$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$$ – k-forma różniczkowa.

$\uparrow$ funkcje gładkie $\mathcal{O} \underset{otw}{\subseteq} \mathbb{R}^n$.

$$d\omega = \sum d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$ k+1 – " — " —

$$df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n$$

Przykład $\mathcal{O} = \mathbb{R}^3$

(a) $f : \mathbb{R}^3 \to \mathbb{R}$ $\quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

(b) $\underset{\Omega^1(\mathbb{R}^3)}{\underset{\cap}{\Theta}} = \Theta_x\, dx + \Theta_y\, dy + \Theta_z\, dz \qquad \Theta_x, \dots, \Theta_z : \mathbb{R}^3 \to \mathbb{R}$.

$$d\Theta = \left(\frac{\partial \Theta_y}{\partial x} - \frac{\partial \Theta_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial \Theta_z}{\partial y} - \frac{\partial \Theta_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial \Theta_x}{\partial z} - \frac{\partial \Theta_z}{\partial x}\right) dz \wedge dx$$

(c) $\omega = \omega_z\, dx \wedge dy + \omega_x\, dy \wedge dz + \omega_y\, dz \wedge dx$

$$d\omega = \left(\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z}\right) dx \wedge dy \wedge dz$$

Stwierdzenie

Niech M będzie rozmaitością gładką, $\dim M = n$; $p \in M$ i $\varphi = (x^1, \dots, x^n)$ & $\psi = (y^1, \dots, y^n)$ będą układami wsp. w otoczeniu p.

Wówczas a) $\frac{\partial}{\partial y^i} = \sum_j \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$; b) $dy^i = \sum \frac{\partial y^i}{\partial x^j} dx^j$

c) $dy^1 \wedge \dots \wedge dy^n = \det\left(\frac{\partial y^i}{\partial x^j}\right)_{i,j} dx^1 \wedge \dots \wedge dx^n$

Dowód (a) $\frac{\partial}{\partial y^i} f\left(x^1(y^1, \dots, y^n), \dots, x^n(y^1, \dots, y^n)\right) = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\partial f}{\partial x^j}$ ✓

(b) $dy^i = \sum \frac{\partial y^i}{\partial x^j} dx^j$ ✓

(c) $$dy^1 \wedge \dots \wedge dy^n = \sum_{i_1, \dots, i_n} \frac{\partial y^1}{\partial x^{i_1}} dx^{i_1} \wedge \dots \wedge \frac{\partial y^n}{\partial x^{i_n}} dx^{i_n} =$$

$$= \sum_{i_1, \dots, i_n} \frac{\partial y^1}{\partial x^{i_1}} \dots \frac{\partial y^n}{\partial x^{i_n}} dx^{i_1} \wedge \dots \wedge dx^{i_n} = \sum_{\sigma \in S_n} \frac{\partial y^1}{\partial x^{\sigma(1)}} \dots \frac{\partial y^n}{\partial x^{\sigma(n)}} \operatorname{sgn}(\sigma)\, dx^1 \wedge \dots \wedge dx^n$$

$$= \det\left(\frac{\partial y^i}{\partial x^j}\right)_{i,j} dx^1 \wedge \dots \wedge dx^n.$$

Cofanie formy różniczkowej przy pomocy gładkiego odwzorowania.

$F : M \to N$. Niech $\omega \in \Omega^k(N)$, $p \in M$ i $v_1, \dots, v_k \in T_pM$.

Formę różniczkową $F^*\omega \in \Omega^k(M)$ zdefiniowaną wzorem

$$(F^*\omega)(p)(v_1, \dots, v_k) := \omega(F(p))(T_pF\, v_1, \dots, T_pF\, v_k).$$

nazywamy cofnięciem ω przy pomocy F.

Stwierdzenie. Niech $F^*: \Omega^*(N) \to \Omega^*(M)$.

(1) $F^*(\overset{k}{\omega} \wedge \overset{l}{\eta}) = F^*\overset{k}{\omega} \wedge F^*\overset{l}{\eta}$.

(2) $F^* d\overset{k}{\omega} = d F^*\overset{k}{\omega}$.

(3) $(F \circ G)^* = G^* \circ F^*$.

<u>Dowód.</u> (1) $F^*(\overset{k}{\omega} \wedge \overset{l}{\eta})(p)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = (\overset{k}{\omega} \wedge \overset{l}{\eta})(F(p))(T_pFv_1, \dots, T_pFv_{k+l}) =$

$\frac{(k+l)!}{k!\,l!} \mathrm{Alt}(\omega(F(p)) \otimes \eta(F(p)))(T_pFv_1, \dots, T_pFv_{k+l}) = \frac{(k+l)!}{k!\,l!} \mathrm{Alt}((F^*\omega)(p) \otimes (F^*\eta)(p))(v_1, \dots, v_{k+l})$

$= ((F^*\omega) \wedge (F^*\eta))(p)(v_1, \dots, v_{k+l})$.

(2) Niech $f: N \to \mathbb{R}$ będzie funkcją gładką. $(f \in \Omega^0(N))$

i $v \in T_pM$: $v = [\gamma]$ gdzie γ jest krzywą zadającą v.

$\langle (F^*df)(p), v \rangle = \langle (df)(F(p)), T_pF[\gamma] \rangle =$

$= \langle df(F(p)), [F \circ \gamma] \rangle = (f \circ (F \circ \gamma))'(0)$

$= ((f \circ F) \circ \gamma)'(0) = \dots = \langle d(f \circ F)(p), v \rangle = \langle dF^*(f)(p), v \rangle$

$\omega \in \Omega^k(N)$ w układzie współrzędnych $\varphi = (y^1, \dots, y^n)$ ma postać

$\omega = \sum \omega_{i_1 \dots i_k} \, dy^{i_1} \wedge \dots \wedge dy^{i_k} =$

$(F^*\omega)(p) = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(F(p)) \, F^*(dy^{i_1} \wedge \dots \wedge dy^{i_k}) =$

$= \sum_{i_1 < \dots < i_k} (\omega_{i_1 \dots i_k} \circ F)(p) \, d(F^*y^{i_1}) \wedge \dots \wedge d(F^*y^{i_k})$

$\boxed{d^2 = 0}$

$d(F^*\omega)(p) = \sum_{i_1 < \dots < i_k} \underbrace{d(\omega_{i_1 \dots i_k} \circ F)(p)}_{F^* d\omega_{i_1 \dots i_k}} \wedge dF^*y^{i_1} \wedge \dots \wedge dF^*y^{i_k}$

$F^* d\omega_{i_1 \dots i_k} \wedge F^*dy^{i_1} \wedge \dots \wedge F^*dy^{i_k} =$

$= F^* \sum d\omega_{i_1 \dots i_k} \wedge dy^{i_1} \wedge \dots \wedge dy^{i_k} = F^*(d\omega)(p)$

(3) $(F \circ G)^*(\omega)(p)(v_1, \dots, v_k) =$

$\omega(F(G(p)))(T_p(F \circ G)v_1, \dots, T_p(F \circ G)v_k)$

$$\begin{tikzcd} M \arrow[r, bend left, "G"] \arrow[rr, bend right, "F \circ G"'] & \tilde{M} \arrow[r, bend left, "F"] & N \end{tikzcd}$$

$= \omega(F(G(p)))(T_{G(p)}F \circ T_pG v_1, \dots, T_{G(p)}F \circ T_pG v_k) =$

$(F^*(\omega))(G(p))(T_pG v_1, \dots, T_pG v_k) =$

$G^*(F^*(\omega))(p)(v_1, \dots, v_k)$

Cofanie form różniczkowych we współrzędnych

$\omega \in \Omega^k(N)$ w układzie współrzędnych $\varphi = (y^1, \dots, y^n)$ ma postać $\omega = \sum \omega_{i_1 \dots i_k} \, dy^{i_1} \wedge \dots \wedge dy^{i_k} =$

$$(F^*\omega)(p) = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(F(p)) \, F^*(dy^{i_1} \wedge \dots \wedge dy^{i_k}) =$$

$$= \sum_{i_1 < \dots < i_k} (\omega_{i_1 \dots i_k} \circ F)(p) \, d(F^* y^{i_1}) \wedge \dots \wedge d(F^* y^{i_k})$$

$$\boxed{d^2 = 0}$$

$$\underbrace{d(F^*\omega)}(p) = \sum_{i_1 < \dots < i_k} \underbrace{d(\omega_{i_1 \dots i_k} \circ F)(p)}_{F^* d\omega_{i_1 \dots i_k}} \wedge dF^* y^{i_1} \wedge \dots \wedge dF^* y^{i_k}$$

$$= F^* \sum_{\dots} d\omega_{i_1 \dots i_k} \wedge dy^{i_1} \wedge \dots \wedge dy^{i_k} = \underbrace{F^*(d\omega)}(p)$$

$$F^* d\omega_{i_1 \dots i_k} \wedge F^* dy^{i_1} \wedge \dots \wedge F^* dy^{i_k} =$$

(3) $\underbrace{(F \circ G)^*(\omega)}(p)(v_1, \dots, v_k) =$

$$\omega(F(G(p)))(T_p(F \circ G) v_1, \dots, T_p(F \circ G) v_k)$$

$$\begin{tikzcd} M \arrow[r, bend left, "G"] \arrow[rr, bend right, "F \circ G"'] & \tilde{M} \arrow[r, bend left, "F"] & N \ni \omega \end{tikzcd}$$

$$= \omega(F(G(p)))(\underbrace{T_{G(p)}F} \circ T_p G v_1, \dots, \underbrace{T_{G(p)}F} \circ T_p G v_k) =$$

$$(F^*(\omega))(G(p))(T_p G v_1, \dots, T_p G v_k) =$$

$$\underbrace{G^*(F^*(\omega))}(p)(v_1, \dots, v_k)$$

$\boxtimes$

Cofanie form różniczkowych we współrzędnych

$M \qquad\qquad N$

$$(x^1, \dots, x^m) \xrightarrow{F} (y^1, \dots, y^n)$$

F we współrzędnych

$$F = \begin{pmatrix} F^1(x^1, \dots, x^m) \\ \vdots \\ F^n(x^1, \dots, x^m) \end{pmatrix}.$$

Niech $\omega \in \Omega^k(N)$ we współrzędnych

$$\omega = \sum_{\dots} \omega_{i_1 \dots i_k}(y^1, \dots, y^n) \, dy^{i_1} \wedge \dots \wedge dy^{i_k}$$

$$(F^*\omega)(x^1, \dots, x^m) = \sum \omega_{i_1 \dots i_k}(F^1(x^1, \dots, x^m), \dots, F^n(x^1, \dots, x^m)) \, dF^{i_1} \wedge \dots \wedge dF^{i_k}$$

Konkretnie: $F(x,y) = xy$ $\qquad \theta \in \Omega^1(\mathbb{R})$ $\qquad \theta = e^{-t^2} dt$.

$F^*\theta \in \Omega^1(\mathbb{R}^2)$ gdzie

$$F^*\theta = e^{-x^2y^2} \, d\, x \cdot y = e^{-x^2y^2}(x\,dy + y\,dx).$$

Zauważmy, że $d\theta = 0$. Obliczmy $d F^*\theta = (F^* d\theta = 0)$

$$= d\left(e^{-x^2y^2}(x\,dy + y\,dx)\right)$$

$$= d e^{-x^2y^2} \wedge (x\,dy + y\,dx) + e^{-x^2y^2} \cdot \overbrace{d(x\,dy + y\,dx)}^{0} =$$

$$= e^{\dots}(-2xy^2\,dx - 2x^2y\,dy) \wedge (x\,dy + y\,dx) + e^{\dots} \underbrace{(dx \wedge dy + dy \wedge dx)}_{0} = e^{\dots}\left((-2x^2y^2 + 2x^2y^2)\,dx \wedge dy\right)$$

$$= 0.$$