Geometrisation of quantum theory beyond pure states and Hilbert spaces

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Motivation

«Unlike the Riemannian manifolds the quantum mechanical unit spheres do not differ one from another: they are all isomorphic. The worlds of the present-day quantum mechanics thus present a picture of structural monotony: they are all 'painted' on the same standard ideally symmetric surface. The formalism of the quantum theory of infinite systems and quantum field theory is not very different from that. (...) the basic structural framework of the theory is conserved at the cost of quantitative multiplication: when meeting a new level of physical reality the quantum theory responds by simply producing infinite tensor products of its basic structure. (...) It may be that present day quantum theory still represents a relatively primitive stage of development and lacks some essential evolutionary steps leading towards structural flexibility. If this were so, further development would involve a programme opposite to the 'quantization of gravity': instead of modifying general relativity to fit quantum mechanics one should rather modify quantum mechanics to fit general relativity.»

Bogdan Mielnik, 1976, *Quantum logic: is it necessarily orthocomplemented?*

«Perhaps the habitual linear structures of quantum mechanics are analogous to the inertial rest frames in special relativity and the geometric description summarized here, analogous to Minkowski's reformulation of special relativity.»

Abhay Ashtekar & Troy Schilling, 1997, *Geometrical formulation of quantum mechanics*

Plan

0. Brief review of Kähler geometrisation of quantum mechanics of pure states

1. Global geometric theory for spaces of general quantum states:
   - **state spaces**: sets of normal states on W^*-algebras
   - **geometry**: Brègman relative entropies & Lie–Poisson structures
   - **causal dynamics**: nonlinear hamiltonian flows
   - **statistical dynamics**: constrained relative entropy maximisation

2. Local theory:
   - spaces of local configurations/effects as tangent/cotangent spaces
   - relative entropies inducing local Codazzi and Weyl structures
   - Brègman relative entropies inducing local hessian structure
   - effective dynamics: geometric path integral with generalised hamiltonian and entropic connection terms, weighted by the metric-dependent measure

Kählerian geometrisation of von Neumann's QM (I)

[Strocchi'66, Kibble'79, Heslot'85, Anandan–Aharonov'90, Cirelli–Manià–Pizzocchero'90, Hughston'95, Ashtekar–Schilling'97, and others...]:

Given a complex Hilbert space $\mathcal{H}$, the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ determines a symplectic form Ω, a riemannian metric G, and a complex structure J:

$$\langle\xi,\zeta\rangle_{\mathcal{H}} =: \frac{1}{2}G(\xi,\zeta) + \frac{i}{2}\Omega(\xi,\zeta), \quad i\langle\xi,\zeta\rangle_{\mathcal{H}} =: \langle\xi,J\zeta\rangle_{\mathcal{H}}$$

satisfying the relationship $G(\xi,\zeta) = \Omega(\xi,J\zeta)$, turning $\mathcal{H}$ into a Kähler manifold.

- observables determine vector fields: $\mathcal{X}_A(\xi) := -JA\xi$ which are Killing w.r.t. G
- Schrödinger equation $\dot{\xi} = -JH\xi$ is Hamilton's equation: $\dot{\xi} = \mathcal{X}_H(\xi)$
- expectation values are real valued functions $f(\xi) := \langle\xi, F\xi\rangle_{\mathcal{H}}$, $f:\mathcal{H}\to\mathbb{R}$
- Poisson brackets are induced from commutator: $\{f,k\}_\Omega(\xi) = \Omega(\mathcal{X}_F,\mathcal{X}_K)(\xi) = \left\langle\xi,(\frac{1}{i}[F,K])\xi\right\rangle_{\mathcal{H}}$

Kählerian geometrisation of von Neumann's QM (II)

$\mathbb{PH}$ is also a Kähler manifold:

- $\mathcal{H}$ is a tangent space at each point
- all above properties of (G, Ω, J) on $\mathcal{H}$ hold for induced (g, ω, j) on $\mathbb{PH}$
- g is a Fubini–Study metric
- $\omega = d\theta$, where θ is a $U(1)$-connection 1-form, with holonomy equal to the Aharonov–Anandan/geometric phase
- observables are characterised as Killing hamiltonian vector fields
- for any $p, p_0 \in \mathbb{PH}$: the 'state vector reduction' due to measurement corresponds to projection $p_0 \mapsto p$ along a geodesic of g
- the transition probability of $p_0 \mapsto p$ reads $\cos^2(d_g(p_0, p)/\sqrt{2})$, where d_g is a geodesic distance of g

1. Global theory

Beyond pure states

- density matrices are trace class operators: $\mathcal{T}(\mathcal{H}) := \{\rho \in \mathfrak{B}(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}_{\mathcal{H}}|\rho| < \infty\}$
- we will consider arbitrary sets of denormalised quantum states: $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$
- more generally, given any W*-algebra [i.e., a C*-algebra $\mathcal{N}$ that is a Banach dual to some Banach space $\mathcal{N}_\star$; $(\mathcal{N}_\star)^\star = \mathcal{N}$]:
  - quantum states are given by the elements of the positive part of $\mathcal{N}_\star$, $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$ (i.e., normal positive linear functionals on $\mathcal{N}$)
  - the pair $(\mathcal{N}_\star^+, \mathcal{N})$ is a generic setting of a noncommutative integration theory, with $\mathcal{N}_\star = L_1(\mathcal{N})$, $\mathcal{N} = L_\infty(\mathcal{N})$, and other noncommutative $L_p(\mathcal{N})$ spaces available
  - $\mathcal{T}(\mathcal{H}) = \mathfrak{B}(\mathcal{H})_\star$ is just a special case of this setting, providing a noncommutative generalisation of $\ell_1(\mathbb{N})$ space.

- Geometric structures on spaces $\mathcal{M}$ of quantum states: relative entropies $D(\cdot,\cdot)$ & Poisson brackets $\{\cdot,\cdot\}$
- Linear operators on Hilbert spaces $\to$ real-valued functions on $\mathcal{M}$
- Unitary evolution $\to$ nonlinear hamiltonian flows on $\mathcal{M}$
- Evolution due to measurement $\to$ constrained relative entropy maximisations on $\mathcal{M}$

Quantum Poisson structure

- Consider the space of self-adjoint trace-class operators: $\mathcal{T}(\mathcal{H})^{\mathrm{sa}} := \mathcal{T}(\mathcal{H}) \cap \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$.
- It can be equipped with a following real Banach smooth manifold structure:
  - tangent spaces: $\mathbf{T}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong \mathcal{T}(\mathcal{H})^{\mathrm{sa}}$
  - cotangent spaces: $\mathbf{T}^{\circledast}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong (\mathcal{T}(\mathcal{H})^{\mathrm{sa}})^\star \cong \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$
- Bóna'91,'00: a Poisson manifold structure on $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$ is defined by a commutator of an algebra:

$$\{h, f\}(\rho) := \mathrm{tr}_{\mathcal{H}}\left(\rho\, \mathrm{i}[\mathbf{d}h(\rho), \mathbf{d}f(\rho)]\right) \,\forall f, h \in \mathrm{C}^\infty(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}; \mathbb{R}) \;\forall \rho \in \mathcal{T}(\mathcal{H})^{\mathrm{sa}}.$$

- So, if $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ is a smooth submanifold of $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$, then every $f \in \mathrm{C}^\infty(\mathcal{M}(\mathcal{H}); \mathbb{R})$ determines a hamiltonian vector field:

$$\mathfrak{X}_f(\rho) = -\{\cdot, f\}(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho\, \mathrm{i}[\mathbf{d}(\cdot), \mathbf{d}f(\rho)]).$$

- More generally, we can choose arbitrary real Banach Lie subalgebra $\mathcal{A}$ of $\mathfrak{B}(\mathcal{H})$ such that: (i) it has a unique Banach predual $\mathcal{A}_\star$ in $\mathcal{T}(\mathcal{H})$; (ii) there exists at least one $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ which is a smooth submanifold of $\mathcal{A}_\star$.

Nonlinear quantum hamiltonian dynamics

For each hamiltonian vector field, the corresponding Hamilton equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\rho(t)) = \{h, f\}(\rho(t)) = \mathrm{i}\,\mathrm{tr}_{\mathcal{H}}\left([\rho(t), \mathbf{d}h(\rho(t))]\mathbf{d}f(\rho(t))\right).$$

The above equation is equivalent to the Bóna equation ['91'00]

$$\boxed{\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [\mathbf{d}h(\rho(t)), \rho(t)].}$$

Hence,

The Poisson structure $\{\cdot,\cdot\}$ induced by a commutator of $\mathfrak{B}(\mathcal{H})$ allows to introduce various nonlinear hamiltonian evolutions on spaces $\mathcal{M}(\mathcal{H})$ of quantum states, generated by arbitrary real-valued smooth functions on $\mathcal{M}(\mathcal{H})$.

The solutions of Bóna equation are state-dependent unitary operators $U(\rho, t)$. They do not form a group, but satisfy a cocycle relationship:

$$U(\rho, t+s) = U((\mathrm{Ad}(U(\rho, t)))(\rho), s)U(\rho, t) \;\; \forall t, s \in \mathbb{R}.$$

In a special case, when $h(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho H)$ for $H \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$,
the Bóna equation turns to the von Neumann equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H, \rho(t)].$$

Quantum relative entropies

$D : \mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}) \to [0, \infty]$ s.t. $D(\rho, \sigma) = 0 \iff \rho = \sigma$.

E.g.

- $D_1(\rho, \sigma) := \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$ [Umegaki'62]
- $D_{1/2}(\rho, \sigma) := 2\|\sqrt{\rho} - \sqrt{\sigma}\|^2_{\mathfrak{G}_2(\mathcal{H})} = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$ (Hilbert–Schmidt norm^2)
- $D_{L_1(\mathcal{N})}(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_{\mathcal{T}(\mathcal{H})} = \frac{1}{2}\mathrm{tr}_{\mathcal{H}}|\rho - \sigma|$ (L_1/predual norm)
- $D_\gamma(\rho, \sigma) := \frac{1}{\gamma(1-\gamma)}\mathrm{tr}_{\mathcal{H}}(\gamma\rho + (1-\gamma)\sigma - \rho^\gamma\sigma^{1-\gamma})$; $\gamma \in \mathbb{R} \setminus \{0, 1\}$ [Hasegawa'93]
- $D_{\alpha,z}(\rho, \sigma) := \frac{1}{1-\alpha} \log \mathrm{tr}_{\mathcal{H}}(\rho^{\alpha/z}\sigma^{(1-\alpha)/z})^z$; $\alpha, z \in \mathbb{R}$ [Audenauert–Datta'14]
- $D_{\mathfrak{f}}(\rho, \sigma) := \mathrm{tr}_{\mathcal{H}}(\sqrt{\rho}\,\mathfrak{f}(\mathfrak{L}_\rho \mathfrak{R}_\sigma^{-1})\sqrt{\rho})$; $\mathfrak{f}$ operator convex, $\mathfrak{f}(1) = 0$ [Kosaki'82,Petz'85]

for $\mathrm{ran}(\rho) \subseteq \mathrm{ran}(\sigma)$, and with all $D(\rho, \sigma) := +\infty$ otherwise.

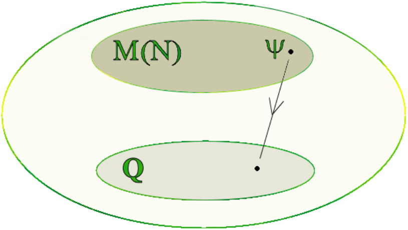
Various "quantum geometries" will arise from different additional conditions imposed on pairs $(\mathcal{M}(\mathcal{H}), D)$

Quantum entropic projections

Let $\mathcal{Q} \subseteq \mathcal{T}(\mathcal{H})^+$ be such that for each $\psi \in \mathcal{M}(\mathcal{H})$ there exists a unique solution

$$\mathfrak{P}^D_{\mathcal{Q}}(\psi) := \arg\inf_{\rho \in \mathcal{Q}} \{D(\rho, \psi)\}\,.$$

It will be called an entropic projection.



E.g.

- for $D_{1/2}(\rho, \sigma) = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$, and $\mathcal{Q}$ given by any closed convex subspace of the positive cone of the (GNS/Hilbert–Schmidt) Hilbert space $\mathcal{K} \cong \mathfrak{G}_2(\mathcal{H})$ via the embedding $\mathcal{T}(\mathcal{H}) \ni \rho \mapsto \sqrt{\rho} \in \mathfrak{G}_2(\mathcal{H})^+$.

  Then the entropic projections $\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}$ coincide with the ordinary projection operators $P_{\sqrt{\mathcal{Q}}}$ in $\mathfrak{B}(\mathcal{K}) \cong \mathfrak{B}(\mathcal{H} \otimes \mathcal{H}^\star)$.
- for $D_1(\rho, \sigma) = \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$ and $\mathcal{M}(\mathcal{H}) = \mathcal{T}(\mathcal{H})^+_1$, $\psi \in \mathcal{T}(\mathcal{H})^+_1$, $h \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$, then [Araki'77, Donald'90]

$$\exists!\ \psi^h := \arg\inf_{\rho \in \mathcal{T}(\mathcal{H})^+_1} \{D_1(\rho, \psi) + \mathrm{tr}_{\mathcal{H}}(\rho h)\}\,.$$

Quantum measurement, bayesianity, and maximum relative entropy

- Lüders' rules:

$$\rho \mapsto \rho_{\mathrm{new}} := \sum_i P_i \rho P_i \quad \text{('weak')}$$

$$\rho \mapsto \rho_{\mathrm{new}} := \frac{P\rho P}{\mathrm{tr}_{\mathcal{H}}(P\rho)} \quad \text{('strong')}$$

- bayesian view on quantum measurement: [Bub'77'79, Caves–Fuchs–Schack'01, Fuchs'02, Jacobs'02]: Lüders' rules should be considered as rules of inference (conditioning) that are quantum analogues of

the Bayes–Laplace rule: $$p(x) \mapsto p_{\mathrm{new}}(x) := \frac{p(x)p(b|x)}{p(b)}.$$

- Williams'80, Warmuth'05, Caticha&Giffin'06: the Bayes–Laplace rule is a special case of entropic projection:

$$p(\chi) \mapsto p_{\mathrm{new}}(\chi) := \arg\inf_{q \in \mathcal{Q}} \{D_1(q,p)\}\,; \quad D_1(q,p) := \int_{\mathcal{X}} \mu(\chi) q(\chi) \log\left(\frac{q(\chi)}{p(\chi)}\right).$$

- Douven&Romeijn'12: the Bayes–Laplace rule is also a special case of another entropic projection:

$$p \mapsto \arg\inf_{q \in \mathcal{Q}} \{D_1(p,q)\} = \mathfrak{P}^{D_0}_{\mathcal{Q}}(p),$$

where $D_0(p,q) = D_0(q,p)$.

Quantum bayesian inference from quantum entropic projections

- Herbut'69: weak Lüders' rule is a special case of $\mathfrak{P}^{d_{L_2(\mathfrak{B}(\mathcal{H}))}}_{\mathcal{Q}}(\rho)$ with $\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i, \sigma] = 0 \; \forall i\}$.
- F.Hellmann–W.Kamiński–RPK'14:
  1. weak Lüders' rule is a special case of $\rho \mapsto \arg\inf_{\sigma \in \mathcal{Q}} \{D_1(\rho, \sigma)\}$ with $\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i, \sigma] = 0 \; \forall i\}$
  2. strong Lüders' rule derived from $\rho \mapsto \arg\inf_{\sigma \in \mathcal{Q}} \{D_1(\rho, \sigma)\}$ with $\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i, \sigma] = 0, \; \mathrm{tr}_{\mathcal{H}}(\sigma P_i) = p_i \; \forall i\}$ under the limit $p_2, \ldots, p_n \to 0$.
  3. hence, weak and strong Lüders' rules are special cases of quantum entropic projection $\mathfrak{P}^{D_0}_{\mathcal{Q}}$ based on relative entropy $D_0(\sigma, \rho) = D_1(\rho, \sigma)$.
  4. more general ("quantum Jeffreys") rule has been also derived, providing a quantum analogue of a standard bayesian generalisation of the Bayes–Laplace rule.

> Bayes–Laplace and Lüders' conditionings are special cases of entropic projections $\Rightarrow$ "quantum bayesianism $\subseteq$ quantum relative entropism".

Meaning: the rule of maximisation of relative entropy (entropic projection on the subspace of constraints) can be considered as a nonlinear generalisation of the dynamics describing "quantum measurement". [RPK'10'11]

- M.Munk-Nielsen'15: partial trace is also a special case of $\mathfrak{P}^{D_0}_{\mathcal{Q}}$, at least for strictly positive states.

D_Ψ

- The choice of the set $\mathcal{Q}$ for which the entropic projection $\mathfrak{P}^D_{\mathcal{Q}}$ exists and is unique depends very strongly on the structure of D: the choice of principle of inference (D) determines the accepted data types ($\mathcal{Q}$).
- What are the general conditions on D that would be sufficient to guarantee existence and uniqueness of $\mathfrak{P}^D_{\mathcal{Q}}$, as well as good composition properties of subsequent projections (to have a category of entropic instruments)?
- RPK'17: It turns out that this can be provided by a class of Brègman relative entropies:

$$D_\Psi(\phi,\omega) := \widetilde{D}_\Psi(\ell(\phi),\ell(\omega))$$
$$\widetilde{D}_\Psi(x,y) := \Psi(x) - \Psi(y) - \mathfrak{D}^{\mathrm{G}}_{+}\Psi(y;x-y)$$
$$\ell : \mathcal{M} \to X$$

where X is a reflexive Banach space, $\Psi : X \to]-\infty,+\infty]$ is convex and lower semi-continuous (with some additional technical conditions), and $\mathfrak{D}^{\mathrm{G}}_{+}$ is a right Gâteaux derivative.
- The existence and uniqueness of $\mathfrak{P}^{D_\Psi}_{\mathcal{Q}}$ is guaranteed for $\mathcal{Q}$ such that $\ell(\mathcal{Q})$ are convex closed subsets of the spaces X.
- One can think of ℓ as a coordinate system on $\mathcal{M}$, and X as the linear parameter space used for specification of the data for entropic projection.

$D_\Psi \iff$ generalised pythagorean equation

A property that is of high importance from information geometric point of view, and is also crucial geometrically, is a generalised (nonsymmetric, nonlinear) pythagorean equation.

- D satisfies a generalised pythagorean theorem at $\mathcal{Q}$ iff [Chencov'68]

$$D(\phi,\psi) = D(\phi,\mathfrak{P}^D_{\mathcal{Q}}(\psi)) + D(\mathfrak{P}^D_{\mathcal{Q}}(\psi),\psi)\;\;\forall(\phi,\psi)\in\mathcal{Q}\times\mathcal{M}.$$

  $\iff$ information distance decomposes additively under a projection onto a suitable subspace $\iff$ a nonlinear **data** $=$ **signal** $+$ **noise** decomposition (!)
- It turns out that all D_Ψ satisfy generalised pythagorean theorem for sets that are affine under ℓ-embeddings.
- Jones–Byrne'90: in commutative case (and under some technical conditions) D_Ψ are characterised by g.p.t.
- Example 1: If $\mathcal{Q}$ forms an affine subset of $\mathfrak{G}_2(\mathcal{H})^+$ under $\rho\mapsto\sqrt{\rho}$, then:

$$\left\|x-\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}(z)\right\|^2_{\mathfrak{G}_2(\mathcal{H})} + \left\|\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}(z)-z\right\|^2_{\mathfrak{G}_2(\mathcal{H})} = \|x-z\|^2_{\mathfrak{G}_2(\mathcal{H})}.$$

- Example 2: If $\mathcal{Q} := \{\phi\in\mathfrak{G}_1(\mathcal{H})^+_1 \mid \phi(h) = \mathrm{const}\}$, then [Donald'90]

$$D_1(\phi,\psi^h) + D_1(\psi^h,\psi) = D_1(\phi,\psi)\;\;\forall(\phi,\psi)\in\mathcal{Q}\times\mathfrak{G}_1(\mathcal{H})^+_1.$$

New kinematics: quantum information geometry

- Main change: Consider expectation values as more fundamental than eigenvalues $\Rightarrow$ foundational role of spectral theory replaced by quantum information geometry
  (1) **spaces**:
  replace: linear Hilbert spaces $\mathcal{H}$ of eigenvectors
  by: sets $\mathcal{M}(\mathcal{N})$ of denormalised expectation functionals on W^*-algebras $\mathcal{N}$.
  (2) **observables**:
  replace: linear functions $\mathcal{H} \to \mathcal{H}$ with real eigenvalues
  by: nonlinear real valued functions $\mathcal{M}(\mathcal{N}) \to \mathbb{R}$.
  (3) **geometry**:
  replace: geometry of Hilbert spaces $\mathcal{H}$ defined by scalar product $\langle\cdot,\cdot\rangle$ *by:*
- **Two fundamental geometric structures on $\mathcal{M}(\mathcal{N})$:**
  a) Quantum Brègman relative entropies $D_\Psi(\cdot,\cdot)$
    - ⋆ represents the convention of a "global" estimation/loss function
    - ⋆ satisfies generalised pythagorean theorem
    - ⋆ allows to derive as special cases: hessian geometry (via $\partial_i\partial_j D_\Psi$, see later slides) and Hilbert space projective geometry (via $\mathfrak{P}_{\mathcal{Q}}^{D_\Psi}$ for $D_\Psi = D_{1/2}$)

  b) Quantum Poisson structures $\{\cdot,\cdot\}$
    - ⋆ represents the choice of a specific algebra of locally conserved quantities
    - ⋆ depends on the choice of a real Banach Lie subalgebra of $\mathcal{N}$
    - ⋆ generalises symplectic geometry
    - ⋆ $\{h,\cdot\}$ represents the choice of a convention of a "global" causality
- No Hilbert spaces, no probability theory in foundations (derived as special cases)

New dynamics: information geometric causal inference

- **Two fundamental dynamic structures on $\mathcal{M}(\mathcal{N})$:**
  a) **Inference:** Entropic projections $\phi \mapsto \arg\inf_{\omega\in\mathcal{Q}} \{D_\Psi(\omega,\phi)\}$ [RPK'10]
     - ⋆ nonlinear and nonlocal
     - ⋆ requires convexity
     - ⋆ represents ("active/external") information dynamics due to learning/measuring
     - ⋆ allows to encode experimental constraints (e.g., $\mathcal{Q}$ can be implemented as $\mathcal{Q}(\eta)$, with η representing additional external control variables)
     - ⋆ reduces in special cases to Lüders', Jeffrey's, Bayes' rules, partial trace, etc.

  b) **Causality:** Hamiltonian flows $\phi \mapsto w^h_t(\phi)$, $\frac{\mathrm{d}}{\mathrm{d}t} f(w^h_t(\phi)) = \{h, f(w^h_t)\}(\phi)$ [Bóna'00]
     - ⋆ nonlinear and local
     - ⋆ requires smoothness
     - ⋆ represents ("passive/internal") information dynamics with no inference
     - ⋆ allows to encode theoretical symmetries
     - ⋆ reduces in a special case to the von Neumann equation
- **Sequential processing ansatz:** consider $\phi \mapsto \mathfrak{P}^{D_\Psi}_{\mathcal{Q}} \circ w^h_t(\phi)$ as an alternative to the paradigm of semigroups of CPTP maps
  - generalises unitary evolution followed by a "projective measurement"
  - allows for arbitrary correlations between subsystems
  - is nonlinear and nonmarkovian
  - from the bayesian perspective, $w^h_t(\phi)$ is a prior for $\mathfrak{P}^{D_\Psi}_{\mathcal{Q}}$-updating

2. Local theory

Some smooth geometries (I)

- The Norden['37]–Sen['44] geometry is a quadruple $(\mathcal{M}, g, \nabla, \nabla^\dagger)$, where $\mathcal{M}$ is a C^∞-manifold, g is a riemannian metric, while ∇ and $\nabla^\dagger$ are torsion-free affine connections s.t.

$$g(u,v) = g(\mathbf{t}_c^{\nabla}(u), \mathbf{t}_c^{\nabla^\dagger}(v)) \;\; \forall u, v \in \mathbf{T}\mathcal{M},$$

where $\mathbf{t}_c^{\cdot}$ is a parallel transport along a smooth curve c.
- $(\nabla + \nabla^\dagger)/2$ is a Levi-Civita connection ∇^g of g.
- A hessian geometry is a Norden–Sen geometry with flat ∇ and $\nabla^\dagger$. In such case there exists a unique pair of functions $\Phi : \mathcal{M} \to \mathbb{R}$, $\Phi^{\mathbf{L}} : \mathcal{M} \to \mathbb{R}$ such that $\mathbf{g}$ is their hessian metric,

$$\mathbf{g}(\rho) = \sum_{i,j} \frac{\partial^2 \Phi(\rho(\theta))}{\partial\theta^i \partial\theta^j} \mathrm{d}\theta^i \otimes \mathrm{d}\theta^j,$$

$$\mathbf{g}(\rho) = \sum_{i,j} \frac{\partial^2 \Phi^{\mathbf{L}}(\rho(\eta))}{\partial\eta^i \partial\eta^j} \mathrm{d}\eta^i \otimes \mathrm{d}\eta^j,$$

where: $\{\theta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla}_{ijk}(\rho(\theta)) = 0 \; \forall \rho \in \mathcal{M}$, $\{\eta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla^\dagger}_{ijk}(\rho(\eta)) = 0 \; \forall \rho \in \mathcal{M}$, and $\Phi^{\mathbf{L}}$ is a Fenchel conjugate of Φ.

Some smooth geometries (II)

- The Codazzi geometry is a triple $(\mathcal{M}, g, \nabla)$, where $\mathcal{M}$ is a C^∞-manifold, g is a riemannian metric, while ∇ is a torsion-free flat affine connection s.t.

$$(\nabla_u g)(v, w) = (\nabla_v g)(u, w) \;\; \forall u, v, w \in \mathbf{T}\mathcal{M}.$$

- $(\mathcal{M}, g, \nabla)$ is Codazzi $\iff$ it is hessian (with $\nabla^\dagger := 2\nabla^g - \nabla$).
- A Weyl geometry is a quadruple $(\mathcal{M}, \tilde{\nabla}, \mathcal{C}, \Theta)$, where $\tilde{\nabla}$ is a torsion-free affine connection (called Weyl connection), $\mathcal{C}$ is a conformal class of (semi-)riemannian metrics, and Θ is a class of one-forms s.t.

$$\tilde{\nabla} g = -2\theta \otimes g,$$

where $\theta \in C^\infty(T^*\mathcal{M})$, g is a riemannian metric, $\mathcal{C} = \{e^{2f} g \mid f \in C^\infty(\mathcal{M})\}$, $\Theta = \{\theta - df \mid f \in C^\infty(\mathcal{M})\}$.

- Bokan–Galley–Simon'96: Every Codazzi geometry $(\mathcal{M}, g, \nabla)$ with $\dim(\mathcal{M}) = n$ determines a Weyl geometry:

$$\begin{aligned} C &:= \nabla^g - \nabla, \\ T(v) &:= \frac{1}{n} \mathrm{trace}\{u \mapsto C(u, v)\}, \\ g(u, T) &:= T(u), \\ \tilde{\nabla}_u v &:= \nabla^g_u v + \frac{n}{n+2}\{T(u) + T(v)u - g(u, v)T\}. \end{aligned}$$

Smooth quantum information geometries

Taylor expansion of D induces a Norden–Sen geometry on $\mathcal{M}(\mathcal{N})$.

- $\mathcal{M}(\mathcal{H}) := \{\rho(\theta) \in \mathcal{T}(\mathcal{H}) \mid \rho(\theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n \ \text{open}, \theta \mapsto \rho(\theta) \ \text{smooth}\}$ is a C^∞-manifold
- Jenčová'05: a general construction of smooth manifold structure on the space of all strictly positive states over arbitrary W^*-algebra, with tangent spaces given by noncommutative Orlicz spaces.
- Eguchi'83/Ingarden et al'82/Lesniewski–Ruskai'99/Jenčová'04: Every smooth relative entropy D with positive definite hessian determines a riemannian metric $\mathbf{g}^D$ and a pair $(\nabla^D, \nabla^{D^\dagger})$ of torsion-free affine connections:

$$
\begin{aligned}
\mathbf{g}_\phi(u, v) &:= -\partial_{u|\phi}\partial_{v|\omega} D(\phi, \omega)|_{\omega=\phi}, \\
\mathbf{g}_\phi((\nabla_u)_\phi v, w) &:= -\partial_{u|\phi}\partial_{v|\phi}\partial_{w|\omega} D(\phi, \omega)|_{\omega=\phi}, \\
\mathbf{g}_\phi(v, (\nabla^\dagger_u)_\phi w) &:= -\partial_{u|\omega}\partial_{w|\omega}\partial_{v|\phi} D(\phi, \omega)|_{\omega=\phi},
\end{aligned}
$$

  which are the Norden–Sen geometry.
- Eguchi equations applied to Brègman relative entropies D_Ψ yield hessian geometries
- Consequently, Weyl geometry of quantum state space arises as a natural local consequence of the Brègman relative entropy.

Smooth quantum information geometries (II)

- Example: $\mathcal{M}(\mathcal{N}) = \mathcal{T}(\mathcal{H}) \cap \{\rho > 0, \mathrm{tr}_{\mathcal{H}}(\rho) = 1\}$
  $D_1(\rho,\sigma) = \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$

  give Mori['55]–Kubo['56]–Bogolyubov['62] riemannian metric:

$$\mathbf{g}^{D_1}_{\rho}(x,y) = \mathrm{tr}_{\mathcal{H}} \left(\int_0^\infty \mathrm{d}\lambda x \frac{1}{\lambda \mathbb{I} + \rho} y \frac{1}{\lambda \mathbb{I} + \rho} \right),$$

  and Nagaoka['94]–Hasegawa['95] affine connections:

$$\mathbf{t}^{\nabla^{D_1}}_{\rho,\omega}(x) = x - \mathrm{tr}_{\mathcal{H}}(\omega x), \quad \mathbf{t}^{\nabla^{D_1 \dagger}}_{\rho,\omega}(x) = x.$$

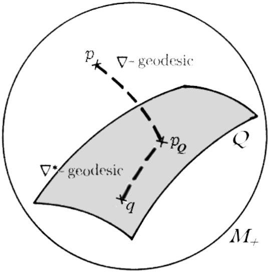
Smooth generalised pythagorean theorem

Let $(\mathcal{M}, \mathbf{g}, \nabla, \nabla^\dagger)$ be a hessian geometry. Then for any $\mathcal{Q} \subseteq \mathcal{M}$ which is:

- $\nabla^\dagger$-autoparallel $:= \nabla^\dagger_u v \in \mathbf{T}\mathcal{Q}\ \forall u, v \in \mathbf{T}\mathcal{Q}$;
- $\nabla^\dagger$-convex $:= \forall \rho_1, \rho_2 \in \mathcal{Q}\ \exists!\ \nabla^\dagger$-geodesics in $\mathcal{Q}$ connecting ρ_1 and ρ_2;

there exists a unique projection

$$\mathcal{M} \ni \rho \mapsto \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho) := \arg\inf_{\sigma \in \mathcal{Q}} \{D_\Phi(\sigma, \rho)\} \in \mathcal{Q}.$$



- it is equal to a unique projection of ρ onto $\mathcal{Q}$ along a ∇-geodesic that is $\mathbf{g}$-orthogonal at $\mathcal{Q}$.
- it satisfies a generalised pythagorean equation

$$D_\Phi(\omega, \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho)) + D_\Phi(\mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho), \rho) = D_\Phi(\omega, \rho) \quad \forall (\omega, \rho) \in \mathcal{Q} \times \mathcal{M}.$$

Hence, for Brègman distances D_Φ the local entropic projections are equivalent with geodesic projections.

New kinematics: local (operational) view

- Consider local embeddings $\ell : U \to X$, $U \subseteq \mathcal{M}$, as local coordinate systems, determining X as a local tangent space of $\mathcal{M}$
- Use the dual space $X^\star$ as a local cotangent space
- resulting structure: locally dually flat information manifold $\mathcal{M}$
- operational local kinematics:
  - local states/preparations: vectors of $T_\psi\mathcal{M} \cong X$ ($\phi(\theta) \to \theta \to \frac{\partial}{\partial\theta^i}$)
  - local effects/observables: vectors of $T^*_\psi\mathcal{M} \cong X^\star$ ($f(\phi) \to \mathbf{d}f(\phi)$)
- Dually flat manifolds are nonlinear generalisation of the euclidean and Hilbert spaces, with the generalised pythagorean theorem playing a key role.
- Locally dually flat manifolds introduce the shift analogous to the shift from Minkowski space to lorentzian manifolds: dual flatness holds only locally, for tangent (and cotangent) spaces.
- If entropic projections are regarded as a form of information dynamics, then introducing the structure of quantum information manifold based on a Brègman distance D_Ψ (equivalently, local dual flatness) amounts to postulating a "free fall principle": if there is no additional causal dynamics or inferential constraints, the local information flows along ∇^{D_Ψ}-geodesics.

Local inference, local causality: free falls and BLP localisation?

- Is it possible to use these geometric structures to localise the 'dynamic ansatz' $\phi \mapsto \mathfrak{P}^{D_\Psi}_{\mathcal{Q}} \circ w^h_t(\phi)$?
- Optimally, one would like to combine locally the entropic geodesic free falls with hamiltonian flows in a single formula, describing an effective information dynamics in which causal and inferential parts are parallely processed:
- Given a hamiltonian observable h and a Brègman relative entropy D_Ψ, the 1-form $\mathbf{d}h_{\mathcal{A}}(\phi) + \mathbf{d}_{\nabla^{D_\Psi}}(\phi)$ would represent a local perturbation of causal dynamics by the information flow along entropic geodesics (and vice versa).
- However, BLP structure and (locally) dually flat structure use different underlying C^∞-manifold structures. This is a serious open problem.
- One way to go would be localisation of a causal dynamics requires to replace a representation of a real Banach Lie algebra $\mathcal{A}$ in terms of a dual pair $(\mathcal{N}^{\mathrm{sa}}_\star, \mathcal{N}^{\mathrm{sa}})$ by a representation in terms of a dual pair of real Banach spaces $(X, X^\star)$. However, it is not clear how to do it in concrete examples, with X is e.g. a noncommutative L_p space.
- Another way, that is not so general, but works, is to give up BLP structure, and use a Haagerup theorem (extended by Jakšić and Pillet from Hilbert space to any noncommutative L_p space), that guarantees an existence of a unique generator of isometries of a positive cone of X that implements the 1-parameter group of automorphisms of a W^*-algebra (an L_p-liouvillean).

Open problems

- Develop a theory of locally dually flat quantum manifolds associated to Brègman distances (an ongoing work with A.Jenčová).
- Construct (or prove impossibility of) local implementation of Banach Lie–Poisson manifold structure as represented in the pairs $(X, X^\star)$, especially in the case when X is a noncommutative Orlicz space.
- In any case, find a way to give an exact meaning to local $\mathbf{d}h_{\mathcal{A}}(\phi) + \mathbf{d}_{\nabla^{D_\Psi}}(\phi)$ ansatz.
- Find the exact range of (linear and nonlinear) CPTP instruments that can be modelled with $\mathfrak{P}^{D_\Psi}_{\mathcal{Q}} \circ w^h_t$ maps.
- Show that one can reconstruct a theory that is locally QM using purely geometric data (hence, characterise local tangent/cotangent spaces as QM state/effect dual pairs without assuming that $\mathcal{M}$ consists of states over a W*-algebra).
- Work out the details of reconstruction of the pure states framework (some aspects are known, but some are missing).
- Investigate in more details the class of quantum Weyl geometries induced by Brègman relative entropies.
- ...

Back to original motivation: some insights

- How we could use this framework for ‘gravitating quantum theory’ instead of ‘quantising gravity’?
- In principle, the choice of geometry of quantum state space corresponds to a specific criteria of statistical inference, and may reflect for example the criteria of coarse graining used in renormalisation procedures.
- Generally, the geometries of quantum manifolds $\mathcal{M}$ are encoding the “effective information geometry”. In particular, in quantum thermodynamics, the curvature $= \infty$ at critical points.
- With the postulate of local dual flatness (hence, local Weyl structure) we leave the global geometry underdetermined, in analogy to local lorentzianity leaving the global structure of space-time underdetermined.
- The tentative postulate is to determine the global structure by means of a variational principle, such as $\delta \int_{\mathcal{M}} dVol\, R = 0$, in analogy to both GR and Weyl’s gravito-electromagnetism. Yet, the detailed meaning of this postulate for particular quantum systems is to be understood.
- The shift from riemannian to lorentzian signature can be performed by Poincarè–Wick rotation of a quantum correlation function g along the vector field of a causal evolution, at least if the latter is globally determined.
- Duch–RPK’2011: Given a choice of a global ’causal’ vector field, and a specific (Kubo–Mori–Bogolyubov) riemannian metric (corresponding to the Umegaki relative entropy D_1), we have determined analytically the space of quantum states that has exactly the 3+1 Schwarzschild geometry.

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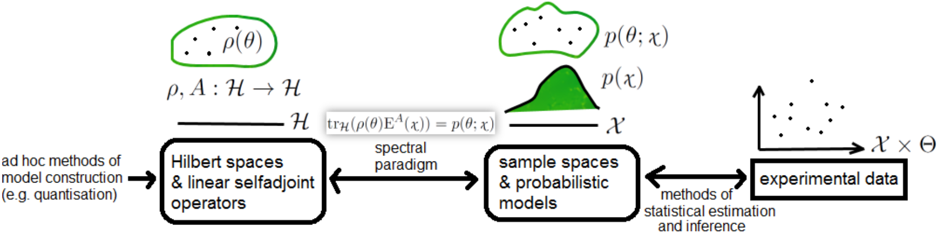
Orthodox quantum mechanical paradigm (von Neumann, 1926-1932):

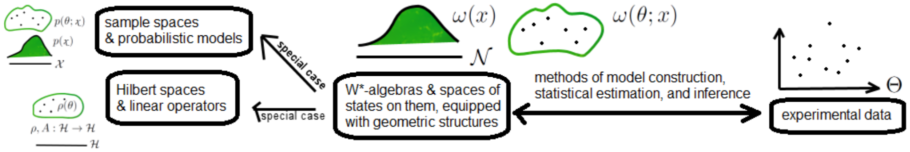


- a solution of a particular problem (solid mathematical framework providing unifying foundations for 'wave mechanics' and 'matrix mechanics')
- von Neumann'1935: "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space anymore."

Some key observations:

- Probability theory is just a special case of integration theory on W^*-algebras.
- From the perspective of this theory, quantum states are **just** integrals, so there is no a priori reason why "general" quantum theory (beyond QM) should depend on probabilities.
- Quantum states (and structures over them) can be associated directly with the epistemic data by generalising the methods of associating epistemic data with probabilities (and with structures over them).

New paradigm:



Basic object of interest: spaces $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$ of states over W^*-algebras $\mathcal{N}$.

- Quantum theoretic kinematics generalises and replaces probability theory.
- Quantum theoretic dynamics generalises and replaces causal statistical inference.
- Nonlinear information geometry of spaces of quantum states replaces the role of (linear) spectral theory of quantum mechanics.
- Replace the use of eigenvalues and expectations of self-adjoint operators on $\mathcal{H}$ (or in $\mathcal{N}$) by observables $f : \mathcal{M}(\mathcal{N}) \to \mathbb{R}$. Given any model construction rule $\mathbb{R}^n \supset \Theta \ni \theta \mapsto \rho(\theta) \in \mathcal{M}(\mathcal{N})$, and the set of experimental functions $f_\Theta : \Theta \to \mathbb{R}$ the set of observables relevant to the problem is given by $\{f : \mathcal{M}(\mathcal{N}) \to \mathbb{R} \mid f_\Theta = f \circ \theta\}$