\right\|^2_{\mathfrak{G}_2(\mathcal{H})} + \left\|\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}(z)-z\right\|^2_{\mathfrak{G}_2(\mathcal{H})} = \|x-z\|^2_{\mathfrak{G}_2(\mathcal{H})}.$$

- Example 2: If $\mathcal{Q} := \{\phi\in\mathfrak{G}_1(\mathcal{H})^+_1 \mid \phi(h)=\mathrm{const}\}$, then [Donald'90]

$$D_1(\phi,\psi^h) + D_1(\psi^h,\psi) = D_1(\phi,\psi) \;\; \forall(\phi,\psi)\in\mathcal{Q}\times\mathfrak{G}_1(\mathcal{H})^+_1.$$

Quantum Poisson structure

- Consider the space of self-adjoint trace-class operators: $\mathcal{T}(\mathcal{H})^{\mathrm{sa}} := \mathcal{T}(\mathcal{H}) \cap \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$.
- It can be equipped with a following real Banach smooth manifold structure:
  - tangent spaces: $\mathbf{T}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong \mathcal{T}(\mathcal{H})^{\mathrm{sa}}$
  - cotangent spaces: $\mathbf{T}^{\circledast}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong (\mathcal{T}(\mathcal{H})^{\mathrm{sa}})^{\star} \cong \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$
- Bóna'91,'00: a Poisson manifold structure on $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$ is defined by a commutator of an algebra:

$$\{h, f\}(\rho) := \mathrm{tr}_{\mathcal{H}}\left(\rho\, \mathrm{i}[\mathbf{d}h(\rho), \mathbf{d}f(\rho)]\right) \; \forall f, h \in \mathrm{C}^\infty(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}; \mathbb{R}) \; \forall \rho \in \mathcal{T}(\mathcal{H})^{\mathrm{sa}}.$$

- So, if $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ is a smooth submanifold of $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$, then every $f \in \mathrm{C}^\infty(\mathcal{M}(\mathcal{H}); \mathbb{R})$ determines a hamiltonian vector field:

$$\mathfrak{X}_f(\rho) = -\{\cdot, f\}(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho\, \mathrm{i}[\mathbf{d}(\cdot), \mathbf{d}f(\rho)]).$$

- More generally, we can choose arbitrary real Banach Lie subalgebra $\mathcal{A}$ of $\mathfrak{B}(\mathcal{H})$ such that: (i) it has a unique Banach predual $\mathcal{A}_\star$ in $\mathcal{T}(\mathcal{H})$; (ii) there exists at least one $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ which is a smooth submanifold of $\mathcal{A}_\star$.

Nonlinear quantum hamiltonian dynamics

For each hamiltonian vector field, the corresponding Hamilton equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\rho(t)) = \{h, f\}(\rho(t)) = \mathrm{i}\,\mathrm{tr}_{\mathcal{H}}\left([\rho(t), \mathbf{d}h(\rho(t))]\mathbf{d}f(\rho(t))\right).$$

The above equation is equivalent to the Bóna equation ['91'00]

$$\boxed{\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [\mathbf{d}h(\rho(t)), \rho(t)].}$$

Hence,

The Poisson structure $\{\cdot,\cdot\}$ induced by a commutator of $\mathfrak{B}(\mathcal{H})$ allows to introduce various nonlinear hamiltonian evolutions on spaces $\mathcal{M}(\mathcal{H})$ of quantum states, generated by arbitrary real-valued smooth functions on $\mathcal{M}(\mathcal{H})$.

The solutions of Bóna equation are state-dependent unitary operators $U(\rho, t)$.
They do not form a group, but satisfy a cocycle relationship:

$$U(\rho, t+s) = U((\mathrm{Ad}(U(\rho, t)))(\rho), s)U(\rho, t) \;\; \forall t, s \in \mathbb{R}.$$

In a special case, when $h(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho H)$ for $H \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$,
the Bóna equation turns to the von Neumann equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H, \rho(t)].$$

New kinematics: quantum information geometry

- Main change: Consider expectation values as more fundamental than eigenvalues $\Rightarrow$ foundational role of spectral theory replaced by quantum information geometry
  (1) **spaces**:
  replace: linear Hilbert spaces $\mathcal{H}$ of eigenvectors
  by: sets $\mathcal{M}(\mathcal{N})$ of denormalised expectation functionals on W^*-algebras $\mathcal{N}$.
  (2) **observables**:
  replace: linear functions $\mathcal{H} \to \mathcal{H}$ with real eigenvalues
  by: nonlinear real valued functions $\mathcal{M}(\mathcal{N}) \to \mathbb{R}$.
  (3) **geometry**:
  replace: geometry of Hilbert spaces $\mathcal{H}$ defined by scalar product $\langle \cdot, \cdot \rangle$
  by: geometry of spaces $\mathcal{M}(\mathcal{N})$ defined by quantum relative entropies $D(\cdot, \cdot)$ and quantum Poisson structures $\{\cdot, \cdot\}$.

- **Two fundamental geometric structures on $\mathcal{M}(\mathcal{N})$:**
  a) Quantum distances $D(\cdot, \cdot)$
    - ⋆ represents the choice of a convention of a "global" (nonasymptotic) estimation/inference/'best fit'
    - ⋆ large variety of choices
    - ⋆ allows to derive as special cases: riemannian geometry (via $\partial_i \partial_j D$, see later slides) and Hilbert space projective geometry (via $\mathfrak{P}^D_\mathcal{Q}$ for $D = D_{1/2}$)

  b) Quantum Poisson structures $\{\cdot, \cdot\}$
    - ⋆ represents the choice of a specific algebra of locally conserved quantities
    - ⋆ depends on the choice of a real Banach Lie subalgebra of $\mathcal{N}$
    - ⋆ generalises symplectic geometry
    - ⋆ $\{h, \cdot\}$ represents the choice of a convention of a "global" causality

- No Hilbert spaces, no probability theory in foundations (derived as special cases)

New dynamics: information geometric causal inference

- **Two fundamental dynamic structures on $\mathcal{M}(\mathcal{N})$:**

  a) **Inference:** Entropic projections $\phi \mapsto \arg\inf_{\omega\in\mathcal{Q}(\eta)}\{D_\Psi(\omega,\phi)\}$ [RPK'10]
    - ★ nonlinear and nonlocal
    - ★ requires convexity
    - ★ represents ("active/external") information dynamics due to learning/measuring
    - ★ allows to encode experimental constraints
    - ★ reduces in special cases to Lüders', Jeffrey's, Bayes' rules

  b) **Causality:** Hamiltonian flows $\phi \mapsto w_t^h(\phi)$, $\frac{\mathrm{d}}{\mathrm{d}t}f(w_t^h(\phi)) = \{h, f(w_t^h)\}(\phi)\ \forall f$ [Bóna'00]
    - ★ nonlinear and local
    - ★ requires smoothness
    - ★ represents ("passive/internal") changes of information states when no inference is made
    - ★ allows to encode theoretical symmetries
    - ★ reduces in a special case to the von Neumann equation

- **Sequential processing postulate:** consider the setting of causal inferences $\phi \mapsto \mathfrak{P}_{\mathcal{Q}}^{D_\Psi}(\eta)\circ w_t^h(\phi)$ as an alternative to the paradigm of semigroups of CPTP maps
  - ▶ it generalises unitary evolution followed by a "projective measurement"
  - ▶ nonlinear and nonmarkovian
  - ▶ allows for arbitrary correlations between subsystems
  - ▶ from the bayesian perspective, $w_t^h(\phi)$ is a prior for $\mathfrak{P}_{\mathcal{Q}}^{D_\Psi}(\eta)$-updating
  - ▶ every CPTP instrument [Davies–Lewis'70] can be decomposed into:
    (1) tensor product of initial state with uncorrelated environment,
    (2) unitary evolution,
    (3) projective measurement,
    (4) partial trace.

  [RPK+M.Munk-Nielsen'15]: (4) is entropic projection for strictly positive states.
  It remains to prove that a join action of (3+4) is an entropic projection.

2. Local theory

- Local geometric structure: riemannian-affine geometry from relative entropy
- Local equivalence of entropic projections and free falls (local dual flatness postulate)
- Tangent/cotangent spaces as configuration/effect spaces with a discrimination functional
- Localisation of the Lie–Poisson structure
- Local hamiltonian dynamics (generalised von Neumann + free falls)
- Local lagrangean dynamics (curvature effects)

Smooth quantum information geometries

Under some conditions, D induces a generalisation of smooth riemannian geometry on $\mathcal{M}(\mathcal{N})$.

- Jenčová'05: a general construction of smooth manifold structure on the space of all strictly positive states over arbitrary W^*-algebra.
- E.g. $\mathcal{M}(\mathcal{H}) := \{\rho(\theta) \in \mathcal{T}(\mathcal{H}) \mid \rho(\theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n \text{ open}, \theta \mapsto \rho(\theta) \text{ smooth}\}$
- Eguchi'83/Ingarden et al'82/Lesniewski–Ruskai'99/Jenčová'04:
  Every smooth distance D with positive definite hessian determines
  a riemannian metric $\mathbf{g}^D$ and a pair $(\nabla^D, \nabla^{D\dagger})$ of torsion-free affine connections:

$$\mathbf{g}_\phi(u,v) := -\partial_{u|\phi}\partial_{v|\omega} D(\phi,\omega)|_{\omega=\phi},$$
$$\mathbf{g}_\phi((\nabla_u)_\phi v, w) := -\partial_{u|\phi}\partial_{v|\phi}\partial_{w|\omega} D(\phi,\omega)|_{\omega=\phi},$$
$$\mathbf{g}_\phi(v, (\nabla^\dagger_u)_\phi w) := -\partial_{u|\omega}\partial_{w|\omega}\partial_{v|\phi} D(\phi,\omega)|_{\omega=\phi},$$

  which satisfy the characteristic equation of the Norden['37]–Sen['44] geometry,

$$\mathbf{g}^D(u,v) = \mathbf{g}^D(\mathbf{t}_c^{\nabla^D}(u), \mathbf{t}_c^{\nabla^{D\dagger}}(v)) \;\; \forall u,v \in \mathbf{T}\mathcal{M}(\mathcal{N}).$$

- A riemannian geometry $(\mathcal{M}(\mathcal{N}), \mathbf{g}^D)$ has Levi-Civita connection $\bar{\nabla} = (\nabla^D + \nabla^{D\dagger})/2$.
- Example 1: $\mathcal{M}(\mathcal{N}) = \mathcal{T}(\mathcal{H}) \cap \{\rho > 0, \mathrm{tr}_\mathcal{H}(\rho) = 1\}$ and $D_1(\rho,\sigma) = \mathrm{tr}_\mathcal{H}(\rho\log\rho - \rho\log\sigma)$ give Mori['55]–Kubo['56]–Bogolyubov['62] $\mathbf{g}^{D_1}$ and Nagaoka['94]–Hasegawa['95] $(\nabla^{D_1}, \nabla^{D_1\dagger})$:

$$\mathbf{g}^{D_1}_\rho(x,y) = \mathrm{tr}_\mathcal{H}\left(\int_0^\infty \mathrm{d}\lambda\, x \frac{1}{\lambda\mathbb{I}+\rho} y \frac{1}{\lambda\mathbb{I}+\rho}\right), \quad \mathbf{t}^{\nabla^{D_1}}_{\rho,\omega}(x) = x - \mathrm{tr}_\mathcal{H}(\omega x), \quad \mathbf{t}^{\nabla^{D_1\dagger}}_{\rho,\omega}(x) = x.$$

Smooth quantum information geometries

Taylor expansion of D induces a generalisation of a smooth riemannian geometry on $\mathcal{M}(\mathcal{N})$.

- $\mathcal{M}(\mathcal{H}) := \{\rho(\theta) \in \mathcal{T}(\mathcal{H}) \mid \rho(\theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n \text{ open}, \theta \mapsto \rho(\theta) \text{ smooth}\}$ is a C^∞-manifold
- Jenčová'05: a general construction of smooth manifold structure on the space of all strictly positive states over arbitrary W^*-algebra, with tangent spaces given by noncommutative Orlicz spaces.
- Eguchi'83/Ingarden et al'82/Lesniewski–Ruskai'99/Jenčová'04:
Every smooth distance D with positive definite hessian determines a riemannian metric $\mathbf{g}^D$ and a pair $(\nabla^D, \nabla^{D^\dagger})$ of torsion-free affine connections:
$$\mathbf{g}_\phi(u, v) := -\partial_{u|\phi}\partial_{v|\omega} D(\phi, \omega)|_{\omega=\phi},$$
$$\mathbf{g}_\phi((\nabla_u)_\phi v, w) := -\partial_{u|\phi}\partial_{v|\phi}\partial_{w|\omega} D(\phi, \omega)|_{\omega=\phi},$$
$$\mathbf{g}_\phi(v, (\nabla^\dagger_u)_\phi w) := -\partial_{u|\omega}\partial_{w|\omega}\partial_{v|\phi} D(\phi, \omega)|_{\omega=\phi},$$
which satisfy the characteristic equation of the Norden['37]–Sen['44] geometry,
$$\mathbf{g}^D(u, v) = \mathbf{g}^D(\mathbf{t}_c^{\nabla^D}(u), \mathbf{t}_c^{\nabla^{D^\dagger}}(v)) \;\; \forall u, v \in \mathbf{T}\mathcal{M}(\mathcal{N}).$$
- A riemannian geometry $(\mathcal{M}(\mathcal{N}), \mathbf{g}^D)$ has Levi-Civita connection $\bar{\nabla} = (\nabla^D + \nabla^{D^\dagger})/2$.

Example

- $\mathcal{M}(\mathcal{N}) = \mathcal{T}(\mathcal{H}) \cap \{\rho > 0, \mathrm{tr}_{\mathcal{H}}(\rho) = 1\}$
  $D_1(\rho, \sigma) = \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$

  give Mori['55]–Kubo['56]–Bogolyubov['62] riemannian metric:

$$\mathbf{g}^{D_1}_{\rho}(x, y) = \mathrm{tr}_{\mathcal{H}} \left(\int_0^{\infty} \mathrm{d}\lambda x \frac{1}{\lambda \mathbb{I} + \rho} y \frac{1}{\lambda \mathbb{I} + \rho} \right),$$

  and Nagaoka['94]–Hasegawa['95] affine connections:

$$\mathbf{t}^{\nabla^{D_1}}_{\rho,\omega}(x) = x - \mathrm{tr}_{\mathcal{H}}(\omega x), \quad \mathbf{t}^{\nabla^{D_1 \dagger}}_{\rho,\omega}(x) = x.$$

Hessian geometries = dually flat Norden–Sen geometries

If $(\mathcal{M}, \mathbf{g}, \nabla, \nabla^{\dagger})$ is a Norden–Sen geometry with flat ∇ and $\nabla^{\dagger}$, then:

1. there exists a unique pair of functions $\Phi : \mathcal{M} \to \mathbb{R}$, $\Phi^{\mathbf{L}} : \mathcal{M} \to \mathbb{R}$ such that $\mathbf{g}$ is their hessian metric,

$$\mathbf{g}(\rho) = \sum_{i,j} \frac{\partial^2 \Phi(\rho(\theta))}{\partial \theta^i \partial \theta^j} \mathrm{d}\theta^i \otimes \mathrm{d}\theta^j,$$

$$\mathbf{g}(\rho) = \sum_{i,j} \frac{\partial^2 \Phi^{\mathbf{L}}(\rho(\eta))}{\partial \eta^i \partial \eta^j} \mathrm{d}\eta^i \otimes \mathrm{d}\eta^j,$$

   where: $\{\theta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla}_{ijk}(\rho(\theta)) = 0 \ \forall \rho \in \mathcal{M}$,
   $\{\eta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla^{\dagger}}_{ijk}(\rho(\eta)) = 0 \ \forall \rho \in \mathcal{M}$,
   and $\Phi^{\mathbf{L}}$ is a Fenchel conjugate of Φ.

2. the Eguchi equations applied to the Brègman distance

$$D_{\Phi}(\rho, \sigma) := \Phi(\rho) + \Phi^{\mathbf{L}}(\sigma) - \sum_i \theta^i(\rho)\eta^i(\sigma)$$

   yield $(\mathbf{g}, \nabla, \nabla^{\dagger})$ above.

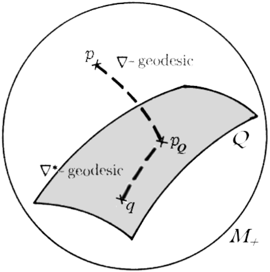
Smooth generalised pythagorean theorem

Let $(\mathcal{M}, \mathbf{g}, \nabla, \nabla^\dagger)$ be a hessian geometry. Then for any $\mathcal{Q} \subseteq \mathcal{M}$ which is:

- $\nabla^\dagger$-autoparallel $:= \nabla^\dagger_u v \in \mathbf{T}\mathcal{Q}\ \forall u, v \in \mathbf{T}\mathcal{Q}$;
- $\nabla^\dagger$-convex $:= \forall \rho_1, \rho_2 \in \mathcal{Q}\ \exists!\ \nabla^\dagger$-geodesics in $\mathcal{Q}$ connecting ρ_1 and ρ_2;

there exists a unique projection

$$\mathcal{M} \ni \rho \mapsto \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho) := \arg\inf_{\sigma \in \mathcal{Q}} \{D_\Phi(\sigma, \rho)\} \in \mathcal{Q}.$$



- it is equal to a unique projection of ρ onto $\mathcal{Q}$ along a ∇-geodesic that is $\mathbf{g}$-orthogonal at $\mathcal{Q}$.
- it satisfies a generalised pythagorean equation

$$D_\Phi(\omega, \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho)) + D_\Phi(\mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho), \rho) = D_\Phi(\omega, \rho) \quad \forall (\omega, \rho) \in \mathcal{Q} \times \mathcal{M}.$$

Hence, for Brègman distances D_Φ the local entropic projections are equivalent with geodesic projections.

New kinematics: local (operational) view

- Consider local embeddings $\ell : U \to X$, $U \subseteq \mathcal{M}$, as local coordinate systems, determining X as a local tangent space of $\mathcal{M}$
- Use the dual space $X^\star$ as a local cotangent space
- Use the Brègman entropic projections (their existence, uniqueness, and composability) as a method of crafting a smooth manifold structure on $\mathcal{M}$ from the local embeddings into X:
  - finite dimensional case: no problem
  - infinite dimensional setting: Jenčová'05 did it for D_1 over faithful (strictly positive) parts of predual of arbitrary W*-algebras, RPK+Jenčová'16-'17 (in progress): extension of this approach to a wide class of D_Ψ
- resulting structure: locally dually flat information manifold $\mathcal{M}$
- operational local kinematics:
  - local states/preparations: vectors of $T_\psi\mathcal{M} \cong X$ $(\phi(\theta) \to \theta \to \frac{\partial}{\partial\theta^i})$
  - local effects/observables: vectors of $T^*_\psi\mathcal{M} \cong X^\star$ $(f(\phi) \to \mathbf{d}f(\phi))$
- basic semantics:
  - the description of possible measurements provided in a local laboratory (as in convex/operational approach to quantum foundations) is expressed in terms of pair $(X, X^\star)$ of spaces
  - the elements of a model $\mathcal{M}$ are "global states of multi-user communication system"
  - the coordinate maps ℓ play a similar role to tetrad systems in GR: they translate between the "multi-user system" states of the manifold, and the local "individual user" operational description

Local inference, local causality: free falls and BLP representation

- Dually flat manifolds are nonlinear generalisation of the euclidean and Hilbert spaces, with the generalised pythagorean theorem playing a key role.
- Locally dually flat manifolds introduce the shift analogous to the shift from minkowskian to lorentzian manifolds: dual flatness holds only locally, for tangent (and cotangent) spaces.
- If entropic projections are regarded as a form of information dynamics, then introducing the structure of quantum information manifold based on a Brègman distance D_Ψ (equivalently, local dual flatness) amounts to postulating a "free fall principle": if there is no additional causal dynamics or inferential constraints, the local information flows along ∇^{D_Ψ}-geodesics.
- On the other hand, a localisation of a causal dynamics requires to replace a representation of a real Banach Lie algebra $\mathcal{A}$ in terms of a dual pair $(\mathcal{N}_\star^{\mathrm{sa}}, \mathcal{N}^{\mathrm{sa}})$ by a representation in terms of a dual pair of real Banach spaces $(X, X^\star)$.
- As a result, one has: $C^\infty(U) \ni h_\mathcal{A} \mapsto \mathbf{d}h_\mathcal{A} \in T^*_\psi U$.

Local effective dynamics: hamiltonian ansatz

- One can combine locally the entropic geodesic free falls with hamiltonian flows in a single formula, describing an effective information dynamics in which causal and inferential parts are parallely processed.
- Given a hamiltonian observable h and a Brègman relative entropy D_Ψ, the 1-form $\mathbf{d}h_{\mathcal{A}}(\phi) + \mathbf{d}_{\nabla^{D_\Psi}}(\phi)$ represents a local perturbation of causal dynamics by the information flow along entropic geodesics (and vice versa).
- In particular, $D_{1/2} = 2\|\sqrt{\rho} - \sqrt{\sigma}\|^2_{\mathcal{H}}$ gives Wigner–Yanase metric $\mathbf{g}^{1/2}$, with tangent spaces given by the GNS Hilbert space bundle $\mathcal{H}_{\rho(t)} \cong \mathfrak{G}_2(\mathcal{H})$, and with $d_{\mathbf{g}^{1/2}}(\rho,\sigma) = 2\arccos(\mathrm{tr}_{\mathcal{H}}(\sqrt{\rho}\sqrt{\sigma}))$. The free fall along the geodesics of Levi-Civita connection $\nabla^{1/2}$ encodes the continuous process of projective measurement.
- The $\nabla^{1/2}$-parallel transport equation of a vector v^a along the trajectory $\rho(t)$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}v^a(t) = -\sum_{b,c}(\Gamma^{\nabla^{1/2}})^a{}_{bc}(\rho(t))v^b(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)\right)^c. \quad (1)$$

Substituting $v = \dot{\rho}(t)$, and integrating out, we get

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\int_{-\infty}^{t}\mathrm{d}t\sum_{b,c}(\Gamma^{\nabla^{1/2}})^a{}_{bc}(\rho(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)\right)^b\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)\right)^c. \quad (2)$$

This equation describes the equation of motion on $\mathcal{M} \subseteq \mathfrak{G}_1(\mathcal{H})^+$ of the free fall along the $\nabla^{1/2}$-geodesic trajectory $\rho(t)$.

Local effective dynamics: hamiltonian ansatz

- An infinitesimal transformation $\rho \mapsto \rho + \eth\rho$ can be decomposed as [Hasegawa'93]:

$$\eth\rho := \widetilde{\eth}\rho + [\rho, W] = \sum_{i=1}^{n} \left(\frac{\partial\rho(\theta)}{\partial\theta^i} + [\rho, W_i] \right) \mathbf{d}\theta^i,$$

where $\widetilde{\eth}\rho = \sum_{i=1}^{n} \frac{\partial\rho(\theta)}{\partial\theta^i}\mathbf{d}\theta^i$ is defined by $[\widetilde{\eth}\rho, \rho] = 0$, and $W = \sum_{i=1}^{n} W_i \mathbf{d}\theta^i$ is an antiself-adjoint operator (hence, $k_i^* = k_i := \mathrm{i}W_i$). The mappings $\eth$, $\widetilde{\eth}$ and $[\,\cdot\,, W]$ are derivations on $\mathfrak{B}(\mathcal{H})$.

- An explicit representation of the tangent space in terms of $\mathfrak{G}_2(\mathcal{H})$ space by means of finite dimensional coordinate parametrisation $\mathbb{R}^n \supsetneq \Theta \ni \theta \mapsto \rho(\theta) \in \mathfrak{G}_1(\mathcal{H})^+$ reads

$$\mathbf{T}_\rho \ell_{1/2}(u) = \sum_{i=1}^{n} u^i \left(\sqrt{\rho}\frac{\partial\rho}{\partial\theta^i} + 2[\sqrt{\rho}, W_i] \right).$$

- Interpreting the ansatz of $\mathbf{d}h_{\mathcal{A}}(\phi) + \mathbf{d}_{\nabla^{D_\Psi}}(\phi)$ as a statement that the effective local dynamics is generated by the sum of vectors $\dot{\rho}(t)$ arising independently from the hamiltonian flow and the geodesic free fall, we receive the following nonlinear evolution of $\rho(t)$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [\mathbf{d}h(\rho(t)), \rho(t)] - \int_{-\infty}^{t} \mathrm{d}t \sum_{b,c} (\Gamma^{\nabla^{1/2}})^a{}_{bc}(\rho(t)) \cdot$$

$$\cdot \left(\sum_i u^i \left(\sqrt{\rho}\frac{\partial\rho}{\partial\theta^i} + 2\,[\sqrt{\rho}, k_i] \right) \right)^b \left(\sum_i u^i \left(\sqrt{\rho}\frac{\partial\rho}{\partial\theta^i} + 2\,[\sqrt{\rho}, k_i] \right) \right)^c.$$

Local effective dynamics: lagrangean ansatz

- Daubechies–Klauder'85 introduced exact continuous-time regularised coherent vectors propagator for the phase space path integral, and proved that under mild assumptions on hamiltonian (square and quadric integrability) one has: $\left\langle z(t=s), \mathrm{e}^{-\mathrm{i}Hs} z(t=0)\right\rangle_{\mathcal{H}} =$

$$= \lim_{\upsilon\to+\infty} \int \mathrm{D}z(\cdot)\mathrm{e}^{\left(\mathrm{i}\int_{\mathrm{i}}\langle z(t),\mathbf{d}z(t)\rangle_{\mathcal{H}}\right)}\mathrm{e}^{\left(-\mathrm{i}\int_0^s \mathrm{d}t\; h(z(t))\right)}\mathrm{e}^{-\frac{1}{2\upsilon}\left(\int_0^s \mathrm{d}t\; \mathbf{g}^{\mathrm{FS}}_{ab}(z(t))\dot{z}^a\dot{z}^b\right)}$$

$$= 2\pi \lim_{\upsilon\to+\infty} \mathrm{e}^{\upsilon s/2} \int \tilde{\mu}^{\upsilon}_{\mathrm{W}}(\mathrm{p}_\Gamma, \mathrm{q}_\Gamma)\mathrm{e}^{\mathrm{i}\int(\mathrm{p}_\Gamma \mathrm{d}\mathrm{q}_\Gamma - H(\mathrm{p}_\Gamma,\mathrm{q}_\Gamma)\mathrm{d}t)},$$

  where $h(z(t))$ is a hamiltonian function with respect to the symplectic form on the pure states, $h(z(t)) := \langle z(t), Hz(t)\rangle_{\mathcal{H}}$ for a given $H \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$, $\mathbf{g}^{\mathrm{FS}}$ is a Fubini–Study riemannian metric on the pure states, while $\tilde{\mu}^{\upsilon}_{\mathrm{W}}(\mathrm{p}_\Gamma, \mathrm{q}_\Gamma)$ is a pinned Wiener measure on a phase space Γ.
- This formulation is mathematically exact, and covariant under canonical transformations of phase space coordinates, what is not the case for most of other approaches to quantisation.
- For finite value of υ the above propagator is not longer unitary [Klauder'95]. This means that the metric structure on the Hilbert space allows (some sort of) quantification of the nonunitary temporal behaviour.

Local effective dynamics: lagrangean ansatz

- This leads us to propose that the langrangean implementation of an ansatz $\mathbf{d}h_{\mathcal{A}}(\phi) + \mathbf{d}_{\nabla^{D_\Psi}}(\phi)$ should be provided in terms of a continuous-time regularised path-integral

$$\lim_{\varepsilon\to+0}\int\mathcal{D}\phi(\cdot)\mathrm{e}^{\mathrm{i}\int_\gamma \mathrm{d}t\left\langle\Omega_{\phi(t)},\mathbf{d}_{\nabla^{D_\Psi}}(\phi(t))\Omega_{\phi(t)}\right\rangle_{\mathcal{H}_{\phi(t)}}}\cdot$$

$$\cdot\mathrm{e}^{-\mathrm{i}\int_\gamma\mathrm{d}t\left\langle\Omega_{\phi(t)},\pi_{\phi(t)}(\mathbf{d}h_{\mathcal{A}}(\phi(t)))\Omega_{\phi(t)}\right\rangle}\mathrm{e}^{-\frac{\varepsilon}{2}\int_\gamma\mathrm{d}t\mathbf{g}^{D_\Psi}_{ab}(\phi(t))\dot{\phi}^a\dot{\phi}^b},$$

- If evaluated for $D_{1/2}$ and $\mathcal{A}=\mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$, only on boundary pure states, and for $h(\phi)=\phi(\mathcal{H})$, it reduces to the Daubechies–Klauder integral.
- For non-Levi-Civita ∇^{D_Ψ} (hence, in any other case than $D_{1/2}$), the corresponding random walk process will not be markovian, but can be well defined.
- The expansion $D_\Psi(\phi+\varepsilon v,\phi)=\frac{\varepsilon^2}{2}\mathbf{g}^{D_\Psi}_{ab}(\phi)v^av^b+\mathcal{O}(\varepsilon^3)$ suggests furthermore to replace the regularising term above by $\exp\left(-\frac{1}{2\varepsilon}\int_\gamma\mathrm{d}t\,D_\Psi(\phi(t)+\varepsilon\dot{\phi}(t),\phi(t))\right)$.

Curvature

- From our perspective, the above propagator formula makes sense only locally, because only locally the manifold structure is assumed to be dually flat, and the ansatz $\mathbf{d}h_{\mathcal{A}}(\phi) + \mathbf{d}_{\nabla^{D_\Psi}}(\phi)$ is valid.
- However, one may ask, to what extent the above formulation can be extended, or is there any specific geometric feature that measures this inextendability?
- In principle, the main object that measures the departure of $\mathcal{M}$ from the dual flatness beyond the local level is the curvature associated with the Levi-Civita connection associated with the riemannian metric $\mathbf{g}$ on $\mathcal{M}$.
- Watson–Klauder'02 observe that if the riemannian geometries of the phase space used for the Wiener measure regularisation have nonconstant scalar curvature, then the weighting of the phase space paths is nonuniform, corresponding to the phase space point dependency of the zero-point energy.
- From the information-theoretic point of view, one can say that the curvature of a connection on $\mathcal{M}$ measures the desynchronisation of the local systems of inference provided by the local dual flat geometries. In other words, it measures how the local tangent/cotangent state/effect pairs are nontrivially changing over the manifold of "global states". If the manifold is globally dually flat (hence the Levi-Civita connection is constant), then all local systems of inference (local notions of free falls) are globally synchronised. Otherwise, their synchronisation is path dependent: the transport of a vector $\dot{\rho}(\theta)$ along two different paths between a given pair of points will give different results (different evolutions), and different corresponding expectation values of "the same" local observables.
- In RPK'16 we show that within the framework of the Jaynes–Mitchell–Favretti source theory the nonzero curvature of a model arises from the renormalisation of a higher-dimensional dually flat geometry which reduces the dimensions corresponding to additional source (control) terms.

Open problems

- Develop a general theory of quantum locally dually flat manifolds associated to Brègman distances (an ongoing work with A.Jenčová).
- Work out the details of local implementation of Banach Lie–Poisson manifold structure as represented in the pairs $(X, X^\star)$, especially in the case when X is a noncommutative Orlicz space. (In the global case, the manifold structure used for the BLP space construction is different from the manifold structure used for the smooth geometries derived from the relative entropies, and it is not clear at all how to relate them.)
- Find the exact range of (linear and nonlinear) CPTP instruments that can be modelled with $\mathfrak{P}^{D_\Psi}_{\mathcal{Q}} \circ w^h_t$ maps.
- Show that one can reconstruct a theory that is locally QM using purely geometric data (hence, characterise local tangent/cotangent spaces as QM state/effect dual pairs without assuming that $\mathcal{M}$ consists of states over a W*-algebra).
- Provide a rigorous brownian motion based mathematical foundation for the generalisation of the Klauder–Daubechies integral introduced here.
- ...

References

Work in progress:

- RPK+A.Jenčová, Dual Banach manifold structure related to a Brègman divergence
- RPK, Quantum information geometric foundations
- RPK, Category-theoretic setting for quantum inference theories
- RPK, 2017, Quantum Brègman distances (IGAIA4 proceedings)

Earlier results and insights:

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