Równanie falowe: $u(x,t)$, $x \in \mathbb{R}^n$, $t \geqslant 0$. $u_{tt} - c^2 \Delta_x u = 0$.

$\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Zagadnienie początkowe $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ $\forall x \in \mathbb{R}^n$, $f \in C^2(\mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n)$.

Dla $n=1$: rozwiązanie ma postać: $u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$ (*)

Struna półskończona: zagadnienie początkowo-brzegowe $n=1$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0. & t>0\ x>0. \\ u(x,0) = f(x) & x \geqslant 0. \\ u_t(x,0) = g(x) & x \geqslant 0. \\ u(0,t) = 0 & \end{cases}$$

$u(x,t) = \alpha(x-ct) + \beta(x+ct)$. Gdy $x - ct \geqslant 0$.

to $u(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

(wyprowadzenie wzoru t. jak wzoru (*)).

Gdy $x - ct < 0$: mamy następujące warunki
$$\begin{cases} \alpha(x) + \beta(x) = f(x) & x \geqslant 0. \\ c\big(\alpha'(x) - \beta'(x)\big) = -g(x) & x \geqslant 0 \\ \alpha(-ct) + \beta(ct) = 0 & t \geqslant 0. \end{cases}$$

$\alpha(0) = \beta(0) = 0$.

bez straty ogólności.

W szczególności dla $s \geqslant 0$ $\boxed{\alpha(-s) = -\beta(s) = -f(s) + \alpha(s)}$.

Ponadto:
$$\begin{cases} -\alpha(x) + \beta(x) = \frac{1}{c}\int_0^x g(s)\,ds. \\ \alpha(x) + \beta(x) = f(x) \end{cases} \quad \underline{\underline{x \geqslant 0.}} \quad \rightarrow \begin{cases} \alpha(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds \\ \beta(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds \end{cases}$$

W szczególności:
$$\alpha(-x) = -f(x) + \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds = -\frac{1}{2}f(x) + \frac{1}{2c}\int_x^0 g(s)\,ds.$$

Zatem $\alpha(x-ct) = -\frac{1}{2}f(ct-x) + \frac{1}{2c}\int_{ct-x}^{0} g(s)\,ds$.

Ostatecznie dla $x - ct < 0$; $u(x,t) = \alpha(x-ct) + \beta(x+ct) = -\frac{1}{2}f(ct-x) + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)\,ds$, Równanie falowe: $u(x,t)$, $x \in \mathbb{R}^3$, $t \geq 0$.

$\Delta_x = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. Zagadnienie początkowe $u_{tt} - c^2 \Delta_x u = 0$.

$u(x,0) = f(x)$ $\quad \forall x \in \mathbb{R}^3$ $\quad f \in C^2(\mathbb{R}^3)$

$u_t(x,0) = g(x)$ $\quad g \in C^1(\mathbb{R}^3)$.

Sprowadźmy ten problem do struny półskończonej:

$h : \mathbb{R}^3 \to \mathbb{R}$; uśredniamy h po sferach $S^2(x,r)$

$$I_h(x,r) = \frac{1}{4\pi r^2} \int_{S^2(x,r)} h(x+z)\, d\sigma_z = \left\{ = \frac{1}{4\pi} \int_{|y|=1} h(x+ry)\, d\sigma_y \right. \xrightarrow[r\to 0]{} h(x)$$

$$\left\{ d\sigma_z = r^2 \underbrace{\sin\theta\, d\theta\, d\tau}_{\text{współrzędne na } S^2(x,r)} \right.$$

Zauważmy:
$$\int_{B(x,R)} h(z)\, dz = \int_0^R 4\pi r^2 I_h(x,r)\, dr$$

$$\Delta_x \int_0^R 4\pi r^2 I_h(x,r)\, dr = \Delta_x \int_{B(0,R)} h(x+z)\, dz = \int_{B(0,R)} (\Delta h)(x+z)\, dz \overset{\text{Tw Stokesa}}{=} \int_{S^2(0,R)} \nabla h(x+z) \cdot \frac{z}{R}\, d\sigma_z$$

$$= R^2 \int_{S^2(0,1)} \sum_{i=1}^{3} \frac{\partial h}{\partial x_i}(x+Ry)\, y_i\, d\sigma_y = \left\{ \frac{\partial}{\partial R} h(x+Ry) = \sum_{i=1}^{3} \frac{\partial h}{\partial x_i} \cdot y_i \right\} = R^2 \frac{\partial}{\partial R} \underbrace{\int_{S^2(0,1)} h(x+Ry)\, d\sigma_y} = 4\pi R^2 \frac{\partial}{\partial R} I_h(x,R)$$

Zróżniczkujmy stronami po R:
$$\Delta_x \cancel{4\pi R^2} I_h(x,R) = \cancel{4\pi} \underbrace{\frac{\partial}{\partial R} R^2 \frac{\partial}{\partial R}} I_h(x,R) = \cancel{4\pi}\cancel{R} \underbrace{\frac{\partial^2}{\partial R^2} R\, I_h(x,R)}.$$

$\Delta_x\, r I_h(x,r) = \frac{\partial^2}{\partial r^2} \underbrace{r I_h(x,r)}$. Niech $\psi_r(h)(x) \overset{def}{=} r I_h(x,r)$. $\quad \frac{\psi_r(h)(x)}{r} \xrightarrow[r\to 0]{} h(x)$

Niech $u(x,t)$ spełnia r-nie falowe, $\frac{\partial^2}{\partial t^2} \psi_r(u) - c^2 \Delta_x \psi_r(u) = \frac{\partial^2}{\partial t^2} \psi_r(u) - c^2 \frac{\partial^2}{\partial r^2} \psi_r(u) = 0$;

$$\begin{cases} \frac{\partial^2}{\partial t^2} \Psi_r(u) - c^2 \frac{\partial^2}{\partial r^2} \Psi_r(u) = 0. \\ \Psi_r(u)(x,0) = (\Psi_r f)(x) \\ \partial_t \Psi_r(u)(x,0) = (\Psi_r g)(x). \\ \Psi_{r=0}(u) = 0. \end{cases}$$

Skoro przechodzimy z $r \to 0$, $r - ct < 0$ i $(\Psi_r u)(x,t) = \frac{1}{2}\left(\Psi_{ct+r} f(x) - \Psi_{ct-r} f(x)\right) + \frac{1}{2c}\int_{ct-r}^{ct+r} (\Psi_s g)(x)\, ds$ dzielimy przez r i $\lim_{r\to 0}$ daje.

$$u(x,t) = \frac{\partial}{\partial r} \Psi_r f(x)\Big|_{r=ct} + \frac{1}{c} \Psi_{ct}(g)(x)$$

$$= \frac{\partial}{\partial r}\left(r \cdot \frac{1}{4\pi} \int_{|y|=1} f(x+ry)\, d\sigma_y \right)\Big|_{r=ct} = \frac{1}{4\pi} \frac{\partial}{\partial t}\, t \cdot \int_{|y|=1} f(x+cty)\, d\sigma_y$$

Wzór Kirchhoffa: $4\pi u(x,t) = \frac{\partial}{\partial t}\, t \cdot \int_{|y|=1} f(x+cty)\, d\sigma_y + t \cdot \int_{|y|=1} g(x+cty)\, d\sigma_y$

Zasada Huygensa: Jeśli $K = \operatorname{supp} f \cup \operatorname{supp} g$ jest zwarty to dla $t \notin [t_1(x), t_2(x)]$ $u(x,t) = 0$, gdzie $t_1(x) = \inf\{t \geq 0 : S^2(x,ct) \cap K \neq \emptyset\}$

$t_2(x) = \sup\{t \geq 0 \;\; S^2(x,ct) \cap K \neq \emptyset\}$.

Uwaga: zasada Huygensa nie obowiązuje dla n-parzystego a obowiązuje dla n nieparzystego.

Ćwiczenie: $\mathbb{R}^2$: $2\pi u(x,t) = \frac{\partial}{\partial t} \int_{D^2(x,ct)} \frac{f(z)\,dz}{\sqrt{c^2t^2 - |z-x|^2}} + \int_{D^2(x,ct)} \frac{g(z)\,dz}{\sqrt{c^2t^2-|z-x|^2}}$