ROBERTSON - WALKER UNIVERSES FROM COLLISION OF PLANE WAVES

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Abstract

For a certain class of metrics which admit two geodesic shearfree, twist-free null congruences k and l the usual Khan - Penrose extension is considered. These metrics describe space - time in which two plane waves collide. We investigate the problem when this collision can produce perfect fluid with four - velocity belonging to the plane spanned by k and l. The particular conformally flat case with metric functions depending on one real variable is solved completely. If we restrict to consider polytropic equation of state, as a result of the collision Friedman - Lemaitre - Robertson - Walker universes with $K = 0$ are obtained. The physical nature of these solutions is puzzling.

1 Introduction

In this paper we study a possibility of constructing Friedman-Lemaitre-Robertson-Walker universes from colliding plane waves. We consider a particular class of metrics for which we perform Khan-Penrose extension [1]. We show that our extended class of metrics can be interpreted as describing spacetime in which two shock waves followed by two planar waves of null dust collide and produce perfect fluid with quite arbitrary equation of state provided that some equations for the metric functions are satisfied (Sections 1, 2, 3). As one of solutions to these equations a spacetime is constructed which in the interaction region is interpreted as a Friedman - Lemaitre - Robertson - Walker spatially flat Universe filled with perfect fluid of polytropic equation of state (Section 4). Depending on the value of parameter γ of the polytrope the Universe created from the waves can either expand or collapse.

Let $A = A(u,v)$ and $B = B(u,v)$ be two *real* functions: we shall assume in the following that A and B are at least C^2 everywhere in the plane u, v, where u, v (*real*) as well as $\zeta, \bar{\zeta}$ (*complex*) are null coordinates. For such functions we can consider the usual Khan - Penrose [1] extension

$$u \to uH(u) \ , \quad v \to vH(v) \ , \tag{1}$$

where $H(u)$ is the Heaviside step function ($H(u) = 1$ for $u \geq 0$, $H(u) = 0$ for $u < 0$); this is accomplished introducing two new functions C, D such that

$$\begin{aligned} C = C(u,v) \equiv A(uH(u), vH(v)) \ , \\ D = D(u,v) \equiv B(uH(u), vH(v)) \ . \end{aligned} \tag{2}$$

Here, we shall consider space - times described by a metric of the form

$$g = 2e^C d\zeta d\bar{\zeta} - 2e^D du dv \ ; \tag{3}$$

such metric was studied in the context of colliding waves by A. H. Taub. Our consideration in this section are closely related to those he gave in the introduction of [2]; however we will present results that directly include the extension (1), and we shall take derivatives with respect to the tetrad (4) instead of coordinate derivatives. Then, with the extension (1), the space - time described by (3) is naturally divided in four regions: region I ($u > 0, v > 0$), which is usually called the interaction region; region II ($u < 0, v > 0$)and region III ($u > 0, v < 0$), in which there usually is some kind of radiation; region IV ($u < 0, v < 0$), in which the spacetime is flat.

Given the above definition of C, D, in the general case we expect singularity of the Riemann tensor of Dirac's delta type on the surfaces $u = 0$ and $v = 0$. Indeed, the extension (1) introduce discontinuities in the derivatives of the metric coefficient: it will be understood in the following that derivatives are computed as shown in [3], page 250. Let us now introduce the null tetrad basis

$$
\begin{array}{rcl}
\omega^1 &=& e^C d\zeta \ , \\
\omega^2 &=& e^C d\bar{\zeta} \ , \\
\omega^3 &=& e^D du \ , \\
\omega^4 &=& e^D dv \ ;
\end{array}
\tag{4}
$$

then its dual is a Newman - Penrose tetrad[1]

$$
\begin{array}{rcrcl}
m &=& \partial_1 &=& e^{-C}\partial_\zeta \ , \\
\overline{m} &=& \partial_2 &=& e^{-C}\partial_{\bar{\zeta}} \ , \\
l &=& \partial_3 &=& e^{-D}\partial_u \ , \\
k &=& \partial_4 &=& e^{-D}\partial_v \ ,
\end{array}
\tag{5}
$$

where the only non - vanishing scalar product are

$$
k_a l^a = -1 \ , \qquad m_a \overline{m}^a = 1 \ . \tag{6}
$$

Denoting derivatives of any function f by

$$
f_3 = \partial_3 f \ , \quad f_4 = \partial_4 f \ , \quad f_u = \partial_u f \ , \quad f_v = \partial_v f \ ,
$$

and using the above quoted prescription to take into account discontinuities in the derivatives of C and D, we have

$$
\begin{gathered}
C_3 = A_3 H(u) \ , \quad C_4 = A_4 H(v) \ , \\
C_{33} = A_{33} H(u) + A_3 e^{-D}\delta(u) \ , \quad C_{44} = A_{44} H(v) + A_4 e^{-D}\delta(v) \ , \\
C_{34} = A_{34} H(u) H(v) \ , \quad C_{43} = A_{43} H(u) H(v) \ , \quad A_{34} \neq A_{43} \ ,
\end{gathered}
\tag{7}
$$

and

$$
\begin{gathered}
D_3 = B_3 H(u) \ , \quad D_4 = B_4 H(v) \ , \\
D_{33} = B_{33} H(u) + B_3 e^{-D}\delta(u) \ , \quad D_{44} = B_{44} H(v) + B_4 e^{-D}\delta(v) \ , \\
D_{34} = B_{34} H(u) H(v) \ , \quad B_{34} \neq B_{43} \ .
\end{gathered}
\tag{8}
$$

Using the tetrad (4) we can compute the connection 1 - forms $\Gamma_{ab} = \Gamma_{[ab]}$; the only non vanishing are:

$$
\begin{array}{rcl}
\Gamma_{13} &=& A_3 H(u)\omega^2 \ , \\
\Gamma_{14} &=& A_4 H(v)\omega^2 \ , \\
\Gamma_{34} &=& B_3 H(u)\omega^3 - B_4 H(v)\omega^4 \ ,
\end{array}
\tag{9}
$$

where we remember that, for any quantity, exchange of indices 1 and 2 implies complex conjugation. From these expressions we derive that the only non identically vanishing spin coefficients are

$$
\varrho = -A_4 H(v) \ , \quad \mu = A_3 H(u) \ , \quad \varepsilon = \frac{1}{2} B_4 \ , \quad \gamma = -\frac{1}{2} B_3 \ . \tag{10}
$$

[1] Here we use the Newman - Penrose formalism as is presented, for example, in [4], but other representation of this formalism are possible (e.g., see [5]).

Since ϱ and μ are real, they just represent the expansion of k, l, and thus these are twist - free; also, k and l are shear - free, because of the vanishing of the spin coefficients σ and λ. Finally, we see from the vanishing of κ and ν that the null congruences k and l are geodesics (which also follows from the fact that k and l are null and hypersurfaces orthogonal). The non vanishing of ϱ in the region $v \geq 0$ (of μ in the region $u \geq 0$) express the fact that null rays of which k (l) is the tangent expand (or contract) when they enter the region where there is a wave propagating along l (k). As long as there is not a divergence in these terms, there is not a physical singularity.
Using the above connection 1 - forms we can compute the curvature 2 - forms

$$\Omega^a{}_b \equiv \frac{1}{2} R^a{}_{bcd} \omega^c \wedge \omega^d = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_d \ , \tag{11}$$

extract from these latter the Riemann tensor components, and compute the Ricci tensor components on the tetrad (4). On doing this we obtain that

$$R_{11} = R_{13} = R_{14} = 0 \ , \tag{12}$$

identically vanish, while the following components can be different from zero

$$\begin{aligned} R_{12} &= \left(A_{34} + A_{43} + 4A_3A_4 + A_4B_3 + A_3B_4\right) H(u)H(v) \ , \\ R_{33} &= -2\left(A_{33} + A_3^2 - A_3B_3\right) H(u) - 2A_3\delta(u) \ , \\ R_{44} &= -2\left(A_{44} + A_4^2 - A_4B_4\right) H(v) - 2A_4\delta(v) \ , \\ R_{34} &= -\left(A_{34} + A_{43} + B_{34} + B_{43} + 2A_3A_4 + 2B_3B_4 + A_3B_4 + A_4B_3\right) H(u)H(v) \ . \end{aligned} \tag{13}$$

From these we immediately have the Ricci scalars: the only non identically vanishing are

$$\begin{aligned} \Phi_{00} &= \frac{1}{2} R_{44} = \left(A_4B_4 - A_{44} - A_4^2\right) H(v) - A_4\delta(v) \ , \\ \Phi_{11} &= \frac{1}{4}\left(R_{12} + R_{34}\right) = \frac{1}{2}\left(A_3A_4 - B_3B_4 - \frac{1}{2}B_{34} - \frac{1}{2}B_{43}\right) H(u)H(v) \ , \\ \Phi_{22} &= \frac{1}{2} R_{33} = \left(A_3B_3 - A_{33} - A_3^2\right) H(u) - A_3\delta(u) \ . \end{aligned} \tag{14}$$

The Weyl scalar can also immediately be inferred from (11): indeed in the tetrad (4) these are

$$\Psi_0 = W_{1414} \ , \quad \Psi_1 = W_{1434} \ , \quad \Psi_2 = \frac{1}{2}(W_{3434} + C_{1234}) \ , \quad \Psi_3 = -W_{2334} \ , \quad \Psi_4 = W_{2323} \ . \tag{15}$$

We find that the only non identically vanishing Weyl scalar is Ψ_2

$$\Psi_2 = \frac{1}{3}(A_{34} + A_{43} - B_{34} - B_{43} + A_3B_4 + A_4B_3 - 2B_3B_4)H(u)H(v) \ , \tag{16}$$

showing that the metric (3) is of Petrov type D in the interaction region. Finally, the vanishing of Ψ_0 and Ψ_4 everywhere shows that there are not plane gravitational waves.

We already noted that the spacetime is flat in region IV; in region II and III the Ricci tensor is of pure radiation type [4] (null dust) i.e.

$$\text{II} \quad R_{ab} \propto \; l_a l_b \; , \tag{17}$$

$$\text{III} \quad R_{ab} \propto \; k_a k_b \; , \tag{18}$$

as it follows from equations (13) and the definitions of l and k. The crucial fact now is that region I is characterized by $R_{11} = R_{13} = R_{14} = 0$, so that, as it will be clarified in the next section, we can associate to the Ricci tensor there an energy - momentum tensor of the perfect fluid type through Einstein equations.

2 Matter: Perfect Fluid

We shall now assume Einstein equations without cosmological constant term: $R_{ab} = \kappa(T_{ab} - \frac{1}{2}g_{ab}T)$ (κ is the gravitational constant). Then the perfect fluid Ricci tensor has the form [4]:

$$R_{ab} = \kappa \left[(\mu + p) w_a w_b + \tfrac{1}{2}(\mu - p) g_{ab} \right] \; , \tag{19}$$

where w^a is the unit four-velocity of the fluid satisfying $w_a w^a = -1$, and μ and p are interpreted as energy density and pressure respectively.

Now, let us write the metric tensor in the form:

$$g = 2\omega^1\omega^2 - 2\omega^3\omega^4 \; , \tag{20}$$

where ω^i $(i = 1, 2, 3, 4)$ is a general null tetrad [not necessarily coinciding with the specific tetrad (4)]. Let ∂_i $(i = 1, 2, 3, 4)$ be its dual. From now on we will restrict our considerations to the fluids for which the four-velocity w has the form

$$w = f\partial_3 + (2f)^{-1}\partial_4 \; , \tag{21}$$

for which w is a unit vector. Inserting the above w in the expression (19) for the Ricci tensor we have:

$$R_{11} = R_{13} = R_{14} = 0 \; , \tag{22}$$

i.e., the same vanishing components (12) we obtained for the metric (3), and

$$\begin{aligned} R_{12} &= \frac{1}{2}\kappa(\mu - p) \; , \qquad R_{34} = \kappa p \; , \\ R_{33} &= \frac{1}{4}\kappa f^{-2}(\mu + p) \; , \qquad R_{44} = \kappa f^2(\mu + p) \; , \end{aligned} \tag{23}$$

The above equation show that the necessary condition for the metric to describe perfect fluid with a four-velocity given by (21) is:

$$R_{33} = \frac{1}{2} f^{-2}(R_{12} + R_{34}) \ , \quad R_{44} = 2f^2(R_{12} + R_{34}) \ . \tag{24}$$

If these equations are satisfied then the pressure p is given by

$$\kappa p = R_{34} \ , \tag{25}$$

and the energy density is

$$\kappa\mu = 2R_{12} + R_{34} \ . \tag{26}$$

It is always assumed that the perfect fluid obeys the weak energy condition $\mu > 0$ so that

$$2R_{12} + R_{34} > 0 \ . \tag{27}$$

Sometimes the strong energy condition $p + \mu > 0$ is assumed; this is equivalent to

$$R_{12} + R_{34} > 0 \ . \tag{28}$$

Equations (22) show that in region I the Ricci tensor of the metric (3) can be of the perfect fluid type with the four - velocity w defined by (21), provided that in this region the equations (24), and inequalities (27) and (28) are satisfied.

3 Colliding Plane Waves

In this section we consider metric tensor (3). We assume that in region I equations (24) and inequalities (27) and (28) are satisfied. It means that in this region the metric (3) describes perfect fluid with four velocity w given by (21) and p, μ given by (25) and (26) respectively. We assume that in region II and III we have respectively

$$R_{44} > 0 \ , \quad R_{33} > 0 \ . \tag{29}$$

This together with automatically satisfied equations (17) and (18) means that in these regions pure radiation (null dust) moving along l in region II and along k in region III is present; however, we already know that this is not gravitational radiation from the vanishing of the Weyl scalars Ψ_0 and Ψ_4, equation (15).

Under such assumptions the metric tensor (3) describes colliding plane-waves which in the region of interaction produce perfect fluid. Summing up informations given so far we can say that the metric tensor (3) describes colliding plane-waves producing perfect fluid with four-velocity w given by (21) if:
i) in region I we have:

$$2(A_{33} + A_3^2 - A_3B_3) + \frac{1}{2} f^{-2}(2A_3A_4 - 2B_3B_4 - B_{34} - B_{43}) = 0 \ , \tag{30}$$

$$2(A_{44}+A_4^2-A_4B_4)+2f^2(2A_3A_4-2B_3B_4-B_{34}-B_{43})=0\ , \tag{31}$$

$$A_{34}+A_{43}-B_{34}-B_{43}+6A_3A_4-2B_3B_4+A_3B_4+A_4B_3>0\ , \tag{32}$$

and

$$2A_3A_4-2B_3B_4-B_{34}-B_{43}>0\ ; \tag{33}$$

ii) in region II

$$A_{44}+A_4^2-A_4B_4<0\ ; \tag{34}$$

iii) in region III

$$A_{33}+A_3^2-A_3B_3<0\ . \tag{35}$$

Equations (30) and (31) correspond to (24); inequalities (32), (33), (34) and (35) correspond to (27), (28), and to(29), respectively [see also (13)]. Finally, in region I the pressure p and the energy density μ of the fluid are given by:

$$\kappa p=-A_{34}-A_{43}-B_{34}-B_{43}-2A_3A_4-2B_3B_4-A_3B_4-A_4B_3\ , \tag{36}$$

$$\kappa\mu=A_{34}+A_{43}-B_{34}-B_{43}+6A_3A_4-2B_3B_4+A_3B_4+A_4B_3\ . \tag{37}$$

[see (25),(26) and (13)].

4 Examples of collisions producing perfect fluid

Let us consider metric (3) satisfying (30)-(35) with

$$A(u,v)=B(u,v)=h(u+v)\ , \tag{38}$$

where h is some real function. Such metric describes colliding plane-waves of null dust which produce perfect fluid with four-velocity given by (21). Assumption (38) shows that space-time is conformally flat. Let us additionally assume that the function f appearing in (21) is such that

$$f^2=\frac{1}{2}\ . \tag{39}$$

It is remarkable that under the assumptions (38) and (39) the conditions (30)-(32) become identities, and (33)-(35) reduce to one inequality

$$h''-(h')^2<0\ , \quad for \quad \eta\equiv u+v>0\ , \tag{40}$$

where primes denote derivatives of h with respect to η. Thus any function h satisfying (40) generates a space-time in which collision of two planar waves occurs, producing a perfect fluid with the pressure p and the energy density μ of the form:

$$\kappa p=-2e^{-2h}\left[2h''+(h')^2\right]\ , \tag{41}$$

$$\kappa\mu=6e^{-2h}(h')^2\ . \tag{42}$$

These equations show that the equation of state for the fluid is:

$$p = -\tfrac{1}{3}\left[2h''(h')^{-2} + 1\right]\mu \; . \tag{43}$$

It is worth noting that any function h such that $h'' < 0$ satisfies (40), so that it gives an example of colliding plane waves producing perfect fluid.

As a more realistic example we will consider now a polytropic equation of state of the form

$$p = (\gamma - 1)\mu \tag{44}$$

where $\gamma > 0$ is a constant. The equation (43) has the form (44) iff

$$h''(h')^{-2} = 1 - \tfrac{3}{2}\gamma \; . \tag{45}$$

Note that for any $\gamma > 0$ function h satisfying (45) satisfies (40). The solution of (45) leads to the following explicit form of the metric:

$$g = 2S^2(d\zeta d\bar{\zeta} - du dv) \; , \tag{46}$$

where

$$S(u,v) = [uH(u) + vH(v) + a]^{\frac{2}{3\gamma-2}} \; , \tag{47}$$

and $a \neq 0$ is some real constant. This metric describes:

1. Minkowski space in region IV ($u, v < 0$);

2. A planar wave of massless particles moving in the direction $l = S\partial_u$ in region II ($u < 0, v > 0$);

3. A planar wave of massless particles moving in the direction $k = S\partial_v$ in region III ($u > 0, v < 0$);

4. Perfect fluid with four-velocity

$$w = \tfrac{\sqrt{2}}{2}(l + k) = \tfrac{\sqrt{2}}{2}S(\partial_u + \partial_v) \; , \tag{48}$$

pressure

$$\kappa p = \frac{24(\gamma - 1)}{(3\gamma - 2)^2}(u + v + a)^{\frac{6\gamma}{2-3\gamma}} \; , \tag{49}$$

and energy density

$$\kappa \mu = \frac{24}{(3\gamma - 2)^2}(u + v + a)^{\frac{6\gamma}{2-3\gamma}} \; , \tag{50}$$

in region I ($u > 0, v > 0$).

It is worth noting that according to ([4] Theorem 32.17 and §22.2) the waves in regions II and III can be interpreted as electromagnetic null fields since it is possible to find solution of the coupled Einstein-Maxwell system in which the metric is of the form (46), (47) and null Maxwell field in regions II and III is a source for Einstein equations there. According to [6] if we admit this interpretation for the solution we should care for matching conditions for the electromagnetic field at the boundaries of the regions. It turns out that they are satisfied at the boundaries between regions II-IV, and III-IV. It is rather impossible to satisfy matching conditions for electromagnetic field between regions II-I and III-I in general case. We believe that the only possible value of parameter γ for which it would be possible is $\gamma = \frac{4}{3}$ since in this case region I is filled with homogeneous gas of photons. Whether this is the case is an open question.

We have some doubts whether the attitude of [6] is applicable to our solutions. This is because at the boundaries of any two regions of spacetime we have Dirac delta singularities in the Riemann tensor. It is an interesting question whether electromagnetic field has to satisfy standard matching conditions along null surfaces with delta singular curvature. If no g can be interpreted as describing two colliding planar electromagnetic waves producing perfect fluid with the polytropic equation of state (44).

5 Conclusions

We showed that metrics (3) can be interpreted as describing two incoming planar waves with energy momentum of null dust which produce a perfect fluid with four-velocity (21). We have never specified (with one exception at the end of Section 4) what kind of radiation is present in regions II and III. The only assumption is that it has energy momentum tensor of null dust. It can be, for example, a mixture of different massless particles moving in a particular null direction. From this point of view our system of equations is not totally determined. We can not specify what equations satisfy fields in regions filled by null dusts. Therefore we can not specify matching conditions for the fields on the boundaries of the regions. Even if we could, it is problematic whether they are necessary since as we noticed in particular case of solution (46), (47) there can be Dirac delta singularity in Riemann tensor along the borders. Such delta behaviour of Riemann tensor is general for all metrics (3) as can be inferred from (13). We can therefore either admit that
1) nongravitational fields can have discontinuities along null surfaces on which curvature is singular
or say that
2) all system of our equations consists only of Einstein equations without specifying from what physical fields energy-momentum tensor originates.
The assumption 1) is more exciting physically. If it is accepted then, for example,

solution (46), (47) can be considered as Friedman-Lemaitre-Robertson-Walker Universe with flat spatial curvature created from a collision of two shock waves followed by two planar electromagnetic waves. It is remarkable that the condition $a \neq 0$ for this solution causes that these Friedman-Lemaitre-Robertson-Walker Universes expand (or collapse) from the non zero value of the scale factor. (Note that condition $a \neq 0$ is imposed by the demand that in region I ($u < 0, v < 0$) the metric g describes the Minkowski space-time. It is an open question on what physical assumption the size of the constant a depends.)
The metric (46)-(47) includes many interesting special cases. For particular values of parameter γ as a result of the collision we obtain the Friedman Universe with zero spatial curvature ($\gamma = 1$), the Tolman Radiation Universe ($\gamma = \frac{4}{3}$), and a stiff matter Universe with $\gamma = 2$. It shows that the metric g or its more realistic counterparts can be used to the description of physics in the early universe.
It is interesting whether there exist spherically symmetric generalizations of solution (46)-(47). By this we mean the following: is it possible (by considering more general metrics than (3)) to construct a Friedman-Lemaitre-Robertson-Walker Universe as a result of a collapse of a spherical bubble of flat spacetime surrounded by a concentric with it spherical shock wave moving radially towards the center of the bubble? (The shock wave can be followed by null dust (possibly Maxwell) also moving radially towards the center of the bubble.)

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