EINSTEIN EQUATIONS AND CAUCHY-RIEMANN GEOMETRY *

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Contents

1 INTRODUCTION

1.1 Historical background of the thesis.

Electromagnetic plane waves are *algebraically special*: their Maxwell tensor has only one, doubly degenerate eigendirection. This direction corresponds to the propagation vector of the wave and generates a congruence of null geodesics without shear, twist and expansion in Minkowski spacetime. Only the shear-free condition of the congruence is a consequence of the Maxwell equations [49], the others are equivalent to the assumption about planar wave fronts. Waves with other than planar fronts define shear-free congruences of null geodesics with vanishing twist and nonvanishing expansion. Electromagnetic fields with twisting congruences of shear-free and null geodesics are generic in the class of algebraically special Maxwell fields. They are radiative in the sense of the energy-momentum tensor but do not admit globally defined wave fronts. A first example of such a field with the so called *Robinson congruence* was constructed by I. Robinson.
Congruences of shear-free and null geodesics appear also in the General Relativity theory, where they are related to the vacuum algebraically special gravitational fields [13]. An assumption about existence of a congruence of shear-free and null geodesics on the spacetime considerably simplifies the Einstein equations [54], [17]. The well known, physically interesting solutions of Schwarzschild [57], Robinson and Trautman [54], Kerr [17], Newman and collaborators [30], [32] give examples of spacetimes satisfying this assumption. Among the quoted solutions only the two first have nontwisting congruence. Congruences of shear-free and null geodesics appearing in the Kerr-Newman and NUT solutions have twist. Thus examples of solutions with twisting congruences were known for a long time, however no systematic geometric approach was used to find them. This work introduces such a geometrical framework and, by using it, constructs new families of solutions.
¿From the geometric point of view twisting shear-free congruences of null geodesics are much more interesting than those without twist. Any spacetime admitting a nontwisting shear-free congruence of null geodesics is locally diffeomorphic to the Cartesian product of a real line (corresponding to the direction of the congruence) and a 3-dimensional manifold $\mathcal{Q}$ diffeomorphic to $\mathbf{R} \times \mathbf{C}$. In the case of twisting congruence the spacetime again has local form $\mathbf{R} \times \mathcal{Q}$ but now the 3-space $\mathcal{Q}$ is more complicated. It is endowed with the so called *nondegenerate Cauchy-Riemann* (CR-)*structure* and as such becomes a 3-dimensional *nondegenerate CR-manifold*. There exists nonequivalent nondegenerate CR-manifolds; to any of them corresponds a non-

twisting congruence of shear-free and null geodesics. In this sense there are many nonequivalent twisting congruences of null geodesics. On the contrary all nontwisting congruences of shear-free and null geodesics are locally equivalent - one can associate with them only $\mathcal{Q} \cong \mathbf{R} \times \mathbf{C}^2$ which is the only existing degenerate CR-structure. The correspondence described above between shear-free congruences of null geodesics and 3-dimensional CR-structures is due to I. Robinson and A. Trautman [53], [52]. Partial results in this context were published earlier by P. Sommers [59], J. Tafel [61] and R. Penrose [42].
The best understood examples of CR-structures are classes of 3-dimensional real surfaces in $\mathbf{C}^2$ equivalent under the biholomorphic mapings of $\mathbf{C}^2$. The study of such structures were initiated by the mathematicians H. Poincare [48], B. Segre [58] and E. Cartan [5], [6]. In particular, Cartan defined a full set of invariants of nondegenerate structures in full analogy with the curvature invariants of Riemannian geometry. His construction of invariants was then successively generalized to the more and more abstractly defined 3-dimensional CR-structures [8], [4], [16]. Expressions for the Cartan invariants in the most general case of a 3-dimensional CR-structure are due to J. Lewandowski [22] (see also [23], [26]).
In Chapter 2 (Sections 2.2-2.5) of this thesis we consider the pure radiation Einstein equations $R_{\mu\nu} = \Phi k_\mu k_\nu$, with k_μ being a vector field tangent to the twisting congruence of shear-free and null geodesics. We formulate them as invariant equations on the underlying nondegenerate CR-structure (Section 2.4). When we restrict ourselves to the CR-structures realized as hypersurfaces in $\mathbf{C}^2$ then the equations reduce to only one real second order equation (Section 2.4) which becomes linear if the so called NUT parameter [32] of the solution vanishes. In this case (Section 2.5) the equation can be solved under certain assumptions. As an example we present solutions to the system with vanishing NUT parameter for CR-structures admitting 3-dimensional symmetry groups of Bianchi types VI_h. The metrics corresponding to these solutions admit no symmetries in general. Most of them are new. Another application of our formulation of the pure radiation Einstein equations is a general solution which admits at least three infinitesimal conformal symmetries (Section 2.3). A short Section 2.6 gives CR-invariant formulation of the twisting type N vacuum Einstein equations. These equations will be studied elsewhere.
The last Section of Chapter 2 provides first explicit solutions describing purely radiative Einstein-Maxwell fields with twisting shear-free geodesic null congruences; also first examples of Einstein-Maxwell fields with twisting congruences which possess both radiative and charged part are presented there.
Chapter 3 deals with orthogonal almost complex structures over a 4-dimensional

Euclidean manifold. In four dimensions such structures are Euclidean analogs of null congruences as described in the last section of the Introduction. In this analogy the integrability conditions for the almost complex structure to be complex correspond to the geodesic and shear-free property of the null congruence [21]. In Sections 3.1, 3.2 we define an Euclidean analog of the Newman-Penrose formalism [31]. As an application of the formalism we discuss integrability conditions for a local existence of orthogonal complex structure on Euclidean 4-manifolds (Section 3.4). In this section it is also showed which orthogonal complex structures are admitted by conformally flat metrics. This gives a part of an Euclidean analog of the Kerr theorem. An Euclidean analog of the Goldberg-Sachs theorem [13] is presented in (Section 3.5). It follows from this theorem and Section 3.4 that an Euclidean 4-manifold satisfying the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ admits orthogonal complex structure if its metric is algebraically special in the sense of an Euclidean analog of the Cartan-Petrov-Penrose classification given in Section 3.3. Such a classification was known in the spinorial language. Our classification refers to the properties of orthogonal almost complex structures.
The results of the whole Chapter 3 seem to be new.

1.2 A note on collaborators.

The thesis is based on research done in collaboration with Jerzy Lewandowski (Warszawa, Syracuse, Gainesville), Ivor Robinson (Trieste, Dallas), Jacek Tafel (Warszawa) and Andrzej Trautman (Warszawa, Trieste). The results of this collaboration are quoted in different parts of the thesis. In particular:
- the last section of the Introduction and the whole Chapter 3 was obtained in collaboration with A. Trautman,
- the small note on relations between integrability conditions for higher dimensional optical geometries and shear-free property of the null congruences quoted in Section 1.4 is a result of a joint work with I. Robinson and A. Trautman,
- the original results of my Master Thesis in SISSA quoted in Section 2.1 were obtained in collaboration with J. Tafel [37] (part on symmetries of CR-structures) and J. Lewandowski [26] (part on Cartan invariants),
- Sections 2.2, 2.4 on invariant formulation of the pure radiation Einstein equations were obtained with J. Lewandowski and J. Tafel [27], [28]; I quote parts of these two papers in Section 2.5 where particular examples of solutions are presented,
- results of Section 2.3 were obtained in collaboration with J. Lewandowski [25],
- the whole Section 2.7.2 is a part of a joint work with J. Tafel [38].

1.3 Conventions and notations.

All our considerations are *local* and concern with nonsingular points of our constructions. We in addition assume, that all real vector fields are *real analytic* in the considered regions.

Bold face letters.
- $\mathbf{K}$ denotes a field of numbers, e.g. $\mathbf{R}$ is the field of real numbers, $\mathbf{C}$ is the field of complex numbers,
- $\mathbf{K}^n$ denotes a vector space of the n-th Cartesian power of $\mathbf{K}$,
- $\mathbf{G}(n, \mathbf{K})$ denotes a Lie subgroup of the general linear group of $\mathbf{K}^n$, e.g. $\mathbf{SL}(2, \mathbf{C})$ is a special linear group of $\mathbf{C}^2$,
- $\mathbf{S}^n$ denotes the n-dimensional sphere.

Vector spaces.
Let V be a vector space and K and L its vector subspaces. Then vector spaces $K+L$ and $K \cap L$ are defined by

$$K + L = \{v \in V : \ v = k + l \text{ and } k \in K, l \in L\},$$

and

$$K \cap L = \{v \in V : \ v \in K \text{ and } v \in L\}.$$

Tensor product of vector spaces is denote by $\otimes$, the exterior product by $\wedge$. In this way $\mathbf{C} \otimes V$ denotes the complexification and $\bigwedge^m V$ denotes the m-th exterior power of a real vector space V. Any vector n in a vector subspace N of $\mathbf{C} \otimes V$ can be decomposed onto $n = v + iv'$, where v and v' are appropriate real vectors from V and $i^2 = -1$. The complex conjugate element to n is $\bar{n} = v - iv'$. A space consisting of all elements w of $\mathbf{C} \otimes V$ such that $\bar{w}$ is in N is denoted by $\bar{N}$. We note that spaces $N + \bar{N}$ and $N \cap \bar{N}$ are both complexifications of real vector spaces.

Manifolds.
If $\mathcal{M}$ denotes a manifold then $\mathrm{T}\mathcal{M}$ (respectively, $\mathrm{T}^*\mathcal{M}$) denotes its tangent (respectively, cotangent) bundle.
If $\mathcal{K}$, $\mathcal{L}$ denote vector bundles over $\mathcal{M}$ then $\Gamma(\mathcal{K})$ denotes the set of all sections of $\mathcal{K}$ and $[\Gamma(\mathcal{K}), \Gamma(\mathcal{L})]$ denotes a set consisting of all commutators of the form $[k, l]$ where k, l are sections of $\mathcal{K}$ and $\mathcal{L}$, respectively.
If $\phi : \mathcal{M} \to \mathcal{M}$' is a diffeomorphism between two manifolds $\mathcal{M}$ and $\mathcal{M}$' then ϕ_*

denotes the transport of contravariant tensor fields from $\mathcal{M}$ to $\mathcal{M}$'; transport of covariant tensor fields from $\mathcal{M}$' to $\mathcal{M}$ is denoted by ϕ^*. Transports of tensor fields of general type is denoted by $\tilde{\phi}$.
The exterior derivative of forms is denoted by d; the inner derivative associated with a vector field X on a manifold $\mathcal{M}$ is denoted by $X\rfloor$; the Lie derivative along X is denoted by $\mathcal{L}_X$.

Metric.
We say that a basis of vector fields (f_μ), $\mu = 1, ..., n$ on a manifold $\mathcal{M}$ is orthonormal with respect to the metric g iff $g(f_\mu,\ f_\nu) = \varepsilon_\mu \delta_{\mu\nu}$, and $\varepsilon_\mu = \pm 1$. If $\varepsilon_\mu = 1$ for any μ then the metric is called Euclidean. The metric is called Lorentzian iff $\varepsilon_\mu = -1$ for exactly one μ.
If X, Y are vector fields on $\mathcal{M}$ then $g(X)$ is a 1-form defined by $< g(X), Y >= g(X, Y)$. On the oriented p-dimensional manifold the volume form is denoted by η. The Hodge dual $*\omega$ of an r-form ω is defined by

$$*\omega(X_{r+1}, X_{r+2}, ..., X_p)\eta := \omega \wedge g(X_{r+1}) \wedge g(X_{r+2}) \wedge ... \wedge g(X_p),$$

where X_i, $i = r+1, ..., p$, are vector fields.
If κ and λ denote one forms on the manifold then $\kappa\lambda = \frac{1}{2}(\kappa \otimes \lambda + \lambda \otimes \kappa)$; such expressions are used in the work to simplify expressions for metrics.
If $\mathcal{K}$ is a vector bundle over $\mathcal{M}$ then $\mathcal{K}^\perp$ denotes the bundle orthogonal to $\mathcal{K}$ with respect to g.

1.4 Complex and optical geometries - an analogy.

In this section we summarize known results on complex and optical geometries that we use in our work.

Let $\mathcal{M}$ be an oriented $2m$-dimensional real manifold equipped with the metric g which has either *Euclidean* or *Lorentzian* signature.
In the Euclidean case we consider an *almost complex structure* i.e. a tensor field $\mathcal{J} : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}$ such that

$$\mathcal{J}^2 = -id.$$

An almost complex structure is called *orthogonal* iff

$$g(\mathcal{J}X,\ \mathcal{J}Y) = g(X,\ Y)$$

for any vector fields X, Y on $\mathcal{M}$.
For any orthogonal $\mathcal{J}$ there exists a set of orthonormal vector fields X_1, ..., X_m such that $(X_1, ..., X_m, \mathcal{J}X_1, ..., \mathcal{J}X_m)$ is an orthonormal basis for vector fields. Any two such bases differ by a linear transformation with determinant equal to 1. This specifies an *orientation associated to* $\mathcal{J}$ on $\mathcal{M}$ which may be in general different from the previous orientation of $\mathcal{M}$. If both these orientations coincide then we say that *$\mathcal{J}$ agrees with the orientation.*
Given an almost complex structure $\mathcal{J}$ we define the Nijenhuis tensor $N_{\mathcal{J}}$ by

$$\frac{1}{2}N_{\mathcal{J}}(X,Y) \stackrel{def}{=} [\mathcal{J}X, \mathcal{J}Y] - [X,Y] - \mathcal{J}[\mathcal{J}X, Y] - \mathcal{J}[X, \mathcal{J}Y], \tag{1}$$

where X and Y are any two real vector fields on $\mathcal{M}$. $\mathcal{J}$ is *integrable* to the *complex* structure on $\mathcal{M}$ iff [33]

$$N_{\mathcal{J}} \equiv 0. \tag{2}$$

In the following we also consider *almost complex structures* in a vector bundle $\mathcal{H}$, with even-dimensional fibers, that is *not necessarily* a subbundle of the tangent bundle. Such a structure is defined by a linear automorphism $\mathcal{J} : \mathcal{H} \to \mathcal{H}$ of the bundle $\mathcal{H}$ which satisfies $\mathcal{J}^2 = -id$. Suppose that $\mathcal{H}$ is equipped with a Euclidean metric g'. $\mathcal{J}$ is called *orthogonal* if it is an orthogonal transformation with respect to this metric.

In the Lorentzian case we consider an *almost optical geometry* [65], [21]. This structure is defined as follows. Consider a real *line* subbundle $\mathcal{K}$ of the tangent bundle T$\mathcal{M}$ which in the metric g has *null (optical)* sections. Since $\mathcal{K} \subset \mathcal{K}^{\perp} =: \mathcal{L}$ then the quotient bundle $\mathcal{H} \stackrel{def}{=} \mathcal{L}/\mathcal{K}$ is well defined. This has even dimensional fibers since dimension of fibers of $\mathcal{K}$ and $\mathcal{L}$ are 1 and $2m-1$, respectively. Moreover, since $\mathcal{L} = \mathcal{K}^{\perp}$ then $\mathcal{K} = \mathcal{L}^{\perp}$ and the metric g descends to a nondegenerate Euclidean metric g' in $\mathcal{H}$. If, in addition, $\mathcal{H}$ is equipped with an almost complex structure $\mathcal{J}$ orthogonal with respect to g', then $\mathcal{O} \equiv (\mathcal{K},\ \mathcal{L},\ \mathcal{J})$ is called an almost optical geometry on $\mathcal{M}$.
The almost optical geometry is called an *optical geometry* if the following *integrability conditions* $a) - d)$ are satisfied [36]:

$$a) \qquad [\Gamma(\mathcal{K}), \Gamma(\mathcal{K})] \subset \Gamma(\mathcal{K}).$$

$$b) \qquad [\Gamma(\mathcal{K}), \Gamma(\mathcal{L})] \subset \Gamma(\mathcal{L}).$$

c) if ϕ_k is a flow generated by any section k of $\mathcal{K}$ then
$\tilde{\phi}_k\mathcal{J} = \mathcal{J}$

To formulate point d) we need some comments. Condition a) says that $\Gamma(\mathcal{K})$ defines a foliation of $\mathcal{M}$ by 1-dimensional manifolds - integral curves of any nonvanishing section $k \in \Gamma(\mathcal{K})$. Any point $x \in \mathcal{M}$ belongs to a precisely one such curve. We define an equivalence relation "$\sim$" on $\mathcal{M}$ identifying points on the same curve. More precisely, any two points $x, x' \in \mathcal{M}$ are in the relation "$\sim$" iff $x' = \phi_k(x)$.

We assume that in a considered region $\mathcal{U}$ of $\mathcal{M}$ a quotient space $\mathcal{Q} \stackrel{def}{=} \mathcal{U}/\sim$ is a $(2m-1)$-dimensional manifold. This quotient manifold has some additional structure. Let π denotes the canonical projection $\pi : \mathcal{U} \to \mathcal{U}/\sim$. If $a) - c)$ holds then this projection defines $\mathcal{H}' \stackrel{def}{=} \pi_*\mathcal{H}$ and $\mathcal{J}' \stackrel{def}{=} \tilde{\pi}\mathcal{J}$. One checks that $\mathcal{H}'$ is a subbundle of a tangent bundle $\mathrm{T}\mathcal{Q}$ with fibers of dimension $2(m-1)$ and $\mathcal{J}'$ is an almost complex structure in $\mathcal{H}'$. The last integrability condition for the optical geometry $\mathcal{O}$ is expressed in terms of any two real sections X', Y' of the bundle $\mathcal{H}'$. This reads

d) $$\mathcal{J}' \; [X' + i\mathcal{J}'X',\ Y' + i\mathcal{J}'Y'] = -i\,[X' + i\mathcal{J}'X',\ Y' + i\mathcal{J}'Y'].$$

Note that X', Y' are now vector fields on $\mathcal{Q}$, and that we extended $\mathcal{J}'$ to $\mathbf{C} \otimes \mathcal{H}'$ by linearity.
This condition generalizes (2), in the sense that if $\mathcal{H}'$ defines a foliation of $\mathcal{Q}$ by $(2m-2)$-dimensional manifolds then $\mathcal{J}'$ restricted to these manifolds and satisfying d) is a complex structure there.

Definition 1.4.1 *A* Cauchy-Riemann (CR) structure *is a real $(2m-1)$-dimensional manifold $\mathcal{Q}$ together with*
(1) a real subbundle $\mathcal{H}' \subset T\mathcal{Q}$ of fibers of dimension $2(m-1)$,
(2) an almost complex structure $\mathcal{J}'$ in $\mathcal{H}'$.
If in addition CR structure satisfies integrability condition d) then it is called an integrable *CR structure.*

The following proposition (see for example [36]) gives another characterization of integrability conditions for a CR-structure.

Proposition 1.4.1 *A CR structure is integrable iff*

$$[\Gamma(\mathcal{H}'^{+}),\ \Gamma(\mathcal{H}'^{+}] \subset \Gamma(\mathcal{H}'^{+}), \tag{3}$$

where $\mathcal{H}'^{\pm} \stackrel{def}{=} \mathcal{H}' \pm i\mathcal{J}'\mathcal{H}'$.

Definition 1.4.2 *Two CR-structures* $(\mathcal{Q},\ \mathcal{H},\ \mathcal{J})$ *and* $(\mathcal{Q}',\ \mathcal{H}',\ \mathcal{J}')$ *are called* (locally) equivalent *iff there exists a (local) diffeomorphism* $\phi : \mathcal{Q} \to \mathcal{Q}'$ *such that*

$$\phi_* \mathcal{H} = \mathcal{H}'$$

and

$$\tilde{\phi} \mathcal{J} = \mathcal{J}'.$$

There exist nonequivalent CR-structures on the same manifold $\mathcal{Q}$.

We have the following theorem.

Theorem 1.4.1
(1) A real $2m$*-dimensional manifold* $\mathcal{M}$ *equipped with an integrable optical geometry is locally diffeomorphic to* $\mathbf{R} \times \mathcal{Q}$ *where* $\mathcal{Q}$ *is a* $(2m-1)$*-dimensional integrable CR structure.*
(2) Given a $(2m-1)$*-dimensional integrable CR-structure* $\mathcal{Q}$ *one can define an integrable optical geometry on* $\mathbf{R} \times \mathcal{Q}$*.*

Proof. Point (1) is obvious in view of what we have said so far. The local factor $\mathbf{R}$ in $\mathbf{R} \times \mathcal{Q}$ is associated with integral curves of any nonvanishing section k of $\mathcal{K}$.
We prove (2) by giving a construction of an integrable optical geometry. Let π denotes the projection $\pi : \mathbf{R} \times \mathcal{Q} \to \mathcal{Q}$. Denote by $\mathcal{S}$ the subbundle of the cotangent bundle $T^*\mathcal{Q}$ which annihilates the bundle $\mathcal{H}'$. It has one dimensional fibers, hence it is generated by a nonvanishing section, say λ'. We define a nonvanishing 1-form λ on $\mathbf{R} \times \mathcal{Q}$ by $\lambda = \pi^*(\lambda')$. We define k as a nonvanishing vector field on $\mathbf{R} \times \mathcal{Q}$ which is tangent to the fibers $\pi^{-1}(x)$, $x \in \mathcal{Q}$. We define $\mathcal{K}$ as a bundle generated by k, and $\mathcal{L} \stackrel{def}{=} ker\lambda$; both $\mathcal{K}$ and $\mathcal{L}$ are subbundles of $T(\mathbf{R} \times \mathcal{Q})$. Due to the construction $\pi^*(\mathcal{L}/\mathcal{K}) = \mathcal{H}'$. Therefore we can define $\mathcal{J}$ in $\mathcal{H} = \mathcal{L}/\mathcal{K}$ by $\mathcal{J}(\ [l]\ ) = \mathcal{J}'(\pi_*(\ [l]\ ))$, where $[l] \in \mathcal{L}/\mathcal{K}$. Let g' be any Euclidean metric in $\mathcal{H}'$ such that $\mathcal{J}'$ is orthogonal in it. We define a Lorentzian metric on $\mathbf{R} \times \mathcal{Q}$ by

$$g = \pi^*(g') - \lambda\alpha,$$

where α is any 1-form on $\mathbf{R} \times \mathcal{Q}$ such that the metric g is nondegenerate. One checks that $(\mathcal{K}, \mathcal{L}, \mathcal{H}, \mathcal{J})$ is an integrable optical geometry with g as a metric on $\mathbf{R} \times \mathcal{Q}$. $\square$
This theorem was first formulated for $m = 2$ in [52].

Note that the metric g in the proof of Theorem 1.4.1 is not uniquely defined. Given an optical geometry one has not only one Lorentzian metric g but the *class* of so called *adapted* metrics. A Lorentzian metric g on $\mathcal{M}$ is said to be adapted to the optical geometry $(\mathcal{K},\ \mathcal{L},\ \mathcal{J})$ if, relative to g
$I)$ $\mathcal{K}$ is null,
$II)$ $\mathcal{L}$ is orthogonal to $\mathcal{K}$,
$III)$ $\mathcal{J}$ is orthogonal with respect to metric g' induced by g in $\mathcal{H}$.

We note that the case $m = 2$ is different from cases $m > 2$ in the following sense. If $m = 2$ and $\mathcal{M}$ is oriented then any almost optical geometry can be defined by choosing a null congruence on $\mathcal{M}$. All vector fields tangent to the congruence form the bundle $\mathcal{K}$, $\mathcal{L}$ equals to $\mathcal{K}^{\perp}$ by definition and, due to the dimension of $\mathcal{M}$, the bundle $\mathcal{H} = \mathcal{L}/\mathcal{K}$ has 2-dimensional fibers. Then the orientation of $\mathcal{M}$ and the metric g' in $\mathcal{H}$ uniquely define a complex structure $\mathcal{J}$ in $\mathcal{H}$. The integrability conditions *a) - d)* for $(\mathcal{K},\ \mathcal{L},\ \mathcal{J})$ are equivalent to the geodesic and shear-free property of the congruence. If they are satisfied in one of the adapted metrics they are also valid in *any other* adapted metric [52], [53].
This is not in general true if $m > 2$. In such cases a null congruence also defines $\mathcal{K}$ and $\mathcal{L}$ as before but, since the fiber dimension of $\mathcal{H} = \mathcal{L}/\mathcal{K}$ is greater or equal than 4, there is no natural way of defining $\mathcal{J}$ in $\mathcal{H}$. Therefore if $m > 2$ the choice of a null congruence does not suffice for a definition of an almost optical geometry. Moreover, the integrability conditions *a) - d)* say nothing about the shear-free property of the congruence generated by sections of the bundle $\mathcal{K}$: it may happen that in some adapted metrics the congruence has shear and in some others no [36].
Geometrical objects which are in one-to-one correspondence with almost optical geometries for arbitrary m are bundles of *maximal totally null vector spaces* over $\mathcal{M}$. They allow for an unified description of almost complex geometries (Euclidean case) and almost optical geometries (Lorentzian case) as follows [21], [66].
In the complexification of the tangent bundle $\mathrm{T}\mathcal{M}$ equipped with the complexification of the real metric g consider a subbundle $\mathcal{N}$ of maximal totally null vector spaces. It turns out that $\mathcal{N} \cap \bar{\mathcal{N}} = \mathbf{C} \otimes \mathcal{K}$ where $\mathcal{K}$ is a real bundle of fiber dimension 0 or 1, depending on whether g is Euclidean or Lorentzian, respectively. Moreover, since $\mathcal{N} + \bar{\mathcal{N}} = \mathbf{C} \otimes \mathcal{K}^{\perp}$ then $\mathcal{L} = \mathcal{K}^{\perp}$ is a subbundle of $\mathrm{T}\mathcal{M}$ with fibers of codimension 0 (in the Euclidean case) or 1 (in the Lorentzian case). In both cases we have a natural almost complex structure $\mathcal{J}$ in $\mathcal{H} \stackrel{def}{=} \mathcal{L}/\mathcal{K}$. To define this we observe that any section l of $\mathcal{L}$ is of the form $l = n + \bar{n}$ where n is some section of $\mathcal{N}$. If $[l]$ denotes

an equivalence class associated with l in $\mathcal{H}$ we define $\mathcal{J}$ by

$$\mathcal{J}(\ [l]\ ) = \mathcal{J}(\ [n+\bar{n}]\ ) \stackrel{def}{=} [-i\ (n-\bar{n})\ ]. \tag{4}$$

One proves that $\mathcal{J}$ is well defined and orthogonal. Therefore $\mathcal{N}$ defines either orthogonal almost complex structure or almost optical geometry over $\mathcal{M}$ depending on whether the signature of g is Euclidean or Lorentzian, respectively. The converse is also true: any orthogonal almost complex structure or almost optical geometry over $\mathcal{M}$ can be obtained in this manner.
Thus we have an *analogy* between almost complex and almost optical geometries. Due to this the integrability conditions (2) and $a) - d)$ have a uniform description. These are equivalent [66] to

$$[\Gamma(\mathcal{N}), \Gamma(\mathcal{N})] \subset \Gamma(\mathcal{N}). \tag{5}$$

Given a maximal totally null space $\mathcal{N}$ we define a maximal totally null space $\mathcal{N}^*$ in $\mathbf{C}\otimes\mathrm{T}^*\mathcal{M}$ by

$$\mathcal{N}^* = \{\alpha \in \Gamma(\mathbf{C}\otimes \mathrm{T}^*\mathcal{M}) : \alpha = g(n) \text{ where } n \in \Gamma(\mathbf{C}\otimes \mathrm{T}\mathcal{M})\}.$$

The space $\Lambda := \bigwedge^m \mathcal{N}^*$ is one dimensional and it turns out that its Hodge dualization is

$$*\Lambda = \pm(i)^{m-\varepsilon}\Lambda,$$

where the parameter ε is 0 in Euclidean case and 1 in Lorentzian case. The sign of right hand side of the above equation, called *helicity*, is an invariant property of $\mathcal{N}$ under the orientation preserving mappings. It also defines the helicity of the corresponding (almost) complex structure or (almost) optical geometry.

2 EINSTEIN EQUATIONS IN 4-MANIFOLDS ADMITTING OPTICAL GEOMETRIES

2.1 Results on 3-dimensional CR-structures and Lorentzian metrics associated with them.

This section is a summary of results on 3-dimensional CR-structures. All of them were presented and proved in [35].

2.1.1 Nondegenerate CR-structures.

A CR-structure on a 3-dimensional real manifold $\mathcal{Q}$ can be defined in terms of equivalence classes of pairs of 1-forms $[(\lambda,\mu)]$ on $\mathcal{Q}$ such that λ is real, μ is complex,

$$\lambda\wedge\mu\wedge\bar{\mu}\neq 0, \tag{6}$$

and pairs (λ,μ), (λ',μ') are equivalent iff there exist nonvanishing functions f (real), h (complex) and a complex function p such that

$$\lambda'=f\lambda,\qquad \mu'=h\lambda+p\mu. \tag{7}$$

A CR-structure can equivalently be defined in terms of a nowhere vanishing complex vector field ∂ (sometimes called a *tangential CR-operator*) such that

$$\partial\rfloor\lambda=\partial\rfloor\bar{\mu}=0.$$

∂ is defined by this equations up to a transformation

$$\partial\to s\partial,$$

where s is a nowhere vanishing complex function on $\mathcal{Q}$.
Any 3-dimensional CR-structure is integrable. In this work we mainly consider *nondegenerate* 3-dimensional CR-structures. They are defined by

$$\lambda\wedge\mathrm{d}\lambda\neq 0. \tag{8}$$

2.1.2 Realizability.

A CR-structure can be realized as a real hypersurface in $\mathbf{C}^2$ (*is realizable*) iff the following *tangential CR-equation*

$$d\xi \wedge \lambda \wedge \mu = 0 \tag{9}$$

or, equivalently,

$$\bar{\partial}\xi = 0, \tag{10}$$

for a complex function ξ on $\mathcal{Q}$ has two *independent* solutions ξ_1 and ξ_2 [1] Not all CR-structures are realizable [34], [16].

The solution $\xi = 0$ of the tangential CR-equation (9) is called *trivial*. If the CR-structure is nondegenerate any nontrivial solution satisfies

$$d\xi \wedge d\bar{\xi} \neq 0.$$

If (x^i) $(i = 1,\ 2,\ 3)$ are coordinates on $\mathcal{Q}$ then any pair of independent solutions $\xi_1(x^i), \xi_2(x^i)$ of the tangential CR-equation (9) define a realization of the CR-structure as a real 3-surface in $\mathbf{C}^2$ by

$$\mathcal{Q} \ni (x^1,\ x^2,\ x^3) \mapsto (\xi_1(x^i),\ \xi_2(x^i)) \in \mathbf{C}^2;$$

another pair of independent solutions $\xi_1',\ \xi_2'$ of (9) define a 3-surface which is related to the former by a biholomorphic transformation of $\mathbf{C}^2$. Conversely, if a real function Ψ of complex variables $\xi_1,\ \xi_2$ defines a 3-surface $\mathcal{Q}$ in $\mathbf{C}^2$ by equation

$$\Psi(\xi_1,\ \bar{\xi}_1,\ \xi_2,\ \bar{\xi}_2) = 0$$

then $\mathcal{Q}$ is endowed with a CR-structure in terms of transformations (7) of forms

$$\lambda = i(\Psi_{\xi_1} d\xi_1 + \Psi_{\xi_2} d\xi_2) \qquad \mu = \begin{cases} d\xi_1 & \text{iff } \Psi_{\xi_2} \neq 0 \\ d\xi_2 & \text{iff } \Psi_{\xi_2} = 0 \end{cases} .$$

Functions ξ_1 and ξ_2 restricted to $\mathcal{Q}$ gives two independent solutions of the tangential CR-equation (9) and any solution of this equation is given by

$$\varphi = \varphi(\xi_1,\ \xi_2), \tag{11}$$

where φ is any holomorphic function of variables $\xi_1,\ \xi_2$. Thus for realizable CR-structures there is no problem in finding general solution of (9).

[1] Independence here means that

$$d\xi_1 \wedge d\xi_2 \neq 0.$$

2.1.3 Cartan invariants.

Cartan invariants of a *nondegenerate* 3-dimensional CR-structures [5], [6] are defined as follows.
Consider a class $[(\lambda,\ \mu)]$ of forms defining a CR-structure on 3-manifold $\mathcal{Q}$. Let $\lambda',\ \mu'$ be forms from this class i.e. they are related to $\lambda,\ \mu$ by (7). We successively restrict this freedom obtaining finally a rigid structure.
First condition we impose on $\lambda',\ \mu'$ is

$$\mathrm{d}\lambda' \wedge \lambda' = i\mu' \wedge \bar{\mu}' \wedge \lambda'. \tag{12}$$

This condition is not very much restrictive; there is still a big freedom

$$\Omega = t\bar{t}\lambda', \qquad \Omega_1 = t(\mu' + z\lambda') \tag{13}$$

in choosing forms Ω and Ω_1 which are in the class $[(\lambda,\ \mu)]$ and satisfy (12). In (13) complex functions $t \neq 0$ and z are arbitrary.

Forms Ω and Ω_1 define 1-forms $\Omega_2,\ \Omega_3$ (complex), Ω_4 (real) and complex functions $\mathcal{R}$, $\mathcal{S}$ which satisfy

$$\begin{aligned}
&\mathrm{d}\Omega = i\Omega_1 \wedge \bar{\Omega}_1 + (\Omega_2 + \bar{\Omega}_2) \wedge \Omega \\
&\mathrm{d}\Omega_1 = \Omega_2 \wedge \Omega_1 + \Omega_3 \wedge \Omega \\
&\mathrm{d}\Omega_2 = 2i\Omega_1 \wedge \bar{\Omega}_3 + i\bar{\Omega}_1 \wedge \Omega_3 + \Omega_4 \wedge \Omega \\
&\mathrm{d}\Omega_3 = \Omega_4 \wedge \Omega_1 + \Omega_3 \wedge \bar{\Omega}_2 + \mathcal{R}\bar{\Omega}_1 \wedge \Omega \\
&\mathrm{d}\Omega_4 = i\Omega_3 \wedge \bar{\Omega}_3 + \Omega_4 \wedge (\Omega_2 + \bar{\Omega}_2) + (\mathcal{S}\Omega_1 + \bar{\mathcal{S}}\bar{\Omega}_1) \wedge \Omega
\end{aligned} \tag{14}$$

These forms and functions are related to Ω and Ω_1 and, as a consequence, depend on t and z from (13). They are defined up to the freedom

$$\Omega_2 \to \Omega_2 + \varrho\lambda'$$

$$\Omega_3 \to \Omega_3 + \varrho\mu'$$

$$\Omega_4 \to \Omega_4 + \mathrm{d}\varrho + \varrho(\Omega_2 + \bar{\Omega}_2) + \varrho^2\lambda'$$

where ϱ is any real function on $\mathcal{Q}$. The explicit expressions for these forms in case of a *general* [2] nondegenerate CR-structure are given in [22], [26].

[2] i.e. even such which does not admit any solution to the tangential CR-equation

The following matrix ω of 1-forms on $\mathcal{Q}$ is called the *Cartan-Chern-Moser connection* [5], [6], [8]

$$\omega = \begin{pmatrix} \frac{1}{3}(2\Omega_2 + \bar{\Omega}_2) & i\bar{\Omega}_3 & -\frac{1}{2}\Omega_4 \\ \mu' & \frac{1}{3}(\bar{\Omega}_2 - \Omega_2) & -\frac{1}{2}\Omega_3 \\ 2\lambda' & 2i\bar{\mu}' & -\frac{1}{3}(2\bar{\Omega}_2 + \Omega_2) \end{pmatrix}. \tag{15}$$

It also depends on $z,\ t$ of (13) and has the property that ω' computed for arbitrary t and z is related to ω computed for $t = 1,\ z = 0$ by

$$\omega' = A^{-1}\omega A + A^{-1}\mathrm{d}A. \tag{16}$$

Here A is a matrix belonging to the group $\mathbf{SU}(2,\ 1)$ and is expressed in terms of t and z as follows

$$A = \begin{pmatrix} |t|e^{i\theta} & i\bar{z}e^{-2i\theta} & -\frac{e^{i\theta}}{4|t|}(\frac{1}{2}\varrho + i|z|^2) \\ 0 & e^{-2i\theta} & -\frac{e^{i\theta}}{2|t|}z \\ 0 & 0 & \frac{e^{i\theta}}{|t|} \end{pmatrix}, \tag{17}$$

where θ is defined by

$$t = |t|e^{3i\theta}.$$

The curvature R of ω is defined by

$$R = \mathrm{d}\omega + \omega \wedge \omega; \tag{18}$$

R' computed for arbitrary t and z of (13) is related to R computed for $t = 1,\ z = 0$ by

$$R' = A^{-1}RA. \tag{19}$$

These transformations show, in particular, that the function $\mathcal{R}'$ of (15) computed for arbitrary t and z is related to $\mathcal{R}$ computed for $t = 1,\ z = 0$ by

$$\mathcal{R}' = \frac{1}{|t|^4}e^{6i\theta}\mathcal{R}. \tag{20}$$

Condition $\mathcal{R}$=0 is invariant under transformations (13) and there is only one (up to equivalence) CR-structure which this satisfies. This is a CR-structure (locally) equivalent to $\mathbf{S}^3 \in \mathbf{C}^2$. For this CR-structure curvature R vanishes and one can always transform the forms $\Omega,\ \Omega_1$ to

$$\Omega = du - \frac{i}{2}\bar{\xi}\mathrm{d}\xi + \frac{i}{2}\xi\mathrm{d}\bar{\xi}, \qquad \Omega_1 = \mathrm{d}\xi \tag{21}$$

in a properly chosen chart $(u,\ \xi,\ \bar{\xi})$.
If $\mathcal{R} \neq 0$ then by means of transformations (20) we achieve

$$\mathcal{R} = 1. \tag{22}$$

To restrict entirely the freedom in choosing Ω and Ω_1 one introduces additional condition

$$\mathrm{d}\Omega = i\Omega_1 \wedge \bar{\Omega}_1. \tag{23}$$

Conditions (22-23) determine Ω and Ω_1 up to the freedom

$$\Omega \to \Omega, \qquad\qquad \Omega_1 \to \pm\Omega_1. \tag{24}$$

They uniquely determine z and specify t up to a sign. For these preferred z and t we compute Ω_2, Ω_3 and Ω_4 which together with already determined Ω and Ω_1 constitute a system of all *(Cartan) invariant forms* of a nondegenerate 3-dimensional CR-structure.
Note that since $\lambda,\ \mu$ satisfy (6) then $(\Omega,\ \Omega_1,\ \bar{\Omega}_1)$ constitute a basis of 1-forms on $\mathcal{Q}$.
In the case $\mathcal{R} \neq 0$ we define *invariant functions* $\alpha,\ \beta,\ \gamma,\ \theta,\ \xi$ by

$$\begin{aligned}
\Omega_2 &= \alpha\Omega_1 - \bar{\alpha}\bar{\Omega}_1 + i\beta\Omega,\\
\Omega_3 &= i\gamma\Omega_1 + \vartheta\bar{\Omega}_1 + \eta\Omega,\\
\Omega_4 &= -\frac{i}{2}\bar{\eta}\Omega_1 + \frac{i}{2}\eta\bar{\Omega}_1 + \xi\Omega.
\end{aligned} \tag{25}$$

This definition implies, in particular, that

$$\mathrm{d}\Omega_1 = \bar{\alpha}\Omega_1 \wedge \bar{\Omega}_1 + i(\beta - \gamma)\Omega \wedge \Omega_1 - \vartheta\Omega \wedge \bar{\Omega}_1. \tag{26}$$

Note that if $\mathcal{R} = 0$ then forms (21) also satisfy (23). We prolong definitions of Cartan invariants to this case by saying that Ω and Ω_1 are given by (21) and all other invariants like $\alpha,\ \beta,\ \gamma,\ \vartheta,\ \xi,\ \Omega_2,\ \Omega_3,\ \Omega_4$ vanish.

2.1.4 Symmetries.

A diffeomorphism $\phi : \mathcal{Q} \to \mathcal{Q}$ is called a *symmetry* of a CR-structure iff the pullbacks $\phi^*\lambda,\ \phi^*\mu$ are related to the forms $\lambda,\ \mu$ by the transformations (7). We say that a real vector field X on $\mathcal{Q}$ is an *infinitesimal symmetry* iff

$$\mathcal{L}_X\lambda = a\lambda, \qquad \mathcal{L}_X\mu = b\lambda + c\mu, \tag{27}$$

where a is a real function and $b,\ c$ are complex functions. X can be equivalently defined in terms of an operator ∂ by

$$[X,\ \partial] = t\partial, \tag{28}$$

where t is some complex function on $\mathcal{Q}$.

It is worth noting that if a nondegenerate CR-structure admit an infinitesimal symmetry X then it follows from the construction of Cartan invariant forms that

$$\mathcal{L}_X\Omega = \mathcal{L}_X\Omega_i = 0 \quad (i = 1,\ 2,\ 3,\ 4) \tag{29}$$

and

$$\mathcal{L}_X\varsigma = 0, \quad \text{where} \quad \varsigma = \alpha,\ \beta,\ \gamma,\ \vartheta \text{ or } \xi. \tag{30}$$

Given a nondegenerate CR-structure we will not always need forms Ω and Ω_1 of definition (13) in the Cartan representation satisfying both conditions (22), (23). Therefore from now on we adopt the following *convention*: Ω and Ω_1 denote always forms (13) which satisfy condition (23). [3] Functions $\alpha,\ \beta - \gamma,\ \vartheta$ are then defined by equation (26). Sometimes, (when explicitly stated) some additional conditions are imposed on these forms. This can, for example, be a condition that these forms are Cartan invariant forms; if a nondegenerate CR-structure admits an infinitesimal symmetry X this additional condition can require that Ω and Ω_1 are invariant when transported along X.

The basic results in the field of symmetries of 3-dimensional CR-structures where obtained by E. Cartan in [5]. In collaboration with Jacek Tafel, [37], we extended his results to the nontransitive (local) groups of symmetries. We also presented the forms $\lambda,\ \mu$ in terms of canonical coordinates in both transitive and nontransitive cases.
A CR-structure can have no symmetries; if it has one, it is always realizable [37]. In

[3] Note that this condition can be imposed independently of (22).

the nondegenerate case the symmetry group is at most 8-dimensional [58]. There is only one (up to the local equivalence) nondegenerate CR-structure with a symmetry group of the highest dimension. This is a CR-structure corresponding to a surface of the 3-dimensional sphere $\mathbf{S}^3 \in \mathbf{C}^2$. Any symmetric CR-structure with a symmetry group G of dimension $D \geq 4$ is locally equivalent to that corresponding to the sphere; hence G is not maximal in such cases [5].

If a CR-structure $\mathcal{Q}$ possesses an infinitesimal symmetry X then there exist coordinates u (real) and ξ, $\bar{\xi}$ on $\mathcal{Q}$ such that $X = \partial_u$ and the CR-structure is equivalent to that defined by forms [37]

$$\mu = \mathrm{d}\xi \quad \text{and} \quad \lambda = \mathrm{d}u + L\mathrm{d}\xi + \bar{L}\mathrm{d}\bar{\xi}, \tag{31}$$

where

$$L = -i\partial_\xi \sigma, \qquad X(\sigma) = 0$$

and σ is a real function such that

$$\tau = 2\partial_{\bar{\xi}}\partial_\xi \sigma > 0.$$

If a CR-structure $\mathcal{Q}$ possesses two infinitesimal symmetries X_1 and X_2 then either they commute or there exists such their linear combination X_1', X_2' that $[X_1',\ X_2'] = X_1'$. Here we concentrate on the commutative case in which coordinates u, $\xi = x + iy$ can be introduced on $\mathcal{Q}$ such that $X_1 = \partial_u$ and $X_2 = \partial_x$ and the the CR-structure can be represented by (31) where now

$$L = -\frac{1}{2}\partial_y \sigma, \qquad \sigma = \sigma(y), \qquad \tau = \frac{1}{2}\partial_y^2 \sigma > 0.$$

Classification of nondegenerate CR-structures with at least 3 infinitesimal symmetries is as follows [37].
One proves that any *finite dimensional* Lie group of dimension $D \geq 3$ includes locally a 3-dimensional subgroup (say G). Given a group with $D > 3$ chose its 3-dimensional subgroup to reduce the problem to the case $D = 3$. The action of this 3-dimensional subgroup G on $\mathcal{Q}$ is locally simply transitive and the forms λ, μ can be transformed to the left invariant forms on G. By virtue of the Bianchi classification [3] of 3-dimensional Lie groups the whole problem reduces to the problem of finding all nonequivalent left invariant frames on these groups. Reduction of parameters is obtained by (i) transformations (7) with constant coefficients and by (ii) linear

transformations of the left invariant frames preserving the structure constants of G. The final result - the classification of nondegenerate homogeneous 3-dimensional CR-structures in terms of the Bianchi type of G is given below. In this classification the CR-structure is parameterized by real coordinates $(u,\ x,\ y)$ and we introduce complex variable ξ by $\xi = x + iy$. Classes of forms $[(\lambda,\ \mu)]$ are generated from left invariant forms λ and μ listed below by (7). We also give forms Ω and Ω_1 which belongs to the class generated by λ and μ, respectively. They are such linear combinations with constant coefficients of λ and μ that (23) is satisfied.

Bianchi type II.

$$\lambda = \Omega = \mathrm{d}u - \frac{i}{2}\bar{\xi}\mathrm{d}\xi + \frac{i}{2}\xi\mathrm{d}\bar{\xi} \qquad \mu = \Omega_1 = \mathrm{d}\xi.$$

Bianchi type IV.

$$\lambda = y^{-1}(\mathrm{d}u - \log y\mathrm{d}x) \qquad \mu = y^{-1}\mathrm{d}\xi$$

$$\Omega = 2\lambda \qquad \Omega_1 = \mu + \lambda$$

Bianchi type $VI_{h'}$.

$$\lambda = y^{d}\mathrm{d}u - y^{-1}\mathrm{d}x \qquad \mu = y^{-1}\mathrm{d}\xi$$

$$\Omega = -\frac{2}{d+1}\lambda \qquad \Omega_1 = \mu + \frac{d}{d+1}\lambda,$$

where $h' \leq 0$ and $d = \frac{1-\sqrt{-h'}}{1+\sqrt{-h'}}$.

Bianchi type $VII_{h'}$.

$$\lambda = \mathrm{d}u + \exp[(i+A)u]\mathrm{d}\xi + \exp[(A-i)u]\mathrm{d}\bar{\xi} \qquad \mu = \exp[(A+i)u]\mathrm{d}\xi$$

$$\Omega = \frac{1}{2}\lambda \qquad \Omega_1 = \mu + \frac{A-i}{2i}\lambda,$$

where $h' \geq 0$ and $A = \sqrt{h'}$.

Bianchi type IX (upper signs) and $VIII$ (lower signs).

$$\lambda = \mathrm{d}u + \frac{\kappa e^{iu} - i\bar{\xi}}{\xi\bar{\xi} \pm 1}\mathrm{d}\xi + \frac{\kappa e^{-iu} + i\xi}{\xi\bar{\xi} \pm 1}\mathrm{d}\bar{\xi} \qquad \mu = \frac{2e^{iu}}{\xi\bar{\xi} \pm 1}\mathrm{d}\xi$$

$$\Omega = \frac{2}{\kappa^2 \pm 1}\lambda \qquad \Omega_1 = \mu - \frac{\kappa}{\kappa^2 \pm 1}\lambda,$$

where $\kappa \geq 0$ and $\kappa \neq 1$ for type $VIII$.

We note that
a) there are no nondegenerate CR-structures with symmetry groups of Bianchi types I and V;
b) there are unique nondegenerate CR-structures with symmetries of Bianchi types II, IV, $VI_{h'}$ and $VII_{h'}$ (for any value of the parameter h');
c) there are one-parameter (with parameter κ) families of nondegenerate CR-structures with symmetry groups of Bianchi types $VIII$ and IX;
d) all structures from the above list are nonequivalent except those of Bianchi types II, VI_{-1}, VI_{-9}, $VIII$ (with $\kappa=\sqrt{2}$) and IX (with $\kappa=0$); only in these cases the group G can be extended to the bigger symmetry group; in such cases the maximal symmetry group is *always* 8-dimensional group isomorphic to $\mathbf{SU}(2,\ 1)$, hence all these structures are equivalent to the sphere $\mathbf{S}^3\in\mathbf{C}^2$.

2.1.5 3-dimensional CR-structures and congruences of null geodesics without shear. Lifting: (CR-structure)→(Lorentzian 4-manifold).

Due to Theorem 1.4.1 any (even degenerate) 3-dimensional CR-structure defines an optical geometry in $\mathcal{M}=\mathbf{R}\times\mathcal{Q}$. The class of metrics adapted to this optical geometry is given by

$$g=p^2(\mu\bar{\mu}-\lambda\varphi), \tag{32}$$

where $[(\lambda,\ \mu)]$ defines a CR structure on $\mathcal{Q}$, φ is any real 1-form, and $p\neq 0$ is any real function on $\mathcal{M}$ provided that g is nondegenerate. Note that we used the same letters in denoting forms on $\mathcal{Q}$ and their natural pullbacks to $\mathbf{R}\times\mathcal{Q}$. A congruence of lines defined on $\mathcal{M}=\mathbf{R}\times\mathcal{Q}$ by a real nonvanishing vector field k such that

$$k\rfloor\lambda=k\rfloor\mu=0 \tag{33}$$

is *geodesic* and *shear-free* [52] in any adapted metric (32). The congruence is *twisting* iff the associated CR-structure is nondegenerate. Any 3-surface $\mathcal{Q}'$ in $\mathcal{M}$ transversal to k becomes naturally a CR-structure. The shear-free, geodesic and null property of the congruence imply that CR-structures on all others 3-surfaces transversal to k are equivalent to that of $\mathcal{Q}'$. Therefore in 4 dimensions, due to Theorem 1.4.1, we have a correspondence: *shear-free congruences of null geodesics* $\longleftrightarrow$ *3-dimensional CR-structures*. In this correspondence $\mathcal{S}^3\in\mathbf{C}^2$ is related to the so called *Robinson congruence* [40]. The procedure of associating a Lorentzian metric (32) with a given CR-structure is called a *lifting of the CR-structure to the Lorentzian 4-manifold*. It is

an interesting question which CR-structures can be lifted to e.g. Ricci flat Lorentzian manifolds. In the case of lifting to the Minkowski space a partial answer to this is given by the Kerr theorem. This is not very much satisfactory since it is not invariant under transformations (7).
Note that the form (32) of adapted metrics is invariant under transformations (7); therefore there are many different *Newman-Penrose (NP)* [31] *tetrads*

$$(p\mu,\ p\bar{\mu},\ p\lambda,\ p\varphi)$$

on $\mathbf{R}\times\mathcal{Q}$ in which the metric g has the form (32). If g is given by (32) and the CR-structure is nondegenerate we define a *preferred* NP tetrad on $\mathbf{R}\times\mathcal{Q}$ by the requirement that $(\lambda,\ \mu)$ in (32) is identical with $(\Omega,\ \Omega_1)$. Note that if, in addition, $(\Omega,\ \Omega_1)$ are Cartan invariant forms then so determined NP tetrad is *uniquely* defined by g.

2.1.6 Fefferman metrics.

For nondegenerate CR-structures there is a conformal subclass of adapted metrics (32) of particular interest. This is the *Fefferman conformal class* [12] generated by a metric

$$g_F = 2\{\ \Omega_1\bar{\Omega}_1 - \Omega(\ dr + \frac{2i}{3}\Omega_2\ )\}, \tag{34}$$

where Ωs are natural pullbacks from $\mathcal{Q}$ to $\mathcal{M}$ of forms defined by (15) with t, z determined by (23), and r is a real coordinate along factor $\mathbf{R}$ in $\mathbf{R}\times\mathcal{Q}$. Metric g_F transforms conformally under transformations (13) preserving (23).
The Fefferman class of metrics can be defined for a CR-structure $\mathcal{Q}$ of any dimension [12], and can be characterized by a properties of the congruence defined on the fibration $\mathcal{M}=\mathbf{S}^1\times\mathcal{Q}$ over $\mathcal{Q}$ by a direction of a vector field tangent to the fibers [60], [22], [26]. Such metrics are also relevant in the twistor theory [43], [45] since they are the only metrics admitting *twisting* congruence of null geodesics which are compatible with existence of solutions to the *twistor equation* [24].

2.2 Pure radiation with twist - Einstein equations in the preferred Newman-Penrose tetrad.

In this section we consider Einstein equations

$$R_{\mu\nu} = \Phi k_\mu k_\nu, \tag{35}$$

in which k_μ generates a *twisting* shear-free congruence of null geodesics and Φ is some real function. Any manifold admitting such congruence is locally equivalent to $\mathcal{M} = \mathbf{R}\times\mathcal{Q}$ with some *nondegenerate* CR-structure $\mathcal{Q}$ and any Lorentzian metric in which this congruence is null geodesic and shear-free has the form

$$g = 2\mathcal{P}^2(\ \Omega_1\bar{\Omega}_1 - \Omega\varphi\ ), \tag{36}$$

where $\Omega,\ \Omega_1$ are pullbacks to $\mathcal{M}$ of natural forms associated with $\mathcal{Q}$, as defined in Section 2.1.6. In (36) $\mathcal{P}$ is any nonvanishing real function on $\mathcal{M}$ and φ is any real 1-form on $\mathcal{M}$ provided that the metric is nondegenerate. Tetrad $(\theta^1,\ \theta^2,\ \theta^3,\ \theta^4) = (\mathcal{P}\Omega_1,\ \mathcal{P}\bar{\Omega}_1,\ \mathcal{P}\Omega,\ \mathcal{P}\varphi)$ is the preferred NP tetrad on $\mathcal{M}$ as discussed in Section 2.1.6. A coordinate r on $\mathcal{M}$ may be chosen such that

$$\varphi = \mathrm{d}r + W\Omega_1 + \bar{W}\bar{\Omega}_1 + H\Omega \quad \text{and} \quad k = \partial_r\ . \tag{37}$$

Forms $(\Omega_1,\ \bar{\Omega}_1,\Omega)$ initially defined on $\mathcal{Q}$ define there dual vector fields $(\partial,\ \bar{\partial},\ \partial_0)$, respectively. We prolong these to $\mathcal{M}$ by the requirement that

$$\partial(r) = \partial_0(r) = 0.$$

Tetrad (e_μ) on $\mathcal{M}$ dual to (θ^μ) is

$$(e_1,\ e_2,\ e_3,\ e_4) = \frac{1}{\mathcal{P}}(\partial - W\partial_r,\ \bar{\partial} - \bar{W}\partial_r,\ \partial_0 - H\partial_r,\ k),$$

where now operators $(\partial,\ \bar{\partial},\ \partial_0)$ are vector fields on $\mathcal{M}$.
Einstein equations (35) written in this tetrad can be partially integrated leading to the metric g in which all the r-dependence is determined. The metric (36) is expressible in terms of unknown functions $p\neq 0,\ s,\ h$ (all real) and $a,\ b,\ \mathcal{C},\ G,\ F$ and f (all complex) as follows [25].

$$\begin{aligned} \mathcal{P} &= \frac{p}{cos[\frac{r+s}{2}]}, \\ W &= 2aie^{ir} + b, \\ H &= Ge^{2ir} + \bar{G}^{-2ir} + Fe^{ir} + \bar{F}e^{-ir} + h. \end{aligned} \tag{38}$$

The unknown functions are subject to the following (Einstein) equations [25].

$$k(p) = k(s) = k(h) = k(a) = k(b) = k(\mathcal{C}) = k(G) = k(F) = k(f) = 0, \tag{39}$$

$$
\begin{aligned}
&a = [\frac{i}{4}(\partial s - b) - \frac{1}{2}\partial\ \log p]e^{is}, && (40)\\
&\mathcal{C} = \partial\ \log p + ae^{-is}, && (41)\\
&\frac{1}{2}(\partial + \alpha)\mathcal{C} - \mathcal{C}^2 + \frac{i}{4}\bar{\vartheta} = 0, && (42)\\
&h + \frac{1}{2}(\Delta\bar{\alpha} + \bar{\Delta}\alpha) + \frac{\beta - \gamma}{2} + 6a\bar{a} + i(\bar{\Delta}b - \Delta\bar{b}) = 0, && (43)\\
&2G = (F - f)e^{is}, && (44)\\
&f = (\bar{\Delta} - i\bar{b})a, && (45)\\
&(F + f)e^{-is} + (\bar{F} + \bar{f})e^{is} - 8\mathcal{C}\bar{\mathcal{C}} + 2(\partial_0 s - h) = 0, && (46)\\
&(\frac{1}{2}\partial + ib + \mathcal{C})\bar{G} = 0. && (47)
\end{aligned}
$$

Here α, $(\beta - \gamma)$ and ϑ are functions defined by (26) and we introduced operator

$$\Delta = \partial - \alpha. \tag{48}$$

Note that due to (39) all unknown functions are independent on r. Hence our equations can be considered as equations on $\mathcal{Q}$ with $(\partial,\ \bar{\partial},\ \partial_0)$ as operators on $\mathcal{Q}$.

Not all unknown functions in equations (39-47) are independent. By *algebraic* elimination of dependent variables equations (39-47) can be simplified very much. This is done in Section 2.4. Here we remain with this form of equations since this is convenient to find solutions with conformal symmetries.

The Weyl tensor $C^{\mu}{}_{\nu\varrho\sigma}$ of g satisfying Einstein equations (35) is expressible in tetrad (θ^{μ}) in terms of the unknown functions as follows [25].

$$
\begin{aligned}
&C_{1214} = C_{1414} = 0, && (49)\\
&C_{3434} = -\frac{2}{\mathcal{P}^2}[Ge^{ir}(e^{ir} + e^{-is}) + \bar{G}e^{-ir}(e^{-ir} + e^{is})], && (50)\\
&C_{1324} = \frac{2}{\mathcal{P}^2}[Ge^{ir}(e^{ir} + e^{-is})], && (51)\\
&C_{1234} = \frac{2}{\mathcal{P}^2}[Ge^{ir}(e^{ir} + e^{-is}) - \bar{G}e^{-ir}(e^{-ir} + e^{is})], && (52)\\
&C_{1334} = \frac{1}{4\mathcal{P}^2}[U + Ye^{ir} + Xe^{2ir} - 24aiGe^{3ir}], && (53)\\
&C_{1313} = \frac{1}{\mathcal{P}^2}[12a^2Ge^{4ir} + Z_3e^{3ir} + Z_2e^{2ir} + Z_{-2}e^{-2ir} +
\end{aligned}
$$

$$+Z_1e^{ir} + Z_{-1}e^{-ir} + Z_0], \tag{54}$$

where we introduced

$$X = -8i(\partial - 2ib + ae^{-is})G, \tag{55}$$

$$Y = -8\delta a + 2i\partial(f - 3F) + 2b(f - 3F) + 8iha, \tag{56}$$

$$U = 4\bar{\alpha}\bar{\vartheta} - 2\bar{\partial}\bar{\vartheta} + 3i(\delta b - \bar{\vartheta}\bar{b}) + \partial(\bar{\Delta}b - \Delta\bar{b} - 4ih) + +8i[\frac{a}{2}(3\bar{F} - \bar{f}) - \partial(a\bar{a})] \tag{57}$$

with

$$\delta = \partial_0 + i(\beta - \gamma); \tag{58}$$

all functions Z_i from eq. (54) satisfy $k(Z_i) = 0$. If

$$G = 0 \tag{59}$$

they are defined by

$$Z_3 = Z_{-2} = 0, \tag{60}$$

$$Z_2 = \frac{i}{2}aY, \tag{61}$$

$$Z_1 = \frac{i}{4}(\partial - ib + \alpha)Y, \tag{62}$$

$$Z_{-1} = (\partial + ib + \alpha)[(\partial + ib)\bar{F} - 2\bar{a}i\bar{\vartheta}] - 2if\bar{f}\bar{\vartheta}, \tag{63}$$

$$Z_0 = (\partial + \alpha)(\partial h - 4a\bar{F} - \delta b + \bar{b}\bar{\vartheta}) - 2a[(\partial + ib)\bar{F} - 2\bar{a}i\bar{\vartheta}] + +[\delta + i(\beta - \gamma) + ih + \Delta\bar{b} - \bar{\Delta}b + 8a\bar{a}i]\bar{\vartheta}. \tag{64}$$

All other components of the Weyl tensor are algebraically expressible in terms of the above.
We note that the metric g satisfying (35) is *never* algebraically general i.e. is at most of Petrov type II. It is of type III iff $G = 0$, and is of type N iff

$$G = U = X = Y = 0. \tag{65}$$

In the case of type N the Einstein equations (35) imply that the metric is *necessarily* vacuum ([20] Theorem ????). Hence in searching for *vacuum* types N it is enough to consider *only* equations (39-47) and (65), and *forget* about the last vacuum Einstein equation $\Phi = 0$.

Note also that not all conditions (65) are independent. This is because equation (47) is equivalent to

$$Ue^{is} - Y + Xe^{-is} + 24aiGe^{-2is} = 0 \tag{66}$$

if equations (39-47) hold.

It turns out that, without loss of generality, we can put $s = 0$ in the equations of this section; the function s has no geometrical meaning. It can be eliminated from the equations by a change of r coordinate through

$$r \to r + s \tag{67}$$

and by appropriate redefinitions of the unknown functions and operators $(\partial,\ \bar{\partial},\ \partial_0)$. If, however, we impose some additional conditions on the coordinate r in its very definition (37), then the freedom (67) in the choice of r can be restricted. In such cases it may be impossible to eliminate s from the equations and it *has* to appear.

2.3 Metrics with at least three conformal symmetries. All twisting vacuum and pure radiation solutions in this case.

An *infinitesimal conformal symmetry* is defined as a real vector field X on $(\mathcal{M},\ g)$ such that

$$\mathcal{L}_X g = \phi g, \tag{68}$$

where ϕ is some real function on $\mathcal{M}$. The commutator of infinitesimal conformal symmetries is also an infinitesimal conformal symmetry; hence all such symmetries form a Lie algebra. In this section we assume that the metric g of (36), (39), (38) admits at least 3 infinitesimal conformal symmetries. Under this assumption we find a general solution to the Einstein equations (39-47).

Theorem 2.3.1 *Suppose that the nonflat Lorentzian manifold $(\mathcal{M},\ g)$:*
1) admits twisting shear-free congruence of null geodesics tangent to the vector field k,
2) possesses at least 3 infinitesimal conformal symmetries that form a Lie algebra A,
3) satisfies the Einstein equations $R_{\mu\nu} = \Phi k_\mu k_\nu$.
Then the CR-structure $\mathcal{Q}$ associated to $\mathcal{M}$ is homogeneous and the algebra of its infinitesimal symmetries is isomorphic to an appropriate Lie subalgebra of A.

A sketch of the proof is as follows. One shows that an infinitesimal conformal symmetry X preserves the congruence. This implies that $\mathcal{L}_k X$ is parallel to k, hence the

projection $\tilde{X}$ of X onto $\mathcal{Q}$ is uniquely defined. It may happen that $\tilde{X} \equiv 0$. However, this is only possible if X is parallel to k. We exclude such situations since any metric compatible with the Einstein equations (35) and the existence of the infinitesimal conformal symmetry X parallel to k is flat [25]. Therefore all projections $\tilde{X}$ of X nowhere vanish. Since any X preserves the CR-structure it generates the symmetry of $\mathcal{Q}$; furthermore X is related to $\tilde{X}$ by $X = \tilde{X} + fk$ with some real function f on $\mathcal{M}$. By virtue of the freedom we have in choosing 3-surface for the CR-structure in $\mathcal{M}$ we can assume that all X's are tangent to $\mathcal{Q}$; hence, in particular, $X \equiv \tilde{X}$ for all X's. Since the dimension n of the Lie algebra A is $n \geq 3$ then the CR-structure has no less then three infinitesimal symmetries and, as such, must be homogeneous.
Thus the CR-structure $\mathcal{Q}$ in $\mathcal{M}$ is a homogeneous CR-structures as listed in the classification of homogeneous CR-structures given in Section 2.1.6; the metric g can be represented by (36) with forms Ω, Ω_1 as defined in this classification. One can always find a 3-dimensional Lie subalgebra A_3 of A and its three generators, say X_1, X_2, X_3, for which

$$\mathcal{L}_{X_i}\Omega = \mathcal{L}_{X_i}\Omega_1 = 0 \quad \text{for} \quad i = 1,\ 2,\ 3. \tag{69}$$

A coordinate r can be chosen in such a way that for any of X_is equation

$$\mathcal{L}_{X_i} r = 0 \tag{70}$$

is satisfied. Equations (70), definition (37) of φ and the condition that all X_is are infinitesimal conformal symmetries imply that the functions W and H of (37) satisfy

$$\mathcal{L}_{X_i} W = \mathcal{L}_{X_i} H = 0, \qquad i = 1,\ 2,\ 3. \tag{71}$$

Imposing the Einstein equations (35) on g we find that the metric functions $\mathcal{P}$, W and H are expressible in terms of variables p, s, a, b, G, F and h defined by (38-39) on $\mathcal{Q}$. Let G_3 will be the local group generated by A_3 on $\mathcal{Q}$. Its action is transitive there. This compared with equations (71) imply that a, b, G, F, h are all *constant*. One checks that for all homogeneous CR-structures the functions α, $\beta - \gamma$, ϑ defined by (26) are also *constant*; therefore the only *a priori* nonconstant functions in the remaining Einstein equations (40-47) are p, $\mathcal{C}$ and s. Note that now (70) restricts the freedom (67) in the choice of r to (67) with *constant* s. Therefore this function can not be *a priori* removed from the equations. One proves [25], however, that if $s \neq const$ then the equations imply that the corresponding metric is flat. Thus, to obtain nonflat metrics we are free to put $s = const$. Then by means of transformation (67) with constant s we eliminate this variable from the equations. The integrability conditions for the remaining nonconstant functions p and $\mathcal{C}$ follow from the commutators

$$[\bar{\partial},\ \partial] = i\partial_0 + \bar{\alpha}\partial - \alpha\bar{\partial}, \qquad [\partial,\ \partial_0] = i(\beta - \gamma)\partial - \bar{\vartheta}\bar{\partial}, \tag{72}$$

which are equivalent to (23), (26). These conditions read

$$\partial_0 \log p = \frac{1}{2}[\bar{\alpha}(b - 4ia) + \alpha(\bar{b} + 4i\bar{a})], \tag{73}$$

$$i(\beta - \gamma)(\mathcal{C} - a) - \bar{\vartheta}(\bar{\mathcal{C}} - \bar{a}) = 0, \tag{74}$$

where we have used the fact that all functions but p and $\mathcal{C}$ are constants.
The system of equations (40-47), (73-74) with $s = 0$ and constant functions a, b, F, G, h can be solved for any homogeneous CR-structure. In Ref. [25] we found which homogeneous CR-structures are admitted by equations (40-47), (73-74). These are CR-structures of Bianchi types II, $VI_{h'}$, $VII_{h'}$, $VIII$ (with $\kappa = 0$) and IX (with $\kappa = 0$). Below we list only solutions which satisfy the positive energy condition $\Phi \geq 0$, where Φ is the function which appears in (35). The spacetime is $\mathcal{M} = \mathbf{R} \times \mathcal{Q}$, the metric is

$$g = \frac{p^2}{\cos^2\frac{r}{2}}[\ \Omega_1\bar{\Omega}_1 - \Omega(dr + W\Omega_1 + \bar{W}\bar{\Omega}_1 + H\Omega)\ ], \tag{75}$$

with

$$W = 2iae^{ir} + b, \tag{76}$$

$$H = Ge^{2ir} + \bar{G}e^{-2ir} + [2G - (\bar{\alpha} + i\bar{b})a]e^{ir} + [2\bar{G} - (\alpha - ib)\bar{a}]e^{-ir} + h, \tag{77}$$

$$h = -6a\bar{a} + \alpha\bar{\alpha} - \frac{\beta - \gamma}{2} + i(\bar{\alpha}b - \alpha\bar{b}), \tag{78}$$

$$2\mathrm{Re}G = h + a(\bar{\alpha} + i\bar{b}) + \bar{a}(\alpha - ib) + 4\mathcal{C}\bar{\mathcal{C}}. \tag{79}$$

$\mathrm{Im}G$ can be any real constant; all other variables appearing above are given below according to the Bianchi type of the corresponding $\mathcal{Q}$. $\mathcal{Q}$ is parameterized by coordinates u (real), $\xi = x + iy$ and $\bar{\xi}$ (complex).

Bianchi type II.

$$\Omega_1 = \mathrm{d}\xi, \qquad \Omega = \mathrm{d}u - \frac{i}{2}\bar{\xi}\mathrm{d}\xi + \frac{i}{2}\xi\mathrm{d}\bar{\xi};$$

$$\alpha = \beta - \gamma = \vartheta = a = b = \mathcal{C} = 0, \quad p = const.$$

This is a Petrov type D *vacuum* solution. It has three conformal symmetries generated by

$$X_1 = \partial_\xi + \partial_{\bar{\xi}} + \frac{i}{2}(\xi - \bar{\xi})\partial_u \qquad X_2 = i(\partial_\xi - \partial\bar{\xi}) + \frac{1}{2}(\xi + \bar{\xi})\partial_u$$

$$X_3 = \partial_u.$$

Bianchi types IX (upper signs) and VIII (lower signs).

$$\Omega_1 = \frac{2e^{iu}}{\xi\bar{\xi} \pm 1}\mathrm{d}\xi \qquad \Omega = \pm 2[\mathrm{d}u + \frac{i}{\xi\bar{\xi} \pm 1}(\xi \mathrm{d}\bar{\xi} - \bar{\xi}\mathrm{d}\xi)]$$

$$\alpha = \vartheta = 0 \quad \beta - \gamma = \pm\frac{1}{2} \quad a = b = \mathcal{C} = 0 \quad p = const.$$

This is a Petrov type D *vacuum* solution. Symmetry transformations generated by X_i are

$$u' = u - i\log(\bar{q}\xi + \bar{w}) + i\log(q\bar{\xi} + w) \qquad \xi' = \frac{w\xi \mp q}{\bar{q}\xi + \bar{w}},$$

where w, q are complex constants such that $w\bar{w} \pm \bar{q}q = 1$.

Bianchi types $VI_{h'}$.

$$\Omega_1 = y^{-1}\mathrm{d}\xi + \frac{d}{d+1}(y^d\mathrm{d}u - y^{-1}\mathrm{d}x) \qquad \Omega = -\frac{2}{d+1}(y^d\mathrm{d}u - y^{-1}\mathrm{d}x),$$

$$\alpha = -\frac{i(1-d)}{2} \quad \beta - \gamma = -\frac{d}{4} \quad \vartheta = -\frac{id}{4},$$

where we introduced

$$d = \frac{1 - \sqrt{-h'}}{1 + \sqrt{-h'}}, \quad h' \leq 0.$$

There are two possibilities:
(i) $G \neq 0$
and
(ii) $G = 0$.
In case (i) we have again to possibilities of solutions. These are:
a) $a = \frac{i}{8} \quad b = \frac{1}{4} \quad \mathcal{C} = -\frac{i}{4} \quad p = \mathrm{const}\ y^{\frac{3}{4}}$;
in this case $\Phi \geq 0$ iff $-\frac{1}{4} \leq d \leq 0$ or $d \geq \frac{1}{2}$.
b) $a = -\frac{id}{8} \quad b = -\frac{d}{4} \quad \mathcal{C} = -\frac{id}{4} \quad p = \mathrm{const}\ y^{\frac{-3}{4d}}$;
in this case $\Phi \geq 0$ iff $d \geq 0$.

In case (ii) we have:

$$a = \frac{i}{4}[-d \pm \sqrt{d(3d+1)}] \quad b = \frac{1}{2}[1 + d \mp \sqrt{d(3d+1)}]$$
$$\mathcal{C} = -\frac{i}{4} \quad p = \text{const } y^{\frac{1-d\pm\sqrt{d(3d+1)}}{2}};$$

- in the upper sign case we have a nonflat vacuum metric for $d = \frac{-1\pm\sqrt{13}}{6}$, and $\Phi > 0$ for $-\frac{1+\sqrt{13}}{6} < d < -\frac{1}{2}$ or $d > -\frac{1-\sqrt{13}}{6}$;
- in the lower sign case we have $\Phi > 0$ iff $-\frac{1}{2} < d < -\frac{1}{3}$; $\Phi = 0$ only for flat metrics.

Bianchi type $VI_{h'}$ solutions are of Petrov type D only in case (i) *b*) for $d = 0$. This solution is vacuum, and coincide with vacuum solutions for Bianchi type $VIII$. All other solutions in (i) case are of Petrov type II. In case (ii) all solutions are of Petrov type III.
Conformal symmetries of the metric are generated by

$$X_1 = \partial_x \quad X_2 = \partial_u \quad X_3 = x\partial_x + y\partial_y - ud \cdot \partial_u$$

in all cases of Bianchi type $VI_{h'}$ solutions.

The above listed solutions give *all* metrics which:
1) admit twisting congruence of shear-free and null geodesics;
2) admit at least three infinitesimal conformal symmetries;
3) satisfy pure radiation Einstein equations (35) with k generating the congruence.
We summarize the results as follows.
All vacuum solutions of Bianchi type II, $VI_{h'}$ [case (i) *b*) $d = 0$] $VIII$ and IX describe Taub-NUT metrics [64], [32]. For these solutions infinitesimal conformal symmetries are simultaneously Killing vectors and can be extended to form a 4-dimensional algebra of Killing symmetries. Other vacuum solutions appearing above are known (see [20] §25.2.1-25.2.4).
Among solutions with $\Phi \neq 0$ there are new solutions. These are, for example, all quoted solutions of Petrov type III. Most of other solutions with $\Phi \neq 0$ are also new.
It is interesting to note that in all cases none of X_is generates a *proper* conformal transformation. They are either Killing or homotetic vector fields on $\mathcal{M}$ and we have not assumed this. In cases when not all of X_is are Killing fields [Bianchi types $VI_{h'}$, except (i) *b*) $d = 0$] X_3 is homotetic, and two other symmetries are Killing.
Finally we note that our results of this section are in accordance and generalize results of Kerr and Debney [18], who considered vacuum fields.

2.4 Einstein equations for pure radiation with twist as invariant equations on a given CR-structure. Partial integration in terms of solutions to the tangential CR-equation.

In Section 2.2 we wrote pure radiation Einstein equations for spacetimes admitting twisting congruence of shear-free and null geodesics aligned with the wave vector of the radiation. These are eqs. (39-47) for the unknown functions p, s, h, b, G, F, f and $\mathcal{C}$. Due to eqs. (39) all the unknown functions from these equations are independent of the r coordinate. Therefore, equations (40-47) can be understood as defined on a given nondegenerate CR-structure $\mathcal{Q}$ with operators $(\partial,\ \bar{\partial},\ \partial_0)$ understood as operators on $\mathcal{Q}$ which are dual to forms $(\Omega_1,\ \bar{\Omega}_1,\ \Omega)$. These forms are naturally defined on $\mathcal{Q}$ by (23) (and sometimes by some additional requirements) and they define functions $\alpha,\ \beta-\gamma,\ \vartheta$ by (26). In this place one can ask which CR-structures are admitted by equations (40-47). We postpone discussion of this question until we write equations (40-47) in a more convenient form.
For simplicity, we put

$$s = 0 \tag{80}$$

in these equations. This does not restrict generality, as discussed in Section 2.2. We note that among three variables a, p and $\mathcal{C}$ only two are independent. Here we eliminate a in terms of p and $\mathcal{C}$. [4] It can be easily checked that all other unknown functions appearing in equations (40- 47) can be expressed in terms of p, $\mathcal{C}$ and G. It is convenient to introduce a variable

$$M = \bar{G}p^4 \tag{81}$$

instead of G [62]. Then, one checks that the functions which appear in metric (36-38) are expressed in terms of variables p, $\mathcal{C}$ and M as follows ($s = 0$):

$$G = \frac{\bar{M}}{p^4} \tag{82}$$

$$a = \mathcal{C} - \partial \log\ p \tag{83}$$

$$b = 4i\mathcal{C} - 2i\partial \log\ p \tag{84}$$

$$f = (\bar{\Delta} - i\bar{b})a \tag{85}$$

[4] A choice of another pair of variables from those three leads to another equations then presented in the remaining part of this section; it is also interesting to consider such possibilities. They will be discussed elsewhere.

$$F = 2G + f \tag{86}$$
$$h = f + \bar{f} - 4\mathcal{C}\bar{\mathcal{C}}. \tag{87}$$

All Einstein equations (40-47) reduce to the following equations for the unknown variables $\mathcal{C}$, M and p:

$$\partial\mathcal{C} + \alpha\mathcal{C} - 2\mathcal{C}^2 + \frac{i}{2}\bar{\vartheta} = 0 \tag{88}$$
$$(\partial - 6\mathcal{C})M = 0 \tag{89}$$
$$[\bar{\Delta}\partial + \Delta\bar{\partial} + \frac{1}{2}(\Delta\bar{\alpha} + \bar{\Delta}\alpha + \beta - \gamma) - 3(\Delta\bar{\mathcal{C}} + \bar{\Delta}\mathcal{C} + 2\mathcal{C}\bar{\mathcal{C}})]p = -\frac{M + \bar{M}}{p^3} \tag{90}$$

These are equations on a *given* nondegenerate CR-structure $\mathcal{Q}$; operators $(\partial,\ \bar{\partial},\ \partial_0)$ are operators on $\mathcal{Q}$ dual to forms $(\Omega_1,\ \bar{\Omega}_1,\ \Omega)$ defined by (23). Functions α, $\beta - \gamma$, ϑ are defined by (26) and operators Δ and δ by (48) and (58), respectively. The system of equations (88-90) is supplemented by the commutators of operators $(\partial,\ \bar{\partial},\ \partial_0)$. These are

$$[\bar{\partial},\ \partial] = i\partial_0 + \bar{\alpha}\partial - \alpha\bar{\partial}, \qquad [\partial,\ \partial_0] = i(\beta - \gamma)\partial - \bar{\vartheta}\bar{\partial}; \tag{91}$$

they are equivalent to (23), (26).

If Ω and Ω_1 are only constrained by (23) then they are defined up to a freedom

$$\Omega \to \Omega' = t\bar{t}\Omega, \qquad \Omega_1 \to \Omega_1' = t[\ \Omega_1 + i\bar{\partial}\log(t\bar{t})\Omega\ ]. \tag{92}$$

As is to be expected equations (88-90) are invariant under these transformations. This can be seen by considering transformations induced by (92). Introducing

$$z = i\bar{\partial}\log(t\bar{t})$$

we find that the other variables transforms under (92) as follows:

$$\partial' = \frac{1}{t}\partial \tag{93}$$
$$\partial_0' = \frac{1}{t\bar{t}}(\partial_0 - z\partial - \bar{z}\bar{\partial}) \tag{94}$$
$$\bar{\alpha}' = \frac{1}{\bar{t}}(\bar{\alpha} - \bar{\partial}\log(\bar{t}t^2)) \tag{95}$$

$$\vartheta' = \frac{1}{\bar{t}^2}(\vartheta + \bar{\alpha} z + \bar{\partial} z + iz^2) \tag{96}$$

$$|t|^2(\beta - \gamma)' = [\beta - \gamma - \bar{\partial}\partial \log t - \partial\bar{\partial}\log \bar{t} - \alpha\bar{\partial}\log t - \bar{\alpha}\partial \log \bar{t} + \\ +2(\bar{\partial}\log t\partial + \partial \log \bar{t}\bar{\partial})\log |t|] \tag{97}$$

$$p' = \frac{p}{|t|}, \qquad M' = \frac{M}{|t|^6}, \qquad \mathcal{C}' = \frac{1}{t}(\mathcal{C} - \partial \log |t|). \tag{98}$$

In view of these transformations one can catch some flavour of the variable $\mathcal{C}$. One checks that $z = 2i\bar{\mathcal{C}}$ transforms ϑ and $\mathcal{C}$ to $\vartheta' = 0,\ \mathcal{C}' = 0$. However, this can be done only if there exists t i.e. if $\partial \log |t| - \mathcal{C} = 0$. One of the integrability conditions for this is that the CR-structure admits a symmetry. This is because the transformed operators ∂' and ∂_0' satisfy

$$[\partial_0',\ \partial'] = -i(\beta - \gamma)'\partial',$$

so that ∂_0' is a symmetry due to definition (28). This case was thoroughly studied in a different context by Ernst and Hauser [11] who found integrability conditions of vacuum type N equations on CR-structures with a symmetry. Interpretation of $\mathcal{C}$ in the general case is discussed below.

Not all CR-structures are admitted by equation (88). Indeed, if $\mathcal{C}$ satisfies this equation then a 1-form

$$\Pi = \Omega_1 + 2i\bar{\mathcal{C}}\Omega,$$

satisfies

$$\Pi \wedge \mathrm{d}\Pi = 0.$$

This means that there exists a complex function ξ on $\mathcal{Q}$ such that $\Pi = q\mathrm{d}\xi$, with some complex function q on $\mathcal{Q}$. Comparing this with the definition of Π one checks that ξ satisfies equation

$$\mathrm{d}\xi \wedge \Omega \wedge \Omega_1 = 0, \tag{99}$$

where we used fact that $q \neq 0$, which is true due to the definition of Π and to the linear independence of Ω and Ω_1 at each point of $\mathcal{Q}$; for the same reasons $\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi} \neq 0$. Equation (99) is the tangential CR-equation for the CR-structure $\mathcal{Q}$ (compare (9)). Therefore only CR-structures which admit at least one nontrivial solution of the tangential CR-equation are admitted by equation (88). Conversely, a nontrivial solution ξ of the tangential CR-equation (99) imply that forms Ω and Ω_1 are related by

$$\Omega_1 = q\mathrm{d}\xi - 2i\bar{\mathcal{C}}\Omega \tag{100}$$

with some (in general) complex functions $\mathcal{C}$ and $q \neq 0$; equation

$$q\mathrm{d}\xi \wedge (q\mathrm{d}\xi) = 0$$

implies that function $\mathcal{C}$ satisfies (88).

Suppose now that $\mathcal{C}$ satisfies (88). We replace equation (89) for M by an equation for a function η related to M by

$$M = [\ (\partial_0 + 2i\mathcal{C}\bar{\partial} - 2i\bar{\mathcal{C}}\partial)\bar{\eta}\ ]^3. \tag{101}$$

Given M such function always exists, since a real operator

$$Z = \partial_0 + 2i\mathcal{C}\bar{\partial} - 2i\bar{\mathcal{C}}\partial$$

can be represented as

$$Z = \partial_u,$$

with an appropriate real function u on $\mathcal{Q}$. Moreover, η is defined by M up to a freedom

$$\eta \to \eta' = \eta + \omega, \tag{102}$$

where function ω satisfies

$$Z\omega = 0. \tag{103}$$

One easily checks that equation (89) for M is equivalent to an equation

$$[\ Z + i(\beta - \gamma) + 4i\mathcal{C}\bar{\mathcal{C}} - 2i\bar{\alpha}\mathcal{C} - 2i\partial\bar{\mathcal{C}}\ ]\partial\bar{\eta} = 0 \tag{104}$$

for η. If η satisfies (104) and corresponds to a given M then there exists η' which defines *the same* M and satisfies

$$\bar{\partial}\eta' = 0. \tag{105}$$

To prove this one has to show that if η satisfies (104) then there exists ω of (103) which obeys

$$\partial(\bar{\eta} + \bar{\omega}) = 0. \tag{106}$$

This equation can always be solved for ω on some real analytic 2-dimensional surface transversal to Z. Since (104) and (103) imply that

$$Z[\ \partial(\bar{\eta} + \bar{\omega})\ ] = -[\ i(\beta - \gamma) + 4i\mathcal{C}\bar{\mathcal{C}} - 2i\bar{\alpha}\mathcal{C} - 2i\partial\bar{\mathcal{C}}\ ]\partial(\bar{\eta} + \bar{\omega})$$

then we can propagate this solution to whole $\mathcal{Q}$. Therefore we can replace equation (89) by

$$M = (\partial_0 \bar{\eta} + 2i\mathcal{C}\bar{\partial}\bar{\eta})^3, \qquad \bar{\partial}\eta = 0. \tag{107}$$

The following theorem summarizes results on $\mathcal{C}$ and M.

Theorem 2.4.1 *Only CR-structures which admit at least one nontrivial solution to the tangential CR-equation are admitted by equations (88-90).*
Any solution $\mathcal{C}$ of (88) is generated by some nontrivial solution ξ of the tangential CR-equation as in (100). If $\mathcal{C}$ satisfies (88) then any solution M of (89) has the form (107) with some (even equal to 0) solution η of the tangential CR-equation (99).

One sees that the Einstein equations (35) are closely related to the tangential CR-equation for an associated CR-structure. They imply that this equation has at least one nontrivial solution. If, for simplicity, one considers the *vacuum* Einstein equations ($\Phi = 0$ in (35)) then these imply that there exist two independent solutions of the tangential CR-equation on $\mathcal{Q}$; hence the CR-structure is realizable. A very technical proof of the following theorem [29] is omitted.

Theorem 2.4.2 *A 3-dimensional CR-structure associated with the vacuum Einstein metric admitting shear-free congruence of null geodesics is realizable.*

In Section 2.1.6 we noted that solutions of the tangential CR-equation are easy to find in case of a realizable CR-structure. Therefore, in view of Theorem 2.4.1, we can easily construct all solutions to eqs. (88-89) on realizable nondegenerate CR-structures. To find a solution to the Einstein equations (35) in such cases one only has to solve equation (90). This equation has interesting geometrical interpretation. The following considerations are also valid in the nonrealizable case.
Equation (90) has linear part which resembles the Laplacian in the complex plane - its highest order part is $2\partial\bar{\partial}p$ if operators ∂ and $\bar{\partial}$ commute. In general case it is equivalent to

$$(\Box_F - \frac{1}{6}R_F + V)p = -\frac{M + \bar{M}}{p^3}, \tag{108}$$

where $\Box_F$ and R_F denote the Laplacian and the Ricci scalar of the Fefferman metric (34) associated with the CR-structure, respectively; the "potential" V is given by

$$V = \frac{1}{2}R_F - 3(\bar{\Delta}\mathcal{C} + \Delta\bar{\mathcal{C}} + 2\mathcal{C}\bar{\mathcal{C}}). \tag{109}$$

When acting on p the Laplacian $\Box_F$ is given by

$$\Box_F p = (\bar{\Delta}\partial + \Delta\bar{\partial})p,$$

and the Fefferman Ricci scalar R_F reads

$$R_F = \frac{3}{2}(\Delta\bar{\alpha} + \bar{\Delta}\alpha + \beta - \gamma).$$

The Fefferman metric (34) transforms conformally under gauge transformations (92)

$$g_F' = |t|^2 g_F. \tag{110}$$

This together with the invariance of equation (90) shows that a 'potential' V has a very simple transformation rule under (92), namely

$$V' = \frac{V}{|t|^2}. \tag{111}$$

We see that $V = 0$ is an invariant condition for gauge transformations (92). It is interesting to consider this as an additional condition to the Einstein equations (88-90). This should be especially useful condition when one considers the vacuum type N i.e. when one postulates $C_{1313} = 0$ in addition to the equations (88-90). The work on this possibility is in progress.

2.5 Vanishing of the NUT parameter - examples of solutions.

In this section we study equations (88-90) in the case of realizable CR-structure. In view of remarks after Theorem 2.4.1 the only difficult equation in such case is (90). To simplify this we assume that the NUT parameter $M + \bar{M}$ satisfies

$$M + \bar{M} = 0. \tag{112}$$

If $M \neq 0$ this is compatible with equation (89) only if the CR-structure possesses a symmetry. Indeed, if

$$M = im \neq 0 \tag{113}$$

with some real function m on $\mathcal{Q}$ then equations (89) and (88) ensure that

$$\partial_0' = \frac{1}{\sqrt[3]{m}}[\partial_0 - 2i\bar{\mathcal{C}}\partial + 2i\mathcal{C}\bar{\partial}]$$

is a symmetry.

Let $\mathcal{Q}$ admit a nontrivial Lie group of symmetries. Then it can be represented by $\mu = \mathrm{d}\xi'$ and $\lambda = \mathrm{d}u' + L'\mathrm{d}\xi' + \bar{L}'\mathrm{d}\bar{\xi}'$, where

$$L' = -i\partial_{\xi'}\sigma, \qquad \partial_{u'}\sigma = 0 \tag{114}$$

and σ is a real function such that

$$\tau = 2\partial_{\bar{\xi}'}\partial_{\xi'}\sigma > 0$$

(compare 31). A general solution of the tangential CR-equation (9) is given by $\xi = f(\xi_1, \xi_2)$, where

$$\xi_1 = \xi', \qquad \xi_2 = u' + i\sigma.$$

It follows from (100) with $\Omega = \lambda$ and $\Omega_1 = (\sqrt{\tau})\mu$ that

$$\bar{\mathcal{C}} = -\frac{i}{2}\frac{\sqrt{\tau}\ \partial_{\xi_2} f}{(\partial_{\xi_1} f - 2L'\partial_{\xi_2} f)}. \tag{115}$$

In the rest of this section we consider $\mathcal{C}$ that satisfies $\partial_{u'}\mathcal{C} = 0$. This is the case when

$$\xi = \xi' \tag{116}$$

or

$$\xi = u' + i\sigma + h(\xi'), \tag{117}$$

where h is a holomorphic function of ξ'. In the first case we can have $M = im \neq 0$, in the latter this is possible only when the symmetry group is at least 2-dimensional.

If a CR-structure admits an Abelian 2-dimensional symmetry group then we can assume that

$$L' = -\frac{1}{2}\partial_{y'}\sigma, \qquad \sigma = \sigma(y'), \qquad \tau = \frac{1}{2}\partial^2_{y'}\sigma > 0, \tag{118}$$

where $\xi' = x' + iy'$. There are two cases for which $m \neq 0$ and $\mathcal{C}$ is independent of u', namely

$$\xi = \xi', \qquad \mathcal{C} = 0, \qquad m = m_0 \tag{119}$$

and

$$\xi = u' + i\sigma + 2a\xi', \qquad \mathcal{C} = \frac{i\sqrt{\tau}}{4(a - \bar{L}')}, \qquad m = \frac{m_0}{(L' - a)^3}, \tag{120}$$

where m_0 and a are real constants.

For ξ given by (119) ($\epsilon = 0$) or (120) ($\epsilon = 1$) equation (90) reads

$$\partial^2_{y'}p + (\partial_{x'} - 2L'\partial_{u'})^2 p + \\ + [-\frac{1}{4}\partial^2_{y'}\log(\partial_{y'}L') + \frac{3}{2}\epsilon\partial^2_{y'}\log(a - L') - \frac{3}{4}\epsilon\frac{(\partial_{y'}L')^2}{(a - L')^2}]p = 0 \quad (121)$$

and has coefficients independent of u' and x'. For certain L', i.e. for certain CR-structures, equation (121) can be reduced to some well known equations of mathematical physics. Here we give examples with the CR-structure corresponding to the Robinson congruence and with the CR-structures with 3-dimensional group of symmetries of Bianchi types $VI_{h'}$.

If a CR-structure corresponds to the Robinson congruence then it can be represented by

$$\lambda = du' + \frac{2}{y'}\mathrm{d}x', \qquad \mu = \mathrm{d}\xi'; \quad (122)$$

i.e.

$$L' = \frac{1}{y'}.$$

Here we focus on solutions to (121) in case $\epsilon = 1$, $a \neq 0$. [5] In this case we can assume $a = 1$ without loss of generality due to the transformation $\xi' \to \frac{\xi'}{a}$ preserving λ. Then

$$\xi = u' - 2i\log\ y' + 2\xi', \qquad \mathcal{C} = \frac{i}{2(y'-1)}, \qquad m = \frac{-m_0 y'^3}{(y'-1)^3}. \quad (123)$$

Let as assume that p is independent of x' and that it admits the Fourier transform with respect to u',

$$p = \int \mathrm{d}k \exp(iku')Y(k, y'). \quad (124)$$

Substituting this into (121) with $L' = \frac{1}{y'}$ and $a = 1$ yields the following equation for Y

$$\ddot{Y} + (\frac{3}{2y'(y'-1)} + \frac{1 - 16k^2}{4y'^2} - \frac{9}{4(y'-1)^2})Y = 0 \quad (125)$$

where a dot denotes the differentiation with respect to y'. Equation (125) becomes the hypergeometric equation

$$y'(1 - y')\ddot{F} + [4k + 1 - (4k + 1 + 2k_0)y']\dot{F} - [(4k+1)k_0 + \frac{3}{2}]F = 0 \quad (126)$$

[5] Cases $\epsilon = 0$ or $a = 0$ are less interesting from the point of view of constructing *new* solutions since they can be easily transformed to the standard equations investigated by relativist for many years [20]. No obvious transformation to these equations exists if $\epsilon = 1$, $a \neq 0$.

under the substitution

$$Y = |y'|^{2k+\frac{1}{2}}|y' - 1|^{k_0}F, \qquad k_0 = \frac{1 \pm \sqrt{10}}{2}. \tag{127}$$

It follows from (124-127) that

$$p = \sqrt{|y'|}|y' - 1|^{k_0}\mathrm{Re}\int \mathrm{d}k\,\exp(iku')|y'|^{2k}A(k)F(k,\ y') \tag{128}$$

where $A(k)$ is a free complex function and $F(k,\ y')$ is any solution of (126) such that $y'^{2k}F(k,\ y')$ and $y'^{-2k}\bar{F}(-k,\ y')$ are linearly independent. For instance, when $y' \in]0,1[$ one can take for $F(k,\ y')$ the hypergeometric function $F(a,\ b,\ c;\ y')$ [1] with

$$a = 2k + k_0 + \sqrt{4k^2 + \frac{3}{4}}, \qquad b = 2k + k_0 - \sqrt{4k^2 + \frac{3}{4}}, \qquad c = 4k + 1. \tag{129}$$

Formulae (113), (123) and (128) define all functions needed to obtain the metric tensor. This satisfies Einstein equations (35) with an indefinite sign of Φ. For sufficiently large m_0 the metric satisfies the Einstein equations with nonnegative energy density ($\Phi > 0$). Finally, we note that there are p expressible in terms of elementary functions among functions defined by (128) e.g.

$$p = y'^{-3/4k_0}(y' - 1)^{k_0}\sin(k_1 u'), \qquad k_1 = \frac{2k_0 + 1}{12}. \tag{130}$$

Theorem 2.5.1 *Forms*

$$\Omega = du' + \frac{2}{y'}dx', \qquad \Omega_1 = \frac{1}{|y'|}d\xi'$$

and functions $\mathcal{C}$, m and p given by (123) and (128) (or (130)) define a solution to the Einstein equations (35). The metric has the form (36-37) with $\mathcal{P}$, W and H as in (38); $M = im$ and other functions are given in terms of $\mathcal{C}$, M and p by (82-87).

If the CR-structure admits a 3-dimensional symmetry group of Bianchi type $VI_{h'}$ then it can be represented by

$$\mu = \mathrm{d}\xi', \qquad \lambda = \mathrm{d}u' - 2(n-1)y'^{n-1}\mathrm{d}x', \quad n \neq 1 \tag{131}$$

where $y' > 0$ and n is a real constant related to h'. Solutions (119) and (120) (with $a = 0$) now reads

$$\xi = \xi', \qquad \mathcal{C} = 0, \qquad m = const \tag{132}$$

and

$$\xi = \begin{cases} u' - 2i \log\ y' & \text{if } n = 0 \\ u' + 2i(1 - \frac{1}{n})y'^n & \text{otherwise} \end{cases}, \qquad \mathcal{C} = \frac{i}{4} y'^{-\frac{n}{2}}, \qquad m = const\ y'^{3-3n}. \tag{133}$$

Then, equation (121) for p takes the form

$$\partial_{y'}^2 p + [\partial_{x'} + 2(n-1)y'^{n-1}\partial_{u'}]^2 p + by'^{-2}p = 0 \tag{134}$$

where

$$b = \frac{1}{4}(n-2) - \frac{3}{4}\epsilon(n^2 - 1)$$

and $\epsilon = 0$ or $\epsilon = 1$ in the case (132) or (133), respectively.

If $n \neq 0$ equation (134) has an interesting property described by the following lemma.

Lemma 2.5.1 *If $n \neq 0$ then the transformation*

$$p(u',\ x',\ y') \mapsto y'^{\frac{(1-\hat{n})}{2}} p(2(1-\hat{n})x',\ \frac{\hat{n}}{2(\hat{n}-1)}u',\ y'^{\hat{n}}) \tag{135}$$

transforms solutions of equation (134) into solutions of (134) with n, ϵ replaced by $\hat{n} = 1/n$ and $1 - \epsilon$, respectively.
Equation (134) for $n = 0$ is equivalent to the equation

$$\partial_y^2 p' + (\partial_x + 2\mathrm{e}^y \partial_u)^2 p' + \frac{3}{4}(\epsilon - 1)p' = 0 \tag{136}$$

under substitution

$$p = \sqrt{y'}\ p', \qquad u = 2x', \qquad x = -\frac{1}{2}u', \qquad y = \log\ y'.$$

Under assumption

$$\partial_u' p = 0 \tag{137}$$

equation (134) becomes

$$\Delta' p + \frac{b}{y'^2} p = 0 \tag{138}$$

where Δ' is the 2-dimensional Laplacian, $\Delta' = 4\partial_{\xi'}\partial_{\bar{\xi}'}$. Equation (138) is the Helmholtz equation

$$*\mathrm{d} * \mathrm{d}p + bp = 0$$

on a *pseudosphere* with the metric $y'^{-2}(\mathrm{d}x'^2 + \mathrm{d}y'2)$. Since this metric is invariant under the following action

$$L_h : \xi' \mapsto \frac{h_{11}\xi' + h_{12}}{h_{21}\xi' + h_{22}}, \qquad h_{ij} \in \mathbf{SL}(2,\ \mathbf{R}) \tag{139}$$

of the group $\mathbf{SL}(2,\ \mathbf{R})$ then the transformation

$$p \mapsto L_h^* p \tag{140}$$

can be used to generate new solutions of (138) from a given one.

For $b = 0$ a general solution of (138) is given by

$$\mathrm{Re}\ f(x' + iy') \tag{141}$$

where f is a holomorphic function of ξ'. This is the case when $\epsilon = 0$, $n = 2$; or $\epsilon = 1$, $3n^2 - n - 1 = 0$. For $b \neq 0$ equation (138) can be separated in various systems of coordinates. Separation of x' and y' leads to complex solutions of the form

$$\sqrt{y'}\ \mathrm{e}^{tx'}\ (A_1 J_\nu(ty') + A_2 Y_\nu(ty')), \qquad (A_3 x' + A_4) y'^{\frac{1}{2}\pm\nu} \tag{142}$$

where t and A_i are complex constants

$$\nu = \sqrt{\frac{1}{4} - b}$$

and $J_\nu(z)$ and $Y_\nu(z)$ are two independent solutions of the Bessel equation

$$z^2\partial_z^2 J + z\partial_z J + (z^2 - \nu^2)J = 0.$$

To get real solutions of (138) one can take e.g.

$$p = \sqrt{y'}\ \mathrm{Re} \int \mathrm{d}t_1 \int \mathrm{d}t_2 \mathrm{e}^{tx'} [A(t_1,\ t_2) J_\nu(ty') + B(t_1,\ t_2) Y_\nu(ty')] \tag{143}$$

where $t = t_1 + it_2$ and A, B are functions or distributions such that the RHS of (143) is well defined.

The following solutions of (134), for $n \neq 0$, are generated from (142) and (141) by virtue of Lemma 2.5.1:

$$\sqrt{y'}\mathrm{e}^{t\hat{u}}[A_1 J_{\nu/n}(ty'^n) + A_2 Y_{\nu/n}(ty'^n)], \qquad (A_3 u' + A_4)y'^{\frac{1}{2}\pm\nu} \tag{144}$$

$$y'^{\frac{1-n}{2}} f(\hat{u} + iy'^n) \qquad (\text{only for } n = \tfrac{-1\pm\sqrt{13}}{2}) \tag{145}$$

where

$$\hat{u} = \frac{n}{2(n-1)}u'.$$

Also for $n = \epsilon = 0$ Lemma 2.5.1 can be applied to obtain u-dependent solutions of (134). For $\epsilon = 0$ and assumption (137) satisfied equation (136) becomes the Helmholtz equation in the plane. This suggests the following form of the function p

$$\sqrt{y'} f(u',\ s) \tag{146}$$

where $s = 2\log y'$. So defined p satisfies (134) with $n = \epsilon = 0$ iff

$$\partial^2_{u'} f + \partial^2_s f = \frac{3}{16} f. \tag{147}$$

It is possible to separate variables in equation (147) in many different ways. A separation of u' and s shows that p can be a linear superposition of the functions

$$u'y'^{\frac{1\pm\sqrt{3}}{2}}, \qquad \sqrt{y'}(\log\ y')\mathrm{e}^{\pm\frac{\sqrt{3}}{4}u'}, \qquad y'^{2k+\frac{1}{2}}\mathrm{e}^{tu'} \tag{148}$$

where $k^2 + t^2 = \frac{3}{16}$.

If $(\lambda,\ \mu)$ are given by (131) we perform a transformation $(\lambda,\ \mu) \mapsto (\hat{\lambda},\ \hat{\mu} = \mathrm{d}\xi)$ with ξ given by (133) and $\hat{\lambda} = y'^{1-n}\lambda$. This is a gauge transformation for the CR-structure since ξ is a nontrivial solution of $\mathrm{d}\xi \wedge \lambda \wedge \mu = 0$. One finds that if $n \neq 0$ then $(\hat{\lambda},\ \hat{\mu})$ has the form (131) with $(u',\ x',\ y')$ replaced by $(2(1-n)x',\ \mathrm{Re}\ \xi,\ \mathrm{Im}\ \xi)$, respectively, and with n replaced by $\hat{n}$. If $n = 0$ then $\hat{\lambda}$ again coincides with λ of (131) with $n = 2$ under the substitution $\xi'' = \mathrm{e}^{\frac{\xi}{2i}}$, $u'' = 2x' + x''y''$. Therefore we could only consider case $\epsilon = 0$ from the very beginning. We considered both cases because by means of Lemma 2.5.1 we generated some u-dependent solutions in case $\epsilon = 0$ from u-independent solutions of case $\epsilon = 1$. We summarize the results in the following theorem.

Theorem 2.5.2 *Forms*

$$\Omega = du' - 2(n-1)y'^{n-1}dx', \qquad \Omega_1 = |n-1|y'^{\frac{n}{2}-1}d\xi', \quad n \neq 0, \quad y' > 0;$$

and functions $\mathcal{C} = \imath$, $m = const \in \mathbf{R}$ and p given by a real linear superposition of the functions given by (142) [(141) if $n = 2$] and (144) [(148) if $n = 0$, (145) if $n = \frac{1}{2}(-1 \pm \sqrt{13})$] with $\nu = \frac{1}{2}\sqrt{3-n}$ define a solution to the Einstein equations (35). The metric has the form (36-37) with Ω and Ω_1 as above; $\mathcal{P}$, W and H is defined by (38); $M = im$ and other functions are given in terms of $\mathcal{C}$, M and p by (82-87). If $\partial_u p \neq 0$ then m can be chosen in such a way that the energy density of matter is nonnegative ($\Phi \geq 0$ in (35)).

The space of admissible p can be extended by means of symmetries (139) possibly followed by transformations (135). Moreover, a separation of variables in (138) can be done in systems of coordinates different from (x', y') and coordinates related to the latter via (139). For instance, equation (134) has solutions of the form $Q(\varphi)\varrho^{\mu}$ and $Q_0(\varphi)\log\ \varrho$, where

$$x' = \varrho \cos\ \varphi, \qquad y' = \varrho \sin\ \varphi,$$

Q satisfies

$$\partial_{\varphi}^2 Q + (\mu^2 + b\sin^{-2}\varphi)Q = 0 \tag{149}$$

and Q_0 satisfies equation (149) with $\mu = 0$. Equation (149) is equivalent to the Legandre equation

$$(1-z^2)\partial_z^2 Q - 2z\partial_z Q - (b + \frac{\mu^2}{1-z^2})Q = 0$$

under the substitution $z = i\cot\ \varphi$. Hence we obtain the following solutions of (134) in terms of the Legendre functions P^{μ}_{β}, Q^{μ}_{β} [1]:

$$|\xi'|^{\mu}(A_1 P^{\mu}_{\nu-\frac{1}{2}}(i\frac{x'}{y'}) + A_2 Q^{\mu}_{\nu-\frac{1}{2}}(i\frac{x'}{y'}))$$

$$\log|\xi'|(A_3 P^{0}_{\nu-\frac{1}{2}}(i\frac{x'}{y'}) + A_4 Q^{0}_{\nu-\frac{1}{2}}(i\frac{x'}{y'})).$$

Transformations (140) and (135) can be applied to the above functions to obtain further solutions of (134). In this way one can obtain e.g.

$$p = y'^{\frac{1-n}{2}}(\hat{u}^2 + y'2n)^{\frac{\mu}{2}}(A_1 P^{\mu}_{\frac{\nu}{n}-\frac{1}{2}}(i\hat{u}y'^{-n}) + A_2 Q^{\mu}_{\frac{\nu}{n}-\frac{1}{2}}(i\hat{u}y'^{-n})) \tag{150}$$

a solution which is expressible in terms of elementary functions for some values of n and μ. For $n = 2$ (CR-structure of the Robinson congruence) the function p (not elementary) given by (150) with $\mu = 1$ corresponds to the Hauser vacuum metric [14]. It is an open and nontrivial question which of the solutions presented in this section describe also vacuum metrics.

2.6 Twisting type N vacuum equations.

In Section 2.5 we linearized equation (90) by assuming a condition $M + \bar{M} = 0$. This condition is invariant under the gauge transformations (92). The stronger condition $M = 0$ is also invariant under these transformations and reflects the fact that function M is related to Petrov type of the metric (36) (compare (82), (49-57)). In particular, vanishing of M means that the metric is of Petrov type III or its specialization. So far we have not formulated the *vacuum* equations for metrics (36). These are equations (88-90) supplemented by an additional equation followed from $R_{33} = 0$ ($\Phi = 0$ in (35)). We failed in writing this equation in a digestible form. Here we restrict only to the type N vacuum equations. Even in this case the last equation is very complicated. The full system of type N vacuum equations reads

$$\partial\mathcal{C} + \alpha\mathcal{C} - 2\mathcal{C}^2 + \frac{i}{2}\bar{\vartheta} = 0 \tag{151}$$

$$M = 0 \tag{152}$$

$$[\bar{\Delta}\partial + \Delta\bar{\partial} + \frac{1}{2}(\Delta\bar{\alpha} + \bar{\Delta}\alpha + \beta - \gamma) - 3(\Delta\bar{\mathcal{C}} + \bar{\Delta}\mathcal{C} + 2\mathcal{C}\bar{\mathcal{C}}]p = 0 \tag{153}$$

$$2\delta a + i(\partial + b)f - 2iha = 0 \tag{154}$$

where functions a, b, f, h and the operator δ are given in terms of functions p and $\mathcal{C}$ by (83), (84), (85), (87) and (58), respectively.

Here we wrote the type N equation (154) in a compact form. This follows from equations (65), since they reduce to only one equation if equations (151-153) are satisfied. The solution corresponds to the Minkowski spacetime iff

$$(\partial + ib + \alpha)[(\partial + ib)\bar{f} - 2\bar{a}i\bar{\vartheta}] - 2i\bar{f}\bar{\vartheta} = 0; \tag{155}$$

otherwise it corresponds to nonflat metrics.

Again all the equations are invariant under the gauge transformations (92). It would be interesting to study integrability conditions of (151-154). This seems to be a difficult problem, but its solution would give the definite answer to the question which CR-structures are admitted by the type N vacuum Einstein equations. Adding condition (155) to the system is also interesting. Integrability conditions in this case would give an invariant characterization of the Kerr theorem i.e. would show which CR-structures can be lifted to the Minkowski spacetime. These questions will be studied elsewhere.

2.7 Einstein-Maxwell equations for algebraically special fields.

In this section we construct examples of solutions of the Einstein-Maxwell fields for which one of the eigendirections of the Maxwell tensor is *aligned* with the null congruence of shear-free geodesics which is not necessarily twisting. We work with the maximally reduced equations that can be found in textbook [20], hence adopt the notations to that of this book. Despite the fact that the methods used here do not refer to the Cauchy-Riemann geometry associated with the congruence we decided to include this section to the thesis since the solutions obtained seem to be new. Among them are first examples of Einstein-Maxwell solutions with purely radiative Maxwell field and a *twisting* congruence. Also first examples of solutions with a twisting congruence and the Maxwell field which have radiative and charged part are presented here [38].

2.7.1 Summary of known results.

We assume that spacetime admits a shear-free congruence of null geodesics, which is *aligned* with one of the eigenvectors of the Maxwell tensor $F_{\mu\nu}$. It is subject to the coupled system of the Einstein-Maxwell equations. It follows from the Goldberg-Sachs theorem [13] that the Weyl tensor is algebraically special. The equations reduce to the equations on a 3-dimensional CR-structure $\mathcal{Q}$ associated with the congruence. Introducing coordinates $(u, \mathrm{Re}\,\xi, \mathrm{Im}\,\xi)$ on $\mathcal{Q}$ the full system of the equations becomes a system for unknown functions $m,\ M,\ P$ (all real) and $L,\ \varphi_1^0,\ \varphi_2^0$ (all complex). It has the following form

$$(\partial - 2L_u)\varphi_1^0 = 0 \tag{156}$$
$$(\partial - L_u)(P^{-1}\varphi_2^0) + \partial_u(P^{-2}\varphi_1^0) = 0 \tag{157}$$
$$P(\partial - 3L_u)(m + iM) = -2\varphi_1^0\bar{\varphi}_2^0 \tag{158}$$
$$P^4(\partial - 2G)\partial(\bar{G}^2 - \bar{\partial}\bar{G}) - P^3[P^{-3}(m + iM)]_u = \varphi_2^0\bar{\varphi}_2^0 \tag{159}$$
$$M = -2P^2\Sigma\mathrm{Re}\,(\partial\bar{G}) + P^2\mathrm{Re}(\partial\bar{\partial}\Sigma - 2\bar{L}_u\partial\Sigma - \Sigma\partial_u\partial\bar{L}) \tag{160}$$

where

$$
\begin{aligned}
&\partial = \partial_\xi - L\partial_u \quad \text{ is a CR-operator on } \mathcal{Q} && (161)\\
&G = L_u - \partial \log\ P && (162)\\
&2i\Sigma = P^2(\bar{\partial}L - \partial\bar{L}). && (163)
\end{aligned}
$$

Let r be an affine parameter along the congruence. Given a solution of equations (156-160) the corresponding metric tensor g and the electromagnetic field 2-form $F = \frac{1}{2}F_{\mu\nu}\vartheta^\mu \wedge \vartheta^\nu$ are defined by

$$
\begin{aligned}
&g = 2(\vartheta^1\vartheta^2 - \vartheta^3\vartheta^4) && (164)\\
&F = (\bar{\varphi}_1 - \varphi_1)\vartheta^1 \wedge \vartheta^2 - (\bar{\varphi}_1 + \varphi_1)\vartheta^3 \wedge \vartheta^4 + \bar{\varphi}_2\vartheta^1 \wedge \vartheta^3 + \varphi_2\vartheta^2 \wedge \vartheta^3 && (165)
\end{aligned}
$$

where the null tetrad is given by

$$
\begin{aligned}
&\vartheta^1 = -\frac{\mathrm{d}\xi}{P\bar{\varrho}} = \bar{\vartheta}^2 \qquad \vartheta^3 = \mathrm{d}u + L\mathrm{d}\xi + \bar{L}\mathrm{d}\bar{\xi}\\
&&& (166)\\
&\vartheta^4 = \mathrm{d}r + W\mathrm{d}\xi + \bar{W}\mathrm{d}\bar{\xi} + H\vartheta^3
\end{aligned}
$$

$$
\begin{aligned}
&\varrho = \frac{-1}{r + i\Sigma} \qquad W = \frac{L_u}{\varrho} + i\partial\Sigma\\
&&& (167)\\
&H = -r(\log\ P)_u - (mr + M\Sigma - \varphi_1^0\bar{\varphi}_1^0)\varrho\bar{\varrho} - P^2\mathrm{Re}\ (\partial\bar{G})
\end{aligned}
$$

and

$$
\varphi_1 = \varrho^2\varphi_1^0 \qquad \varphi_2 = \varrho\varphi_2^0 + \varrho^2 P(2\bar{L}_u - \bar{\partial})\varphi_1^0 + 2i\varrho^3 P(\Sigma\bar{L}_u - \bar{\partial}\Sigma)\varphi_1^0. \tag{168}
$$

The energy momentum tensor of the electromagnetic field is given by

$$
T_{12} = T_{34} = 2\bar{\varphi}_1\varphi_1 \qquad T_{13} = \bar{T}_{23} = 2\bar{\varphi}_2\varphi_1 \qquad T_{33} = 2\bar{\varphi}_2\varphi_2; \tag{169}
$$

all other components vanish. Equations (158-160) with φ_1^0 are equivalent to the (reduced) Einstein equations with a pure radiation field. [6] If, in addition, φ_2^0, they reduce to the vacuum equations [17]. In the following we say that a solution is *vacuum*

[6]Equations (158) and (160) correspond to equations (88-90). However they are written in a specific tetrad (ϑ^μ) that (in general) does not satisfies (23). Equation (159) is a definition of φ_2^0 and corresponds to the nonnegativity of the energy of matter.

iff $\varphi_1 = \varphi_2 = 0$, *charged vacuum* iff $\varphi_2 = 0$, *radiative* iff $\varphi_2 \neq 0$, *purely radiative* iff it is radiative and $\varphi_1 = 0$; in the generic case when $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$ we call a solution *generic.*

Equations (156-160) are invariant under a coordinate transformation

$$u' = U(\xi,\ \bar{\xi},\ u) \qquad \xi' = h(\xi) \qquad r' = \frac{r}{U_u} \tag{170}$$

which induces the following transformation laws of dependent variables

$$L' = \frac{-\partial U}{h_\xi} \qquad (m+iM)' = \frac{m+iM}{(U_u)^3} \qquad P' = \frac{|h_\xi|}{U_u}P \tag{171}$$

$$(\varphi_1^0)' = \frac{\varphi_1^0}{(U_u)^2} \qquad (\varphi_2^0)' = \sqrt{\frac{h_\xi}{\bar{h}_{\bar{\xi}}}}\frac{\varphi_2^0}{(U_u)^2} \tag{172}$$

$$\Sigma' = U_u\Sigma \qquad h_\xi G' = G - \frac{1}{2}\,\frac{h_{\xi\xi}}{h_\xi}. \tag{173}$$

We see that the gauge transformations (170) of the system do not allow for changing variable ξ into another solution of the tangential CR-equation $\mathrm{d}\eta \wedge \vartheta^3 \wedge \mathrm{d}\xi = 0$; given a CR-structure a class of solutions $h(\xi)$ of this equation is distinguished by Einstein equations. Hence this formulation of the equations is not CR-invariant. However, working within this framework we partially integrate the system and construct first examples of purely radiative solutions with *twisting* congruences and first examples of generic twisting Petrov type II solutions [38].

2.7.2 Radiative solutions admitting a special Killing vector.

Introducing variables

$$q = \frac{\varphi_1^0}{P^2} \qquad f = \frac{\varphi_2^0}{P^2} \qquad S = -2\frac{\Sigma}{P} \qquad \tilde{m} + i\tilde{M} = \frac{m+iM}{P^3}, \tag{174}$$

following the methode of Ref. [50] we assume that

$$G_u = q_u = f_u = \tilde{m}_u = \tilde{M}_u = 0. \tag{175}$$

This conditions are invariant under transformations (170), and we can use these transformations to obtain

$$P_u = 0. \tag{176}$$

Condition (176) still admits the following changes of the coordinates u, ξ, r

$$u' = U_1(\xi,\ \bar{\xi})u + U_2(\xi,\ \bar{\xi}) \qquad \xi' = h(\xi) \qquad r' = \frac{r}{U_1}. \tag{177}$$

In addition we assume that the Maxwell field is radiative and that

$$\gamma \equiv i(\bar{\partial}G - \partial\bar{G}) = 0. \tag{178}$$

In this case ∂_u is a Killing vector since transformation (177) can be used to obtain

$$\partial_u L = 0. \tag{179}$$

In such case we reduce the Einstein-Maxwell equations (156-160) to equations for P and Σ. Equations (156-158) yield

$$\varphi_1^0 = \bar{a} \qquad \varphi_2^0 = \bar{\partial}\bar{b}P \qquad m + iM = -2\bar{a}b - \bar{c} \tag{180}$$

where a, b, c are holomorphic functions of ξ and $b \neq$constant. Equations (159-160) reduce to

$$\partial\bar{\partial}(P^2\partial\bar{\partial}\log\ P) = |\partial b|^2 \tag{181}$$

$$\partial\bar{\partial}\Sigma + 2\Sigma\partial\bar{\partial}\log\ P = \frac{\mathrm{Im}(2a\bar{b} + c)}{P^2}. \tag{182}$$

Given P and Σ, the function L can be found by integrating equation (163). Up to a transformation (177) (with $U_1 = 1$) L is given by

$$L = i\int \frac{\Sigma}{P^2}\mathrm{d}\bar{\xi}. \tag{183}$$

Double integration of equation (181) yields

$$P^2\partial\bar{\partial}\log\ P = b\bar{b} + e + \bar{e} \tag{184}$$

where $e \equiv e(\xi)$. Equation (182) can be replaced by the equation for the variable Σ_0 related to Σ by

$$\Sigma = \Sigma_0 + \mathrm{Im}(\partial a_0 - 2a_0\partial\log\ P) \tag{185}$$

where

$$a_0 = \frac{a}{\partial b}. \tag{186}$$

The function Σ_0 is subject to the equation

$$\partial\bar{\partial}\Sigma_0 + 2\Sigma_0\partial\bar{\partial}\log\ P = \frac{\mathrm{Im}(c_0)}{P^2} \tag{187}$$

where

$$c_0 = c + 2a_0\partial e. \tag{188}$$

The splitting (185) of Σ corresponds to the following decomposition of L:

$$L = L_0 - \frac{\bar{a}_0}{P^2} \tag{189}$$

where

$$2i\Sigma_0 = P^2(\bar{\partial}L_0 - \partial\bar{L}_0). \tag{190}$$

Given P and Σ_0 function L_0 can be defined by a formula analogous to (183).

Theorem 2.7.1 *Let a, b ($b \neq$constant), c_0 and e be holomorphic functions of ξ. If real functions P, Σ_0 of ξ and $\bar{\xi}$ satisfy equations (184) and (187) then functions*

$$L = i\int\frac{\Sigma_0}{P^2}d\bar{\xi} - \frac{\bar{a}}{\bar{\partial}\bar{b}P^2}$$

$$\varphi_1^0 = \bar{a} \qquad \varphi_2^0 = \bar{\partial}\bar{b} \qquad m + iM = -2\bar{a}b - \bar{c}_0 + 2\frac{\bar{a}\bar{\partial}\bar{e}}{\bar{\partial}\bar{b}}$$

define an algebraically special Einstein-Maxwell field via relations (164-168), (185).

For $e = 0$ equation (184) is equivalent to the Liouville equation, which is exactly soluble. Due to the freedom (177) (restricted by (185) and $U_1 = 1$) it is sufficient to take only one solution of the Liouville equation, say

$$P = |b|(1 + \xi\bar{\xi}). \tag{191}$$

Given this P, equation (187) takes the form

$$(1 + \xi\bar{\xi})^2\partial\bar{\partial}\Sigma_0 + 2\Sigma_0 = \frac{\mathrm{Im}c}{b\bar{b}}. \tag{192}$$

Solutions of (192) have the form

$$\Sigma_0 = Y + S \tag{193}$$

where S is a particular solution of (192) and Y satisfies the homogeneous equation

$$(1+\xi\bar{\xi})^2\partial\bar{\partial}Y = -2Y. \tag{194}$$

Equation (194) is the eigenfunction equation (with eigenvalue equal to -2) for the Laplace operator on the 2-dimensional sphere. The variable ξ can be interpreted as the complex projective coordinate on the sphere

$$\xi = \tan\frac{1}{2}\vartheta e^{i\varphi} \tag{195}$$

where ϑ, φ are the usual spherical angles. Separation of ϑ and φ in (194) leads to Y being a linear combination of functions

$$(\sqrt{\frac{\xi}{\bar{\xi}}})^\mu P_1^\mu(z) \qquad (\sqrt{\frac{\xi}{\bar{\xi}}})^\mu Q_1^\mu(z) \tag{196}$$

where P_1^μ Q_1^μ are the associated Legendre functions [1] and

$$z = \frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}}. \tag{197}$$

Solutions which are regular on the whole sphere are given by

$$Y = \frac{\alpha - \alpha\xi\bar{\xi} + \beta\xi + \bar{\beta}\bar{\xi}}{1+\xi\bar{\xi}} \tag{198}$$

where α and β are constants, real and complex, respectively.

A particular solution of (192) can be found e.g. in a case when

$$c = c_1 b + c_2 \qquad c_2 \in \mathbf{R} \tag{199}$$

where $c_1,\ c_2$ are constants. In this case

$$S = \mathrm{Im}\frac{c_1}{2\bar{b}}. \tag{200}$$

Theorem 2.7.2 *Let a and b, $b \neq$constant, be arbitrary holomorphic functions of ξ and let P, M, m, L, φ_1^0, φ_2^0 be given by*

$$P = |b|(1+\xi\bar{\xi}) \tag{201}$$

$$m + iM = c_1 b + c_2 \tag{202}$$

$$L = i\int\frac{\Sigma_0}{P^2}d\bar{\xi} - \frac{\bar{a}}{\bar{\partial}\bar{b}P^2} \tag{203}$$

$$\varphi_1^0 = \bar{a} \qquad \varphi_2^0 = \bar{\partial}\bar{b}P \tag{204}$$

where

$$\Sigma_0 = Y + Im(\frac{c_1}{2\bar{b}}) \tag{205}$$

Y satisfies (194) (e.g. it is given by (198)) and c_2, α (both real), c_1, β (both complex) are constants. The above relations, together with (164-168), (185), define an algebraically special solution of the Einstein-Maxwell equations. The Weyl tensor of the solution is of Petrov type II iff $m + iM \neq 0$ or $a \neq 0$, and it is of Petrov type III iff $m + iM = a = 0$.

Metric functions P, m, M, L, φ_1^0, φ_2^0 given by (201-205) with $a = 0 \neq \Sigma_0$ describe purely radiative Einstein-Maxwell fields with twisting rays.

Below we give two examples of solutions covered by Theorem 2.7.2.

If $Y = c_1 = 0$ then the metric corresponding to (201-205) reads:

$$g = 2\frac{(r^2+\Sigma^2)\mathrm{d}\xi\mathrm{d}\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2} - 2\vartheta^3\ [\ \mathrm{d}r + i(\partial\Sigma\mathrm{d}\xi - \bar{\partial}\Sigma\mathrm{d}\bar{\xi}) + (b\bar{b} + \frac{a\bar{a} - c_2 r}{r^2+\Sigma^2})\vartheta^3\ ] \tag{206}$$

where

$$\vartheta^3 = \mathrm{d}u - \frac{\bar{a}_0\mathrm{d}\xi + a_0\mathrm{d}\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2} \qquad a_0 = \frac{a}{\partial b} \tag{207}$$

$$\Sigma = \mathrm{Im}\{\partial a_0 - a_0\partial\log[b(1+\xi\bar{\xi})^2]\}. \tag{208}$$

The electromagnetic field is given by (165) with

$$\varphi_1 = \bar{a}\varrho^2 \tag{209}$$

$$\varphi_2 = \varrho|b|(1+\xi\bar{\xi})\{\bar{\partial}\bar{b} - \varrho\bar{\partial}\bar{a} - 2i\varrho^2\bar{a}\bar{\partial}\Sigma\} \tag{210}$$

where

$$r = \frac{-1}{r+i\Sigma}. \tag{211}$$

Solution (206-211) with $a \neq 0$ is twisting, charged and radiative. In the limit $a \to 0$ it yields a non-twisting pure radiation (no charge) Einstein-Maxwell field. The latter solution belongs to the class of solutions obtained by Bartrum [2]. In the limit $b \to \frac{1}{\sqrt{2}}$ it reduces to the Schwarzschild solution with mass $m = c_2$.

An example of solution with a purely radiative ($\varphi_1 = 0$) Maxwell field follows from Theorem 2.7.2 if, e.g. $a = c_1 = 0$ and

$$Y = \alpha\frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}}. \tag{212}$$

In this case the metric and the electromagnetic field read

$$g = 2\frac{(r^2+\Sigma^2)\mathrm{d}\xi\mathrm{d}\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2} - 2\vartheta^3\ [\ \mathrm{d}r + 2\alpha i\frac{\xi\mathrm{d}\bar{\xi}-\bar{\xi}\mathrm{d}\xi}{(1+\xi\bar{\xi})^2} + (b\bar{b} - \frac{c_2 r}{r^2+\Sigma^2})\vartheta^3\ ] \tag{213}$$

$$F = \vartheta^3 \wedge (\partial b\mathrm{d}\xi + \bar{\partial}\bar{b}\mathrm{d}\bar{\xi}) \tag{214}$$

where

$$\Sigma = \alpha\frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}} \qquad \vartheta^3 = \mathrm{d}u + L\mathrm{d}\xi + \bar{L}\mathrm{d}\bar{\xi} \tag{215}$$

$$L = i\alpha\frac{\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2} + i\alpha\int\frac{\bar{\xi}\bar{\partial}\log\ \bar{b}}{b\bar{b}(1+\xi\bar{\xi})^2}\mathrm{d}\bar{\xi}. \tag{216}$$

This solution is twisting if $\alpha \neq 0$. It is of Petrov type II if $c_2 \neq 0$ and it is of Petrov type III if $c_2 = 0$. In the limit $b \to \frac{1}{\sqrt{2}}$ it reduces to the Kerr metric with mass parameter $m = c_2$ and angular momentum parameter equal to α. In the limit $\alpha \to 0$ it coincides with solution (206-211) with $a \to 0$.

3 ORTHOGONAL COMPLEX GEOMETRY OVER 4-DIMENSIONAL EUCLIDEAN MANIFOLD.

3.1 Newman - Penrose coefficients for 4-dimensional Euclidean manifolds.

Let $\mathcal{M}$ be a 4-dimensional manifold equipped with an Euclidean metric g. Any orthonormal tetrad $(f_\mu) = (f_1,\ f_2,\ f_3,\ f_4)$ defines a *Newman-Penrose complex null tetrad* $(e_\mu) = (m, \bar{m}, n, \bar{n})$ by

$$m = \frac{f_1 + if_2}{\sqrt{2}} \qquad n = \frac{f_3 + if_4}{\sqrt{2}}.$$

The dual to (e_μ) is $(e^\mu) = (M, \bar{M}, N, \bar{N})$, respectively, and the metric takes the form

$$g = 2(M\bar{M} + N\bar{N}). \tag{217}$$

We define (modified) Newman - Penrose coefficients [31] by

$$\mathrm{d}M = \alpha M\wedge\bar{M} + (\bar{\varrho}-\bar{\beta})M\wedge N + (\beta-\bar{\lambda})M\wedge\bar{N} + \bar{\sigma}\bar{M}\wedge N - \bar{\mu}\bar{M}\wedge\bar{N} - (\bar{\tau}+\bar{\pi})N\wedge\bar{N}, \tag{218}$$

$$\mathrm{d}\bar{M} = -\bar{\alpha}M\wedge\bar{M} - \mu M\wedge N + \sigma M\wedge\bar{N} + (\bar{\beta}-\lambda)\bar{M}\wedge N + (\varrho-\beta)\bar{M}\wedge\bar{N} + (\tau+\pi)N\wedge\bar{N}, \tag{219}$$

$$\mathrm{d}N = (\varrho+\bar{\lambda})M\wedge\bar{M} + (\bar{\gamma}+\tau)M\wedge N + \kappa M\wedge\bar{N} - (\gamma+\bar{\pi})\bar{M}\wedge N - \bar{\nu}\bar{M}\wedge\bar{N} + \varepsilon N\wedge\bar{N}, \tag{220}$$

$$\mathrm{d}\bar{N} = -(\bar{\varrho}+\lambda)M\wedge\bar{M} - \nu M\wedge N - (\bar{\gamma}+\pi)M\wedge\bar{N} + \bar{\kappa}\bar{M}\wedge N + (\gamma+\bar{\tau})\bar{M}\wedge\bar{N} - \bar{\varepsilon}N\wedge\bar{N}. \tag{221}$$

It follows from these equations that the commutators of the tetrad vector fields are given by

$$[m,\bar{m}] = -\alpha m + \bar{\alpha}\bar{m} - (\varrho+\bar{\lambda})n + (\bar{\varrho}+\lambda)\bar{n}, \tag{222}$$

$$[m,n] = (\bar{\beta}-\bar{\varrho})m + \mu\bar{m} - (\bar{\gamma}+\tau)n + \nu\bar{n}, \tag{223}$$

$$[m,\bar{n}] = (\bar{\lambda}-\beta)m - \sigma\bar{m} - \kappa n + (\bar{\gamma}+\pi)\bar{n}, \tag{224}$$

$$[n,\bar{m}] = \bar{\sigma}m + (\bar{\beta}-\lambda)\bar{m} - (\gamma+\bar{\pi})n + \bar{\kappa}\bar{n}, \tag{225}$$

$$[\bar{n},\bar{m}] = -\bar{\mu}m + (\varrho-\beta)\bar{m} - \bar{\nu}n + (\gamma+\bar{\tau})\bar{n}, \tag{226}$$

$$[n,\bar{n}] = (\bar{\tau}+\bar{\pi})m - (\tau+\pi)\bar{m} - \varepsilon n + \bar{\varepsilon}\bar{n}. \tag{227}$$

3.2 (Modified) Newman-Penrose equations.

Levi-Civita connection associated with g is defined in terms of "connection 1-forms" $\Gamma_{\mu\nu} = g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu}$ which satisfy

$$\mathrm{d}e^{\mu} + \Gamma^{\mu}{}_{\nu} \wedge e^{\nu} = 0, \qquad \Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0. \tag{228}$$

It follows from (218-221) and (228) that

$$\Gamma_{12} = \bar{\alpha}M - \alpha\bar{M} + \bar{\beta}N - \beta\bar{N}, \tag{229}$$

$$\Gamma_{13} = \mu M + \lambda\bar{M} + \nu N + \pi\bar{N}, \tag{230}$$

$$\Gamma_{14} = -\sigma M - \varrho\bar{M} - \tau N - \kappa\bar{N}, \tag{231}$$

$$\Gamma_{23} = -\bar{\varrho}M - \bar{\sigma}\bar{M} - \bar{\kappa}N - \bar{\tau}\bar{N}, \tag{232}$$

$$\Gamma_{24} = \bar{\lambda}M + \bar{\mu}\bar{M} + \bar{\pi}N + \bar{\nu}\bar{N}, \tag{233}$$

$$\Gamma_{34} = \bar{\gamma}M - \gamma\bar{M} + \bar{\varepsilon}N - \varepsilon\bar{N}. \tag{234}$$

Defining curvature 2-forms by

$$\Omega_{\mu\nu} = \frac{1}{2}R_{\mu\nu\varrho\sigma}e^{\varrho} \wedge e^{\sigma} = \mathrm{d}\Gamma_{\mu\nu} + \Gamma_{\mu\sigma} \wedge \Gamma^{\sigma}{}_{\nu}, \tag{235}$$

one computes the Riemann tensor $R_{\mu\nu\varrho\sigma}$, its totally traceless part - the Weyl tensor $C_{\mu\nu\varrho\sigma}$, and traces $R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma}$ and $R = R^{\nu}{}_{\nu}$, where we lowered and raised indices in terms of the metric g.

If $n = n^{\mu}e_{\mu}$, etc., one defines (modified) Newman-Penrose curvature coefficients by

$$\Phi_{00} = \frac{1}{2}R_{\mu\nu}n^{\mu}n^{\nu}, \tag{236}$$

$$\Phi_{01} = \frac{1}{2}R_{\mu\nu}n^{\mu}m^{\nu}, \tag{237}$$

$$\Phi_{02} = \frac{1}{2}R_{\mu\nu}m^{\mu}m^{\nu}, \tag{238}$$

$$\Phi_{11} = \frac{1}{4}R_{\mu\nu}(m^{\mu}\overline{m^{\nu}} - n^{\mu}\overline{n^{\nu}}), \tag{239}$$

$$\Phi_{12} = \frac{1}{2}R_{\mu\nu}n^{\mu}\overline{m^{\nu}}, \tag{240}$$

$$\Psi_0 = C_{\mu\nu\varrho\sigma} n^\mu m^\nu n^\varrho m^\sigma, \tag{241}$$

$$\Psi_1 = C_{\mu\nu\varrho\sigma} n^\mu \overline{n^\nu} n^\varrho m^\sigma, \tag{242}$$

$$\Psi_\pm = \frac{1}{2} C_{\mu\nu\varrho\sigma} n^\mu \overline{n^\nu} (n^\varrho \overline{n^\sigma} \pm m^\varrho \overline{m^\sigma}), \tag{243}$$

$$\Psi_3 = C_{\mu\nu\varrho\sigma} n^\mu \overline{n^\nu} n^\varrho \overline{m^\sigma}, \tag{244}$$

$$\Psi_4 = C_{\mu\nu\varrho\sigma} n^\mu \overline{m^\nu} n^\varrho \overline{m^\sigma}. \tag{245}$$

All other coefficients of $R_{\mu\nu\varrho\sigma}$, $C_{\mu\nu\varrho\sigma}$, and $R_{\nu\sigma}$ are algebraically expressible in terms of Φs, Ψs and R; in particular

$$\begin{aligned} \Omega_{12} &= (\Psi_+ + \Psi_- - 2\Phi_{11} - \frac{R}{12}) M \wedge \bar{M} - (\Psi_1 + \Phi_{01}) M \wedge N - (\bar{\Psi}_3 + \bar{\Phi}_{12}) M \wedge \bar{N} + \\ &+ (\Psi_3 + \Phi_{12}) \bar{M} \wedge N + (\bar{\Psi}_1 + \bar{\Phi}_{01}) \bar{M} \wedge \bar{N} + (\Psi_+ - \Psi_-) N \wedge \bar{N}, \end{aligned} \tag{246}$$

$$\begin{aligned} \Omega_{34} &= (\Psi_+ - \Psi_-) M \wedge \bar{M} + (\Phi_{01} - \Psi_1) M \wedge N + (\bar{\Psi}_3 - \bar{\Phi}_{12}) M \wedge \bar{N} + \\ &+ (\Phi_{12} - \Psi_3) \bar{M} \wedge N + (\bar{\Psi}_1 - \bar{\Phi}_{01}) \bar{M} \wedge \bar{N} + \\ &+ (\Psi_+ + \Psi_- + 2\Phi_{11} - \frac{R}{12}) N \wedge \bar{N}, \end{aligned} \tag{247}$$

$$\begin{aligned} \Omega_{13} &= -(\Psi_1 + \Phi_{01}) M \wedge \bar{M} + \Psi_0 M \wedge N + \Phi_{02} M \wedge \bar{N} + \\ &+ \Phi_{00} \bar{M} \wedge N + (\Psi_+ + \frac{R}{12}) \bar{M} \wedge \bar{N} + (\Phi_{01} - \Psi_1) N \wedge \bar{N} = \\ &= \bar{\Omega}_{24}, \end{aligned} \tag{248}$$

$$\begin{aligned} \Omega_{14} &= -(\bar{\Psi}_3 + \bar{\Phi}_{12}) M \wedge \bar{M} + \Phi_{02} M \wedge N + \bar{\Psi}_4 M \wedge \bar{N} + \\ &+ (\Psi_- + \frac{R}{12}) \bar{M} \wedge N + \bar{\Phi}_{00} \bar{M} \wedge \bar{N} + (\bar{\Psi}_3 - \bar{\Phi}_{12}) N \wedge \bar{N} = \\ &= \bar{\Omega}_{23}. \end{aligned} \tag{249}$$

The Weyl 2-forms

$$C_{\mu\nu} = \frac{1}{2} C_{\mu\nu\varrho\sigma} e^\varrho \wedge e^\sigma, \tag{250}$$

can be formally obtained from (246-250) by putting R and all Φs equal to zero in the formulae for $\Omega_{\mu\nu}$, e. g.

$$\begin{aligned} C_{12} &= (\Psi_+ + \Psi_-) M \wedge \bar{M} - \Psi_1 M \wedge N - \bar{\Psi}_3 M \wedge \bar{N} + \\ &+ \Psi_3 \bar{M} \wedge N + \bar{\Psi}_1 \bar{M} \wedge \bar{N} + (\Psi_+ - \Psi_-) N \wedge \bar{N}, \quad etc. \end{aligned} \tag{251}$$

If the orientation is defined by a volume form $\eta = M \wedge \bar{M} \wedge N \wedge \bar{N}$ then the Hodge dual of $C_{\mu\nu}$s is given by

$$
\begin{aligned}
*C_{13} &= -C_{13}, \\
C_{23} &= C_{23}, \\
C_{43} &= C_{12}.
\end{aligned}
\tag{252}
$$

Equations (253) imply the following Proposition.

Proposition 3.2.1
(i) g is self-dual iff $\Psi_0 = \Psi_1 = \Psi_+ = 0$
(ii) g is antiself-dual iff $\Psi_3 = \Psi_4 = \Psi_- = 0$.

If we denote operators m and n by

$$m = D \qquad n = \Delta \tag{253}$$

then we can express Ψs, Φs and R in terms of Newman-Penrose coefficients and their derivatives obtaining (modified) Newman-Penrose equations. These are

$$\bar{D}\nu - \Delta\lambda = \lambda(\bar{\varepsilon} + \lambda) + \nu(2\gamma + \bar{\pi} + \alpha) - \mu\bar{\sigma} - \pi\bar{\kappa} + \Phi_{00}, \tag{254}$$

$$\bar{\Delta}\mu - D\pi = \mu(2\beta - \bar{\lambda} + \varepsilon) + \pi(\bar{\alpha} - \pi) + \sigma\lambda + \kappa\nu - \Phi_{02}, \tag{255}$$

$$\Delta\pi - \bar{\Delta}\nu = \mu(\bar{\tau} + \bar{\pi}) - \lambda(\tau + \pi) - \nu(2\varepsilon + \beta) - \bar{\beta}\pi - \Psi_1 + \Phi_{01}, \tag{256}$$

$$\bar{D}\mu - D\lambda = \nu(\varrho + \bar{\lambda}) - \pi(\bar{\varrho} + \lambda) + \mu(2\alpha + \gamma) + \bar{\gamma}\lambda + \Psi_1 + \Phi_{01}, \tag{257}$$

$$\Delta\bar{\alpha} - D\bar{\beta} = \mu(\alpha - \bar{\pi}) + \nu(\bar{\lambda} + \beta) + \tau(\bar{\varrho} + \bar{\beta}) + \bar{\alpha}(\bar{\varrho} - \bar{\beta}) - \sigma\bar{\kappa} + \bar{\gamma}\bar{\beta} + \Psi_1 + \Phi_{01}, \tag{258}$$

$$\Delta\bar{\gamma} - D\bar{\varepsilon} = \mu(\gamma - \bar{\pi}) + \nu(\bar{\lambda} + \varepsilon) + \bar{\gamma}(\bar{\varrho} + \bar{\varepsilon} - \bar{\beta}) + \tau(\bar{\varepsilon} - \bar{\varrho}) + \sigma\bar{\kappa} + \Psi_1 - \Phi_{01}, \tag{259}$$

$$
\begin{aligned}
\Delta\varepsilon + \bar{\Delta}\bar{\varepsilon} &= 2\varepsilon\bar{\varepsilon} + \pi\bar{\pi} + \tau\bar{\tau} - \nu\bar{\nu} - \kappa\bar{\kappa} + \\
&\quad -\bar{\gamma}(\bar{\tau} + \bar{\pi}) - \gamma(\tau + \pi) - \Psi_+ - \Psi_- - 2\Phi_{11} + \frac{R}{12},
\end{aligned}
\tag{260}
$$

$$
\begin{aligned}
D\alpha + \bar{D}\bar{\alpha} &= 2\alpha\bar{\alpha} + \lambda\bar{\lambda} + \varrho\bar{\varrho} - \mu\bar{\mu} - \sigma\bar{\sigma} + \\
&\quad +\bar{\beta}(\bar{\lambda} + \varrho) + \beta(\lambda + \bar{\varrho}) - \Psi_+ - \Psi_- + 2\Phi_{11} + \frac{R}{12},
\end{aligned}
\tag{261}
$$

$$\Delta\beta + \bar{\Delta}\bar{\beta} = \pi\bar{\pi} - \nu\bar{\nu} - \tau\bar{\tau} + \kappa\bar{\kappa} - \bar{\alpha}(\bar{\tau} + \bar{\pi}) - \alpha(\tau + \pi) + \beta\bar{\varepsilon} + \bar{\beta}\varepsilon - \Psi_+ + \Psi_-, \tag{262}$$

$$D\gamma+\bar{D}\bar{\gamma}=\lambda\bar{\lambda}-\mu\bar{\mu}-\varrho\bar{\varrho}+\sigma\bar{\sigma}+\bar{\varepsilon}(\varrho+\bar{\lambda})+\varepsilon(\bar{\varrho}+\lambda)+\gamma\bar{\alpha}+\bar{\gamma}\alpha-\Psi_{+}+\Psi_{-}, \quad (263)$$

$$\Delta\alpha+\bar{D}\bar{\beta}=-\bar{\sigma}(\bar{\alpha}+\tau)+\alpha(\bar{\beta}-\lambda)+\bar{\beta}(\gamma+\bar{\pi})+\bar{\kappa}(\beta+\varrho)+\lambda\bar{\pi}-\bar{\mu}\nu+\Psi_3+\Phi_{12}, \quad (264)$$

$$\Delta\gamma+\bar{D}\bar{\varepsilon}=\bar{\sigma}(\tau-\bar{\gamma})+\bar{\kappa}(\varepsilon-\varrho)+\gamma(\bar{\beta}+\bar{\varepsilon}-\lambda)+\bar{\pi}\bar{\varepsilon}+\lambda\bar{\pi}-\bar{\mu}\nu-\Psi_3+\Phi_{12}, \quad (265)$$

$$\Delta\bar{\tau}-\bar{\Delta}\bar{\kappa}=\bar{\varrho}(\bar{\tau}+\bar{\pi})-\bar{\sigma}(\tau+\pi)+\bar{\kappa}(\beta-2\varepsilon)+\bar{\tau}\bar{\beta}+\Psi_3-\Phi_{12}, \quad (266)$$

$$\bar{D}\sigma-D\varrho=\tau(\varrho+\bar{\lambda})-\kappa(\bar{\varrho}+\lambda)-\sigma(\gamma-2\alpha)-\varrho\bar{\gamma}-\bar{\Psi}_3-\bar{\Phi}_{12}, \quad (267)$$

$$D\nu-\Delta\mu=\mu(2\bar{\beta}+\lambda+\bar{\varepsilon}-\bar{\varrho})+\nu(\pi-\tau-\bar{\alpha}-2\bar{\gamma})+\Psi_0, \quad (268)$$

$$\Delta\bar{\sigma}-\bar{D}\bar{\kappa}=\bar{\kappa}(\bar{\tau}-2\gamma-\bar{\pi}+\alpha)+\bar{\sigma}(\bar{\varrho}-\lambda-\bar{\varepsilon}+2\bar{\beta})+\Psi_4, \quad (269)$$

$$\begin{aligned}&D\bar{\pi}+\bar{D}\pi-\Delta\bar{\lambda}-\bar{\Delta}\lambda=2\mu\bar{\mu}+2\nu\bar{\nu}+\\&+\bar{\pi}(\bar{\alpha}-\tau)+\pi(\alpha-\bar{\tau})-\lambda(\varepsilon+\varrho)-\bar{\lambda}(\bar{\varepsilon}+\bar{\varrho})+2\Psi_{+}+\frac{R}{6},\end{aligned} \quad (270)$$

$$\begin{aligned}&\Delta\varrho+\bar{\Delta}\bar{\varrho}-D\bar{\tau}-\bar{D}\tau=2\sigma\bar{\sigma}+2\kappa\bar{\kappa}+\\&+\bar{\varrho}(\varepsilon-\bar{\lambda})+\varrho(\bar{\varepsilon}-\lambda)-\tau(\alpha+\bar{\pi})-\bar{\tau}(\bar{\alpha}+\pi)+2\Psi_{-}+\frac{R}{6},\end{aligned} \quad (271)$$

$$\Delta\bar{\varrho}-D\bar{\kappa}=\bar{\varrho}(\bar{\varrho}-\bar{\varepsilon})+\bar{\kappa}(2\bar{\gamma}+\tau-\bar{\alpha})-\bar{\sigma}\mu-\bar{\tau}\nu+\Phi_{00}, \quad (272)$$

$$\bar{D}\bar{\tau}-\bar{\Delta}\bar{\sigma}=\bar{\sigma}(2\beta-\varrho-\varepsilon)-\bar{\tau}(\alpha+\bar{\tau})+\bar{\mu}\bar{\varrho}+\bar{\nu}\bar{\kappa}-\bar{\Phi}_{02}. \quad (273)$$

In this language the Bianchi identities for the Riemann tensor are given by

$$\begin{aligned}&-\Delta\Psi_1-\bar{D}\Psi_0+D\Phi_{00}-\Delta\Phi_{01}-3\Psi_{+}\nu+\Psi_0(2\alpha+2\gamma+\bar{\pi})-\Psi_1(4\lambda+\bar{\beta}+\bar{\varepsilon})+\\&+\Phi_{01}(2\bar{\varrho}-\bar{\beta}-\bar{\varepsilon})-\Phi_{02}\bar{\kappa}+\Phi_{00}(2\bar{\gamma}+\tau)-2\Phi_{12}\mu+2\Phi_{11}\nu=0,\end{aligned} \quad (274)$$

$$\begin{aligned}&-\bar{\Delta}\Psi_1+D\Psi_{+}-\bar{\Delta}\Phi_{01}-\bar{D}\Phi_{02}+\frac{1}{12}DR+\Psi_1(\beta+\varepsilon-2\bar{\lambda})+\Psi_0\bar{\nu}-2\bar{\Psi}_1\mu+\\&-3\Psi_{+}\pi+\Phi_{01}(2\varrho+\beta+\varepsilon)+\Phi_{02}(2\alpha-\bar{\tau})-2\bar{\Phi}_{12}\lambda+\Phi_{00}\kappa+2\Phi_{11}\pi=0,\end{aligned} \quad (275)$$

$$\begin{aligned}&\bar{\Delta}\Psi_0-D\Psi_1+D\Phi_{01}-\Delta\Phi_{02}+\Psi_0(\bar{\lambda}-2\beta-2\varepsilon)+\Psi_1(4\pi-\bar{\alpha}-\bar{\gamma})-3\Psi_{+}\mu+\\&+\Phi_{01}(2\tau+\bar{\gamma}+\bar{\alpha})+\Phi_{02}(\bar{\varrho}-2\bar{\beta})-\Phi_{00}\sigma-2\Phi_{11}\mu-2\bar{\Phi}_{12}\nu=0,\end{aligned} \quad (276)$$

$$\begin{aligned}&\bar{D}\Psi_1+\Delta\Psi_{+}-\bar{D}\Phi_{01}-\bar{\Delta}\Phi_{00}+\frac{1}{12}\Delta R-\Psi_1(\gamma+\alpha+2\bar{\pi})-\Psi_0\bar{\mu}-2\bar{\Psi}_1\nu+3\Psi_{+}\lambda+\\&+\Phi_{01}(\gamma-\alpha-2\bar{\tau})+\Phi_{00}(2\varepsilon+\varrho)+2\Phi_{12}\pi-\Phi_{02}\bar{\sigma}+2\Phi_{11}\lambda=0,\end{aligned} \quad (277)$$

$$-\Delta\Psi_3 - D\Psi_4 + \bar{D}\Phi_{00} - \Delta\Phi_{12} + \Psi_4(2\bar{\alpha} - 2\bar{\gamma} - \tau) + \Psi_3(4\bar{\varrho} + \bar{\beta} - \bar{\varepsilon}) + 3\Psi_-\bar{\kappa} +$$
$$+\Phi_{12}(\bar{\beta} - \bar{\varepsilon} - 2\lambda) - \Phi_{00}(\bar{\pi} + 2\gamma) + 2\Phi_{01}\bar{\sigma} - 2\Phi_{11}\bar{\kappa} + \bar{\Phi}_{02}\nu = 0, \tag{278}$$

$$-\bar{\Delta}\Psi_3 + \bar{D}\Psi_- - \bar{\Delta}\Phi_{12} - D\bar{\Phi}_{02} + \frac{1}{12}\bar{D}R + \Psi_3(2\varrho + \varepsilon - \beta) - \Psi_4\kappa + 2\bar{\Psi}_3\bar{\sigma} + 3\Psi_-\bar{\tau} +$$
$$+\Phi_{12}(\varepsilon - \beta - 2\bar{\lambda}) + \bar{\Phi}_{02}(2\bar{\alpha} + \pi) + 2\bar{\Phi}_{01}\bar{\varrho} - \Phi_{00}\bar{\nu} - 2\Phi_{11}\bar{\tau} = 0, \tag{279}$$

$$\bar{\Delta}\Psi_4 - \bar{D}\Psi_3 + \bar{D}\Phi_{12} - \Delta\bar{\Phi}_{02} + \Psi_4(2\beta - \varrho - 2\varepsilon) + \Psi_3(\gamma - \alpha - 4\bar{\tau}) + 3\Psi_-\bar{\sigma} +$$
$$+\Phi_{12}(\alpha - \gamma - 2\bar{\pi}) + \bar{\Phi}_{02}(2\bar{\beta} - \lambda) + \Phi_{00}\bar{\mu} + 2\Phi_{11}\bar{\sigma} + 2\bar{\Phi}_{01}\bar{\kappa} = 0, \tag{280}$$

$$D\Psi_3 + \Delta\Psi_- - D\Phi_{12} - \bar{\Delta}\Phi_{00} + \frac{1}{12}\Delta R + \Psi_3(\bar{\gamma} - \bar{\alpha} + 2\tau) + \Psi_4\sigma - 3\Psi_-\bar{\varrho} + 2\bar{\Psi}_3\bar{\kappa} +$$
$$+\Phi_{12}(2\pi - \bar{\gamma} - \bar{\alpha}) + \Phi_{00}(2\varepsilon - \bar{\lambda}) - 2\Phi_{01}\bar{\tau} - 2\Phi_{11}\bar{\varrho} + \bar{\Phi}_{02}\mu = 0, \tag{281}$$

$$\bar{\Delta}\Phi_{01} + \Delta\bar{\Phi}_{12} + D\Phi_{11} + \bar{D}\Phi_{02} - \frac{1}{8}DR + \Phi_{02}(\bar{\tau} - \bar{\pi} - 2\alpha) - \Phi_{01}(2\varrho + \varepsilon + \beta - \bar{\lambda}) +$$
$$+\bar{\Phi}_{12}(2\lambda + \bar{\beta} - \bar{\varepsilon} - \bar{\varrho}) + 2\Phi_{11}(\tau - \pi) - \Phi_{12}\sigma - \Phi_{00}\kappa + \bar{\Phi}_{00}\nu + \bar{\Phi}_{01}\mu = 0, \tag{282}$$

$$\bar{D}\Phi_{01} + D\Phi_{12} - \Delta\Phi_{11} + \bar{\Delta}\Phi_{00} - \frac{1}{8}\Delta R + \Phi_{00}(\bar{\lambda} - \varrho - 2\varepsilon) + \Phi_{01}(2\bar{\tau} - \alpha - \gamma - \bar{\pi}) +$$
$$+\Phi_{12}(\bar{\gamma} - \bar{\alpha} + \tau - 2\pi) + 2\Phi_{11}(\bar{\varrho} - \lambda) + \Phi_{02}\bar{\sigma} + \bar{\Phi}_{12}\bar{\kappa} - \bar{\Phi}_{02}\mu - \bar{\Phi}_{01}\nu = 0. \tag{283}$$

Equations (222-227), (254-273), (274-283) constitute the all system of Newman-Penrose equations for the metric (217). For the completeness they can be supplemented by the identities which are implied by the identities $d^2M = 0$, $d^2N = 0$. However we do not write them here.

Transformations

$$D \longleftrightarrow \Delta \tag{284}$$

leave invariant the form of the metric (217). They induces the following transformations of the Newman-Penrose coefficients

$$\alpha \longleftrightarrow \varepsilon, \quad \mu \longleftrightarrow -\nu, \quad \bar{\sigma} \longleftrightarrow -\kappa, \quad \beta \longleftrightarrow \gamma, \quad \bar{\varrho} \longleftrightarrow -\tau, \quad \lambda \longleftrightarrow -\pi;$$
$$\Phi_{00} \longleftrightarrow \Phi_{02}, \quad \Phi_{01} \longleftrightarrow \Phi_{01}, \quad \Phi_{11} \longleftrightarrow -\Phi_{11}, \quad \Phi_{12} \longleftrightarrow \bar{\Phi}_{12};$$
$$\Psi_0 \longleftrightarrow \Psi_0, \quad \Psi_1 \longleftrightarrow -\Psi_1, \quad \Psi_\pm \longleftrightarrow \Psi_\pm, \quad \Psi_3 \longleftrightarrow -\bar{\Psi}_3, \quad \Psi_4 \longleftrightarrow \bar{\Psi}_4.$$

Similarly transformations

$$D \longleftrightarrow \bar{D}, \tag{285}$$

which also preserves the form of the metric (217), induce the following transformations of the Newman-Penrose coefficients:

$$\alpha \longleftrightarrow \bar{\alpha}, \quad \bar{\sigma} \longleftrightarrow -\mu, \quad \bar{\kappa} \longleftrightarrow -\nu, \quad \varepsilon \longleftrightarrow \varepsilon, \quad \bar{\pi} \longleftrightarrow -\tau, \quad \gamma \longleftrightarrow -\gamma,$$
$$\beta \longleftrightarrow -\beta, \quad \lambda \longleftrightarrow -\bar{\varrho};$$
$$\Phi_{00} \longleftrightarrow \Phi_{00}, \quad \Phi_{01} \longleftrightarrow \Phi_{12}, \quad \Phi_{02} \longleftrightarrow \bar{\Phi}_{02}, \quad \Phi_{11} \longleftrightarrow \Phi_{11};$$
$$\Psi_0 \longleftrightarrow \Psi_4, \quad \Psi_1 \longleftrightarrow \Psi_3, \quad \Psi_\pm \longleftrightarrow \Psi_\mp.$$

3.3 Orthogonal almost-complex structures on 4-dimensional Euclidean manifolds in Newman-Penrose formalism. Cartan-Petrov-Penrose classification.

Given an Euclidean metric g on an oriented 4-manifold $\mathcal{M}$ we consider an orthogonal almost complex structure J. We define a special class of Newman-Penrose tetrads which are *(+)adapted* to the pair $(g,\ J)$. Let f_1 be an unit vector. Define f_2 by $f_2 = J(f_1)$. Let f_3 be an unit vector orthogonal to both f_1 and f_2. Define f_4 by $f_4 = J(f_3)$. By virtue of properties of g and J the basis (f_μ) $(\mu = 1,\ 2,\ 3,\ 4)$ is orthonormal. The (+)adapted Newman-Penrose tetrad consists of complex vectors m, $\bar{m}$, n and $\bar{n}$ which are defined by

$$m = \frac{f_1 + if_2}{\sqrt{2}} \qquad n = \frac{f_3 + if_4}{\sqrt{2}}. \tag{286}$$

If (ϑ^μ)=$(M,\ \bar{M},\ N,\ \bar{N})$ is a dual to $(m,\ \bar{m},\ n,\ \bar{n})$ then g and J read [7]

$$g = 2(M\bar{M} + N\bar{N}), \tag{287}$$

$$J = i(\bar{M} \otimes \bar{m} - M \otimes m + \bar{N} \otimes \bar{n} - N \otimes n) \tag{288}$$

[7]Note that according to Proposition 3.2.1 if J agrees (respectively, disagrees) with the orientation of $\mathcal{M}$ then equations $\Psi_0 = \Psi_1 = \Psi_+ = 0$ mean that the metric is self-dual (respectively, anti-self-dual).

The $(+)$adapted Newman-Penrose tetrad is not unique; it is defined up to the following freedom

$$M' = \frac{e^{iu}}{\sqrt{1+\xi\bar{\xi}}}[M + \xi N]$$
$$N' = \frac{e^{iv}}{\sqrt{1+\xi\bar{\xi}}}[N - \bar{\xi} M] \tag{289}$$

where u, v (real) and ξ (in general complex) are some functions on $\mathcal{M}$. The integrability conditions for J expressed in terms of the $(+)$adapted Newman-Penrose tetrad read

$$dM \wedge M \wedge N = 0 \qquad dN \wedge M \wedge N = 0.$$

Rewriting this in terms of the Newman-Penrose coefficients of $(+)$adapted tetrad we have

$$J \text{ is integrable} \Leftrightarrow \mu = \nu = 0. \tag{290}$$

As can be expected these conditions are invariant under transformations (289); more explicitly in tetrad (289) we have

$$\mu' = \frac{\mu + \bar{\xi}\nu}{\sqrt{1+\xi\bar{\xi}}} \qquad \nu' = \frac{\nu - \xi\mu}{\sqrt{1+\xi\bar{\xi}}}. \tag{291}$$

The Newman-Penrose equation (268) shows that if J is integrable then $\Psi_0 = 0$. This condition is also invariant under transformations (289). It can be considered independently of integrability conditions (290). Given g we, therefore, ask for all orthogonal almost complex structures which in some (therefore any) $(+)$adapted Newman-Penrose tetrad satisfy

$$\Psi_0 = 0.$$

Consider now a triple $(g,\ J,\ \vartheta^\mu)$ consisting of: Euclidean metric g, an orthogonal almost complex structure J and an $(+)$adapted to them Newman-Penrose tetrad ϑ^μ. Any other orthogonal almost complex structure with the same helicity as J is given by

$$J(x,\eta) = xJ + \eta K + \bar{\eta}\bar{K}, \tag{292}$$

where $\eta = y + iz$, $x^2 + y^2 + z^2 = 1$ and K is given in terms of tetrad ϑ^μ by

$$K = \bar{N} \otimes m - \bar{M} \otimes n. \tag{293}$$

The definition of $J(x,\eta)$ is independent of the choice of the tetrad since

$$K' = \mathrm{e}^{-i(v+u)}K$$

under transformations (289). This shows that the set of all orthogonal almost complex structures of the same helicity forms a sphere $\mathbf{S}^2$. It is convenient to introduce a stereographic variable

$$\zeta = \frac{iy+z}{1+x}.$$

In this way we write $J(\zeta)$ instead of $J(x,\eta)$ in the following. To any point $\zeta \sim (x,\ y,\ z)$ of the sphere there exists an $(+)$adapted Newman-Penrose tetrad associated with the corresponding $J(\zeta)$ and g. If $(M, \bar{M}, N, \bar{N})$ is an $(+)$adapted Newman-Penrose tetrad corresponding to $J(0)$ then the $(+)$adapted Newman-Penrose tetrad corresponding to $J(\zeta)$ reads

$$\begin{aligned} M(\zeta) &= \frac{M - \bar{\zeta}\bar{N}}{\sqrt{1+\zeta\bar{\zeta}}}, \\ N(\zeta) &= \frac{N + \bar{\zeta}\bar{M}}{\sqrt{1+\zeta\bar{\zeta}}}. \end{aligned} \tag{294}$$

If $\Psi_0,\ \Psi_1,\ \Psi_+$ denote the Newman-Penrose Weyl curvature coefficients computed in the tetrad with $\zeta = 0$ then the Weyl coefficient $\Psi_0(\zeta)$ reads

$$(1+\zeta\bar{\zeta})^2\Psi_0(\zeta) = \Psi_0 - 4\zeta\Psi_1 + 6\zeta^2\Psi_+ + 4\zeta^3\bar{\Psi}_1 + \zeta^4\bar{\Psi}_0. \tag{295}$$

¿From this one finds that if $\Psi_0\Psi_1\Psi_+ \neq 0$ then there are at most four values of ζ, say ζ_i, $(0 < i \leq 4)$, for which $\Psi_0(\zeta)$ vanishes. The solutions of equation

$$\Psi_0(\zeta) = 0, \tag{296}$$

are divided into two pairs for if ζ is a solution then $-\bar{\zeta}^{-1}$ also is. The points $(\zeta,$ $-\bar{\zeta}^{-1})$ describes a pair of antipodal points on the sphere; the corresponding almost complex structures are conjugated to each other i.e. $J(\zeta) = -J(-\bar{\zeta}^{-1})$. The almost complex structures $J(\zeta_i)$ are, therefore, divided into two pairs of mutually conjugated structures. They are called *(*)principal*, where in place of the sign "*" one puts either "+" or "-" depending on whether one of them (therefore any) agrees or disagrees with the orientation of $\mathcal{M}$, respectively. It follows from the construction that the number of such structures is an invariant property of the metric itself.

If equation (296) has multiple roots then the two pairs of almost complex structures reduce to only one doubly degenerated pair. In such case the metric is called *(*)algebraically special*, where the sign "*" is as before.

Consider now orthogonal almost complex structures $J'(\zeta)$ of an opposite helicity than J of (288). They form another sphere which is formally obtained from definitions (292-293) by means of transformation (285) which replaces M into $\bar{M}$, m into $\bar{m}$ and vice versa. By virtue of this transformation we find that

$$J'(0) = i(M \otimes m - \bar{M} \otimes \bar{m} + \bar{N} \otimes \bar{n} - N \otimes n) \tag{297}$$

is integrable iff $\kappa = \sigma = 0$. [8] This, in particular, implies that $\Psi_4 = 0$. This property is invariant under the possible transformations of the Newman-Penrose tetrad which preserves the form of the metric and the form (297) of $J'(0)$. If J' has the same helicity as $J'(0)$ then any Newman-Penrose tetrad in which the metric and J' have the form (287) and (297), respectively, is called *(-)adapted*. One can find (-)adapted Newman-Penrose tetrads for any of $J'(\zeta)$ and ask about such ζ's for which $\Psi_4(\zeta) = 0$. These are defined as the roots of the following equation

$$\Psi_4 - 4\zeta\Psi_3 + 6\zeta^2\Psi_- + 4\zeta^3\bar{\Psi}_3 + \zeta^4\bar{\Psi}_4 = 0. \tag{298}$$

The corresponding $J'(\zeta)$'s are called *(*)principal*, where in place of the sign "*" one puts the opposite sign to that which was attributed to structures $J(\zeta_i)$. Again, for general g, we have two pairs of mutually conjugated such structures. If these pairs degenerate to only one pair the metric is called (*)algebraically special.

The Euclidean metric over 4-manifold is called *properly algebraically special* if it is either (+) or (-)algebraically special. It is called *algebraically special* if it is either properly algebraically special or its Weyl tensor is either self- or anti-self-dual. The similar terminology is applied also to an Euclidean manifold itself.

Thus, considering orthogonal almost complex structures over Euclidean 4-manifold we arrived at the algebraic classification of Euclidean metrics, similar to the Cartan-Petrov-Penrose classification in Lorentzian case [7], [46], [41]. Let a denotes a one pair of mutually conjugated (+)principal orthogonal almost complex structures, b denotes a one pair of mutually conjugated (-)principal orthogonal almost complex structures, **a**, **b** denote doubly degenerated pairs of orthogonal almost complex structures (+)

[8] Note that transformation (285) implies $(\mu = \nu = 0) \longleftrightarrow (\kappa = \sigma = 0)$.

and (-)principal, respectively, and let signs "+" and "-" denote that the metric is self- or anti-self-dual, respectively. With such a notation we have the following Cartan-Petrov-Penrose types of the metric: $aabb$, $\mathbf{a}bb$, $+bb$, $aa\mathbf{b}$, $\mathbf{ab}$, $+\mathbf{b}$, $aa-$, $\mathbf{a}-$, $+-$; in the last case the metric is conformally flat.

In the following theorem, which is implied by equations (296) and (298), the sign "$\Leftrightarrow$" means "iff in some Newman-Penrose tetrad".

Theorem 3.3.1 *The metric is of Cartan-Petrov-Penrose type*

$$\mathbf{a}bb \;\Leftrightarrow\; \Psi_0 = \Psi_1 = 0 \tag{299}$$
$$+bb \;\Leftrightarrow\; \Psi_0 = \Psi_1 = \Psi_+ = 0 \tag{300}$$
$$aa\mathbf{b} \;\Leftrightarrow\; \Psi_3 = \Psi_4 = 0 \tag{301}$$
$$\mathbf{ab} \;\Leftrightarrow\; \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{302}$$
$$+\mathbf{b} \;\Leftrightarrow\; \Psi_0 = \Psi_1 = \Psi_+ = \Psi_3 = \Psi_4 = 0 \tag{303}$$
$$aa- \;\Leftrightarrow\; \Psi_3 = \Psi_4 = \Psi_- = 0 \tag{304}$$
$$\mathbf{a}- \;\Leftrightarrow\; \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = \Psi_- = 0 \tag{305}$$
$$+- \;\Leftrightarrow\; \Psi_0 = \Psi_1 = \Psi_+ = \Psi_3 = \Psi_4 = \Psi_- = 0. \tag{306}$$

We note, that the matric is (*)algebraically special iff in some Newman-Penrose tetrad $\Psi_0 = \Psi_1 = 0,\ \Psi_+ \neq 0$ or $\Psi_4 = \Psi_3 = 0,\ \Psi_- \neq 0$.

3.4 Towards integrability conditions for an existence of an orthogonal complex structure on a 4-dimensional Euclidean manifold.

Given an orthogonal almost complex structure J we know that it is complex iff in a (+)adapted Newman-Penrose tetrad it satisfies $\mu = \nu = 0$. These equations can be equivalently written in terms of the tetrad 1-forms as

$$\mathrm{d}M \wedge M \wedge N = 0 \qquad \mathrm{d}N \wedge M \wedge N = 0. \tag{307}$$

Equations (307) imply that $\Psi_0 = 0$ i.e. if both self-dual and anti-self-dual parts of the Weyl tensor do not vanish *only (*)principal orthogonal almost complex structures can* *be complex.* [9] To answer whether in (anti-)self-dual case one also can have orthogonal complex structures we do as follows.

Consider a sphere of almost complex structures (292-293) of a given helicity written in a (+)adapted Newman-Penrose tetrad. Without any particular assumption on Euclidean metric g we ask what are necessary and sufficient conditions for ζ in order to describe an integrable $J(\zeta)$. It is clear that these conditions reads

$$\mathrm{d}(M-\bar{\zeta}\bar{N})\wedge(M-\bar{\zeta}\bar{N})\wedge(N+\bar{\zeta}\bar{M})=0,\qquad \mathrm{d}(N+\bar{\zeta}\bar{M})\wedge(M-\bar{\zeta}\bar{N})\wedge(N+\bar{\zeta}\bar{M})=0, \tag{308}$$

i.e. follow from conditions (307) after replacing the tetrad $(M,\ \bar{M},\ N,\ \bar{N})$ by the Newman-Penrose tetrad (294) (+)adapted to $J(\zeta)$. Rewriting these equations in terms of the Newman-Penrose coefficients of $(M,\ \bar{M},\ N,\ \bar{N})$ we obtain the following two differential equations for ζ:

$$\bar{D}\bar{\zeta}-\bar{\zeta}\Delta\bar{\zeta}+\bar{\mu}+\bar{\zeta}(\alpha-\bar{\pi}+\gamma)+\bar{\zeta}^2(\bar{\beta}+\lambda+\bar{\varepsilon})-\bar{\zeta}^3\nu=0 \tag{309}$$

$$\bar{\Delta}\bar{\zeta}+\bar{\zeta}D\bar{\zeta}+\bar{\nu}+\bar{\zeta}(\bar{\lambda}+\beta+\varepsilon)+\bar{\zeta}^2(\pi-\bar{\gamma}-\bar{\alpha})+\bar{\zeta}^3\mu=0. \tag{310}$$

Integrability conditions for these two equations follows from the successive use of the commutators (226), (222), (227), (217), (218), (223) and read:

$$\bar{\Psi}_0-4\bar{\zeta}\bar{\Psi}_1+6\bar{\zeta}^2\Psi_++4\bar{\zeta}^3\bar{\Psi}_1+\bar{\zeta}^4\bar{\Psi}_0=0. \tag{311}$$

These are exactly conditions (296) saying that if the corresponding $\Psi_0\Psi_1\Psi_+\neq 0$ then $J(\zeta)$ is (*)principal, as we already predicted. We note, however, that our conditions (311) say also something more: since they are the *integrability conditions* of equations (309-310) and are *identically satisfied* if $\Psi_0=\Psi_1=\Psi_+=0$ then in such case the equations (309-310) always can be solved. Thus if $\Psi_0=\Psi_1=\Psi_+=0$ i.e. if the metric is (anti-)self-dual there always exists at least one orthogonal *complex* structure on the manifold.

Theorem 3.4.1 *A (anti-)self-dual Euclidean 4-manifold (locally) always admits orthogonal integrable complex structure.*

[9] Note the similarity to the Lorentzian case in which only principal null directions can be simultaneously geodesic and shear-free if the Weyl tensor does not vanish [45], [54]. This similarity is due to the analogy described in the Introduction!

It is an interesting question how many such structures can exist on a (anti-) self-dual manifold. We failed in answering this question in the general half conformally flat case. In the more special case of totally conformally flat Euclidean 4-manifold there are two families of orthogonal complex structures described as follows.
First we note that any conformally flat Euclidean 4-manifold admits the same orthogonal complex structures that the flat one. Therefore we can restrict ourselves to the flat case when the metric has the form

$$g = 2(\mathrm{d}z\mathrm{d}\bar{z} + \mathrm{d}w\mathrm{d}\bar{w}). \tag{312}$$

The Newman-Penrose tetrad $(\mathrm{d}z,\ \mathrm{d}\bar{z},\ \mathrm{d}w,\ \mathrm{d}\bar{w})$ defines a family of Newman-Penrose tetrads $(m(\zeta),\ \overline{m(\zeta)},\ n(\zeta),\ \overline{n(\zeta)})$ with duals $(M(\zeta),\ \overline{M(\zeta)},\ N(\zeta),\ \overline{N(\zeta)})$ by

$$M(\zeta) = \frac{\mathrm{d}z - \bar{\zeta}\mathrm{d}\bar{w}}{\sqrt{1+\zeta\bar{\zeta}}},$$
$$N(\zeta) = \frac{\mathrm{d}w + \bar{\zeta}\mathrm{d}\bar{z}}{\sqrt{1+\zeta\bar{\zeta}}}.$$

Any such tetrad defines an orthogonal almost complex structure $J(\zeta)$ by

$$J(\zeta) = i(\overline{M(\zeta)} \otimes \overline{m(\zeta)} - M(\zeta) \otimes m(\zeta) + \overline{N(\zeta)} \otimes \overline{n(\zeta)} - N(\zeta) \otimes n(\zeta)). \tag{313}$$

Let $\mu(\zeta)$ denotes Newman-Penrose coefficient μ in the tetrad parameterized by ζ, etc. By applying (290) we find that $J(\zeta)$ is integrable iff ζ satisfies the following equations

$$\nu(\zeta) = \mu(\zeta) = 0$$

or, equivalently, iff

$$\mathrm{d}M(\zeta) \wedge M(\zeta) \wedge N(\zeta) = 0 \qquad \mathrm{d}N(\zeta) \wedge M(\zeta) \wedge N(\zeta) = 0.$$

The last two conditions written explicitly in terms of z and w read

$$\mathrm{d}\bar{\zeta} \wedge \mathrm{d}\bar{w} \wedge \mathrm{d}(z - \bar{\zeta}\bar{w}) \wedge \mathrm{d}(w + \bar{\zeta}\bar{z}) = 0 \qquad \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}(z - \bar{\zeta}\bar{w}) \wedge \mathrm{d}(w + \bar{\zeta}\bar{z}) = 0.$$

Under the assumption that ζ is analytic they can be easily integrated. The general solution for ζ is implicitly defined by

$$\bar{\zeta} = f(z - \bar{\zeta}\bar{w},\ w + \bar{\zeta}\bar{z}) \tag{314}$$

where f is any holomorphic function of variables $z-\bar{\zeta}\bar{w},\ w+\bar{\zeta}\bar{z}$. Any such solution defines an orthogonal integrable almost complex structure by (313).
Another family of orthogonal complex structures $J'(\zeta')$ for the metric (312) has an opposite helicity than $J(\zeta)$. It is given in terms of the Newman-Penrose tetrad $(M(\zeta'),\ \overline{M(\zeta')},\ N(\zeta'),\ \overline{N(\zeta')})$ defined by

$$M(\zeta')=\frac{\mathrm{d}z-\zeta'\mathrm{d}w}{\sqrt{1+\zeta'\bar{\zeta}'}},$$
$$N(\zeta')=\frac{\mathrm{d}w+\bar{\zeta}'\mathrm{d}z}{\sqrt{1+\zeta'\bar{\zeta}'}}.$$

In this tetrad complex structures read

$$J'(\zeta')=i(M(\zeta')\otimes m(\zeta')-\overline{M(\zeta')}\otimes\overline{m(\zeta')}+\overline{N(\zeta')}\otimes\overline{n(\zeta')}-N(\zeta')\otimes n(\zeta')),\quad (315)$$

where ζ' satisfies the equation

$$\bar{\zeta}'=f(\bar{z}-\bar{\zeta}'\bar{w},\ w+\bar{\zeta}'z)\quad (316)$$

with some holomorphic function f of variables $\bar{z}-\bar{\zeta}'\bar{w},\ w+\bar{\zeta}'z$.

One checks that all possible orthogonal complex structures in conformally flat metrics $g'=\varphi^2 g$ are given by either $J(\zeta)$ or $J'(\zeta')$ with ζ and ζ' defined in terms of some holomorphic function f of two variables by (314) or (316), respectively. This is a part of an Euclidean analog of the Kerr theorem [61].[10]

Finally, we note that, on the ground of our considerations, it is in principle possible to find the necessary and sufficient conditions for the Euclidean metric to admit an orthogonal complex structure: a possible strategy is to solve *algebraic* equations (311) for ζ and insert them to equations (309) and (310). As a result one obtains (very complicated) first order in derivative conditions for the Weyl tensor. Only if these conditions are satisfied the metric can admit an orthogonal complex structure. We have not a geometric interpretation of these conditions yet. We will publish them elsewhere.

[10]In Lorentzian case all congruences of shear-free and null geodesics in Minkowski space are given in terms of holomorphic function f of two variables by a relation analogous to (314) (or (316)). The Robinson congruence corresponds to a function f which is linear in both of its arguments. It is therefore interesting to study $J(\zeta)$ (or $J'(\zeta')$) with $\bar{\zeta}=a(z-\bar{\zeta}\bar{w})+b(w+\bar{\zeta}\bar{z})$ (or $\bar{\zeta}'=a(\bar{z}-\bar{\zeta}'\bar{w})+b(w+\bar{\zeta}'z)$), where a and b are some complex constants. These are the Euclidean analogs of the Robinson congruence.

3.5 Einstein 4-manifolds with Euclidean metric admitting orthogonal complex structures. Goldberg-Sachs theorem.

We say that Euclidean manifold is *Einstein* if its metric g satisfies Einstein equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{317}$$

In this section we find necessary and sufficient conditions for Einstein 4-manifolds to admit an orthogonal complex structure.
Let $(M,\, \bar{M},\, N,\, \bar{N})$ be a Newman-Penrose tetrad for g with Newman-Penrose coefficients as defined in Section 3.1 and 3.2

Theorem 3.5.1 *Suppose that g satisfies the Einstein equations (317).*
(i) If $\Psi_0 = \Psi_1 = 0,\ \Psi_+ \neq 0$ then $\mu = \nu = 0$.
(ii) If $\mu = \nu = 0$ then $\Psi_0 = \Psi_1 = 0$.
(iii) If $\Psi_3 = \Psi_4 = 0,\ \Psi_- \neq 0$ then $\kappa = \sigma = 0$.
(iv) If $\kappa = \sigma = 0$ then $\Psi_3 = \Psi_4 = 0$.

PROOF.
Einstein equations (317) when written in terms of the Newman-Penrose coefficients read

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = \Phi_{12} = 0. \tag{318}$$

ad (i). If we insert equations (318) and assumptions $\Psi_0 = \Psi_1 = 0$ to the Bianchi identities (274) and (276) we get

$$3\Psi_+\nu = 0 \quad \text{and} \quad 3\Psi_+\mu = 0,$$

respectively. This, compared with the assumption $\Psi_+ \neq 0$, yields the required result.
ad (ii). The assumption

$$\mu = \nu = 0 \tag{319}$$

and the Newman-Penrose equation (268) gives

$$\Psi_0 = 0, \tag{320}$$

as we already noted. Then equations (319), (320), (318) inserted to the Bianchi identities (274) and (276) give

$$\Delta\Psi_1 = -(4\lambda + \bar{\beta} + \bar{\varepsilon})\Psi_1 \tag{321}$$

and

$$D\Psi_1 = (4\pi - \bar{\alpha} - \bar{\gamma})\Psi_1, \tag{322}$$

respectively. The commutator (223) of Δ and D when applied to Ψ_1 gives

$$\begin{aligned}
&(\bar{\beta} - \bar{\varrho})(4\pi - \bar{\alpha} - \bar{\gamma})\Psi_1 + (\bar{\gamma} + \tau)(4\lambda + \bar{\beta} + \bar{\varepsilon})\Psi_1 = \\
&= -(4D\lambda + D\bar{\beta} + D\bar{\varepsilon})\Psi_1 - (4\lambda + \bar{\beta} + \bar{\varepsilon})(4\pi - \bar{\alpha} - \bar{\gamma})\Psi_1 + \\
&-(4\Delta\pi - \Delta\bar{\alpha} - \Delta\bar{\gamma})\Psi_1 + (4\pi - \bar{\alpha} - \bar{\gamma})(4\lambda + \bar{\beta} + \bar{\varepsilon})\Psi_1,
\end{aligned}$$

where we used equations (321-322). We rewrite this equation to the form

$$[\ 4(D\lambda + \Delta\pi) + D\bar{\beta} - \Delta\bar{\alpha} + D\bar{\varepsilon} - \Delta\bar{g} + (\bar{\beta} - \bar{\varrho})(4\pi - \bar{\alpha} - \bar{\gamma}) + (\bar{\gamma} + \tau)(4la + \bar{\beta} + \bar{\varepsilon})\ ]\Psi_1 = 0. \tag{323}$$

Now we express $D\lambda$ in terms of equation (257), $\Delta\pi$ in terms of (256), $(D\bar{\beta} - \Delta\bar{\alpha})$ in terms of (258) and $(D\bar{\varepsilon} - \Delta\bar{\gamma})$ in terms of (259). Inserting this to equation (323) we find that (323) is equivalent to

$$-10\Psi_1^2 = 0,$$

which together with (320) finishes the proof.
ad (*iii*) and (*iv*). The results are obtained from (*i*) and (*ii*) by means of transformation (285).

Theorem 3.5.1 (when compared with Theorem 3.3.1) means that an existence of an orthogonal complex structure on the Einstein 4-manifold implies algebraic speciality of the metric. Moreover if the Einstein 4-manifold is properly algebraically special then the multiple almost complex structure is always complex. Another implication of the theorem is that in the Einstein 4-manifold which is neither self- nor anti-self-dual there are at most two orthogonal complex structures. The case of two such structures can happen only if the metric is of type **ab** in the Cartan-Petrov-Penrose classification of the preceeding section. Recalling the analogy between orthogonal almost complex structures and almost optical geometries defined in the introduction we find that Theorem 3.5.1 is an Euclidean analog of the Goldberg-Sachs theorem [13].

4 CONCLUSIONS

Research on Lorentzian and Euclidean 4-dimensional manifolds has been carried on by both mathematicians and physicists for a long time. 4-dimensional Lorentzian manifolds satisfying Einstein equations are of particular interest for physicists since they correspond to the possible physical spacetimes. One of the efficient methods of constructing such manifolds (or, more precisely, of finding local forms of the metrics that satisfy Einstein equations) is based on an assumption that the manifold admits a congruence of shear-free and null geodesics. The relation between manifolds satisfying this assumption and 3-dimensional CR-structures is established quite recently [52] and Chapter 2 of this thesis is one of the first application of this fact.

It was known for a long time that Einstein equations

$$R_{\mu\nu} = \Phi k_\mu k_\nu \tag{324}$$

for spacetimes $\mathcal{M}$ admitting a congruence of shear-free and null geodesics tangent to the vector field k_μ reduce to the equations on some 3-dimensional submanifold $\mathcal{Q}$ of $\mathcal{M}$. If the congruence is nontwisting further simplifications occur and one finally ends with the only condition

$$\Phi = \Phi(P, m) \geq 0 \tag{325}$$

for two real functions P and m. This is the positive energy condition for the pure radiation, and is either an equation (in the vacuum case) or an inequality (in the general case).

In this work we have only considered equations (324) in the case of twisting k_μ; this situation is much more complicated than the nontwisting one.

Manifold $\mathcal{Q}$ to which Einstein equations (324) reduce is naturally endowed with a 3-dimensional CR-structure. Complexity of equations in the case of twisting k_μ corresponds to the nondegeneracy of this structure. However, this nondegeneracy has also a fortunate feature. This can be described as follows.

Any metric on $\mathcal{M} = \mathbf{R} \times \mathcal{Q}$ admitting congruence of shear-free and null geodesics k_μ can be written in the form

$$g = 2p^2(\mu\bar{\mu} - \lambda\phi), \tag{326}$$

where λ (real) and μ (complex) are 1-forms defined on $\mathcal{Q}$ and annihilated by k. These forms are determined up to the following freedom

$$\lambda \to \lambda' = f\lambda, \qquad \mu \to \mu' = h\mu + s\lambda. \tag{327}$$

Transformations (refeq:4c) preserves the form (refeq:3c) of the metric. To fix a gauge in this freedom all the authors were using a corollary from Einstein equations (324) which guaranteed that there existed a form μ' among forms generated by (327) such that

$$\mu' \wedge \mathrm{d}\mu' = 0. \tag{328}$$

Gauge-fixing condition (328) allowed for the introduction of a coordinate system $(\xi, \bar{\xi}, u)$ on $\mathcal{Q}$ such that ξ was a complex function satisfying $\mu' \wedge \mathrm{d}\xi = 0$ and u was a real function such that $\lambda' = \mathrm{d}u + L\mathrm{d}\xi + \bar{L}\mathrm{d}\bar{\xi}$. The reduced Einstein equations were then written in these coordinates.
The main objection against this approach is that the gauge-fixing condition (328) does not follow from the CR-geometry of Q, which was given *before* Einstein equations (324) were imposed; it does not suit to the CR geometry, hence the final equations are not CR-invariant.
A gauge fixing procedure which agrees with the CR geometry and totally fixes λ, μ, p and ϕ of (326) is only possible if

$$\lambda \wedge \mathrm{d}\lambda \neq 0 \tag{329}$$

i.e. if the CR-structure is nondegenerate. We described this procedure in Section 2.1.3 and identified the resulting forms

$$\Omega = f_o\lambda, \qquad \Omega_1 = h_o\mu + s_o\lambda \tag{330}$$

with a part of Cartan invariant forms (23-26).

The main result of the whole Chapter 2 is a formulation of the reduced Einstein equations (324) as CR-invariant equations on the CR-structure $\mathcal{Q}$ associated with the congruence k. This is done in Sections 2.2 and 2.4 (see also Section 2.6 for type N vacuum equations) by using Cartan invariant forms Ω and Ω_1 instead of λ and μ in (326). The final equations are (88-90) for one real and two complex functions on $\mathcal{Q}$. (These are supplemented by an additional equation (154) in the case of the vacuum type N). An unsolved and very interesting problem here is to find integrability conditions of equations (88-90), (154) in terms of the Cartan invariants α, β, γ, ϑ, $,\eta$ and ξ of the CR-structure (see (26) for definitions). This would answer the following question of A. Trautman: which nondegenerate CR-structures can be lifted to the type N vacuum spacetimes. Even in the simpler case of lifting to the Minkowski space the answer is known in special cases only [55].

A comment on what we mean by a CR-invariant formulation of the reduced equations (324) is worthwhile. Suppose that we have a CR-structure $\mathcal{Q}$ which is defined in terms of two different pairs of forms $(\lambda', \mu`)$ from the class given by (327). Our formulation obeys such a condition that regardless of which forms $(\lambda',\ \mu`)$ we started, after computation of forms Ω and Ω_1, the reduced Einstein equations (324) are given by (88-90). Another interpretation of these equations is as follows. Suppose that we have forms Ω and Ω_1 being in the class generated by (327) and satisfying

$$\mathrm{d}\Omega = i\Omega_1 \wedge \bar{\Omega}_1. \tag{331}$$

Then equations (88-90) with $\alpha,\ \beta - \gamma.\ \vartheta$ given by (26) are invariant under transformations (327) preserving (331) (see transformations (92-98)).

An interesting feature of equations (88-90) is that they are closely related to the tangential CR-equation (TCRE) of the CR-structure $\mathcal{Q}$. In particular, solutions of the TCRE generate all solutions to (88- 89) (see equations (100, (101) and (105)). If the CR-structure is realizable all solutions of the TCRE are known. Thus, in such case, there is no problem in finding solutions of(88- 89) and finally one ends with only one real equation (90) for one real function p. This, when written in the form (108), is

$$(\Box - \frac{1}{6}R + V)p = -\frac{M + \bar{M}}{p^3} \tag{332}$$

Its part $\Box - \frac{1}{6}R$ is a purely geometrical operator on the CR-structure and consists of the restricted to $\mathcal{Q}$ D'Alambert operator $\Box$ and Ricci scalar R of the Fefferman conformal class of metrics (34).

We were not able to find a general solution to equation (332). However, our methode allowed for a construction of many new solutions. In the table below, which is taken from [20] and supplemented by new results of [10] and [69], we recall all known classes of solutions of Einstein equations admitting twisting congruence of shear-free and null geodesics k. Symbols appearing in this table have the following meaning:
EM - solutions to the Einstein-Maxwell equations with k being simultaneous eigendirection of both Weyl and Maxwell tensor; further specification agrees with the terminology of Section 2.7.1,
PR - solutions to pure radiation Einstein equations (324) with $\Phi > 0$,
V - solutions to the vacuum Einstein equations,
A - all solutions are known,
S - some solutions are known,
$\nexists$ - the equations do not admit such solutions,
empty space - nothing is known about corresponding solutions.

ALGEBRAIC TYPE	II	D	III	N
V	S	A Kinnersley [19]	S	S Hauser [14]
EM generic			$\nexists$	$\nexists$
EM charged vacuum	S	S	$\nexists$	$\nexists$
EM purely radiative		$\nexists$		$\nexists$
PR	S	$\nexists$		$\nexists$

Our solutions, presented in this thesis, fill all the empty spaces in the above table with the sign "S". In particular:
1) pure radiation fields of Cartan-Petrov-Penrose type III are given by solutions of Section 2.3 with 3-dimensional conformal symmetry group of Bianchi types $VI_{h'}$,
2) solutions of Theorems 2.5.1, 2.5.2 are in general pure radiation solutions of type II; if $m = 0$ and the positive energy condition is satisfied then they are of type III,
3) Einstein-Maxwell solutions of Theorem 2.7.2 can be generic, charged vacuums or radiative; in the purely radiative case they can be either of type III (if $m + iM = 0$) or of type II; in the charged or generic case they are of type II, or, if certain equation is satisfied, of type D,
4) explicit form of generic Einstein-Maxwell solutions of type II ($c_2 \neq 0$) and type III ($c_2 = 0$) are given by (206-211),
5) explicit form of purely radiative Einstein-Maxwell solutions of type II ($c_2 \neq 0$) and type III ($c_2 = 0$) are given by (213-216).

An interesting application of our methode would be to study the vacuum type N equations of Section 2.6. We hope that this could lead to a construction of some new solutions enlarging the only known family of Hauser [14].

As we have showed in Section 1.4 an orthogonal almost complex structure in 4-dimensional Euclidean manifold is an analog of a null congruence in Lorentzian 4-manifold. Integrability conditions for the almost complex structure are analogous to the shear-free and geodesic condition for the congruence. In Chapter 3 of the thesis we further studied this analogy constructing Euclidean counterparts of theorems and procedures well known by relativists in Lorentzian case. A convenient tool in such studies was a Euclidean analog of the Newman-Penrose formalism defined in Sections 3.1, 3.2. Using this we proved the following theorem.

Theorem *Suppose that Euclidean manifold $\mathcal{M}$ satisfies Einstein equations*

$$R_{\mu\nu} = \Lambda g_{\mu\nu}.$$

Then $\mathcal{M}$ locally admits an orthogonal integrable complex structure if and only if its metric is algebraically special.

This theorem follows from Theorems 3.3.1, 3.5.1 and integrability conditions (311). It constitutes Euclidean analog of the Goldberg-Sachs theorem valid for Lorentzian 4-manifolds. According to R. Penrose [44] this result is new.
The above theorem implies that Einstein manifold may not admit, even locally, an orthogonal complex structure. A similar conclusion may be obtained even in general (not Einstein) case of Euclidean manifold. Indeed, suppose that the Weyl tensor of an arbitrary Euclidean manifold has nonvanishing selfdual and antiselfdual part. Then the only candidates for orthogonal complex structures are principal almost ccomplex structures (see integrability condition (311)). They are complex if and only if certain, first order in differentials, conditions for the Weyl tensor are satisfied. These conditions are given, in implicit form, by equations (309) -(310), (refeq:paw) (and analogous equations for principal almost complex structures of opposite helicity as discussed at the end of Section 3.4). If they are not satisfied then the manifold does not admit any orthogonal complex structure. We note that this is a purely local result.

Another Euclidean counterpart of Lorentzian theorems is our construction of all locally integrable orthogonal complex structures in flat Euclidean manifold. Any such structure is given by a holomorphic function of two variables as described by equations (314) and (316). This is an analog of the Lorentzian Kerr theorem (see also [15]).

In this thesis we have not exhausted all possibilities of the analogy between complex and optical structures. Further results will be presented in [39].

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