

Przykład
 Rozważmy funkcję $\mathbb{R}^3 \setminus \{0\} \ni \vec{r} \mapsto \frac{1}{\sqrt{x^2+y^2+z^2}} \in \mathbb{R}$ ozn. $\frac{1}{r}$.
 Funkcja $\frac{1}{r}$ definiuje dyfuzyjną regularną na $D(\mathbb{R}^3)$
 w sensie $D(\mathbb{R}^3) \ni g \xrightarrow{f} \int_{\mathbb{R}^3} \frac{1}{r} g(\vec{r}) dx dy dz \stackrel{\text{wzrost sfer.}}{=} \int_{\mathbb{R}^3} \frac{1}{r} g(\vec{r}) r^2 \sin \theta d\theta d\varphi dr$
 $= \int_{\mathbb{R}^3} r g(\vec{r}) \sin \theta d\theta d\varphi dr$

Wykażemy, że $\Delta \frac{1}{4\pi r} = \delta_0$. Rozważmy pierwszy $\Delta \frac{1}{4\pi r} = \delta_0$
 Niech $r \neq 0$ obliczamy $\frac{\partial^2}{\partial x^2} \frac{1}{r} = -\frac{2x}{r^3} + \frac{3x^2}{r^5}$
 Zatem $\Delta \frac{1}{r} = -\frac{3}{r^3} + 3 \frac{x^2+y^2+z^2}{r^5} = 3 \left(\frac{1}{r^3} - \frac{1}{r^3} \right) = 0$.

Niech teraz $g \in D(\mathbb{R}^3)$. Obliczamy $\text{div } \vec{F} = \left(\frac{\partial F_x}{\partial x}, \frac{\partial F_y}{\partial y}, \frac{\partial F_z}{\partial z} \right)$
 $(*) \Delta \left(\frac{1}{4\pi r} \right) (g) = \int_{\mathbb{R}^3} \frac{1}{4\pi r} \Delta(g) dV \stackrel{(*)}{=} g(0) = \delta_0(g)$.
 Dlaczego?

Skorzystajmy z tw. Stokesa dla $\partial M \subset \mathbb{R}^3$ gdzie $\dim M = 2$
 (na przykład $\partial M = S^2$). Niech $w \in \Omega^2(\mathbb{R}^3)$:

$\int_M dw = \int_{\partial M} w$ Warje dla pól wektorowych.
 Niech $\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ - pole wektorowe $\int_M \text{div } \vec{F} dV = \int_{\partial M} \vec{F} \cdot d\vec{A}$.
 $\vec{F} = (F_x, F_y, F_z) \Rightarrow w = F_x dy dz + F_y dz dx + F_z dx dy$
 $dw = \text{div } \vec{F} dx dy dz$

gdzie $dA = \vec{n} \cdot (w)$ $f, g \in C^\infty(\mathbb{R}^3)$, \vec{F} g.w. gradient.
 $\text{div}(g\vec{F}) = g \text{div } \vec{F} + \nabla g \cdot \vec{F}$ (il. Stokesa)
 $\text{div}(g \nabla f) = g \text{div } \nabla f + \nabla g \cdot \nabla f$
 zatem $g \Delta f - f \Delta g = \text{div}(g \nabla f) - \text{div}(f \nabla g)$
 $(*) = -\frac{1}{4\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B_\epsilon} \frac{1}{r} \Delta g dV = -\frac{1}{4\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B_\epsilon} (\text{div}(f \nabla g) - \text{div}(g \nabla f)) dV$
 gdzie R takie aby $g|_{r \geq R} = 0$
 $M = \{ \vec{r} \mid r \geq \epsilon \text{ i } r \leq R \}$
 $\partial M = \{ \vec{r} \mid r = \epsilon \} \cup \{ \vec{r} \mid r = R \}$
 (całki weźmiemy)

Na $\{ \vec{r} \mid r = \epsilon \}$ $dA = \epsilon^2 \sin \theta d\theta d\varphi$
 $\vec{n} = \frac{\vec{r}}{r} = \frac{x^2+y^2+z^2}{r} = \frac{\vec{r}}{r}$
 $\int_{\partial M} \vec{F} \cdot d\vec{A} = \int_{\partial M} r^2 \sin \theta d\theta d\varphi = r^2 \sin \theta d\theta d\varphi$
 $(*) = -\frac{1}{4\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^R \left(\frac{1}{r} (-\vec{n}) \cdot \nabla g + g \vec{n} \cdot \nabla \frac{1}{r} \right) \epsilon^2 \sin \theta d\theta d\varphi = \epsilon \rightarrow 0 \rightarrow 0$
 $\left. \begin{aligned} \nabla \frac{1}{r} &= -\frac{\vec{r}}{r^3} \\ \vec{n} \cdot \nabla \frac{1}{r} &= -\frac{\vec{r}}{r} \cdot \frac{\vec{r}}{r^3} = -\frac{1}{r^2} = -\frac{1}{\epsilon^2} \end{aligned} \right\} = -\frac{1}{4\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^R \int_{S^2} -\vec{n} \cdot \nabla g \epsilon^2 \sin \theta d\theta d\varphi + \frac{1}{4\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^R \int_{S^2} g(\epsilon, \theta, \varphi) \sin \theta d\theta d\varphi$
 $= \frac{1}{4\pi} g(0) \int_{S^2} \sin \theta d\theta d\varphi = g(0) = \delta_0(g) \Rightarrow \Delta \frac{1}{4\pi r} = \delta_0$

TRANSFORMATA FOURIERA.

Przykład.
 Niech $a > 0, k \in \mathbb{R}$. Obliczmy $\int_{\mathbb{R}} e^{-ax^2} e^{-ikx} dx =$
 $\int_{\mathbb{R}} e^{-a(x + \frac{ik}{2a})^2} e^{-\frac{k^2}{4a}} dx = e^{-\frac{k^2}{4a}} \int_{\mathbb{R}} e^{-a(x + \frac{ik}{2a})^2} dx$
 $\int_{\mathbb{R}} e^{-az^2} dz = \int_{-N}^N e^{-a(x + \frac{ik}{2a})^2} dx$ gdzie $\gamma_N = \left\{ x + \frac{ik}{2a} \mid x \in [-N, N] \right\}$
 typ. $\int_{C_N} e^{-az^2} dz = \int_{-N}^N e^{-az^2} dz + \int_N^{N + \frac{ik}{2a}} e^{-az^2} dz + \int_{N + \frac{ik}{2a}}^{-N + \frac{ik}{2a}} e^{-az^2} dz + \int_{-N + \frac{ik}{2a}}^{-N} e^{-az^2} dz \rightarrow 0$
 $\int_{C_N} e^{-az^2} dz \xrightarrow{N \rightarrow \infty} 0$

Zatem $\lim_{N \rightarrow \infty} \int_{-N}^N e^{-a(x + \frac{ik}{2a})^2} dx = \lim_{N \rightarrow \infty} \int_{-N}^N e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$
 $\int_{\mathbb{R}} e^{-ax^2} e^{-ikx} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a}}$ w szczególności $\int_{\mathbb{R}} e^{-\frac{ax^2}{2}} e^{-ikx} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a}}$
 $S(\mathbb{R}^n)$, $\|f\|_{KL} = \sup_{x \in \mathbb{R}^n} |x^k \frac{\partial^{|k|}}{\partial x^k} f(x)| < \infty \quad \forall k \in \mathbb{N}$
 W szczególności jeśli $f \in S(\mathbb{R}^n)$ to istnieje $C > 0$ t. że
 $|f(x)| \leq \frac{C}{(1+x_1^2)(1+x_2^2) \dots (1+x_n^2)}$ $\int_{\mathbb{R}^n} |f(x)| dx \leq C \pi^{n/2} < \infty$

Definicje.
 Niech $f \in S(\mathbb{R}^n)$ funkcje $\mathbb{R}^n \ni k \mapsto \int_{\mathbb{R}^n} f(x) e^{-ikx} dx \in \mathbb{R}$
 $\int_{\mathbb{R}^n} |f(x)| dx$

nowym transformacie Fouriera f oznaczam $\mathcal{F}f$.
 Przykład $\mathcal{F}(e^{-\frac{ax^2}{2}})(k) = \left(\frac{2\pi}{a}\right)^{\frac{n}{2}} e^{-\frac{k^2}{2a}}$ $\left\{ \begin{array}{l} a > 0 \\ k, x \in \mathbb{R}^n \end{array} \right.$

Własności \mathcal{F} .
 ① $k_j \mathcal{F}f(k) = \int_{\mathbb{R}^n} f(x) k_j e^{-ikx} dx = \int_{\mathbb{R}^n} f(x) i \frac{\partial}{\partial x_j} e^{-ikx} dx$ cożk
 $= \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} f \right) e^{-ikx} dx = \mathcal{F}\left(-i \frac{\partial}{\partial x_j} f\right)$ prze-
nieści
w tej
zmiętej
 $|k_j \mathcal{F}f(k)| \leq \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_j} f \right| dx$, Ogólniej $|k^\alpha \mathcal{F}f(k)| \leq \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| dx$

② $\frac{\partial}{\partial k_j} (\mathcal{F}f)(k) = \mathcal{F}(-ix_j f)(k)$

$$\left| \frac{\partial}{\partial k_j} (\mathcal{F}f)(k) \right| \leq \int_{\mathbb{R}^n} |x_j| |f| dx < \infty$$

Wniosek $\mathcal{F}f \in S(\mathbb{R}^n)$ oraz $\mathcal{F}: S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$ jest odwracalnym liniowym.

Twierdzenie Transformacji Fouriera jest odwracalnym odwracalnym oraz $\mathcal{F}^{-1}(f)(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} f(k) e^{+ikx} dk$

Przykład $\mathcal{F}(e^{-\frac{ax^2}{2}}) = \left(\frac{2\pi}{a}\right)^{\frac{n}{2}} e^{-\frac{k^2}{2a}}$
 $\mathcal{F}^{-1}\left(\left(\frac{2\pi}{a}\right)^{\frac{n}{2}} e^{-\frac{k^2}{2a}}\right) = \left(\frac{1}{2\pi}\right)^n \left(\frac{2\pi}{a}\right)^{\frac{n}{2}} \left(\frac{2\pi}{a}\right)^{\frac{n}{2}} e^{-\frac{ax^2}{2}} = e^{-\frac{ax^2}{2}}$