COMPACT AND NONCOMPACT QUANTUM GROUPS. I

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Lecture notes written by

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0 Preface

In February 1995 S.L. Woronowicz delivered a series of lectures on 'Compact and noncompact quantum groups' at SISSA, Trieste. The lectures started from such basic C^*-algebra notions and facts as Gelfand-Neimark theorems and definition of morphisms. Then the theory of topological compact quantum groups was developed. In particular, quantum counterparts of the Haar measure, right regular representation and the decomposition theory of representations was presented. Also the relation between this approach and the Hopf-algebraic one was explained.

The second part of the lectures was more advanced and served as an introduction to the current research on noncompact quantum groups. This subject is not fully developed yet. Such topics as the Pontryagin duality, multiplicative unitaries, pentagonal relations and their links to the Weyl form of the Heisenberg commutation rules were discussed.

The lectures focused on topological and functional analytic aspects of the quantum group theory. This makes them interesting for both physicists and mathematicians. Since a review of such an approach is not avaliable in the literature, two of us (L. D. and P. N.) decided to register the lectures in the present notes. We tried to preserve as much as possible the style of the lectures. However, we had to put the material in linear order. L. D. and P. N. are entirely responsabile for any errors, imprecisions, abuse of notation, etc. that could arise in such a process. We refer to the papers [4–7] for more details and accuracy.

The present text covers the first part of the lectures. The second part is in preparation.

1 C^*-algebras

The purpose of these lectures is to present a description of topological groups in terms of the corresponding algebras of functions and an extension of this theory to noncommutative algebras.

Definition 1.1 *An algebra is a structure $(\mathcal{A},\ +,\ \cdot;\ \bullet)$, where*
1) $(\mathcal{A},\ +,\ \cdot)$ is a complex vector space
2) a map $\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is bilinear and associative.
If, in addition, there exists $I_{\mathcal{A}} \in \mathcal{A}$ which is a neutral element for $\bullet$ then $(\mathcal{A},\ +,\ \cdot;\ \bullet)$ is called a unital algebra.

The notion of a subalgebra is self-explanatory.

Definition 1.2 *A subalgebra $\mathcal{I}$ of $\mathcal{A}$ is called a left (right) ideal iff $\mathcal{A} \bullet \mathcal{I} \subset \mathcal{I}$ ($\mathcal{I} \bullet \mathcal{A} \subset \mathcal{I}$, respectively). A left ideal which is also a right ideal is called simply an ideal.*

Remark 1.3 The quotient $\mathcal{A}/\mathcal{I}$ of $\mathcal{A}$ by a left ideal $\mathcal{I}$, ie. the set of equivalence classes $a+\mathcal{I}$ is a complex vector space with the quotient operations $\alpha(a+\mathcal{I}) = \alpha a+\mathcal{I}$ and $(a+\mathcal{I})+(b+\mathcal{I}) = (a+b)+\mathcal{I}$. If $\mathcal{I}$ is an ideal then $A/\mathcal{I}$ is an algebra with the quotient product $(a+\mathcal{I}) \bullet (b+\mathcal{I}) = (a \bullet b)+\mathcal{I}$.

In the sequel, an algebraic operation such as $+$, etc. defined in different algebraic structures will be denoted by the same symbol.

Definition 1.4 *Let $(\mathcal{A}_1,\ +,\ \cdot;\ \bullet)$ and $(\mathcal{A}_2,\ +,\ \cdot;\ \bullet)$ be algebras. A vector space homomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an algebra homomorphism if*

$$\phi(a \bullet b) = \phi(a) \bullet \phi(b)\ . \tag{1.1}$$

A bijective homomorphism is called isomorphism.
If both the algebras are unital then a homomorphism ϕ is called unital iff $\phi(I_{\mathcal{A}_1}) = I_{\mathcal{A}_2}$.

Example 1.5 Denote $\mathbf{SL}(2,\mathbf{C})$ the space of all 2×2 matrices g with entries $g_{ij} \in \mathbf{C}$, $i,j = 1,2$ and with $\det g = 1$. (The group structure on $\mathbf{SL}(2,\mathbf{C})$ is irrelevant in this example). Elements g of $\mathbf{SL}(2,\mathbf{C})$ act on vectors $z = (z_1, z_2)^t \in \mathbf{C}^2$ by $z \mapsto gz$. There are two standard quadratic forms Q_1, Q_2 on $\mathbf{C}^2$ defined by

$$Q_1(z) = |z_1|^2 + |z_2|^2, \qquad Q_2(z) = |z_1|^2 - |z_2|^2.$$

Correspondingly one has the following two subspaces (real forms) of $\mathbf{SL}(2,\mathbf{C})$

$$\mathbf{SU}(2) \ = \{g \in \mathbf{SL}(2,\mathbf{C}) \mid Q_1(gz) = Q_1(z)\ ,\ \forall z \in \mathbf{C}^2\}\ ,$$

$$\mathbf{SU}(1,1) = \{g \in \mathbf{SL}(2,\mathbf{C}) \mid Q_2(gz) = Q_2(z)\ ,\ \forall z \in \mathbf{C}^2\}\ .$$

Another well known real form,

$$\mathbf{SL}(2,\mathbf{R}) = \{g \in \mathbf{SL}(2,\mathbf{C}) | g_{ij} \in \mathbf{R}\} \ ,$$

is known to be homeomorphic to $\mathbf{SU}(1,1)$; however $\mathbf{SU}(2)$ and $\mathbf{SU}(1,1)$ are not homeomorphic.
On $\mathbf{SL}(2,\mathbf{C})$ we define four functions α, β, γ and δ by

$$\alpha(g) = g_{11}, \quad \beta(g) = g_{12}, \quad \gamma(g) = g_{21}, \quad \delta(g) = g_{22}.$$

Consider now a free algebra generated by α, β, γ and δ. A quotient of this algebra modulo the ideal generated by $\alpha\delta - \beta\gamma - I$ and all the commutators of generators (e.g. $\alpha\beta - \beta\alpha$, etc) is denoted by $\mathcal{A}_{\mathbf{SL}(2,\mathbf{C})}$.
Given a subspace Λ of $\mathbf{SL}(2,\mathbf{C})$ the functions α, β, γ and δ can be restricted to Λ. Combinations of the so restricted functions generate an algebra (by applying the addition, multiplication by numbers and multiplication). Conversely, any element of the restricted algebra can be uniquelly extended to $\mathbf{SL}(2,\mathbf{C})$; the extension being just the same combination of the 'unrestricted' generators. We denote by $\mathcal{A}_{\mathbf{SU}(2)}$, $\mathcal{A}_{\mathbf{SU}(1,1)}$ and $\mathcal{A}_{\mathbf{SL}(2,\mathbf{R})}$ the restricted algebras corresponding, respectively, to $\mathbf{SU}(2)$, $\mathbf{SU}(1,1)$ and $\mathbf{SL}(2,\mathbf{R})$. They are all isomorphic

$$\mathcal{A}_{\mathbf{SU}(2)} \simeq \mathcal{A}_{\mathbf{SU}(1,1)} \simeq \mathcal{A}_{\mathbf{SL}(2,\mathbf{R})} \simeq \mathcal{A}_{\mathbf{SL}(2,\mathbf{C})} \ . \qquad \diamond$$

One of our tasks is to encode topological information about the spaces in terms of corresponding algebras. However, the above example shows that the 'naked' notion of an algebra is too rough to distinguish between $\mathbf{SU}(2)$ and $\mathbf{SU}(1,1)$. This suggest that we should supplement $(\mathcal{A}, +, \cdot; \bullet)$ with some additional structure. One possibility is to add a *-operation.

Definition 1.6 *A *-operation is an antilinear, antimultiplicative and involutive map $^* : \mathcal{A} \to \mathcal{A}$. An algebra endowed with a *-operation is called *-algebra.*

Definition 1.7 *Let $(\mathcal{A}_1, +, \cdot; \bullet, {}^*)$ and $(\mathcal{A}_2, +, \cdot; \bullet, {}^*)$ be *-algebras. An algebra homomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a *-algebra homomorphism if*

$$\phi(a^*) = (\phi(a))^*. \tag{1.2}$$

Such notions like unital *–homomorphism, *–isomorphism, etc. are self-explanatory.

Example 1.8 (Continuation of Example 1.5)
Corresponding to $\mathbf{G} = \mathbf{SU}(2)$, $\mathbf{SU}(1,1)$ or $\mathbf{SL}(2,\mathbf{R})$, the algebra $\mathcal{A}_{\mathbf{SL}(2,\mathbf{C})}$ can be equiped with three *-operations which are defined by the following steps:
i) restrict an element $a \in \mathcal{A}_{\mathbf{SL}(2,\mathbf{C})}$ to $\mathbf{G}$
ii) to such a restricted element apply the complex conjugation
iii) extend so obtained element to $\mathcal{A}_{\mathbf{SL}(2,\mathbf{C})}$.

By applying these rules we get the following three *-operations on generators

$$\begin{array}{ll} \alpha^* = \delta,\ \beta^* = -\gamma,\ \gamma^* = -\beta,\ \delta^* = \alpha & \text{for}\ \ \mathbf{G} = \mathbf{SU}(2) \\ \alpha^* = \delta,\ \beta^* = \gamma,\ \gamma^* = \beta,\ \delta^* = \alpha & \text{for}\ \ \mathbf{G} = \mathbf{SU}(1,1) \\ \alpha^* = \alpha,\ \beta^* = \beta,\ \gamma^* = \gamma,\ \delta^* = \delta & \text{for}\ \ \mathbf{G} = \mathbf{SL}(2,\mathbf{R}) \end{array}$$

It can be seen that $\mathcal{A}_{\mathbf{SL}(2,\mathbf{C})}$ equipped either with the first or second *-operation are not isomorphic as *-algebras, although they are isomorphic as algebras. ($\mathcal{A}_{\mathbf{SL}(2,\mathbf{C})}$ equipped with the second or third *-operation are isomorphic also as *-algebras). $\diamond$

We have already learned that the *-algebra structure can distinguish between different spaces, even if they have the same complexification. In the sequel we shall be mainly interested not just in topological spaces but in topological groups. In particular, we would like to incorporate the unitary representations of groups in our approach. It is known, that unitary representations of noncompact groups can not be encoded in terms of polynomials only. To admit more general functions we have to complete the algebra of polynomials. This leads to endowing $(\mathcal{A},\ +,\ \cdot;\ \bullet,\ ^*)$ with a structure of C^*–algebra.
Recall that a norm on $\mathcal{A}$ is a non-negative function such that $\forall\ a, b \in A$, $\forall\ \alpha \in \mathbf{C}$, $\|\alpha a\| = |\alpha|\|a\|$, $\|a+b\| \leq \|a\| + \|b\|$ and if $\|a\| = 0$ then $a = 0$.

Definition 1.9 *A C^*–algebra is a structure $(A,\ +,\ \cdot;\ \bullet,\ ^*,\ \|\ \|)$, where*
i) $(A,\ +,\ \cdot;\ \bullet,\ ^*)$ is a *-algebra
ii) $(A,\ +,\ \cdot,\ \|\ \|)$ is a Banach space, complete in the norm $\|\ \|$ satisfying the following conditions
a) $\forall\ a,\ b \in A\ \ \|a \bullet b\| \leq \|a\|\|b\|$
b) $\forall\ a \in A\ \ \|a^*\| = \|a\|$
c) $\forall\ a \in A\ \ \|a^* \bullet a\| = \|a\|^2$.

Notions of a subalgebra and of an ideal in a C^*-algebra are self-explanatory.

Definition 1.10 *An ideal $\mathcal{I}$ of a C^*-algebra A is called*
i) essential iff for all $b \in A$ it holds that $(b \bullet a = 0\ \ \forall a \in \mathcal{I}) \Rightarrow\ b = 0$.
ii) modular iff there exist $b \in A$ such that $a - a \bullet b \in \mathcal{I}$ and $a - b \bullet a \in \mathcal{I}$ for all $a \in A$

In view of the following theorem *-homomorphisms are adequate for C^*-algebras as no additional consistency condition with respect to the norm $\|\ \|$ is needed. (Nevertheless, in Section 3 we shall come back to the question of morphisms between C^*–algebras).

Theorem 1.11 *Let A and B be two C^*–algebras and $\phi : A \to B$ a *-homomorphism. Then $\phi : A \to B$ is norm decreasing, i.e. $\|\phi(a)\| \leq \|a\|$ and $\phi(A)$ is a C^*-subalgebra of B. Moreover, if ϕ is injective then $\|\phi(a)\| = \|a\|$.*

In order to simplify notation, from now on, A will stand for $(A,\ +,\ \cdot\ ;\ \bullet,\ ^*,\ \|\ \ \|)$ and the dots of the products will be often omitted. Also, unless stated differently, A is assumed to be separable, ie. to contain a countable $\|\ \|$-dense subset.

2 Gelfand-Naimark theorems

We recall some standard notation and facts. In the following Λ denotes a locally compact space. Let $C(\Lambda)$ be the set of all continuous complex functions on Λ, $C_b(\Lambda)$ be the subset of bounded functions

$$C_b(\Lambda) = \{a \in C(\Lambda) | \; \exists N \; \forall \lambda \in \Lambda \quad |a(\lambda)| \leq N\}$$

and $C_\infty(\Lambda)$ the subset of functions vanishing at infinity

$$C_\infty(\Lambda) = \{a \in C_b(\Lambda) \mid \forall \varepsilon > 0 \text{ the set } \{\lambda \in \Lambda : |a(\lambda)| \geq \varepsilon\} \text{ is compact}\} \; .$$

$C_b(\Lambda)$ can naturally be endowed with a C^*–algebra structure as follows. Operations $(+, \;\; \cdot \;\; ; \; \bullet)$ are respectively the addition, the multiplication by a number and the multiplication of two functions; *-operation is the complex conjugation. A norm $\|\cdot\|$ is defined for any $a \in C_b(\Lambda)$ by

$$\|a\| = \sup\{\lambda \in \Lambda \mid a(\lambda)| < \infty\} \; .$$

Remark 2.1 It is easy to see that $C_\infty(\Lambda)$ is a C^*-subalgebra of $C_b(\Lambda)$. Moreover, $C_\infty(\Lambda)$ is an essential ideal in $C_b(\Lambda)$ (see Definition 1.10) and $C_b(\Lambda)$ is the largest C^*-algebra which contains $C_\infty(\Lambda)$ as an essential ideal.

It turns out that any commutative C^*–algebra is of the form $C_\infty(\Lambda)$ for some Λ. This can be seen via a canonical Gelfand-Naimark representation. Let Λ be the space of characters on a commutative C^*–algebra A, ie. all nonzero *-homomorphisms $\chi : A \to \mathbf{C}$. (Such χ is automatically continuous). It can be shown that Λ is a locally compact topological space with a (restriction to Λ of) weak topology on A' (a basis of open neighbourhoods of 0 is $U_{\epsilon,a} = \{\chi \mid \chi(a) < \epsilon\}$, where $\epsilon > 0$ and $a \in A$). For any $a \in A$ the function $\hat{a} : \Lambda \to \mathbf{C}$, $\hat{a} : \chi \mapsto \chi(a)$ belongs in fact to $C_\infty(\Lambda)$. Moreover the assignment of $\hat{a}$ to a is a *-homomorphism, which is injective and due to Stone-Weierstrass theorem also surjective. In addition if A is isomorphic to $C_\infty(\Lambda')$, where Λ' is another space, then Λ' can be shown to be homeomorphic to Λ.

The above statements imply the first of structure theorems due to Gelfand and Naimark

Theorem 2.2

$$\left(\; A \text{ is a commutative } C^*\text{−algebra} \; \right) \Leftrightarrow \left(\begin{array}{c} \text{There exists a unique locally compact} \\ \text{space } \Lambda \text{ such that } A \simeq C_\infty(\Lambda) \end{array} \right).$$

Remark 2.3 Here the uniqueness holds up to a homeomorphism and $\simeq$ denotes a *-isomorphism. $\diamond$

Remark 2.4 The kernel of a character χ on abelian A is a maximal modular ideal in A and, vice versa, any maximal modular ideal is the kernel of a unique character χ on A. Thus, the points in Λ can be also thought of as maximal modular ideals in A. For unital A any ideal is modular (set just $b = I$ in Definition 1.10), hence in that case the points in Λ can be thought of as maximal ideals. $\diamond$

Remark 2.5 The C^*–algebra $C_b(\Lambda)$ is not adequate enough as there are $\Lambda \neq \Lambda'$ such that $C_b(\Lambda) \simeq C_b(\Lambda')$. In fact, let Λ be noncompact. We have $C_b(\Lambda) \simeq C_\infty(\Lambda')$ for some Λ'. Since $C_b(\Lambda)$ is unital then Λ' is compact and thus non-homeomorphic to Λ. Moreover, $C_\infty(\Lambda') \simeq C_b(\Lambda')$. $\diamond$

Theorem 2.2 shows that classical spaces can be encoded in commutative C^*–algebras. It shows also a way to develop a non-commutative geometry. The main idea is to consider non-commutative C^*-algebras as a counterpart of something which could be a non-commutative (quantum) space, cf [4]. In such an approach one works with non-commutative C^*–algebras and tries to mimic on them such structures that in commutative C^*–algebras correspond to certain properties of classical spaces (e.g. group structure).

Given a (separable) Hilbert space H let $B(H)$ denote the C^*-algebra of all linear operators on H, which are bounded in the norm

$$\|a\| = \sup\{\|ax\| \mid x \in K\},$$

where $a \in B(H)$ and $K = \{x \in H \mid \|x\| \leq 1\}$. The *-operation in $B(H)$ is just the usual adjoint of operators.

Definition 2.6 *A *-homomorphism $\pi : A \to B(H)$ is called a representation of a C^*-algebra A in H. Moreover, π is said to be*
i) faithful iff π is *-isomorphism between A and $\pi(A)$, ie. $\ker \pi = \{0\}$
ii) nondegenerate iff ($\psi \in H$ and $\pi(a)\psi = 0 \quad \forall a \in A \;\Rightarrow\; \psi = 0$)
iii) cyclic iff $\exists \Omega \in H$ cyclic for π, ie.

$$\{\pi(a)(\Omega) \mid a \in A\} \text{ is dense in } H \tag{2.1}$$

iv) irreducible iff $\pi(A)$ is irreducible on H, ie. the only closed subspaces of H which are invariant under $\pi(A)$ are $\{0\}$ and H
v) unitarily equivalent to $\pi' : A \to B(H')$ iff $\exists\, V : H \to H'$ which is unitary and $\pi(a) = V^*\pi'(a)V$, $\forall a \in A$.

Remark 2.7 Such notions as trivial representation, subrepresentation and direct sum of representations are defined in the usual manner. $\diamond$

We denote by $\mathrm{Rep}\,(A, H)$ the set of nondegenerate representations of A in H and recall one useful theorem known in the theory of C^*-algebras.

Theorem 2.8 *Let $A_1 \subset A_2$ be an ideal in a C^*–algebra A_2 and $\pi \in \mathrm{Rep}\,(A_1, H)$. Then, there exist a unique $\tilde{\pi} \in \mathrm{Rep}\,(A_2, H)$ such that $\tilde{\pi}|_{A_1} = \pi$.*

In order to see that there are nontrivial representations it is helpful to consider states. Let A' be the topological dual to A, ie. the space of all continuous linear complex valued functionals on A. The norm of $\phi \in A'$ is given by

$$\|\phi\| = \sup\{|\phi(a)| \mid \|a\| = 1\}\ .$$

Definition 2.9 *A linear functional ω on A is said to be positive iff $\omega(a^*a) \geq 0$, $\forall a \in A$. If in addition $\omega \in A'$ and $\|\omega\| = 1$ then ω is said to be a state. A state ω is faithful iff $\omega(a^*a) = 0$ implies that $a = 0$.*

Remark 2.10 A positive linear functional ω can be seen to have the following properties (cf. [1]):
i) $\omega \in A'$
ii) $\omega(a^*) = \overline{\omega(a)}$, $\forall a \in A$
iii) $|\omega(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b)$, $\forall a, b \in A$
iv) $\omega((ab)^*ab) \leq \|a\|^2\omega(b^*b)$, $\forall a, b \in A$
v) if A is unital then $\omega(I) = 1$.

Remark 2.11 It can be shown that
i) $\forall a \in A$ there exist a state ω such that $\omega(a^*a) = 1$
ii) on a (separable) C^*–algebra A there always exist a faithful state. $\diamond$

Example 2.12 Given a Hilbert space H with a scalar product $\langle \ | \ \rangle$, a representation $\pi \in \mathrm{Rep}\,(A, H)$ and a (normalized) vector $\Omega \in H$, $\|\Omega\| = 1$, the expression

$$\omega(a) = \langle \Omega \mid \pi(a)\Omega \rangle \tag{2.2}$$

provides an important example, known as a 'vector state'. $\diamond$

It turns out that any state on A is a vector state for some representation, as will be clear from the following construction of Gelfand-Naimark-Segal (GNS). Given a state ω on A the GNS construction provides a canonical representation $\pi_\omega : A \to B(H_\omega)$. We describe first the underlying Hilbert space H_ω. Start with the underlying (Banach) space A and using ω introduce a semi-definite hermitean product

$$\langle a \mid b \rangle = \omega(ab^*) \ .$$

Then, by Remark 2.10 iv), $\mathcal{I} = \{a \in A \mid \omega(aa^*) = 0\}$ is a left ideal in A. The quotient $A/\mathcal{I}$ is a complex vector space with a nondegenerate hermitean product

$$\langle a + \mathcal{I} \mid b + \mathcal{I} \rangle = \omega(ab^*) \ . \tag{2.3}$$

(which is independent of a represenative in each class). The Hilbert space H_ω is just the completion of $A/\mathcal{I}$ in this norm.

Next, given $a \in A$, $\pi_\omega(a)$ defined by $\pi_\omega(a)(b + \mathcal{I}) = ab + \mathcal{I}$ is a (well defined) linear operator in H_ω, which is bounded due to

$$\|\pi_\omega(a)(b + \mathcal{I})\|^2 = \langle ab + \mathcal{I} | ab + \mathcal{I} \rangle = \omega(b^*a^*ab) \leq \|a\|^2\omega(bb^*) = \|a\|^2\|b + \mathcal{I}\|^2 \ .$$

Denoting its bounded closure still by $\pi_\omega(a)$, we verify that

$$\pi_\omega(a_1)\pi_\omega(a_2)(b + \mathcal{I}) = a_1a_2b + \mathcal{I} = \pi_\omega(a_1a_2)(b + \mathcal{I}) \ ,$$

ie. $\pi_\omega(a_1)\pi_\omega(a_2) = \pi_\omega(a_1 a_2)$. Thus $\pi_\omega : A \to B(H_\omega)$ is a representation (quotient of the left multiplication).

Now it remains only to construct $\Omega_\omega \in H_\omega$. If A is a unital algebra, consider $\Omega_\omega = I + \mathcal{I}$. Then

$$\langle \Omega_\omega \mid \pi(a)\Omega_\omega \rangle = \langle I + \mathcal{I} \mid a + \mathcal{I} \rangle = \omega(a) \ . \tag{2.4}$$

Moreover, $\{\pi(a)\Omega_\omega | a \in A\} = \{a + \mathcal{I} | a \in A\}$ is obviously dense in H_ω and thus Ω_ω is cyclic for π_ω.

If A is not a unital algebra, take instead the C^*-algebra $\tilde{A} = \mathbf{C} \times A$ with

$$\begin{aligned}
(\alpha, a) + (\beta, b) &= (\alpha + \beta, a + b) \\
(\alpha, a)(\beta, b) &= (\alpha\beta, \alpha b + \beta a + ab) \\
(\alpha, a)^* &= (\bar{\alpha}, a^*) \\
\|(\alpha, a)\| &= \sup\{\|\alpha b + ab\| \mid b \in A, \|b\| = 1\} \ .
\end{aligned}$$

Clearly, identifying a with $(0, a)$, A is just a C^*-subalgebra of $\tilde{A}$. Sometimes $\tilde{A}$ is denoted $\mathbf{C}I + A$, and $(\alpha, a) = \alpha I + a$ (the identity being $(1, 0)$). Moreover, extend ω in a canonical way to $\tilde{\omega}$ by $\tilde{\omega}(\alpha I + a) = \alpha + \omega(a)$. ($\tilde{\omega}$ can be seen to be a state on $\tilde{A}$ due to Remark 2.10 iii).

At this point one essentially repeats the GNS construction of H_ω and π_ω using $\tilde{A}$ and $\tilde{\omega}$. The only delicate point of the construction is the cyclicity of Ω_ω for π_ω. In fact it does hold since $\pi_\omega(\mathbf{C}I + A)\Omega_\omega = \mathbf{C}\Omega_\omega + \pi_\omega(A)\Omega_\omega$ is dense and Ω_ω can be shown to belong to the closure of $\pi_\omega(A)\Omega_\omega$.

It turns out that GNS representation is unique up to a unitary equivalence. Indeed, given another state ω', there exist $V : H_\omega \to H_{\omega'}$, defined (using the cyclicity of Ω_ω) by

$$V\pi_\omega(a)\Omega_\omega = \pi_{\omega'}(a)\Omega_{\omega'} \ .$$

The operator V is unitary

$$\langle V\pi_\omega(a)\Omega_\omega \mid V\pi(b)\Omega_\omega \rangle = \langle \pi_{\omega'}(a)\Omega_{\omega'} \mid \pi_{\omega'}(b)\Omega_{\omega'} \rangle = \omega(a^* b) = \langle \pi_\omega(a)\Omega_\omega \mid \pi(b)\Omega_\omega \rangle \ ,$$

and intertwines the representations, $V^{-1}\pi_{\omega'}(a)V = \pi_\omega(a)$, $\forall a \in A$, and the cyclic states, $V(\Omega_\omega) = \Omega_{\omega'}$.

Using Remark 2.11 i) and GNS representations, it can be shown that a faithful representation of any C^*-algebra can be always constructed as a direct sum $\pi = \bigoplus_\omega \pi_\omega$ on $H = \bigoplus_\omega H_\omega$. This leads to the second theorem due to Gelfand and Naimark which gives a characterization of not (necessarily) commutative C^*–algebras.

Theorem 2.13

$$\left(\begin{array}{c} A \text{ is a C}^*\text{–algebra} \\ \ \end{array} \right) \Leftrightarrow \left(\begin{array}{c} A \text{ is } {}^* - \text{isomorphic to a} \\ C^*\text{–subalgebra of } B(H) \end{array} \right).$$

Remark 2.14 The association to A of a C^*-subalgebra of $B(H)$ is nonunique (in the sense of unitary equivalence). We mention also that in case A is nonseparable, H may be nonseparable as well. ◇

We refer eg. to [1], [2], [3] for more details.

3 Morphisms

In this section we define morphisms between C^*–algebras in such a way that they correspond to the morphisms (continuous maps) between the associated spaces. At a first glance $*$-homomorphisms are natural candidates (see Theorem 1.11).

Theorem 3.1 *Let ϕ be a $*$-homomorphism from $C_\infty(\Lambda_1)$ to $C_\infty(\Lambda_2)$. If there exists a continuous map $\phi_* : \Lambda_2 \to \Lambda_1$ such that*

$$(\phi(a))(\lambda) = a(\phi_*(\lambda)),\ \forall \lambda \in \Lambda_2\ ,$$

then ϕ_* is proper, i.e. the inverse image of any compact set is compact.

It turns out that some important continuous maps are not proper. For example, the group multiplication

$$\cdot : \mathbf{G} \times \mathbf{G} \to \mathbf{G};\quad (h, g) \mapsto h \cdot g \tag{3.1}$$

on a locally compact group $\mathbf{G}$, is proper only if $\mathbf{G}$ is compact, since the inverse image of the neutral element is $\{(g^{-1}, g)\}$, which is homeomorphic to $\mathbf{G}$. Thus, the algebraic counterpart of (3.1) may not be a $*$-homomorphism from $C_\infty(\mathbf{G})$ to $C_\infty(\mathbf{G}\times\mathbf{G})$. This shows that $*$-homomorphisms do not satisfy our requirement that morphisms should encode information about all continuous maps. We therefore need to find another candidate for a morphism between C^*–algebras. The choice will be motivated by a sequence of observations at the commutative level.
Given a function $a \in C_\infty(\Lambda_1)$ and a continuous map $\phi_* : \Lambda_2 \to \Lambda_1$ we note that a pull-back $\phi(a)$ of a defined by $\phi(a)(\lambda) = a(\phi_*(\lambda))$ is a function on Λ_2 which is bounded but not necessarily vanishing at infinity. As an example take such ϕ_* that maps a sequence of points $\lambda_n \to \infty$ tending to infinity in Λ_2 (ie. $\forall K \subset \Lambda$, K compact, $\exists N$ such that $\forall n > N$ it holds that $\lambda_n \notin K$) to a sequence $\phi_*(\lambda_n)$ converging to some $\lambda \in \Lambda_1$). This fact together with Remark (2.1) suggests that target space of a morphism between C^*–algebras A_1 and A_2 should be the largest algebra that contains A_2 as an essential ideal. The construction of such an algebra proceeds along the following steps.
First, note that any element of $C_b(\Lambda)$ can be canonically viewed as a (bounded) operator acting by the left multiplication on $C_\infty(\Lambda)$. Correspondingly, given a C^*–algebra A we embed it in the algebra $B(A)$ of bounded operators on A. By this we mean that $b \in A$ is viewed as an operator $b \in B(A)$ acting as $b : a \mapsto ba$ for all $a \in A$. Note, that in contrast to A, the algebra $B(A)$ does not have any natural *-operation (A is not a Hilbert space, in general). However, in $B(A)$ there is a subset $M(A)$ that has a natural *-structure. To see this note the following identity

$$a_2^* b^* a_1 = (ba_2)^* a_1\ ,$$

which holds in A. Analogously, given two arbitrary elements $b, b^* \in B(A)$, we say that b^* is a Hermitean conjugate of b iff

$$\forall a_1, a_2 \in A\ , \qquad a_2^* b^*(a_1) = (b(a_2))^* a_1. \tag{3.2}$$

Now we define

$$M(A) = \{b \in B(A) \mid \exists \text{ Hermitean conjugate of } b\}. \tag{3.3}$$

It is clear that $M(A)$ equipped with a star operation being the Hermitean conjugate and with the operatorial norm is a C^*–algebra. Moreover, identifying $a \in A$ with an operator of a left multiplication by a, A can be viewed as a closed linear subspace of $M(A)$. Next consider an operator $L_a b$ for $a \in A$ and $b \in M(A)$. For any $c \in A$ we have $L_a b(c) = L_a(b(c)) = a(b(c)) = (b^*(a^*))^* c = L_{b^*(a^*)}(c)$, where in the third equality we use (3.2) with $a_1 = c$, $a_2 = a^*$ and $b = b^*$. Hence $L_a b = L_{b^*(a^*)}$, which shows that A is a right ideal in $M(A)$. Similarly, for $a, c, d \in A$ and $b \in M(A)$ we have $(ad)^* b^*(c) = d^* a^* b^*(c) = (b(ad))^*\ c$, where in the last equality we use (3.2) with $a_1 = c$ and $a_2 = ad$. Hence $b(ad) = b(a)d$, and in turn, $bL_a = L_{b(a)}$, which shows that A also is a left ideal in $M(A)$. It can be also verified that A is an essential ideal and $M(A)$ is the maximal C^*–algebra that contains A as an essential ideal. Moreover, since the identity operator in $B(A)$ is its own Hermitean conjugate then it belongs to $M(A)$. Thus $M(A)$ is unital. We shall denote I_A the unit of $M(A)$. (In the theory of C^*–algebras, $M(A)$ is known as the multiplier algebra of A).

Example 3.2 One can check that for $A = C_\infty(\Lambda)$ this construction yields $M(A) = C_b(\Lambda)$, as it should. $\diamond$

If $A \subset B(H)$, which can be always assumed due to Theorem 2.13, we shall be more specific and 'adapt' the Hilbert space H to A. Namely, if there exists a vector in H that is annihilated by all elements of A, we can pass to the orthogonal complement of such a vector. Thus, without the loss of generality, we assume from now on that $A \subset B(H)$ is nondegenerate, i.e. $\bigcap_{a\in A}(\ker\ a) = \{0\}$. Then we have the following Proposition which motivates the name 'multiplier algebra' for $M(A)$ in the general case.

Proposition 3.3 $M(A) = \{b \in B(H) \mid bA \subset A \text{ and } Ab \subset A\}$.

Proof. We prove first that $M(A) \supset \{b \in B(H) \mid bA \subset A \text{ and } Ab \subset A\}$. If $bA \subset A$ then the left multiplication $L_b : A \to A$, $L_b(a) = ba$, is a bounded operator $L_b \in B(A)$. Moreover $(L_b)^* := L_{b^*}$, which is an element of $B(A)$ since $b^* A = (Ab)^* \subset A^* = A$, satisfies the conditions (3.2). In addition, because of the nondegeneracy assumption the assignment $b \mapsto L_b$ is injective, since otherwise $ba = 0\ \forall a \in A$ for some b would mean that $b\psi = 0$, $\forall \psi \in H_0 = \{\psi = a\phi \mid \phi \in H, a \in A\}$. Clearly, $H_0 \neq H$ and we can take $\chi \in H_0^\perp$. Then $0 = \langle \chi \mid a\phi \rangle = \langle a^*\chi \mid \phi \rangle$ shows that $a^*\chi = 0$, $\forall a \in A$, which contradicts the nondegeneracy assumption. Therefore the $\supset$ part is proven.
To prove the $\subset$ part we apply Theorem 2.8 by specifying $A_2 = M(A)$ and π to be the embedding of A in $B(H)$. Then for any element $\tilde{b} \in M(A)$ we have $\tilde{b} = L_{\tilde{\pi}(\tilde{b})}$. Indeed $\forall a \in A, \quad L_{\tilde{\pi}(\tilde{b})} a = (\tilde{\pi}(\tilde{b}))\pi(a) = \tilde{\pi}(\tilde{b}a) = \tilde{b}(a)$.

$\square$

Example 3.4 Let $CB(H)$ be a C^*–algebra of compact operators on H. This is an ideal of $B(H)$ which is in fact essential. Thus $M(CB(H)) = B(H)$. Clearly, $CB(H)$ is not unital unless H is finite dimensional. $\diamond$

Suppose now that A is a unital C^*–algebra with unit I. Then the map

$$M(A) \ni b \mapsto bI \in A$$

is an isomorphisms of C^*–algebras A and $M(A)$. Taking into account that $M(A)$ has always unity I_A, we have

$$(\ M(A) = A\ ) \iff (\ A \text{ is unital }) \ .$$

This is a noncommutative version of the first part of

$$(\ C(\Lambda) = C_\infty(\Lambda)\ ) \iff (\ C_\infty(\Lambda) \text{ is unital }) \iff (\ \Lambda \text{ is compact }) \ .$$

Extending the second part (last $\iff$) to the non-commutative case we say that unital C^*–algebras correspond to compact quantum spaces. Instead non-compact quantum spaces correspond to non-unital C^*–algebras. Hence, we have the following correspondence

$$\text{(compact spaces)} \longleftrightarrow \text{(unital algebras)}$$

$$(\text{non} - \text{compact spaces}) \longleftrightarrow (\text{non} - \text{unital algebras})$$

It can be shown that if $M(A)$ is separable then A is unital. Thus it is evident that $M(A)$ is rarely separable in the norm topology. However, there is another natural topology on $M(A)$.

Definition 3.5 *We say that a sequence (b_n) of operators $b_n \in M(A)$ converges strictly to zero iff*

$$\forall a \in A \quad \|b_n a\| \to 0 \text{ and } \|ab_n\| \to 0. \tag{3.4}$$

This equips $M(A)$ with a strict topology. We have the following Proposition.

Proposition 3.6 *A is strictly dense in $M(A)$.*

Proof. It can be seen (cf. [1]) that there exists an approximate unity $e_n \nearrow I_A$ for A, i.e. a strictly converging to $I_A \in M(A)$ increasing sequence $(e_{n+1} - e_n > 0)$ of elements in A with $\|e_n\| \leq 1$. If $b \in M(A)$ then $be_n \in A$. Moreover, $\forall a \in A$ $e_n a \to a$ in norm and since L_b is continuous, $be_n a \to ba$ in norm. In addition, since $ab \in A$ then $\forall a \in A$ $abe_n \to ab$ in norm. As a consequence, $be_n \to b$ strictly.

$\square$

Remark 3.7 This proposition shows that $M(A)$ has a countable strictly dense subset or, what is the same, $M(A)$ is separable in the strict topology. In particular, Example 3.4 shows that $B(H)$ is separable in the strict topology. In this special case it coincides with the (operatorial) *-strong topology, according to which $a_n \to 0$ *-strongly iff $a_n \to 0$ strongly and $(a_n)^* \to 0$ strongly.
Moreover, on $C_b(\Lambda)$ the strict topology is not much different from the uniform topology. Namely, $f_n \to 0$ strictly iff f_n are bounded by the same number [3] and $f_n|_K \to 0$ uniformly for every compact subset K of Λ [4]. $\diamond$

[3] $\exists M\ \forall n\ \forall \lambda\ |f_n(\lambda)| < M$

[4] $\forall K \subset \Lambda,\ K$ compact $\forall \epsilon > 0\ \exists N\ \forall n > N,\ \lambda \in K\ |f_n(\lambda)| < \epsilon$

We return to the problem of finding the right notion of a morphism. As we have argued, the target of a morphism from A_1 to A_2 should be $M(A_2)$ rather than A_2 itself. Then however another trouble arises. Suppose that ϕ is a *-homomorphism from $C_\infty(\Lambda_1)$ to $M(C_\infty(\Lambda_2)) = C_b(\Lambda_2)$. Given a point in Λ_1, it is impossible that all functions in $C_\infty(\Lambda_1)$ vanish at this point. Thus for arbitrary $\lambda \in \Lambda_2$, $\phi(a)(\lambda)$, which is equal to $a(\phi_*(\lambda))$, also can not vanish for all a in $C_\infty(\Lambda_1)$. A correct algebraic translation of this fact is that the set

$$\phi(C_\infty(\Lambda_1)) \cdot C_\infty(\Lambda_2) \text{ is norm dense in } C_\infty(\Lambda_2)\ . \tag{3.5}$$

Indeed, suppose that for some $\lambda \in \Lambda_2$,

$$\{\phi(a)(\lambda) \mid a \in C_\infty(\Lambda_1)\} = \lambda(\phi(C_\infty(\Lambda_1))) = \{0\}\ .$$

Then also $\lambda\left(\phi(C_\infty(\Lambda_1)) \cdot C_\infty(\Lambda_2)\right) = \{0\}$ and hence (3.5) does not hold. Conversely, if (3.5) does not hold then $\phi(C_\infty(\Lambda_1)) \cdot C_\infty(\Lambda_2)$ is obviously a proper ideal in $C_\infty(\Lambda_2)$. It can be seen to be included in a maximal modular ideal $\mathcal{I}_\phi$ which corresponds to some $\lambda \in \Lambda_2$ (λ is a character, the kernel of which is $\mathcal{I}_\phi$; see Remark 2.4). The functions in $\phi(C_\infty(\Lambda_1)) \cdot C_\infty(\Lambda_2)$ vanish at λ, but clearly $C_\infty(\Lambda_2)$ contains at least one function which is not zero at λ in Λ_2. This means that all the functions in $\phi(C_\infty(\Lambda_1))$ should vanish at λ, which is precisely the drawback we are going to avoid.

In view of the above discussion, we have to exclude those ϕ's which do not possess the property (3.5). Consequently, we define the space $\mathrm{Mor}(A_1, A_2)$ of morphisms from A_1 to A_2 as follows.

Definition 3.8 *A map ϕ is a morphism between C^*-algebras A_1 and A_2 if it belongs to a set*

$$\mathrm{Mor}(A_1, A_2) = \left\{ \phi : A_1 \to M(A_2) \;\middle|\; \begin{array}{c} \phi \text{ is a } {}^*\text{−algebra homomorphism} \\ \text{such that } \phi(A_1)A_2 \text{ is dense in } A_2 \end{array} \right\}.$$

Specifying for example to $A_2 = CB(H)$ and using Example 3.4 and Definition 2.6 ii), we have the following proposition.

Proposition 3.9

$$\mathrm{Mor}(A, CB(H)) = \mathrm{Rep}\,(A, H).$$

A more careful analysis of Definition 3.8 shows that the problems caused by Theorem 3.1 are resolved. Actually we have now the following theorem.

Theorem 3.10 *Let $\phi \in \mathrm{Mor}(C_\infty(\Lambda_1), C_\infty(\Lambda_2))$. Then there exists a continuous map $\phi_* : \Lambda_2 \to \Lambda_1$ such that $\forall a \in C_\infty(\Lambda_1)\ \ \forall \lambda \in \Lambda_2$*

$$\phi(a)(\lambda) = a(\phi_*(\lambda))\ .$$

The converse is also true.

The Definition 3.8 is more 'symmetric' that it seems due to the fact that $(\phi(A_1)A_2)^* = A_2\phi(A_1)$. However, still the domain and the target of a morphism are not of the same type. This drawback can be cured by noting that in the commutative case not only functions in C_∞ can be pull-back, but also those in C_b. So, one should be able to extend the domain of a homomorphism from A to $M(A)$. This is indeed possible.

Theorem 3.11 *Let $\phi : A_1 \to M(A_2)$. Then we have*

$$(\phi \in \mathrm{Mor}(A_1, A_2)) \Leftrightarrow \left(\begin{array}{c} \text{there exists a strictly continuous} \\ {}^*\text{−algebra homomorphism } \tilde{\phi} : M(A_1) \to M(A_2), \\ \text{such that } \tilde{\phi}(I_{A_1}) = I_{A_2} \text{ and } \tilde{\phi}|_{A_1} = \phi \end{array} \right).$$

Proof. The $\Leftarrow$ part is obvious except that $\tilde{\phi}(A_1)A_2$ is strictly dense in A_2. But if $A_1 \ni e_n \to I_{A_1}$ strictly then $\tilde{\phi}(e_n) \to I_{A_2}$ strictly, i.e. $\tilde{\phi}(e_n)b \to b$ in norm $\forall b \in A_2$.
In order to show the $\Rightarrow$ part note that nondegenerate inclusion $A_2 \subset B(H)$ means that $\phi : A_1 \to M(A_2)$ is a (nondegenerate) representation. Applying Theorem 2.8 with A_2 specified to be $M(A_1)$ we are guaranteed that there exists $\tilde{\phi} : M(A_1) \to B(H)$. The image of $M(A_1)$ by $\tilde{\phi}$ is actually contained in $M(A_2)$.

□

We remark that $\phi \in \mathrm{Mor}(A_1, A_2)$ are in fact norm-continuous and that this theorem allows for a fully 'symmetric' reformulation of Definition 3.8 of $\mathrm{Mor}(A_1, A_2)$. We have

$$\mathrm{Mor}(A_1, A_2) = \left\{ \phi : M(A_1) \to M(A_2) \,\middle|\, \begin{array}{c} \phi \text{ is a unital strictly continuous} \\ {}^*\text{−algebra homomorphism} \end{array} \right\}. \quad (3.6)$$

Now from this last formulation it is obvious that the morphisms can be composed. Thus, together with C^*–algebras, they form a category.

4 Compact quantum groups

Before discussing groups in the algebraic context we need some preparations.
Given two *–algebras A and B, their algebraic tensor product is defined as the tensor product of vector spaces $A \otimes_{alg} B$ equipped with a *–algebra structure by
i) $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$
ii) $(a \otimes b)^* = a^* \otimes b^*$.
Next, given two (nondegenerate) C^*–algebras $A \subset B(H)$ and $B \subset B(K)$ their tensor product $A \otimes B$ is defined as a C^*–algebra which is a norm closed linear span of a set

$$\{a \otimes b \in B(H \otimes K) : a \in A \text{ and } b \in B\}.$$

Remark 4.1 It can be shown that $A \otimes B$, the completion of $A \otimes_{alg} B$ with respect to the operator norm in $B(H \otimes K)$, does not depend on the (not unique) way of embedding A in $B(H)$ and B in $B(K)$. ◇

Remark 4.2 In the classical case the C^*–algebra corresponding to direct (Cartesian) product of two spaces is just the tensor product of the C^*–algebras corresponding to these spaces (this differs from the notion of a free product in algebraic geometry). ◇

As far as the tensor product $\phi \otimes \psi$ of two morphisms $\phi \in \mathrm{Mor}(A, A_1)$ and $\psi \in \mathrm{Mor}(B, B_1)$ is concerned, it is first defined on the algebraic product $A \otimes_{alg} B$, i.e.

$$\phi \otimes \psi \; : \; A \otimes_{alg} B \to M(A_1) \otimes_{alg} M(B_1),$$

and then extended by continuity to $A \otimes B$. The resulting $\phi \otimes \psi$ is a map

$$\phi \otimes \psi \; : \; A \otimes B \to M(A_1) \otimes M(B_1).$$

Since $M(A_1) \otimes M(B_1)$ is naturally included in $M(A_1 \otimes B_1)$, then finally

$$\phi \otimes \psi \in \mathrm{Mor}(A \otimes B, A_1 \otimes B_1) \; .$$

Now given a group $\mathbf{G}$ with the neutral element e we shall use the following maps:

$$\begin{array}{rll} \cdot : \mathbf{G} \times \mathbf{G} \to \mathbf{G} \; ; \;\; (g_1, g_2) \mapsto g_1 \cdot g_2 & \text{for the multiplication in } \mathbf{G}, & (4.1) \\ \varepsilon : \{e\} \to \mathbf{G} \; ; \;\; e \mapsto e & \text{for the inclusion of } e \text{ in } \mathbf{G}, & \\ \mathrm{diag} : \mathbf{G} \to \mathbf{G} \times \mathbf{G} \; ; \;\; g \mapsto (g, g) & \text{for the diagonal map,} & (4.2) \\ \mathrm{inv} : \mathbf{G} \to \mathbf{G} \; ; \;\; g \mapsto g^{-1} & \text{for the inverse operation} & (4.3) \\ \star : \mathbf{G} \to \{e\} \; ; \;\; g \mapsto e & \text{for the point map.} & \end{array}$$

The axioms of a usual group can be rewriten in terms of commutative diagrams as follows:
i) the associativity

$$\begin{array}{ccc} \mathbf{G} \times \mathbf{G} \times \mathbf{G} & \xrightarrow{\mathrm{id} \times \cdot} & \mathbf{G} \times \mathbf{G} \\ {\scriptstyle \cdot \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \cdot} \\ \mathbf{G} \times \mathbf{G} & \xrightarrow{\cdot} & \mathbf{G} \end{array} \qquad (4.4)$$

ii) the neutral element

$$\begin{array}{ccc} \mathbf{G} \times \{e\} & \xrightarrow{\mathrm{id} \times \varepsilon} & \mathbf{G} \times \mathbf{G} \\ \| & & \downarrow {\scriptstyle \cdot} \\ \mathbf{G} & \xrightarrow{\mathrm{id}} & \mathbf{G} \end{array} \qquad \begin{array}{ccc} \{e\} \times \mathbf{G} & \xrightarrow{\varepsilon \times \mathrm{id}} & \mathbf{G} \times \mathbf{G} \\ \| & & \downarrow {\scriptstyle \cdot} \\ \mathbf{G} & \xrightarrow{\mathrm{id}} & \mathbf{G} \end{array} \qquad (4.5)$$

iii) the inverse

$$\begin{array}{ccccc} \mathbf{G} & \xrightarrow{\mathrm{diag}} & \mathbf{G} \times \mathbf{G} & \xrightarrow{\mathrm{inv} \times \mathrm{id}} & \mathbf{G} \times \mathbf{G} \\ {\scriptstyle \star} \downarrow & & & & \downarrow {\scriptstyle \cdot} \\ \{e\} & & \xrightarrow{\varepsilon} & & \mathbf{G} \end{array} \qquad \begin{array}{ccccc} \mathbf{G} & \xrightarrow{\mathrm{diag}} & \mathbf{G} \times \mathbf{G} & \xrightarrow{\mathrm{id} \times \mathrm{inv}} & \mathbf{G} \times \mathbf{G} \\ {\scriptstyle \star} \downarrow & & & & \downarrow {\scriptstyle \cdot} \\ \{e\} & & \xrightarrow{\varepsilon} & & \mathbf{G} \end{array} \qquad (4.6)$$

If $\mathbf{G}$ is a matrix group, its matrix elements generate a (commutative) *–algebra A of 'polynomial' functions on $\mathbf{G}$ (see eg. Example 1.5). The multiplication in A defines a map

$$m : A \otimes A \to A \; , \quad m : a \otimes b \mapsto ab \; , \qquad (4.7)$$

dual to the diagonal map (4.2). The algebraic counterpart of the above diagrams requires an introduction of the following important algebra homomorphisms

$$\begin{aligned} \Delta : A &\rightarrow A \otimes A \quad &&\text{(coproduct)}, \\ \epsilon : A &\rightarrow \mathbf{C} \quad &&\text{(counit) and} \\ S : A &\rightarrow A \quad &&\text{(coinverse)}, \end{aligned}$$

that are dual to the product $\cdot$, the embedding ε and the inverse map (4.3), respectively. We have then the following commutative diagrams, corresponding to (4.4)-(4.6)

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\mathrm{id}\otimes\Delta"'] \\ A \otimes A \arrow[u, "\Delta\otimes\mathrm{id}"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{4.8}$$

$$\begin{tikzcd} A \otimes \mathbf{C} \arrow[d, equal] & A \otimes A \arrow[l, "\mathrm{id}\otimes\epsilon"'] \\ A & A \arrow[l, "\mathrm{id}"'] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C} \otimes A \arrow[d, equal] & A \otimes A \arrow[l, "\epsilon\otimes\mathrm{id}"'] \\ A & A \arrow[l, "\mathrm{id}"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{4.9}$$

$$\begin{tikzcd} A & A \otimes A \arrow[l, "m"'] & A \otimes A \arrow[l, "S\otimes\mathrm{id}"'] \\ \mathbf{C} \arrow[u] & & A \arrow[ll, "\epsilon"'] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} A & A \otimes A \arrow[l, "m"'] & A \otimes A \arrow[l, "\mathrm{id}\otimes S"'] \\ \mathbf{C} \arrow[u] & & A \arrow[ll, "\epsilon"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{4.10}$$

A generalization of (4.8) to a noncommutative algebra A leads to the notion of a quantum semigroup. While counit won't be so interesting in the quantum case, the notion of coinverse is somewhat delicate. The multiplication map m employed in (4.10) is not a homomorphism for noncommutative A (it does not preserve the products) and thus it is not a map in the category we are considering. In other words, we don't have a quantum analogue of the diagonal map (4.2) in our setup [5]. In fact (4.9) and (4.10) will be replaced by another axiom.

In the following the superscript "$^{c.l.s.}$" stands for "closed linear span".

Definition 4.3 *A compact quantum group is a pair $G = (A, \Delta)$, such that A is a unital separable C^*–algebra and $\Delta \in \mathrm{Mor}(A, A \otimes A)$ (coproduct) satisfies*
i)

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{4.11}$$

ii)

$$\{(b \otimes I)\Delta(a) \mid a, b \in A\}^{c.l.s.} = A \otimes A \tag{4.12}$$

and

$$\{(I \otimes b)\Delta(a) \mid a, b \in A\}^{c.l.s.} = A \otimes A \ . \tag{4.13}$$

[5]Such a map exists for the free product, cf. Remark 4.2

Remark 4.4 In a more adequate terminology a compact quantum group should be rather called a C^*–algebra of 'quantum functions on a compact quantum group'. $\diamond$

Remark 4.5 For a unital C^*–algebra A, since $M(A) = A$ and $M(A \otimes A) = A \otimes A$, the coproduct is actually a homomorphism $\Delta : A \to A \otimes A$. In Definition 4.3 we require that Δ is a morphism to allow a future extension to the noncompact case. For nonunital A the conditions (4.12) and (4.13) mean in particular that $(b \otimes I_A)\Delta(a)$ and $(I_A \otimes b)\Delta(a)$, which generally are elements of $M(A \otimes A)$, must belong to $A \otimes A$. $\diamond$

If A is commutative then the Gelfand – Naimark Theorem 2.2 provides a compact space $\mathbf{G}$, such that $A = C(\mathbf{G})$. Also, by Theorem 3.10 we have a continuous map $\Delta_* : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, which in the usual notation $\Delta_*(\lambda_1, \lambda_2) = \lambda_1 \cdot \lambda_2$, defines a continuous binary operation (multiplication) in $\mathbf{G}$. Its properties follow from conditions (4.11) - (4.13). For instance, (4.11) means that

$$(\lambda_1 \cdot \lambda_2) \cdot \lambda_3 = \lambda_1 \cdot (\lambda_2 \cdot \lambda_3) \tag{4.14}$$

for any λ_1, λ_2, $\lambda_3 \in \mathbf{G}$. We claim that (4.12) is equivalent to the cancellation law from the left

$$(\lambda \cdot \lambda_1 = \lambda \cdot \lambda_2) \Longrightarrow (\lambda_1 = \lambda_2) \tag{4.15}$$

for any λ, λ_1, $\lambda_2 \in \mathbf{G}$. In fact, observe that $(b \otimes I)\Delta(a)(\lambda_1, \lambda_2) = b(\lambda_1)a(\lambda_1 \cdot \lambda_1)$. Recall also that by Stone-Weierstrass theorem a set of functions which separate points of Λ is dense in $C(\Lambda)$. Now assume that (4.15) holds. Take $(\lambda_1, \lambda_2) \neq (\lambda_1', \lambda_2')$. In order that $(b \otimes I)\Delta(a)(\lambda_1, \lambda_2) \neq (b \otimes I)\Delta(a)(\lambda_1', \lambda_2')$ it suffices to take $a = I$ and b such that $b(\lambda_1) \neq b(\lambda_1')$ if $\lambda_1 \neq \lambda_1'$, and a such that $a(\lambda_1 \cdot \lambda_2) \neq a(\lambda_1' \cdot \lambda_2')$ and $b = I$ if $\lambda_1 = \lambda_1'$ but $\lambda_2 \neq \lambda_2'$, so $\lambda_1 \cdot \lambda_2 \neq \lambda_1' \cdot \lambda_2'$ by (4.15) (clearly one of the two alternatives must hold).
Next we show the converse. Assume that (4.15) does not hold so there are some λ_1 and $\lambda_2 \neq \lambda_2'$ such that $\lambda_1 \cdot \lambda_2 = \lambda_1 \cdot \lambda_2'$. Take some $w \in A \otimes A$ such that $w(\lambda_1, \lambda_2) \neq w(\lambda_1, \lambda_2')$. Such w can not be approximated by elements of the type $(b \otimes I)\Delta(a)$ since for them always holds $(b \otimes I)\Delta(a)(\lambda_1, \lambda_2) = (b \otimes I)\Delta(a)(\lambda_1, \lambda_2')$.
Similarly (4.13) is equivalent to the cancellation law from the right

$$(\lambda_1 \cdot \lambda = \lambda_2 \cdot \lambda) \Longrightarrow (\lambda_1 = \lambda_2) \tag{4.16}$$

for any λ, λ_1, $\lambda_2 \in \mathbf{G}$.

It turns out that any compact space $\mathbf{G}$ endowed with a continuous multiplication satisfying conditions (4.14) – (4.16) is a topological group (in particular there exist the neutral element and the inverse). Namely, choose some $\lambda \in \mathbf{G}$. Then, by compactness, the sequence $\lambda^n = \lambda \cdot \lambda \ldots \lambda$ (n times) has a convergent subsequence, say λ^{n_k}. Call its limit λ'. It may happen that $n_{k+1} - n_k \to \infty$ but $\lambda^{n_{k+1}} = \lambda^{n_k} \cdot \lambda^{n_{k+1}-n_k}$ in the limit yields $\lambda' = \lambda' \cdot e'$, where e' is the limit of $\lambda^{n_{k+1}-n_k}$. Multiplying from the right by arbitrary λ'' we get $\lambda' \cdot \lambda'' = \lambda' \cdot e' \cdot \lambda''$, which by (4.15) yields $\lambda'' = e' \cdot \lambda''$, $\forall \lambda'' \in \mathbf{G}$, ie. e' is the neutral element.
Analogously it can be seen that $\lambda^{n_{k+1}-n_k-1}$ converges to λ^{-1}, the inverse of λ.

Remark 4.6 Applying the Hermitean conjugation to the elements of (4.12) and (4.13) we show that

$$\{\Delta(a)(b\otimes I)\mid a,b\in A\}^{c.l.s.}=A\otimes A \tag{4.17}$$

and

$$\{\Delta(a)(I\otimes b)\mid a,b\in A\}^{c.l.s.}=A\otimes A\ . \tag{4.18}$$

◇

5 Haar measure

We pass now to the question of integration over a compact quantum group. Recall that clasically this requires a measure on a compact group $\mathbf{G}$, ie. linear positive (normalized) functional on $C(\mathbf{G})$. More specifically we need integrals with respect to the Haar measure dg, which is, say, right invariant,

$$\forall g_1\in\mathbf{G}\ ,\quad \forall a\in C(\mathbf{G}\quad \int_{\mathbf{G}} a(gg_1)dg=\int_{\mathbf{G}} a(g)dg\ .$$

In the context of C^*–algebras the above requirements have an immediate extension to the quantum case.

Definition 5.1 *On a compact quantum group $G=(A,\Delta)$ right (respectively left) invariant Haar measure is a state h on A such that for any $a\in A$*

$$(h\otimes\mathrm{id})\Delta(a)=h(a)I, \tag{5.1}$$

respectively,

$$(\mathrm{id}\otimes h)\Delta(a)=h(a)I\ . \tag{5.2}$$

Let ξ, $\xi'\in A'$ be continuous linear functionals on A and let $a\in A$. We shall use the following convolution products

$$\xi * a=(\mathrm{id}\otimes\xi)\Delta(a)\in A \tag{5.3}$$

$$a*\xi'=(\xi'\otimes\mathrm{id})\Delta(a)\in A \tag{5.4}$$

$$\xi'*\xi=(\xi'\otimes\xi)\Delta\in A' \tag{5.5}$$

Due to the coassociativity condition (4.11), the convolution product is associative:

$$\begin{aligned}(a*\xi')*\xi'' &= a*(\xi'*\xi'')\\ (\xi*a)*\xi' &= \xi*(a*\xi')\\ (\xi''*\xi)*a &= \xi''*(\xi*a)\\ (\xi*\xi')*\xi'' &= \xi*(\xi'*\xi'')\end{aligned}$$

for any ξ, ξ', $\xi''\in A'$ and $a\in A$. Moreover

$$(\xi'*\xi)(a)=\xi(a*\xi')=\xi'(\xi*a) \tag{5.6}$$

for any ξ, $\xi' \in A'$ and $a \in A$.

In terms of the convolution product we can rewrite (5.1) and (5.2) as, respectively,

$$h * a = h(a)I \tag{5.7}$$

$$a * h = h(a)I, \tag{5.8}$$

for any $a \in A$. Applying a functional ξ we also have

$$(h * \xi)(a) = h(a)\xi(I) \tag{5.9}$$

$$(\xi * h)(a) = h(a)\xi(I). \tag{5.10}$$

This allows for a reformulation of Definition 5.1. Namely, h is right (resp. left) invariant Haar measure on G if for any functional ξ,

$$h * \xi = \xi(I)h \tag{5.11}$$

$$\xi * h = \xi(I)h. \tag{5.12}$$

Remark 5.2 If there exist a right invariant Haar measure h_R and a left invariant Haar measure h_L then $h_R = h_L$. To see this just set $\xi = h_L$ in (5.11) and $\xi = h_R$ in (5.12) and use $h_R(I) = 1 = h_L(I)$. $\diamond$

The construction of the Haar measure proceeds along the following lines. First we show that we can weaken condition (5.11). It turns out that it is enough to verify (5.11) for at least one faithful state ξ in order to infer that h is a Haar measure.

Lemma 5.3 *Let $G = (A, \Delta)$ be a compact quantum group and h, ρ be states on A. Assume that ρ is faithful and*

$$h * \rho = \rho * h = h \tag{5.13}$$

Then h is the Haar measure.

Proof. Let

$$L_{h\otimes\rho} = \{q \in A \otimes A : \ (h \otimes \rho)(q^* q) = 0\}$$

be the left ideal related to the state $h \otimes \rho$ (compare the GNS construction). For any $c \in A$ we set

$$\Psi_L(c) = h * c - h(c)I \tag{5.14}$$

Clearly Ψ_L is a continuous linear mapping acting on A which is even completely bounded (it is the difference of two completely positive mappings) [8]. This guarantees that $\mathrm{id} \otimes \Psi_L$ is also a continuous linear mapping which therefore extends to $A \otimes A$. We shall show that

$$(\mathrm{id} \otimes \Psi_L)\Delta(c) \in L_{h\otimes\rho} \tag{5.15}$$

for any $c \in A$. Indeed, denoting the above element by q we have:

$$\begin{aligned} q &= (\mathrm{id} \otimes \Psi_L)\Delta(c) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes h)(\mathrm{id} \otimes \Delta)\Delta(c) - (\mathrm{id} \otimes h)\Delta(c) \otimes I \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes h)(\Delta \otimes \mathrm{id})\Delta(c) - (\mathrm{id} \otimes h)\Delta(c) \otimes I \\ &= \Delta(h * c) - (h * c) \otimes I. \end{aligned}$$

Therefore

$$q^* q = \Delta((h*c)^*(h*c)) - \Delta(h*c)^*[(h*c) \otimes I] - [(h*c)^* \otimes I]\Delta(h*c) + [(h*c)^*(h*c)] \otimes I$$

and

$$(h \otimes \rho)(q^* q) = \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV},$$

where

$$\begin{aligned} \mathrm{I} &= (h \otimes \rho)\Delta((h*c)^*(h*c)) = (h * \rho)((h*c)^*(h*c)), \\ \mathrm{II} &= \mathrm{III}^*, \\ \mathrm{III} &= (h \otimes \rho)\left\{[(h*c)^* \otimes I]\,\Delta(h*c)\right\} = h((h*c)^*(\rho * h * c)), \\ \mathrm{IV} &= (h \otimes \rho)((h*c)^*(h*c) \otimes I) = h((h*c)^*(h*c)). \end{aligned}$$

Now, using (5.13) we get $\mathrm{I} = \mathrm{II} = \mathrm{III} = \mathrm{IV}$, so $(h \otimes \rho)(q^* q) = 0$ and (5.15) follows.

Let $a \in A$. Using the density of (4.12), for any $\epsilon > 0$, one can find $b_1, b_2, \ldots b_n$, $c_1, c_2, \ldots c_n \in A$ such that

$$\|I \otimes a - \sum (b_k \otimes I)\Delta(c_k)\| \leq \epsilon\ .$$

Since Ψ_L is completely bounded (with the bound ≤ 2) we have

$$\|I \otimes \Psi_L(a) - \sum (b_k \otimes I)(\mathrm{id} \otimes \Psi_L)\Delta(c_k)\| \leq 2\epsilon\ .$$

Using (5.15) and remembering that $L_{h\otimes\rho}$ is a closed left ideal we get $I \otimes \Psi_L(a) \in L_{h\otimes\rho}$. It means that

$$(h \otimes \rho)(I \otimes \Psi_L(a)^* \Psi_L(a)) = \rho(\Psi_L(a)^* \Psi_L(a)) = 0\ .$$

We assumed that ρ is faithful. Therefore $\Psi_L(a) = 0$ and (cf (5.14)) $h * a = h(a)I$ for any $a \in A$.

Introducing in a similar way another completely bounded mapping $\Psi_R : A \to A$ by the formula

$$\Psi_R(c) = c * h - h(c)I$$

one can show that $(\Psi_R \otimes \mathrm{id})\Delta(c) \in L_{\rho\otimes h}$. Then using the density of (4.13) one obtains $\Psi_R(a) = 0$ for any $a \in A$. The latter means that $a * h = h(a)I$. Combining the two results we see that h is a Haar measure.

□

We are ready to prove the following theorem.

Theorem 5.4 *Given compact quantum group $G = (A, \Delta)$ there exists a unique left and right invariant Haar measure h on G.*

Proof. Recall (Remark 2.11 ii) that there is always a faithful state ρ on separable A. Let $\rho^{*n} = \rho * \rho * \ldots * \rho$ be the convolution product of n-copies of ρ,

$$h_n = \frac{1}{n}\sum_{k=1}^{n} \rho^{*k}$$

be the Cesaro sum and h be a weak accumulation point of the sequence $(h_n)_{n=1,2,\ldots}$ (the set of states of any unital C^*-algebra is compact with respect to the weak topology, so the accumulation point always exists). On a corresponding subsequence we see easily that

$$h_n * \rho = \rho * h_n = h_n + \frac{1}{n}(\rho^{*(n+1)} - \rho).$$

Taking the weak limit yields thus $h * \rho = \rho * h = h$ and by Lemma 5.3, h is a Haar measure.

To prove the uniqueness assume that h and h' are Haar measures. Then (cf (5.6)) for any $a \in A$ we have

$$h'(a) = h(h'(a)I) = h(h' * a) = h'(a * h) = h'(h(a)I) = h(a).$$

□

Remark 5.5 In this way we showed that the sequence $(h_n)_{n=1,2,\ldots}$ has only one accumulation point. Therefore h_n converges weakly to h. In other words, h is the Cesaro weak limit of ρ^{*n} (cf. formula (4.18) in [6]). ◇

6 Representations

Let $G = (A, \Delta)$ be a compact quantum group and $U = (u_{kl})_{k,l=1,2,\ldots N}$ be an $N \times N$ matrix with entries belonging to A.

Definition 6.1 *We say that U is an N–dimensional matrix representation of G if U is an invertible element of* $\mathrm{Mat}_{N\times N}(A) = \mathrm{Mat}_{N\times N}(\mathbf{C}) \otimes A$ *and*

$$\Delta u_{kl} = \sum_{r=1}^{N} u_{kr} \otimes u_{rl} \tag{6.1}$$

for all $k, l = 1, 2, \ldots N$. In addition, U is said to be unitary if

$$\sum_r u_{kr} {u_{lr}}^* = \delta_{kl} I \ , \qquad \sum_r {u_{rk}}^* u_{rl} = \delta_{kl} I \ . \tag{6.2}$$

Remark 6.2 The invertibility of U is required in order to delete some bad representations like $u_{kl} = 0 \ \ \forall k, l$. ◇

Remark 6.3 If A is commutative then (6.1) simply means that

$$u_{kl}(g_1 g_2) = \Delta u_{kl}(g_1, g_2) = \sum_r u_{kr}(g_1)\ u_{rl}(g_2),$$

i.e. U is a usual representation in terms of matrices. ◇

Remark 6.4 Equations (6.2) can be written as $U\ U^* = \mathbf{1}_N \otimes I$ and $U^*\ U = \mathbf{1}_N \otimes I$ in $\mathrm{Mat}_{N\times N}(\mathbf{C}) \ \otimes \ A$. ◇

In the sequel we shall need also infinite–dimensional representations. This requires a generalization of Definition 6.1. Recall, that in the classical case a unitary representation is a map

$$U : \mathbf{G} \ni g \mapsto U_g \in B(H)$$

such that U_g is unitary and

$$U_g U_h = U_{gh}, \quad \forall g, h \in \mathbf{G}. \tag{6.3}$$

In addition,
i) U is norm continuous iff $\|U_g - I\| \to 0$ as $g \to e$
ii) U is strong continuous iff $\forall x \in H \ \|U_g x - x\| \to 0$ as $g \to e$
The requirement i) is often too strong; for instance for the Lorentz group there are no unitary norm continuous representations (except the trivial one).

Now we shall formulate these properties algebraically. Obviously, U belongs to some class of functions on $\mathbf{G}$ with values in $B(H)$. Denote by $C_\infty(\mathbf{G}, B)$ the space of vanishing at infinity continuous functions from $\mathbf{G}$ to a C^*–algebra B. It is known that one has an identification $C_\infty(\mathbf{G}, B) \simeq B \otimes C_\infty(\mathbf{G})$. We also claim that elments of $M(B \otimes C_\infty(\mathbf{G}))$ can be viewed as $M(B)$-valued functions on $\mathbf{G}$. One way to verify this is to 'evaluate' $q \in M(B \otimes C_\infty(\mathbf{G}))$ at a point $g \in \mathbf{G}$. For usual (complex-valued) functions such an evaluation corresponds to the functional $\xi_g : C_\infty(\mathbf{G}) \ni f \mapsto f(g) \in \mathbf{C}$. Thus $\xi_g \in \mathrm{Mor}(C_\infty(\mathbf{G}), \mathbf{C})$ and $\mathrm{id}_B \otimes \xi_g \in \mathrm{Mor}(B \otimes C_\infty(\mathbf{G}), B)$. According to our definition of Mor, $\mathrm{id}_B \otimes \xi_g$ can be extended to $q \in M(B \otimes C_\infty(\mathbf{G}))$ and then $q(g) \doteq (\mathrm{id}_B \otimes \xi_g) q \in M(B)$. In fact, it can be seen that

$$M(B \otimes C_\infty(\mathbf{G})) = C_b(\mathbf{G}, M(B)), \tag{6.4}$$

where on the right hand side $M(B)$ is equipped with the strict topology.
Take now $B = B(H)$ in (6.4). Then $M(B(H)) = B(H)$ with the norm topology, hence when requiring the norm continuity we assume that

$$U \in M(B(H) \otimes C_\infty(\mathbf{G}))\ .$$

Another possibility in (6.4) is $B = CB(H)$. Then $M(CB(H)) = B(H)$ with the *–strong topology, cf. Remark 3.7. Since for unitary representations $U_g^* = U_{g^{-1}}$, this corresponds simply to the strong continuity, for which we thus require

$$U \in M(CB(H) \otimes C_\infty(\mathbf{G})\ .$$

Now, for a compact quantum group $G = (A, \Delta)$ we note that the homomorphism $\mathrm{id} \otimes \Delta : CB(H) \otimes A \to CB(H) \otimes A \otimes A$ extends to a homomorphism between the multipliers. Given $U \in M(CB(H) \otimes A)$ we use the following notation. U_{12} is such an element of $M(CB(H) \otimes A \otimes A)$ which, when viewed as an operator on $CB(H) \otimes A \otimes A$, acts as U on the first two factors and as an identity on the last factor. Similarly, $U_{13} \in M(CB(H) \otimes A \otimes A)$ acts as U on the first and last factor and as an identity on the middle factor.

Definition 6.5 *$U \in M(CB(H) \otimes A)$ which is unitary (i.e. $U^*U = UU^* = I_{CB(H)\otimes A}$ in the sense of *–algebras) and satisfies*

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}\ . \tag{6.5}$$

is called a strongly continuous unitary representation of a compact quantum group $G = (A, \Delta)$ acting on a Hilbert space H.

Remark 6.6 A norm continuous unitary representation is similarly defined as a unitary $U \in M(B(H) \otimes A)$ which satisfies (6.5). If G is compact ($I_A \in A$) then for a norm continuous U we have $U \in B(H) \otimes A$.
In the classical situation (6.5) means just (6.3) .
The finite matrix representations 6.1 correspond to the case $H = \mathbf{C}^N$ and
$M(CB(\mathbf{C}^N) \otimes A) = M(\mathrm{Mat}_{N\times N}(\mathbf{C}) \otimes A) = \mathrm{Mat}_{N\times N}(\mathbf{C}) \otimes A.$ $\diamond$

As we have shown, the framework of C^*–algebras has enough tools to distinguish quite a subtle difference between the strong and norm continuous representations.

7 Right regular representation

Now we shall introduce an important right regular representation. This representation acts in a Hilbert space H, denoted sometimes $L^2(h)$, where h is the Haar measure on G. More precisely, H is obtained via the GNS construction using the Haar measure as a reference state $\omega = h$. Recall also that GNS representation $\pi = \pi_h$ of A assures the existence of a special vector $\Omega = \Omega_h \in H$, which is cyclic (see (2.1)) and satisfies (2.2). In the sequel we shall often use the Dirac 'bra' and 'ket' notation.

Definition 7.1 *The right regular representation* $U \in M(CB(H) \otimes A)$, $H = L^2(h)$, *is defined by extending the following action of* U*:*

$$U(\pi(a)|\Omega\rangle\langle x| \otimes c) = ((\pi \otimes \mathrm{id})\Delta(a))(|\Omega\rangle\langle x| \otimes c) \ , \tag{7.1}$$

for all $x \in H$ *and* $a, c \in A$.

In order to explain this definition we recall that U can be viewed as an operator on $CB(H) \otimes A$ and that for $a \in A \subset M(A) \ni b$, $b(a) = ba$, i.e. the action of b (as an operator) on a is just the multiplication of b and a in $M(A)$. Then, it is shown by the following sequence of affirmations, that the action (7.1) of U is well defined:

$$\begin{aligned}
\pi &\in \mathrm{Rep}\,(A, H) = \mathrm{Mor}(A, CB(H)) \ , \\
\pi \otimes \mathrm{id} &\in \mathrm{Mor}(A \otimes A, CB(H) \otimes A) \ , \\
(\pi \otimes \mathrm{id}) \circ \Delta &\in \mathrm{Mor}(A, CB(H) \otimes A) \ , \\
(\pi \otimes \mathrm{id}) \circ \Delta(a) &\in M(CB(H) \otimes A) \ .
\end{aligned}$$

Thus, since $|\Omega\rangle\langle x| \otimes c$ belongs to $CB(H) \otimes A$, then also the r.h.s. of (7.1) belongs to $CB(H) \otimes A$.

In additon, due to the cyclicity of Ω, the set

$$\{\pi(a)|\Omega\rangle\langle x| \otimes c\} \text{ is dense in } CB(H) \otimes A \ . \tag{7.2}$$

Thus, in order to extend U as operator to the whole $CB(H) \otimes A$, we have to show that U is norm continuous. For that aim we search for a Hermitean conjugate of U. Let $q = \pi(a)|\Omega\rangle\langle x| \otimes c$ and $q' = \pi(a')|\Omega\rangle\langle x' \mid \otimes c'$. We have:

$$\begin{aligned}
(Uq)^* Uq' &= (|x\rangle\langle\Omega| \otimes c^*)(\pi \otimes \mathrm{id})\Delta(a^*)(\pi \otimes \mathrm{id})\Delta(a')(|\Omega\rangle\langle x' \mid \otimes c') \\
&= (|x\rangle\langle\Omega| \otimes c^*)(\pi \otimes \mathrm{id})\Delta(a^* a')(|\Omega\rangle\langle x' \mid \otimes c') \\
&= |x\rangle\langle x'| \otimes c^*(h \otimes \mathrm{id})\Delta(a^* a')c' \\
&= |x\rangle\langle x'| \otimes c^* h(a^* a')c'
\end{aligned} \tag{7.3}$$

where in the second equality we use the fact that $\pi \otimes \mathrm{id}$ is homomorphism, in the third equality we use (2.2) and in the fourth equality we use a property of Haar measure. This means that

$$(Uq)^* Uq' = q^* q' \tag{7.4}$$

and hence $\|Uq\| = \|q\|$, so U is a (partial) isometry. In fact we have furthermore

$$\begin{aligned}
U(CB(H) \otimes A) &= \{(\pi \otimes \mathrm{id})\Delta(a)(|\Omega\rangle\langle x| \otimes bc)\}^{c.l.s.} \\
&= \{(\pi \otimes \mathrm{id})\Delta(a)(I \otimes b)(|\Omega\rangle\langle x| \otimes c)\}^{c.l.s.} \\
&= \{(\pi \otimes \mathrm{id})(a \otimes b)(|\Omega\rangle\langle x| \otimes c)\}^{c.l.s.} \\
&= \{\pi(a)(|\Omega\rangle\langle x| \otimes bc)\}^{c.l.s.} \\
&= CB(H) \otimes A \ ,
\end{aligned}$$

where in the third equality we used the compact quantum group axiom (4.18) and in the last equality we applied (7.2). Thus, U extends to a unitary operator on $CB(H) \otimes A$ which has an inverse U^{-1}.
Next, setting $q' = U^{-1}s$ in (7.4) we have $q^*(U^{-1}s) = (Uq)^*s$, which compared with (3.2) shows that the Hermitean conjugate of U is just $U^* = U^{-1}$. Thus U is a (unitary) element in $M(CB(H) \otimes A)$.

Now, in order to complete the explanation of Definition 7.1 we have still to show the representation property (6.5). Take $\xi \in A'$ and compute $(\mathrm{id} \otimes \xi)U\ (\pi(a)\Omega)$. Since in (7.1) the action of the operator U can be regarded as a multiplication in $M(CB(H)) = B(H)$ we can substitute for c the unity I_A obtaining

$$U\pi(a)|\Omega\rangle\langle x| \otimes I_A = (\pi \otimes \mathrm{id})\Delta(a)|\Omega\rangle\langle x| \otimes I_A \ .$$

Next since $(\mathrm{id} \otimes \xi)U \in M(CB(H)) = B(H)$, then acting with $\mathrm{id} \otimes \xi$ we obtain

$$\begin{aligned}(\mathrm{id} \otimes \xi)U\pi(a)|\Omega\rangle\langle x| &= (\pi \otimes \xi)\Delta(a)|\Omega\rangle\langle x| = \pi(\mathrm{id} \otimes \xi)\Delta(a)|\Omega\rangle\langle x| \\ &= \pi(\xi * a)|\Omega\rangle\langle x| \ ,\end{aligned}$$

which shows that

$$(\mathrm{id} \otimes \xi)U\pi(a)\Omega = \pi(\xi * a)\Omega \ . \tag{7.5}$$

Therefore, for arbitrary $\xi_1, \xi_2 \in A'$ applying $\mathrm{id} \otimes \xi_1 \otimes \xi_2$ to l.h.s. (respectively, r.h.s.) of (6.5), we have

$$\begin{aligned}(\mathrm{id} \otimes \xi_1 \otimes \xi_2)(\mathrm{id} \otimes \Delta)U\pi(a)\Omega &= (\mathrm{id} \otimes (\xi_1 * \xi_2))U\pi(a)\Omega = \pi((\xi_1 * \xi_2) * a)\Omega \\ &= \pi(\xi_1 * (\xi_2 * a))\Omega\end{aligned} \tag{7.6}$$

and

$$(\mathrm{id} \otimes \xi_1 \otimes \xi_2)U_{12}U_{13}\pi(a)\Omega = (\mathrm{id} \otimes \xi_1)U(\mathrm{id} \otimes \xi_2)U\pi(a)\Omega = \pi(\xi_1 * (\xi_2 * a))\Omega \ . \tag{7.7}$$

This proofs that U is a unitary representation.

Remark 7.2 The left regular representation is similar but technically more complicated; indeed it is simpler to introduce the left regular anti–representation. $\diamond$

The construction of right regular representation works also in the noncompact case and U is 'onto', but one has to employ weights rather than Haar measures and, so far, to postulate, rather than deduce, its strong right invariance.
In the classical situation, $H = L^2(\mathbf{G})$, $\Omega = 1$ (a constant function), $\pi(a)$ is just a multiplication by a, $\xi_g a = a(g)$ and $U_g \doteq (\mathrm{id} \otimes \xi_g) \in B(H)$. Then, (7.5) with $\xi = \xi_g$ shows that $U_g : a \mapsto \xi_g * a$ and

$$(U_g a)(g') = (\mathrm{id} \otimes \xi_g)\Delta(a)(g') = \Delta(a)(g, g') = a(gg') \ ,$$

i.e. U_g is just the right shift by g.

8 Decomposition theory

We shall investigate now unitary representations of compact quantum groups in more detail. We have quantum analogues of such classical concepts like a direct sum, decomposability, irreducibility of representations, etc. Let $U_1 \in M(CB(H_1) \otimes A)$ and $U_2 \in M(CB(H_2) \otimes A)$ be two strongly continuous unitary representations of a compact quantum group $G = (A, \Delta)$ in H_1 and H_2, respectively. The Hilbert spaces H_1, H_2 and $H_1 \oplus H_2$ can be connected by the following cannonical embedings and projections.

$$H_1 \xrightarrow{\iota_1} H_1 \oplus H_2 \xrightarrow{\pi_1} H_1$$
$$H_2 \xrightarrow{\iota_2} H_1 \oplus H_2 \xrightarrow{\pi_2} H_2.$$

We define a direct sum of representations U_1 and U_2 by

$$U_1 \oplus U_2 = (\iota_1 \otimes I)U_1(\pi_1 \otimes I) + (\iota_2 \otimes I)U_2(\pi_2 \otimes I).$$

It follows that $U_1 \oplus U_2 \in M(CB(H_1 \oplus H_2) \otimes A)$. Moreover it is unitary and satisfies representation property (6.5).

An important role is played by intertwining operator (intertwiner for short).

Definition 8.1 *Let $U_1 \in M(CB(H_1) \otimes A)$ and $U_2 \in M(CB(H_2) \otimes A)$ be two strong continuous unitary representations of a compact quantum group A in H_1 and H_2, respectively. A linear operator T from H_1 to H_2 is an intertwiner between U_1 and U_2 if*

$$(T \otimes I)U_1 = U_2(T \otimes I) \ .$$

We say that U_1 and U_2 are equivalent representations iff there exists invertible intertwiner between them.

Let now K be a Hilbert subspace of a Hilbert space H and U be a strongly coninuous unitary representation of a compact quantum group G in H. Consider sequence

$$K \xrightarrow{\iota} H \xrightarrow{P} K,$$

where ι and P are canonical inclusion and projection, respectively.

Definition 8.2 *We say that K is a U-invariant subspace of H iff P intertwins U with itself, i.e.*

$$(P \otimes I)U = U(P \otimes I).$$

In such case $U_K = (P \otimes I)U(\iota \otimes I)$ is called a subrepresentation of U in K.

Remark 8.3 It can be seen that $U_K \in M(CB(K) \otimes A)$ is a strongly continuous unitary representation in K. $\diamond$

Definition 8.4 *A strongly continuous unitary representation U of G is called irreducible iff the only U-invariant subspaces are $\{0\}$ and H.*

Consider now an operator T that intertwines a strongly continuous unitary representation U of G with itself. Suppose that T has a discrete spectral decomposition with distinct eigenvalues. This means that $T = \sum \lambda^i P_i$ where $P_i : H \to H$ and $P_i P_k = \delta_{ik} P_k$. We claim that spaces $H_k = P_k H$ are invariant subspaces for U and define subrepresentations U_{H_k} of U. Indeed, given T we easily see that its positive powers T^k as well as polynomials $w(T)$ are intertwiners for U. Moreover, it follows from the properties of P_js that any polynomial $w(T)$ has the form $w(T) = \sum w(\lambda^i) P_i$. Thus if we take a polynomial $w_j(\lambda)$ such that $w_j(\lambda^i) = \delta_j{}^i$ we find that $(w_j(T) \otimes I)U = U(w_j(T) \otimes I)$. Comparing this with $w_j(T) = \sum w_j(\lambda^i) P_i = P_j$ we see that all P_j are intertwiners for U. Thus, all U_{H_k} define subrepresentations of U. It further follows that U is a direct sum of these (sub)representations. We denote this fact by $U = \sum^{\oplus} U_{H_k}$.

Given $x \in H$ we introduce a useful operator Q acting on H by

$$Q = (\mathrm{id} \otimes h) U(|x\rangle\langle x| \otimes I) U^*. \tag{8.1}$$

It is easy to see that $CB(H) \ni Q \geq 0$. Moreover,

$$\begin{aligned}
U(Q \otimes I)U^* &= (\mathrm{id} \otimes \mathrm{id} \otimes h)(U_{12} U_{13} (|x\rangle\langle x| \otimes I \otimes I) U_{13}^* U_{12}^*) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes h)(\mathrm{id} \otimes \Delta)(U(|x\rangle\langle x| \otimes I)U^*) \\
&= (\mathrm{id} \otimes h)(U(|x\rangle\langle x| \otimes I)U^*) = Q \otimes I \ .
\end{aligned}$$

i.e. Q intertwines U with U itself. Consequently, the eigenspaces of Q are invariant under the action of U. Since Q is a compact operator, the eigenspaces corresponding to positive eigenvalues are finite dimensional.

Remark 8.5 In the classical case

$$Q = \int_{\mathbf{G}} U_g |x\rangle\langle x| U_g^* \ dg$$

and it is a priori not obvious why it is compact. But in fact, it is even trace class with trace equal to the normalization of the Haar measure. In the quantum case Q is not trace class, in general. $\diamond$

Now, we show that we can choose such x that Q is nonvanishing. Take an orthonormal basis $(x_n)_{n=1,2,\ldots}$ in H. Then

$$\sum_{n=1}^{N} |x_n\rangle\langle x_n| \longrightarrow I$$

for $N \to \infty$ in the *–strong topology (strict topology in $M(CB(H))$). Since the multiplication is continuous with respect to strict topology on bounded sets

$$U \left[\sum_{n=1}^{N} |x_n\rangle\langle x_n| \otimes I \right] U^*$$

converges (for $N \to \infty$) to $I \in M(CB(H) \otimes A)$ strictly and

$$(\mathrm{id} \otimes h)U \left[\sum_{n=1}^{N} |x_n\rangle\langle x_n| \otimes I \right] U^* \tag{8.2}$$

converges to $I \in M(CB(H))$ strictly. Therefore at least one term in (8.2) must be nonzero and $\exists x \in H$ such that $Q \neq 0$.

Clearly, given a representation U of a compact quantum group $G = (A, \Delta)$ in H we can not 'evaluate it at a point'. But, since there exist measures, we can instead smear U, obtaining an operator in $B(H)$. For instance the classical Fourier transform

$$\mathcal{F}f(k) = \int e^{-ikx} f(x) dx \tag{8.3}$$

can be thought of as a smearing of a (one dimensional) unitary representation e^{-ikx} by the Haar measure dx on $\mathbf{R}$.

Now, if h is the Haar measure on G let for any $a \in A$ the product ha be defined by $ha(b) = h(ab)$. (Similarly, we have ah defined by $ah(b) = h(ba)$; in general $ah \neq ha$. For any functional $\xi \in A'$ the products ξa and $a\xi$ are defined in an analogous way). Then, corresponding to (8.3), the quantum Fourier transform is

$$\mathcal{F}_U(a) = (\mathrm{id} \otimes ha)U^* \ . \tag{8.4}$$

We introduce

$$B_U \doteq \{\mathcal{F}_U(a) \mid a \in A\}, \tag{8.5}$$

and formulate without proof the following results.

Proposition 8.6
i) B_U is a C^*–algebra
ii) An operator $X \in B(H)$ intertwins U with itself if and only if X commutes with all elements of B_U.
iii) If U is finite dimensional irreducible, $B_U = B(H)$.

Remark 8.7 Proposition 8.6 iii) means that for a unitary irreducible (matrix) representation U, matrix elements u_{jk} $(j, k = 1, 2 \ldots N)$ are linearly independent elements of A. Otherwise the correspopnding B_U would not yield all the $N \times N$ matrices in $H = \mathbf{C}^N$. Similarly, if U is reducible and $U = U^{++} \oplus U^{--}$, where U^{++} and U^{--} are inequivalent irreducible of dimension N_1 and N_2 respectively, then B_U is an algebra of block diagonal matrices with two blocks consisting of all the numerical matrices of dimension N_1 and N_2. Thus, $u^{++}_{i,j}$ $(j, k = 1, 2 \ldots N_1)$ and $u^{--}_{k,\ell}$ $(j, k = 1, 2 \ldots N_2)$ are linearly independent. $\diamond$

Let now U^α, $\alpha \in \hat{G}$ be a complete family of all pairwise inequivalent unitary irreducible representations of G, ie. each representation enters only once and any unitary irreducible representation of G is equivalent to U^α for some $\alpha \in \hat{G}$. Then, as a result, $u^\alpha_{k\ell}$, $\alpha \in \hat{G}; k, \ell = 1, 2 \ldots N_\alpha$ are linearly independent elements of A.

We state now a basic theorem:

Theorem 8.8 *If U is strong continuous unitary representation of a compact quantum group G then $U = \sum^{\oplus} U_k$, where U_k are finite dimensional representations*

Proof. Take a nonvanishing Q of the form (8.1). Let P be the spectral projection of Q corresponding to some strictly positive eigenvalue. Clearly, P is a nonvanishing finite-rank projector which intertwins U with U itself. Moreover, PH is a finite-dimensional subspace of H, $H = PH \oplus (I-P)H$, PH and $(I-P)H$ are U-invariant and $U = U_1 \oplus U_2$, where U_1 and U_2 are restrictions of U to PH and $(I-P)H$ respectively. Using the transfinite induction we obtain the desired decomposition.

$\square$

It is a straightforward collorary of Theorem 8.8 that the irreducible strongly continuous unitary representations of compact quantum groups are necessarily finite dimensional. As a matter of fact, it is interesting to know if in Theorem 8.8 the adjective 'finite dimensional' can be replaced by 'irreducible' (such a much stronger result holds in the classical theory of compact groups). For that, it would be enough that the orthogonal complement of an invariant subspace is also invariant. This is unfortunately not always the case as the following example shows.

Example 8.9 Take a unitary $U \in B(\mathbf{C}^n) \otimes A = \mathrm{Mat}_{n\times n}(A)$ of the block form

$$U = \begin{pmatrix} U^{++} & U^{+-} \\ 0 & U^{--} \end{pmatrix}. \tag{8.6}$$

If $A = \mathbf{C}$ then necessarily $U^{+-} = 0$, but not in general. For instance let H be Hilbert space with orthonormal basis e_k labelled by 'integers plus a half', i.e. $k - \frac{1}{2} \in \mathbf{Z}$. Denoting H^+ and H^- the Hilbert spaces spanned by e_k with $k \geq \frac{1}{2}$ and $k \leq -\frac{1}{2}$, respectively, we have a decomposition $H = H^+ \oplus H^-$. Next, identifying H^- with H^+ by $e_{-k} \mapsto e_k$, to any unitary operator in $B(H)$ we can associate a unitary element of $\mathrm{Mat}_{2\times 2}(B(H^+))$. In particular to a 'shift by one', $e_k \mapsto e_{k+1}$, there corresponds a unitary element U of the form (8.6) with the following blocks

$$\begin{aligned} U^{++}e_k &= e_{k+1}, \quad k \geq \frac{1}{2} \\ U^{+-}e_{\frac{1}{2}} &= e_{\frac{1}{2}}\,, \quad U^{+-}e_k = 0, \quad k \geq \frac{3}{2} \\ U^{--}e_{\frac{1}{2}} &= 0\,, \quad U^{--}e_k = e_{k-1}, \; k \geq \frac{3}{2}\,. \end{aligned}$$

Thus, in $B(H^+) \oplus B(H^+)$ the first copy of $B(H^+)$ is an invariant subspace while its orthogonal complement (the second copy of $B(H^+)$) is not. $\diamond$

Despite this counterexample, using Theorem 8.6 and the fact that any finite dimensional representation of a C^*-algebra decomposes into a direct sum of irreducible representations, one can nevertheless prove the desired stronger version of Theorem 8.8.

Theorem 8.10 *Any unitary representation U of a compact quantum group $G = (A, \Delta)$ is a direct sum of finite dimensional irreducible unitary representations.*

9 Compact quantum group versus Hopf algebra

We recall the notion of a Hopf *-algebra.

Definition 9.1 *Let $\mathcal{A}$ be a unital *-algebra with a multiplication $m : \mathcal{A} \otimes_{\rm alg} \mathcal{A} \to \mathcal{A}$. Let $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\rm alg} \mathcal{A}$ be a unital *-algebra homomorphism such that*

$$(\Delta \otimes {\rm id}) \circ \Delta = ({\rm id} \otimes \Delta) \circ \Delta \qquad \text{(coassociativity)}.$$

A pair $(\mathcal{A}, \Delta)$ is called a Hopf *-algebra if there exist linear mappings $\epsilon : \mathcal{A} \to \mathbf{C}$ and $S : \mathcal{A} \to \mathcal{A}$ such that for any $a \in \mathcal{A}$,

$$(\epsilon \otimes {\rm id})\Delta(a) = a \tag{9.1}$$

$$({\rm id} \otimes \epsilon)\Delta(a) = a \tag{9.2}$$

$$m(S \otimes {\rm id})\Delta(a) = \epsilon(a)I \tag{9.3}$$

$$m({\rm id} \otimes S)\Delta(a) = \epsilon(a)I \ . \tag{9.4}$$

Remark 9.2 A commutative $*$–algebra of polynomial functions on matrix group $\mathbf{G}$ equipped with m, Δ, ϵ, S as in (4.7) - (4.10) provides us with such a structure.

The following three remarks are direct consequences of the definition.

Remark 9.3 For a given Hopf *–algebra ϵ (called counit) and S (called coinverse or antipode) are uniquely determined.
Counit ϵ is a unital *–algebra homomorphism, ie. $\epsilon(a^*) = \overline{\epsilon(a)}$, where bar denotes the complex conjugation in $\mathbf{C}$).
Coinverse S is antimultiplicative, anticomultiplicative and satisfies

$$S\left(S(a^*)^*\right) = a$$

for any $a \in \mathcal{A}$. $\diamond$

We can state now an important result.

Theorem 9.4 *Let $\mathcal{A}$ be the set of all finite linear combinations of matrix elements of all unitary matrix representations of a compact quantum group $G = (A, \Delta)$. Then*
i) $\mathcal{A}$ is a dense *-subalgebra of A
ii) $\Delta(\mathcal{A}) \subset \mathcal{A} \otimes_{\rm alg} \mathcal{A}$
iii) $(\mathcal{A}, \Delta|_{\mathcal{A}})$ is a Hopf *-algebra.

Proof. To prove that $\mathcal{A}$ is dense in A we essentially have to show that G has sufficiently many representations. As in the classical case, this follows from the properties of the right regular representation of G. Using (7.5) we have

$$\langle \pi(b)\Omega \mid ({\rm id} \otimes \xi)U\pi(a)\Omega\rangle = h(b^*(\xi * a)) = h(b^*)(\xi * a) = \xi(a * hb^*).$$

So, the matrix elements of U are just

$$\langle \pi(b)\Omega \mid U\pi(a)\Omega\rangle = a * hb^* = (hb^* \otimes \mathrm{id})\Delta(a) = (h \otimes \mathrm{id})(b^* \otimes I)\Delta(a)\ .$$

The set of such elements is dense in A due to the axiom (4.12).
Let $(U^\alpha)_{\alpha\in\hat{G}}$ be the complete family of pairwise inequivalent irreducible unitary representations of G (see Remark 8.7). Let N_α be the dimension of U^α. To show that $\mathcal{A}$ is closed under the multiplication we observe that given U^α and U^β, we have

$$\Delta(u^\alpha_{kl}u^\beta_{mn}) = \sum_{r=1}^{N_\alpha}\sum_{s=1}^{N_\beta} u^\alpha_{kr}u^\beta_{ms} \otimes u^\alpha_{rl}u^\beta_{sn}. \tag{9.5}$$

Thus the product of two matrix elements of two representations is a matrix element of another matrix representation U^γ. It further follows that U^γ is unitary.
To complete the proof of *i)* we still need to show that if $\alpha \in \hat{G}$ then $\forall k,l\ \ {u^\alpha_{kl}}^* \in \mathcal{A}$. Using the first equation of (6.2) (unitarity of U^α), by applying the Haar measure, we get

$$\sum_{k,l=1}^{N_\alpha} h(u^\alpha_{kl}{u^\alpha_{kl}}^*) = N_\alpha h(I) = N_\alpha.$$

We already know that $\mathcal{A}$ is dense in A. Therefore there exist $\beta \in \hat{G}$ and integers $1 \le m,n \le N_\beta$ such that

$$E_{mk} = h(u^\alpha_{kl}u^\beta_{mn}) \neq 0 \tag{9.6}$$

Applying $\mathrm{id} \otimes h$ to the both sides of (9.5) and using (5.2) we obtain

$$E_{mk} = \sum_{rs} u^\alpha_{kr}u^\beta_{ms}E_{sr}\ .$$

Contracting from the left with ${u^\alpha_{kr}}^*$, we get

$$\sum_k E_{mk}{u^\alpha_{kr}}^* = \sum_s u^\beta_{ms}E_{sr} \tag{9.7}$$

for any $m = 1,2,\dots N_\beta$ and $n = 1,2,\dots N_\alpha$.

Let λ_r ($r = 1,2,\dots N_\alpha$) be complex numbers such that $\sum_r E_{sr}\lambda_r = 0$ for any $s = 1,2,\dots N_\beta$. Then $\sum_{rk} E_{mk}{u^\alpha_{kr}}^*\lambda_r = 0$ and using the linear independence of the elements u^α_{kl} (cf. Remark 8.7) and relation (9.6) we get $\lambda_r = 0$ for all r. It shows that $\mathrm{Rank}\, E = N_\alpha$. Similarly if μ_m ($m = 1,2,\dots N_\beta$) are complex numbers such that $\sum_m \mu_m E_{mk} = 0$ for any $k = 1,2,\dots N_\alpha$, then $\sum_{sm}\mu_m u^\beta_{ms}E_{sr} = 0$ and $\mu_m = 0$ for all m. Therefore $\mathrm{Rank}\, E = N_\beta$.

In this way we have shown that E is a quadratic invertible matrix and (9.7) implies that

$${u^\alpha_{kr}}^* = \sum_{sm}(E^{-1})_{km}u^\beta_{ms}E_{sr} \tag{9.8}$$

for any $k,r = 1,2,\dots N_\alpha$.
Point *ii)* is straightforward.
Point *iii)*. We easily verify on $u^\alpha_{k\ell}$ that $\Delta|_{\mathcal{A}}$ is a coproduct in $\mathcal{A}$ due to Definition

6.1. To equip $\mathcal{A}$ with a *–Hopf algebra structure we need still the counit and the coinverse. These are defined on the independent generators $u^\alpha_{k,\ell}$ by

$$\begin{aligned} \epsilon(u^\alpha_{kl}) &= \delta_{kl} && (9.9)\\ S(u^\alpha_{kl}) &= u^{\alpha\,*}_{lk} && (9.10) \end{aligned}$$

for any $\alpha \in \hat{G}$ and $k, l = 1, 2, \dots N_\alpha$. We extend these relations to any $a \in \mathcal{A}$ by the requirement that ϵ and S have the properties mentioned in Remarks 9.3. One can easily check relations (9.1) – (9.4).

□

Remark 9.5 It can be shown that any unitary irreducible representation U^α appears N_α-times in the right regular representation U. ◇

10 Example: $SU_\mu(2)$

Given a parameter $\mu \in [-1, 1]$ let A be a C^*-algebra with unit I generated by two elements α, γ satisfying the commutation relations

$$\alpha\gamma = \mu\gamma\alpha, \quad \alpha\gamma^* = \mu\gamma^*\alpha, \quad \gamma^*\gamma = \gamma\gamma^*, \quad \alpha^*\alpha + \gamma^*\gamma = I, \quad \alpha\alpha^* + \mu^2\gamma^*\gamma = I\ . \tag{10.1}$$

More precise definition of A goes as follows. Consider the free (noncommutative) *-algebra with unity $\mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]$ generated by symbols α, γ. We say that a *–representation $\pi : \mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]] \to B(H)$ on a Hilbert space H is admissible if the operators $\pi(\alpha), \pi(\gamma)$ satisfy the above commutation relations. Then for any $a \in \mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]$ we set

$$\|a\| = \sup_\pi \{\|\pi(a)\|\}\ , \tag{10.2}$$

where π runs over all admissible representations of $\mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]$. Clearly, $\|a\| < \infty$ and $\|\ \|$ is a C^*-seminorm. Thus the set

$$N = \{a \in \mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]] \mid \|a\| = 0\} \tag{10.3}$$

is a two-sided ideal in $\mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]$ and the seminorm (10.2) provides a C^*-norm on the quotient algebra $\mathcal{A} = \mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]/N$. Then A is just the completion of $\mathcal{A}$ with respect to this norm.

Remark 10.1 A is a C^*-algebra with unity and two distinguished elements α, γ satisfying relations (10.1) (We use the same symbols α, γ to denote the N-equivalence class of α, γ in $\mathcal{A}$). The *-subalgebra $\mathcal{A}$ generated by α, γ is dense in A. For any two $\alpha', \gamma' \in B(H)$ satisfying the relations (10.1) there exists exactly one representation $\pi : A \to B(H)$ such that $\pi(\alpha) = \alpha'$ and $\pi(\gamma) = \gamma'$. ◇

We introduce a representation π of A in a Hilbert space with an orthonormal basis $\psi_{n,k}$, $n = 0, 1, 2\ldots$; $k \in \mathbf{Z}$, by

$$\begin{aligned} \pi(\alpha)\psi_{n,k} &= \sqrt{1-\mu^{2n}}\ \psi_{n-1,k} \\ \pi(\gamma)\psi_{n,k} &= \mu^n\ \psi_{n,k+1} \end{aligned} \tag{10.4}$$

for $-1 < \mu < 1$, and by

$$\begin{aligned} \pi_t(\alpha)\psi_{n,k} &= \sqrt{1-t^2}\ \psi_{n-1,k} \\ \pi_t(\gamma)\psi_{n,k} &= t\mu^n\ \psi_{n,k+1} \end{aligned} \tag{10.5}$$

for $\mu = \pm 1$.
Using the commutation relations 10.1 it is not difficult to guess that

$$a_{kmn} = \begin{cases} \alpha^k \gamma^m \gamma^{*n} & \text{for } k = 0, 1, 2\ldots \\ (\alpha^*)^{-k}\gamma^m\gamma^{*n} & \text{for } k = -1, -2\ldots\ , \end{cases} \tag{10.6}$$

where $k \in \mathbf{Z}$ and $m, n = 0, 1, 2\ldots$ form a (linear) basis in $\mathcal{A}$. Indeed, one computes that if $c = \sum_{kmn} c_{kmn} a_{kmn}$ is a (finite) sum with coefficients $c_{kmn} \in \mathbf{C}$, such that at least one $c_{kmn} \neq 0$, then

$$c_{k,s-n,n} = \begin{cases} \lim_{r\to\infty} \mu^{sr} \langle \psi_{r+k,s-2n} \mid \pi(c)\psi_{r,0}\rangle & \text{for } |\mu| < 1 \\ \lim_{t\to 0} t^s \langle \psi_{k,s-2n} \mid \pi_t(c)\psi_{0,0}\rangle & \text{for } \mu = \pm 1 \end{cases},$$

where s is the smallest integer such that $c_{kmn} \neq 0$, for some k, m, n satisfying $m+n = s$. Thus c can not be zero and the linear independence follows.

Remark 10.2 The ideal N actually coincides with the ideal N' generated by the following seven elements $\alpha\gamma - \mu\gamma\alpha$, $\alpha\gamma^* - \mu\gamma^*\alpha$, $\gamma^*\gamma - \gamma\gamma^*$, $\gamma\alpha^* - \mu\alpha^*\gamma$, $\gamma^*\alpha^* - \mu\alpha^*\gamma^*$, $\alpha^*\alpha + \gamma^*\gamma - 1$ and $\alpha\alpha^* + \mu^2\gamma^*\gamma - 1$. In fact such elements are mapped to 0 by any admissible representation and thus $N' \subset N$. Conversely, let T be a subspace consisting of all linear combinantions of the basis (10.6). Since any element of $\mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]$ is equivalent modulo N' to some element of T we have $\mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]] = T + N'$. Also, $T \cap N = \{0\}$ since no nonzero element of T is mapped to 0 by by the canonical projection onto $\mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]/N$. Hence, for $a = t + n \in N$, with $t \in T, n \in N'$ we have $t = a - n \in T \cap N = 0$ and consequently $a = n \in N'$. $\diamond$

Now, the following theorem shows that for $\mu \in [-1, 1]$, $\mu \neq 0$, a pair (A, u), where

$$u = \begin{pmatrix} \alpha, -\mu\gamma^* \\ \gamma, \alpha^* \end{pmatrix} \in \mathrm{Mat}_{\,2\times 2}(\mathbf{C}) \otimes A\ , \tag{10.7}$$

is a quantum matrix group denoted (in the sense of [6], where terminology 'matrix pseudogroup' is used).

Theorem 10.3 *Given a parameter* $\mu \in [-1, 1]$ *let* A *be a* C^**-algebra with unit* I *generated by two elements* α, γ *satisfying the commutation relations (10.1) and let* u *be given by (10.7). Then*
i) The **-algebra* $\mathcal{A}$ *generated by* α, γ *is dense in* A.
ii) There exist a C^**-algebra homomorphism (coproduct)*

$$\Delta : A \to A \otimes A$$

such that

$$(\mathrm{id} \otimes \Delta)u = u \otimes_{\bullet} u \ , \tag{10.8}$$

where $\otimes_{\bullet}$ *denotes the usual product of matrices in which the product of matrix elements is replaced by their tensor product*
iii) u *is an invertible element of* $\mathrm{Mat}_{2\times 2}(\mathbf{C}) \otimes A$*. If* $\mu \neq 0$ *then there exists a linear antimultiplicative mapping (coinverse)*

$$S : \mathcal{A} \to \mathcal{A}$$

such that $S(S(a^*)^*) = a$*,* $\forall a \in \mathcal{A}$ *and*

$$(\mathrm{id} \otimes S)u = u^{-1} \ . \tag{10.9}$$

Proof. The statement i) is covered by Remark 10.1. To prove the statement ii) notice that (10.8) is equivalent to

$$\Delta(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma \ , \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma \ . \tag{10.10}$$

Now in order to show the existence of Δ assume that $A \subset B(H)$. Then, one verifies that the operators occuring on the r.h.s of equations (10.10) satisfy the relations (10.1). Thus by the last statement of Remark 10.1, there exists an admissible representation $\pi = \Delta : A \to B(H \otimes H)$. Clearly, the image $\Delta(A)$ is a C^*-subalgebra of $A \otimes A \subset B(H \otimes H)$.
To prove iii) one checks directly that u is a unitary element of $\mathrm{Mat}_{2\times 2}(\mathbf{C}) \otimes A$. Thus

$$u^{-1} = u^* = \begin{pmatrix} \alpha^*, \gamma^* \\ -\mu\gamma, \alpha \end{pmatrix}$$

and (10.9) becomes equivalent to

$$S(\alpha) = \alpha^*, \ S(-\mu\gamma^*) = \gamma^*, \ S(\gamma) = -\mu\gamma, \ S(\alpha^*) = \alpha \ . \tag{10.11}$$

Assume $\mu \neq 0$. In order to show the existence of a linear antimultiplicative mapping $S : \mathcal{A} \to \mathcal{A}$, consider $\alpha' = \alpha^*, \ \gamma' = -\mu\gamma, \ \alpha^{*\prime} = \alpha, \ \gamma^{*\prime} = -1/\mu \ \gamma^*$. Regarded as elements of $\mathcal{A}^{op}$, which is isomorphic to $\mathcal{A}$ as far as the linear structure is concerned but has the opposite product (in the reversed order with respect to $\mathcal{A}$), they satisfy the relations corresponding to the ideal N' of Remark 10.2. Thus, the linear multiplicative mapping $S : [[\alpha, \gamma, \alpha^*, \gamma^*]] \to \mathcal{A}^{op}$ defined by sending $\alpha, \gamma, \alpha^*, \gamma^*$ respectively to $\alpha', \gamma', \alpha^{*\prime}, \gamma^{*\prime}$ (such S exists since $\mathbf{C}[[\alpha, \gamma, \alpha^*, \gamma^*]]$ is free) passes to the quotient modulo $N' = N$ and thus defines an antimultiplicative mapping $S : \mathcal{A} \to \mathcal{A}^{op}$ satisfying

$$S(\alpha) = \alpha^{*\prime}, \ S(-\mu\gamma^*) = \gamma^{*\prime}, \ S(\gamma) = -\mu\gamma', \ S(\alpha^*) = \alpha' \ . \tag{10.12}$$

Since (10.11) coincide with (10.12) and an antimultiplicative mapping $S : \mathcal{A} \to \mathcal{A}^{op}$ corresponds to a multiplicative mapping $S : \mathcal{A} \to \mathcal{A}$ (which in our case satisfies $S(S(a^*)^*) = a, \forall a \in \mathcal{A}$) we have shown the existence of the coinverse S.

$\square$

Remark 10.4 The above quantum matrix group may be defined also for $|\mu| \geq 1$, but the transformation $\mu \to 1/\mu$, $\alpha \to \alpha^*$ and $\gamma \to -\mu\gamma$, shows that it is isomorphic to the one with $|\mu| \leq 1$. $\diamond$

The case $\mu \to 1$ in $SU_\mu(2)$ is very special. Note that for $\mu = 1$ relations (10.1) imply that the corresponding algebra A is commutative. It therefore defines a compact group $\mathbf{G}$ with a multiplication Δ_*. Explicitely, given A we have a space

$$\mathbf{G} = \{(\mathrm{id} \otimes \chi)u \mid \chi \text{ is a character of the algebra } A\}$$

whose elements are 2×2 matrices. $\mathbf{G}$ with the usual multiplication of matrices forms a group, isomorphic to $\mathbf{SU}(2)$. To see this, observe that since any character χ of A satisfies $\chi(\alpha^*) = \overline{\chi(\alpha)}$, $\chi(\gamma^*) = \overline{\chi(\gamma)}$ and $|\chi(\alpha)|^2 + |\chi(\gamma)|^2 = 1$ then the matrix

$$(\mathrm{id} \otimes \chi)u = \begin{pmatrix} \chi(\alpha), -\overline{\chi(\gamma)} \\ \chi(\gamma), \overline{\chi(\alpha)} \end{pmatrix}$$

is unitary and unimodular. To show that any element of $\mathbf{SU}(2)$ is of the above form we take a generic element

$$g = \begin{pmatrix} \alpha', -\overline{\gamma'} \\ \gamma', \overline{\alpha'} \end{pmatrix} \tag{10.13}$$

of $\mathbf{SU}(2)$. Its matrix elements α', γ' are complex numbers such that $|\alpha'|^2 + |\gamma'|^2 = 1$. Thus they satisfy relations (10.1) for $\mu = 1$. According to Remark 10.1 we then have a 1-dimensional representation (i.e. character) χ of A such that $\chi(\alpha) = \alpha'$ and $\chi(\gamma) = \gamma'$. Then $(\mathrm{id} \otimes \chi)u = g$, which proofs the isomorphism between $\mathbf{G}$ and $\mathbf{SU}(2)$. Knowing that $\mathbf{G} = \mathbf{SU}(2)$, we easily identify a quantum matrix group A for $\mu = 1$ with the C^*-algebra $C(\mathbf{SU}(2))$ of all continuous functions on the group $\mathbf{SU}(2)$, in which coproduct is defined by $(\Delta a)(g_1, g_2) = a(g_1 g_2)$, where $a \in C(\mathbf{SU}(2))$ (compare with Example 1.8). These observations explain in which sense the quantum matrix group (A, u) provides a deformation of a classical group $\mathbf{SU}(2)$.

We claim that the quantum matrix group (A, u) described above can be regarded as a compact quantum group $SU_\mu(2) = (A, \Delta)$ in the sense of Definition 4.3. The only points which require a verification are (4.12) and (4.13). We demonstrate them by showing that the left hand side of (4.12) and of (4.13) contain $b \otimes a$ for arbitrary $a, b \in A$. In fact, by taking $b_j = b\ S(a_j^{(1)})$ and $a_j = a_j^{(2)}$ we obtain $b \otimes a = \sum_j (b_j \otimes I)\Delta(a_j)$. Analogously, by taking $b_j = a\ S^{-1}(b_j^{(2)})$ and $a_j = b_j^{(1)}$ we obtain $b \otimes a = \sum_j (I \otimes b_j)\Delta(a_j)$, see Theorem 4.9 of [6]. (Here, for $a \in A$ we used the notation $\Delta(a) = \sum_j a_j^{(1)} \otimes a_j^{(2)}$ and similarly for b).
It is also easy to check that u given by (10.7) is an (irreducible) two-dimensional

unitary matrix representation of $SU_\mu(2)$.

It turns out that the Haar measure on $SU_\mu(2)$ is defined on the basis (10.6) by

$$h(a_{kmn}) = \delta_{k0}\ \delta_{mn}\ \frac{(1-\mu^2)}{(1-\mu^{2m+2})}\ , \tag{10.14}$$

where δ is the Kronecker symbol. With the help of representation (10.4) the Haar mesure can be also written as

$$h(a) = (1-\mu^2)\sum_{n=0}^{\infty} \mu^{2n}\langle\psi_{n,0} \mid \pi(a)\psi_{n,0}\rangle\ . \tag{10.15}$$

¿From this, it can be inferred that the representation $\tilde{\pi}(a) = \bigoplus_{n=0}^{\infty} \pi(a)$ acting on $\tilde{H} = \bigoplus_{n=0}^{\infty} H$, is equivalent to the GNS representation associated with the state h. The only point to be checked is the cyclicity of $\Omega = (1-\mu^2)\bigoplus_{n=0}^{\infty}\mu^n\psi_{n,0}$ in $\tilde{H}$. For this purpose let P be the projection onto $\tilde{\pi}(A)\Omega$. Then $P\tilde{\pi}(a)P = \tilde{\pi}(a)P$. Replacing a by a^* we have also $P\tilde{\pi}(a)P = P\tilde{\pi}(a)$. Thus

$$[P, \tilde{\pi}(a)] = 0\ , \quad \forall\ a \in A\ . \tag{10.16}$$

We introduce the components P_{ij} of P, $\tilde{\pi}(a)_{ij} = \delta_{ij}\pi(a)$ of $\tilde{\pi}(a)$ and Ω_j of Ω with respect to the decomposition $\tilde{H} = \bigoplus_{n=0}^{\infty} H$. Then from (10.16) it follows that

$$[P_{ij}, \pi(a)] = 0, \quad \forall\ a \in A\ ,$$

ie. P_{ij} are in the commutant of $\pi(A)$. Now, note that the representation (10.4) is reducible. In fact, using the Fourier transform between the first factor of $l^2(\mathbf{Z})\otimes l^2(\mathbf{N})$ and of $L^2(S^1)\otimes l^2(\mathbf{N})$, it can be seen that

$$\pi = \bigoplus_{|\lambda|=1} \pi_\lambda\ ,$$

where the irreducible representations π_λ are given by

$$\begin{aligned} \pi_\lambda(\alpha)\psi_n &= \sqrt{1-\mu^{2n}}\ \psi_{n-1} \\ \pi_\lambda(\gamma)\psi_n &= \lambda\mu^n\ \psi_n\ . \end{aligned} \tag{10.17}$$

Thus, P_{ij} must have the form $P_{ij}\psi_{nk} = \sum_\ell p_{ij}^{k-\ell}\psi_{n\ell}$, where $p_{ij}^{k-\ell} \in \mathbf{C}$. Next, using $P\Omega = \Omega$ we get $p_{ij}^{k-\ell} = \delta_{ij}$. Thus $P_{ij} = \delta_{ij}I$, ie. $P = I$ in $\tilde{H}$, and Ω is cyclic.

More details can be found in [5, 6].

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TO BE ADDED (??!!........):

?– REPS. of SU_μ
??– LEFT REG. REP. U