On representation theory of topological groups: from SU(2) through compact groups to semisimple Lie groups

Jacek Kenig, Michał Pietruszka

September 2020

Contents

1 Introduction **2**

2 SU(2) and its representation theory **3**

2.1 Definition and properties of SU(2) 3
2.2 The Lie algebra of SU(2) and its properties 6
2.3 Homomorphisms and isomorphisms 8
2.4 Representations 10
2.5 Representations of SU(2) 13

3 Basic representation theory of compact groups **23**

3.1 Compact groups and their representations 23
3.2 Lie groups 29

4 Basic theory of representations of semisimple Lie groups and Borel-Weil theorem **32**

4.1 Basic terminology 32
4.2 Some of the most important notions 34
4.3 More on the structure of Lie algebras and Lie groups 38
4.4 Borel-Weil theorem 42
4.5 Borel-Weil theorem for SU(2) 47

Chapter 1

Introduction

This work is a review article, which arose as a student project at the Faculty of Physics at University of Warsaw. It's main purpose is to present basic results of the theory of representations of the group SU(2) and its Lie algebra, $\mathfrak{su}(2)$, as well as some fundamental properties and notions of the theory of representations of general compact groups and semisimple groups. Although the very first part of this work is focused on SU(2), the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ will turn out to be crucial in our presentation, because, as we shall see, it is naturally isomorphic to the complexification of $\mathfrak{su}(2)$, and passing to the complexified Lie algebra will make our calculations easier. We focus on SU(2) for many reasons. Firstly, it is important from the point of view of physics: the representations of complexified $\mathfrak{su}(2)$ appear in the theory of angular momentum in quantum mechanics, and secondly, theory of its representations is rather trivial, but despite this fact, it can still be seen as the very special (and the simplest) case of more general and sophisticated theory. As noted above, this work is not a research paper and should be treated as a script and a handy source of some information about the elements of the theory. We will be mathematically rigorous in presenting the material, but we skip most of proofs and give many references instead. We desired this work to be friendly for both mathematics and physics students as well (even for those physics students which are not keen on mathematics - yes, unfortunately, there are many of them!), but we assume that the Reader is familiar with the definition of a group, basics of analysis and linear algebra, especially the theory of matrices and linear operators on Hilbert spaces. It would be also helpful for the Reader to be familiar with basic notions of differential geometry, functional analysis, Lebesgue integration and topology, but to understand most part of this work, this is not necessary. There is, however, one exception. This is chapter 4, which is actually the most essential part, and where we enter the world of algebraic geometry, and where Borel-Weil theorem is formulated and proven. This is definitely not for beginners.

We would like to thank our project managers, prof. Piotr Sołtan and dr hab. Paweł Kasprzak, for Their appreciated help and patience.

Chapter 2

SU(2) and its representation theory

This chapter is a warm-up, and it is entirely focused on SU(2). We start with the definition of SU(2) and write down its basic properties, and then we come to its Lie algebra $\mathfrak{su}(2)$ and its complexification, $\mathfrak{sl}(2,\mathbb{C})$. Then we formulate definitions of homomorphism and finite-dimensional representations of SU(2), $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{C})$. We introduce the notion of equivalence of representations and give examples of how one uses given representations to construct new ones. Then we find finite-dimensional irreducible representations of SU(2) and claim that these are all irreducible finite-dimensional representations of SU(2), up to equivalence. We end this chapter by analyzing the structure of finite-dimensional, irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$, which are in one-to-one correspondence with finite-dimensional, irreducible (complex) representations of SU(2). In our presentation of the material in this chapter, we mainly follow [5].

2.1 Definition and properties of SU(2)

Definition 2.1.1. SU(2) *is the set of all 2×2 unitary matrices with determinant one.*

We say that $n\times n$ complex matrix U is called **unitary** if and only if $UU^* = U^*U = \mathbb{1}$, which is equivalent to $U^* = U^{-1}$, where, by definition, U^* is an adjoint[1] of U: $(U^*)_{ij} = \overline{(U)_{ji}}$ (matrix transpose + complex conjugate of the entries). An equivalent definition says that $n\times n$ complex matrix U is unitary if and only if

$$\langle Ux|Uy\rangle = \langle x|y\rangle \tag{2.1}$$

for all $x,y\in\mathbb{C}^n$, where $\langle\cdot|\cdot\rangle : \mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ is the standard inner product in $\mathbb{C}^n$:

$$\langle x|y\rangle = \sum_{i=1}^{n}\overline{x_i}y_i. \tag{2.2}$$

If (2.1) holds for $n\times n$ complex matrix U and all $x,y\in\mathbb{C}^n$, then we say that U preserves inner product, or that inner product is invariant under the action of

[1] U^* is also called **Hermitian conjugate** of U. Physicists often write $U^\dagger$ in place of U^*.

U. Moreover, it can be shown that $n \times n$ complex matrix is unitary if and only if its column vectors are orthonormal with respect to the standard inner product on $\mathbb{C}^n$ (2.2).

Definition 2.1.2. *The group of $n \times n$ invertible complex (real) matrices is called* ***general linear group*** *and it is denoted* $\mathrm{GL}(n, \mathbb{C})$ $(\mathrm{GL}(n, \mathbb{R}))$.

Since SU(2) is, by definition, the set of those 2×2 unitary matrices which have determinant one, all matrices in SU(2) are invertible, and $\mathrm{SU}(2) \subset \mathrm{GL}(2,\mathbb{C})$. Moreover, it can be easily seen that if $U \in \mathrm{SU}(2)$, then U^{-1} has determinant one and preserves inner product, so $U^{-1} \in \mathrm{SU}(2)$. The same result holds for the product of two matrices, namely, if $A, B \in \mathrm{SU}(2)$, then $AB \in \mathrm{SU}(2)$. Of course, $\mathbb{1} \in \mathrm{SU}(2)$. Thus we have the following:

Theorem 2.1.1. SU(2) *is a subgroup of* GL(2,$\mathbb{C}$) *and, in particular, it is itself a group.*

The name "SU" refers to "special (determinant one) unitary", hence SU(2) (or SU(n), in general) is often called **special unitary group**. We know so far that SU(2) is a group, but in fact we can show even more. Namely, if we identify $M(k, \mathbb{C})$ (the set of all $k \times k$ complex matrices with $k \in \mathbb{N}$) with the space $\mathbb{C}^{k^2} \cong \mathbb{R}^{2k^2}$, then we can use the usual topological structure on $\mathbb{C}^{k^2} \cong \mathbb{R}^{2k^2}$ and introduce the notion of convergence: a sequence of matrices $A_n \in M(k, \mathbb{C})$ converges to some matrix $A \in M(k, \mathbb{C})$ if and only if, for all $1 \leq i, j \leq k$, $(A_n)_{ij}$ converges to $(A)_{ij}$, that is, entries of the sequence A_n converge to the corresponding entries of A (standard convergence of complex numbers). In this case, we write simply $A_n \to A$. Now, if U_n is a sequence of matrices in SU(2), and $U_n \to U$, then one can show that $U \in \mathrm{SU}(2)$, because the inner product (2.2) and determinant are continuous.

Definition 2.1.3. *A* ***matrix Lie group***[2] *is a subgroup H of* $\mathrm{GL}(n, \mathbb{C})$ *with the property that if $A_n \in H$ and $A_n \to A$ for some matrix A, then either $A \in H$, or $A \notin$* $\mathrm{GL}(n, \mathbb{C})$. *In other words, H is a matrix Lie group if it is a subgroup and a closed subset of* $\mathrm{GL}(n, \mathbb{C})$ *(this is not the same as saying that it is closed in* $M(n, \mathbb{C})$*).*

Note that SU(2) is thus a matrix Lie group as a subgroup of GL(2,$\mathbb{C}$), but with the stronger property that is required from a subgroup to be matrix Lie group: if $\mathrm{SU}(2) \ni U_n \to U$, then *always* $U \in \mathrm{SU}(2)$. Let us now list some topological properties of SU(2). For proofs, see for example [5] and [6]. They are rather elementary, but, as we shall see, they will play a crucial role in our study and classification of irreducible representations of SU(2), that's why we list them below.

[2]Some authors, for example [12], call a matrix Lie group simply a *matrix group*. We use the name "matrix Lie group" to emphasize that a matrix Lie group is a Lie group, see Chapter 3, especially Proposition 3.2.1.

Important topological properties of SU(2):

1. SU(2) is a closed subset of $\mathbb{C}^4 \cong \mathbb{R}^8$.
   This is precisely the result obtained in the above discussion.

2. SU(2) is a bounded subset of $\mathbb{C}^4 \cong \mathbb{R}^8$.
   Recall that $\forall U \in \mathrm{SU}(2)$ the column vectors of U are orthonormal, so each component of each vector has absolute value no greater than one. This means that SU(2) is bounded.

3. SU(2) is a compact subset of $\mathbb{C}^4 \cong \mathbb{R}^8$.
   Recall that we say that a subset of a topological space is compact if its every open cover has a finite subcover. The above property is in fact a corollary from the previous two properties, since it is a standard result from elementary analysis that a subset of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is compact if and only if it is closed and bounded.

4. SU(2) is homeomorphic to 3-dimensional real unit sphere $S^3 \subset \mathbb{R}^4$, $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$. This result basically comes from solving five equations: $\det U = 1$ and $UU^* = \mathbb{1}$ for arbitrary complex 2×2 matrix U. It turns out that $U \in SU(2)$ if and only if it is of the form

   $$U = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

   where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2+|\beta|^2= 1$. By putting $\alpha = x_1+ix_2$ and $\beta = x_3+ix_4$, we obtain $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ The map

   $$\mathrm{SU}(2) \ni U = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \mapsto (x_1, x_2, x_3, x_4) \in S^3 \tag{2.3}$$

   is clearly a bijection. See [6] for a proof that (2.3) is actually a homeomorphism[3] (that is, it is continuous and the inverse map is also continuous). The idea of the proof is to rewrite U from (2.3) as

   $$\mathrm{SU}(2) \ni U = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} = \mathbf{x} \cdot \Sigma,$$

   where $\mathbf{x} = (x_1, x_2, x_3, x_4) \in S^3$ (that is, $\|\mathbf{x}\|_{\mathbb{R}^4} = 1$) and $\Sigma = (\mathbb{1}, i\sigma_3, i\sigma_2, i\sigma_1)$ with

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

   Matrices $\sigma_1, \sigma_2, \sigma_3$ are called **Pauli matrices**, they arise naturally in the theory of spin in quantum mechanics.

[3] It turns out that this map is even the diffeomorphism when one treats SU(2) as a differentiable manifold and S^3 as a differentiable submanifold embedded in $\mathbb{R}^4$.

5. SU(2) is connected and path-connected.
   By connectedness we mean that SU(2) cannot be divided into two disjoint non-empty open sets, and by path-connectedness we mean that for all $A, B \in$ SU(2), there exists a continuous map $\gamma : \mathbb{R} \supset [a, b] \to$ SU(2), such that $\gamma(a) = A$ and $\gamma(b) = B$. In other words, every two elements from SU(2) can be connected to each other by a continuous path lying in SU(2). Moreover, it can be shown that every matrix Lie group (hence also SU(2)) is connected if and only if it is path-connected.

6. SU(2) is simply connected.
   By simple connectedness we mean that every loop in SU(2) may be contracted continuously to an element of SU(2), if we treat SU(2) as a subset of $\mathbb{C}^4 \cong \mathbb{R}^8$. Property 4 says that SU(2) is topologically equivalent to 3-dimensional sphere $S^3 \subset \mathbb{R}^4$, which is simply connected.

We will mention more properties of SU(2) later. Now, let us turn to the general definition of Lie algebra of a matrix Lie group and to the Lie algebra of SU(2).

2.2 The Lie algebra of SU(2) and its properties

Definition 2.2.1. *Given a matrix Lie group G, the* ***Lie algebra*** *of G, denoted $\mathfrak{g}$, is a set of matrices X such that $e^{tX} \in G \ \forall t \in \mathbb{R}$.*

In the above definition, e^X (or $\exp(X)$) is the exponential of a matrix X. It is given by the usual power series:

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}. \tag{2.5}$$

Here are some properties of the matrix exponential. Again, see [5] for proofs and discussion of this issue as well as for a proof of convergence of matrix exponential.

Theorem 2.2.1. *Let X, Y be arbitrary $n \times n$ matrices.*

1. $e^0 = \mathbb{1}$ *(here 0 is the zero matrix).*

2. e^X *is invertible and* $(e^X)^{-1} = e^{-X}$.

3. *If Z is invertible matrix, then* $e^{ZXZ^{-1}} = Ze^XZ^{-1}$.

4. *Define the* ***matrix commutator*** $[\cdot, \cdot]$ *to be* $[X, Y] = XY - YX$. *If* $[X, Y] = 0$ *(we say that X and Y commute) then* $e^{X+Y} = e^Xe^Y = e^Ye^X$. *In general, it is not true that* $e^{X+Y} = e^Xe^Y = e^Ye^X$*, unlike in the case of complex numbers which the reader is familiar with.*

5. *For any $\alpha, \beta \in \mathbb{C}$ we have* $e^{(\alpha+\beta)X} = e^{\alpha X}e^{\beta X}$.

6. $e^{X+Y} = \lim_{n\to\infty}(e^{\frac{X}{n}}e^{\frac{Y}{n}})^n$. *This is called the Lie product formula and it is a special, finite-dimensional case of the Trotter product formula in operator theory. See, for example, [11].*

7. $\det(e^X) = e^{\mathrm{Tr}(X)}$.

8. e^X *is a continuous function of* X *and* $\frac{d}{dt}\big|_{t=0} e^{tX} = X$.

9. $\frac{d}{dt}\big|_{t=0}\left(e^{tX}Ye^{-tX}\right) = [X,Y]$, *the commutator of* X *and* Y.

Let us go back to the Lie algebra $\mathfrak{g}$ of a matrix Lie group G. We defined it to be: $\mathfrak{g} = \{X \mid e^{tX} \in G\ \forall t \in \mathbb{R}\}$. Note that we require t to be real, not complex, even if X is a complex matrix and G is a group of complex matrices, and that if G is a group of $n \times n$ matrices, then $\mathfrak{g}$ is a set of $n \times n$ matrices. It turns out that $\mathfrak{g}$ is a **real vector space**, which is **closed under taking matrix commutator**, that is, if $X, Y \in \mathfrak{g}$, then $[X,Y] \in \mathfrak{g}$. Moreover, for any $A \in G$ and any $X \in \mathfrak{g}$, we have $AXA^{-1} \in \mathfrak{g}$ (see property 3. of the matrix exponential). The commutator is bilinear, skew-symmetric ($[X,Y] = -[Y,X]$) and the following identity, called **Jacobi identity**, holds:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{2.6}$$

We will now determine the Lie algebra of $\mathrm{SU}(2)$, which is denoted $\mathfrak{su}(2)$. Recall the property $\det(e^X) = e^{\mathrm{Tr}(X)}$. Clearly, if $\mathrm{Tr}(X) = 0$, then $\det(e^{tX}) = 1$ for all real t. On the other hand, if $\forall t \in \mathbb{R}$ we have $\det(e^{tX}) = 1$, then $\forall t \in \mathbb{R}$ also $e^{t\mathrm{Tr}(X)} = 1$, so $\forall t \in \mathbb{R}$: $t\mathrm{Tr}(X) = 2i\pi n$ for some $n \in \mathbb{Z}$, and this is possible only if $\mathrm{Tr}(X) = 0$. Thus the condition $\mathrm{Tr}(X) = 0$ is necessary and sufficient for $\det(e^{tX}) = 1$, so all X in $\mathfrak{su}(2)$ must have trace zero. But this is not the end, for if $U \in \mathrm{SU}(2)$, then $UU^* = \mathbb{1}$. Take $X \in \mathfrak{g}$. We see that we must have

$$e^{tX^*} = \left(e^{tX}\right)^* = \left(e^{tX}\right)^{-1} = e^{-tX},$$

where the first equality is obtained by taking adjoints term by term, and the last equality comes from property 2. of the matrix exponential. The sufficient condition is that $X^* = -X$, but if the above holds for all real t, then by differentiating at the point $t = 0$ and using property 8., we see that this is also necessary.

Definition 2.2.2. *The Lie algebra of* $\mathrm{SU}(2)$, *denoted* $\mathfrak{su}(2)$, *is the set of* 2×2 *complex matrices given by*

$$\mathfrak{su}(2) = \{X \in M(2,\mathbb{C}) \mid \mathrm{Tr}(X) = 0, X^* = -X\}. \tag{2.7}$$

It is a simple matter to check that this is a real vector space which is closed under matrix commutator. Using similar techniques we can determine the Lie algebras $\mathfrak{u}(2)$ and $\mathfrak{sl}(2,\mathbb{C})$ of the matrix Lie groups $\mathrm{U}(2)$ of 2×2 unitary matrices and $\mathrm{SL}(2,\mathbb{C})$ of matrices with determinant one, respectively. We have:

$$\mathfrak{u}(2) = \{X \in M(2,\mathbb{C}) \mid X^* = -X\}$$

and

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in M(2,\mathbb{C}) \mid \mathrm{Tr}(X) = 0\}. \tag{2.8}$$

In particular, $\mathfrak{sl}(2,\mathbb{C})$ will play a crucial role in our analysis, as we shall finally see.

2.3 Homomorphisms and isomorphisms

Definition 2.3.1. *Let G, H be arbitrary groups. A map $\Phi : G \to H$ is called a* ***group homomorphism*** *if $\forall g_1, g_2 \in G$ we have $\Phi(g_1g_2) = \Phi(g_1)\Phi(g_2)$. If in addition, Φ is a bijective map, it is called an* ***isomorphism*** *of groups G and H. If there exists an isomorphism between groups G and H, then they are said to be* ***isomorphic****, and we denote this property by $G \cong H$. An isomorphism of a group to itself is called an* ***automorphism****.*

Two groups which are isomorphic should be thought of as being the same group. For any group G, it can be trivially checked that the set $\mathrm{Aut}(G)$ of automorphisms of G is itself a group with the group product being composition of maps. The following proposition reveals further properties of group homomorphisms.

Proposition 2.3.1. *Let G, H be arbitrary groups, e_G the identity of G and e_H the identity of H, and let $\Phi : G \to H$ be homomorphism. Then $\Phi(e_G) = e_H$ and for all $g \in G$: $\Phi(g^{-1}) = \Phi(g)^{-1}$.*

Proof. Take $g \in G$. We have

$$e_H = \Phi(g)^{-1}\Phi(g) = \Phi(g)^{-1}\Phi(ge_G) = \Phi(g)^{-1}\Phi(g)\Phi(e_G) = \Phi(e_G).$$

In light of this result we can compute $e_H = \Phi(e_G) = \Phi(g^{-1}g) = \Phi(g^{-1})\Phi(g)$, and using the standard theorem from group theory which says that if an element of a group multiplied by another element from left or right side gives the identity, then one of these elements is the unique inverse of the other (see, for example, [5], Proposition 1.4.), we conclude that $\Phi(g^{-1}) = \Phi(g)^{-1}$.

□

In the case of matrix Lie groups, which have the natural notion of convergence, we demand an additional property:

Definition 2.3.2. *Let G, H be matrix Lie groups. A map $\Phi : G \to H$ is called* ***matrix Lie group homomorphism*** *if Φ is a group homomorphism and Φ is continuous. A matrix Lie group homomorphism is called* ***matrix Lie group isomorphism*** *if it is a bijective map and the inverse map is continuous. Matrix Lie group isomorphism of a matrix Lie group to itself is called* ***matrix Lie group automorphism****. If there exists a matrix Lie group isomorphism between two matrix Lie groups G and H, then they are said to be* ***isomorphic****, and this is denoted $G \cong H$.*

We now turn to the corresponding definitions for Lie algebras of matrix Lie groups.

Definition 2.3.3. *Let G, H be matrix Lie groups, and $\mathfrak{g}, \mathfrak{h}$ the corresponding Lie algebras. A map $\phi : \mathfrak{g} \to \mathfrak{h}$ is said to be **Lie algebra homomorphism** if ϕ is real linear map and*

$$\phi([X_1, X_2]) = [\phi(X_1), \phi(X_2)] \tag{2.9}$$

*for all $X_1, X_2 \in \mathfrak{g}$. If in addition ϕ is a bijection, then it is called a **Lie algebra isomorphism**, and we say that $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic. This property is denoted by $\mathfrak{g} \cong \mathfrak{h}$. One defines a **Lie algebra automorphism** of $\mathfrak{g}$ to be a Lie algebra isomorphism from $\mathfrak{g}$ to itself.*

Note that the commutator on the left-hand side of (2.9) is a commutator of elements from $\mathfrak{g}$, but the commutator on the right-hand side - of elements from $\mathfrak{h}$, but we denoted them by the same symbol because, in the case of matrices, this commutator is always $[A, B] = AB - BA$.

Definition 2.3.4. *The **complexification** of $\mathfrak{su}(2)$ (as a real vector space) is the complex vector space $\mathfrak{su}^{\mathbb{C}}(2) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, with the scalar multiplication $\mathbb{C} \times \mathfrak{su}^{\mathbb{C}}(2) \to \mathfrak{su}^{\mathbb{C}}(2)$ defined by*

$$(x + iy)(X_1, X_2) = x(X_1, X_2) + y(-X_2, X_1),$$

for $x, y \in \mathbb{R}$ and $X_1, X_2 \in \mathfrak{su}(2)$.

Define the commutator on $\mathfrak{su}^{\mathbb{C}}(2)$, denoted by the same symbol $[\cdot, \cdot]$, as

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1] - [X_2, Y_2], [X_1, Y_2] + [X_2, Y_1]). \tag{2.10}$$

It is straightforward to show that it is $\mathbb{C}$-bilinear, skew-symmetric and the Jacobi identity (2.6) holds. The above formulas take more intuitive forms when one defines $(X_1, X_2) \in \mathfrak{su}^{\mathbb{C}}(2)$ to be the formal linear combination $X_1 + iX_2$. One often writes $\mathfrak{su}^{\mathbb{C}}(2) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$ for that matter. The above arguments show that $\mathfrak{su}^{\mathbb{C}}(2)$, as a complex vector space, can be given a complex algebra structure with the product given by (2.10). It is an example of algebraic structure called **complex Lie algebra**.

Despite the fact that $\mathfrak{sl}(2, \mathbb{C})$ is a real vector space as a Lie algebra of a matrix Lie group (see (2.8)), it may easily be given a complex vector space structure since matrix with trace zero multiplied by a complex number still has trace zero. Moreover, the matrix commutator $[X, Y] = XY - YX$ is clearly $\mathbb{C}$-bilinear. Thus $\mathfrak{sl}(2, \mathbb{C})$ is naturally a Lie algebra over $\mathbb{C}$ (complex Lie algebra).

In further analysis, we will need the following theorem.

Theorem 2.3.1. $\mathfrak{su}^{\mathbb{C}}(2) \cong \mathfrak{sl}(2, \mathbb{C})$.

Proof. Rewrite $X \in \mathfrak{sl}(2,\mathbb{C})$ as $X = X_1 + iX_2$, where $X_1 = (X - X^*)/2$ and $X_2 = (X + X^*)/(2i)$. Note that both $X_1, X_2 \in \mathfrak{su}(2)$, and it is easy to see that this decomposition is unique. Thus $\mathfrak{su}^{\mathbb{C}}(2) \cong \mathfrak{sl}(2,\mathbb{C})$ as a complex vector space, but this is in fact an isomorphism of (complex) Lie algebras since one can show that in both cases (2.10) holds.

$\square$

2.4 Representations

Definition 2.4.1. *Let $\mathcal{H}$ be a finite-dimensional, complex Hilbert space. Denote the set of all linear operators on $\mathcal{H}$ by $\mathfrak{gl}(\mathcal{H})$, and consider the complex Lie algebra structure on $\mathfrak{gl}(\mathcal{H})$. A* ***finite-dimensional representation*** *of $\mathfrak{su}(2)$ on $\mathcal{H}$ is the $\mathbb{R}$-linear Lie algebra homomorphism $\pi_1 : \mathfrak{su}(2) \to \mathfrak{gl}(\mathcal{H})$, such that for all $X \in \mathfrak{su}(2)$ we have $\pi(X)^* = -\pi(X)$. A* ***finite-dimensional representation*** *of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathcal{H}$ is the $\mathbb{C}$-linear Lie algebra homomorphism $\pi_2 : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(\mathcal{H})$ with $\pi(X)^* = -\pi(X)$ for all $X \in \mathfrak{sl}(2,\mathbb{C})$.*

Proposition 2.4.1. *Every finite-dimensional representation π of $\mathfrak{su}(2)$ has a unique extension to the ($\mathbb{C}$-linear) representation of $\mathfrak{su}^{\mathbb{C}}(2) \cong \mathfrak{sl}(2,\mathbb{C})$, which will be also denoted π. Then, as a representation of $\mathfrak{sl}(2,\mathbb{C})$, it satisfies $\pi(X + iY) = \pi(X) + i\pi(Y)$ for all $X, Y \in \mathfrak{su}(2)$.*

We will not prove the above proposition, but it will be used in the next section. We now turn to the definition of a representation of SU(2).

Definition 2.4.2. *A* ***finite-dimensional representation*** *of* SU(2) *is a matrix Lie group homomorphism $\Pi : \mathrm{SU}(2) \to \mathrm{U}(\mathcal{H})$, where $\mathcal{H}$ is a finite-dimensional, complex Hilbert space and $\mathrm{U}(\mathcal{H})$ is a group of unitary operators on $\mathcal{H}$.*

It is possible to consider a finite-dimensional representation acting on general finite-dimensional, real or complex vector space V, but in the case of SU(2) and $\mathfrak{su}(2)$ (and general compact topological groups, see Chapter 3) it is desirable to restrict the definition to some complex, finite-dimensional Hilbert space $\mathcal{H}$. The common abuse of terminology, which takes place in many lectures and textbooks, refers to $\mathcal{H}$ as to the representation, without an explicit reference to maps π or Π. Although convenient in many situations (especially when one deals with irreducible representations, see Definition 2.4.3 below), this terminology may be confusing for some readers, so we try to use it as rarely as possible. The number $n = \dim_{\mathbb{C}}(\mathcal{H})$ is also called a **dimension** or **degree of a representation**. Throughout this chapter, we will sometimes denote the finite-dimensional representation shortly by $(\pi, \mathcal{H})$ or $(\Pi, \mathcal{H})$, and $\mathcal{H}$, the so-called **action space**, will *always* be finite-dimensional, complex Hilbert space for that matter, just as stated in Definitions 2.4.1 and 2.4.2. Of particular interests to us will be irreducible representations:

Definition 2.4.3. *Let $(\theta, \mathcal{H})$ be the finite-dimensional representation of A, where A is* $\mathrm{SU}(2)$*,* $\mathfrak{su}(2)$ *or* $\mathfrak{sl}(2, \mathbb{C})$*. A subspace $\mathcal{H}_0 \subset \mathcal{H}$ is called **invariant** (under the action of A via the representation θ) if $\theta(X)\psi \in \mathcal{H}_0$ for all $\psi \in \mathcal{H}_0$ and all $X \in A$. An invariant subspace is **non-trivial** (or **proper**) if $\mathcal{H}_0 \neq \{0\}$ and $\mathcal{H}_0 \neq \mathcal{H}$. A representation θ is called **irreducible**, if $\mathcal{H}$ has no non-trivial invariant subspaces. In other words, the only non-zero invariant subspace of $\mathcal{H}$ is the whole $\mathcal{H}$.*

It is important to note that $\mathcal{H}$ is complex Hilbert space space by definition, and we are talking only about *complex* invariant subspaces in this case.

Assume that $(\theta, \mathcal{H})$ is a finite-dimensional representation of A, as in the above definition. Assume that θ is not irreducible. By definition, there exists $\mathcal{H}_0 \subset \mathcal{H}$ such that $\mathcal{H}_0 \neq \{0\}$ and $\mathcal{H}_0 \neq \mathcal{H}$, and such that $\theta(X)\psi \in \mathcal{H}_0$ for all $\psi \in \mathcal{H}_0$ and all $X \in A$. But then, the restriction $\theta|_{\mathcal{H}_0}$ defined by the formula: $\theta|_{\mathcal{H}_0}(X)\psi = \theta(X)\psi$ for all $X \in A$ and all $\psi \in \mathcal{H}_0$ is itself a representation of A, acting on $\mathcal{H}_0$. If there is no non-trivial invariant $\mathcal{H}_0$, then θ cannot be restricted in such a way (of course, it can still be restricted to the subspace $\{\mathbf{0}\}$ on which it acts trivially, but we find this case uninteresting). Hence the name "irreducible". This allow us to think of an irreducible representations as of building blocks, or atoms of the world of representations (gr. *atomos* means "uncuttable"). Fortunately, as we shall see momentarily, if $\mathcal{H}_0 \subset \mathcal{H}$ is invariant under the action of some finite-dimensional representation of $\mathrm{SU}(2)$, and it is also non-triavial, there exists invariant and non-trivial subspace $\mathcal{H}_1 \subset \mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. This result has far-reaching consequences, but before we establish them, we present two methods of constructing new representations from the old ones. Firstly, we consider direct sums.

Definition 2.4.4. *Let $\Pi_1, \Pi_2, ..., \Pi_k$ be finite-dimensional representations of* $\mathrm{SU}(2)$ *acting on $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$, respectively. Define $\Pi_1 \oplus \Pi_2 \oplus \cdots \oplus \Pi_k$ to be a new representation of* $\mathrm{SU}(2)$ *acting on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$ by the formula*

$$\Pi_1 \oplus \Pi_2 \oplus \cdots \oplus \Pi_k(g)(\psi_1, \psi_2, ..., \psi_k) = (\Pi_1(g)\psi_1, \Pi_2(g)\psi_2, ..., \Pi_k(g)\psi_k), \quad (2.11)$$

*with $\psi_i \in \mathcal{H}_i$ and $g \in \mathrm{SU}(2)$. $\Pi_1 \oplus \Pi_2 \oplus \cdots \oplus \Pi_k$ is called the **direct sum** of representations $\Pi_1, \Pi_2, ..., \Pi_k$. Similarly, let $\mathfrak{g}$ be $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{C})$, and let $\pi_1, \pi_2, ..., \pi_k$ be representations of $\mathfrak{g}$ acting on $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$, respectively. Then we can define $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ to be a new representation of $\mathfrak{g}$ acting on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$ via*

$$\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k(X)(\psi_1, \psi_2, ..., \psi_k) = (\pi_1(X)\psi_1, \pi_2(X)\psi_2, ..., \pi_k(X)\psi_k), \quad (2.12)$$

*where $\psi_i \in \mathcal{H}_i$ and $X \in \mathfrak{g}$. This new representation is called the **direct sum** of representations $\pi_1, \pi_2, ..., \pi_k$.*

Recall that, given two finite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and operators A, B acting on $\mathcal{H}_1, \mathcal{H}_2$ respectively, then we can define tensor product of A and B, denoted $A \otimes B$, to be the operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ given by

$(A \otimes B)(\psi_1 \otimes \psi_2) := (A\psi_1 \otimes B\psi_2)$ for $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$. This formula may be easily extended to the cases of tensor products of more than two vector spaces. See, for example, [11] for discussion of this and for generalization to bounded (and unbounded) operators on infinite-dimensional Hilbert spaces.

Definition 2.4.5. *Let $\Pi_1, \Pi_2, ..., \Pi_k$ be representations of $\mathrm{SU}(2)$ acting on $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$, respectively. Define $\Pi_1 \otimes \Pi_2 \otimes \cdots \otimes \Pi_k$ to be the representation acting on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k$ called **tensor product** of representations $\Pi_1, \Pi_2, ..., \Pi_k$, given by*

$$\Pi_1 \otimes \Pi_2 \otimes \cdots \otimes \Pi_k(g) = \Pi_1(g) \otimes \Pi_2(g) \otimes \cdots \otimes \Pi_k(g) \tag{2.13}$$

*Now let $\mathfrak{g}$ be $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{C})$, and let $\pi_1, \pi_2, ..., \pi_k$ be representations of $\mathfrak{g}$ acting on $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$, respectively. Then define **tensor product** of representations $\pi_1, \pi_2, ..., \pi_k$, denoted $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_k$, as the new representation acting on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$ and given by*

$$\begin{aligned}\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_k(X) = \pi_1(X) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes \pi_2(X) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\ + \cdots + \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \pi_k(X).\end{aligned} \tag{2.14}$$

Let us now introduce extremely important notion of equivalence of representations.

Definition 2.4.6. *Let A be $\mathrm{SU}(2)$, $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{C})$, and let θ_1, θ_2 be representations of A acting on $\mathcal{H}_1, \mathcal{H}_2$, respectively. If there exists unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that, for all $X \in A$ we have $\theta_1(X) = U^*\theta_2(X)U$, then U is called an (uunitary) **isomorphism** of representations θ_1 and θ_2, and we say that θ_1 and θ_2 are **equivalent** (or **isomorphic**), which is denoted $\theta_1 \cong \theta_2$.*

Two representations which are isomorphic should be thought of as being essentially the same representation.

Definition 2.4.7. *A finite-dimensional representation of A acting on $\mathcal{H}$, where A is $\mathrm{SU}(2)$, $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{C})$, is called **completely reducible**, if given an invariant subspace $\mathcal{H}_1$ of $\mathcal{H}$, there is an invariant subspace $\mathcal{H}_2 \subset \mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.*

Theorem 2.4.1.

1. *Let $(\Pi, \mathcal{H})$ be a finite-dimensional representation of $\mathrm{SU}(2)$. Then Π is completely reducible.*

2. *A finite-dimensional completely reducible representation of $\mathrm{SU}(2)$, $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{C})$ is equivalent to the direct sum of irreducible representations.*

Proof of 1. By definition, there exists an inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{H}$ which is invariant under Π. Suppose $\mathcal{H}_1$ is an invariant subspace of $\mathcal{H}$. Define $\mathcal{H}_2 = \mathcal{H}_1^\perp$. Then, because $\mathcal{H}$ is a Hilbert space, we have $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Take $\psi_2 \in \mathcal{H}_2$ and $\psi_1 \in \mathcal{H}_1$ and compute

$$\langle\psi_1|\Pi(g)\psi_2\rangle = \langle\Pi(g^{-1})\psi_1|\Pi(g^{-1})\Pi(g)\psi_2\rangle = \langle\Pi(g^{-1})\psi_1|\psi_2\rangle,$$

but the right-hand side is 0 since $\mathcal{H}_1$ is invariant by assumption. That means $\Pi(g)\psi_2 \in \mathcal{H}_2$ for all $g \in \mathrm{SU}(2)$, and it follows that $\mathcal{H}_2$ is also invariant.

We skip the proof of 2. It is actually nothing more than a contemplation of definitions and reasoning by induction on dimension of a representation space.

□

Definition 2.4.8. *Given a representation* Π *of* $\mathrm{SU}(2)$*, we define the* ***character*** $\chi_\Pi : \mathrm{SU}(2) \to \mathbb{C}$ *of* Π *by the formula*

$$\chi_\Pi(g) := \mathrm{Tr}(\Pi(g)) \tag{2.15}$$

for $g \in \mathrm{SU}(2)$*.*

Definition 2.4.9. *Let* $\mathcal{H}$ *be separable, infinite-dimensional, complex Hilbert space. An* ***infinite-dimensional representation*** *of SU(2) is a strongly continuous*[4] *group homomorphism* $\Pi : \mathrm{SU}(2) \to U(\mathcal{H})$*, where* $U(\mathcal{H})$ *is the group of unitary operators on* $\mathcal{H}$*.*

The corollary from Chapter 3 will be that every finite-dimensional representation of SU(2) is equivalent to a direct sum of finite-dimensional irreducible representations, and this decomposition is unique, up to equivalence. Moreover, every finite-dimensional representation of SU(2) is determined uniquely, up to equivalence, by its character. We will see also that every properly defined infinite-dimensional representation of SU(2) is also completely reducible, and hence is equivalent to infinite direct sum of finite-dimensional irreducible representations. It turns out that if a representation of SU(2) is irreducible, then it must be finite-dimensional. Thus, finite-dimensional irreducible representations are indeed kind of building blocks, from which all representations of SU(2) are build. Our next goal is to classify all finite-dimensional irreducible representations of SU(2), up to equivalence.

2.5 Representations of SU(2)

Let V_m be the complex vector space of homogeneous polynomials in two complex variables $z := (z_1, z_2) \in \mathbb{C}^2$ of degree m for any non-negative integer m. If $f \in V_m$, then f is of the form

$$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \cdots + a_m z_2^m, \tag{2.16}$$

[4] Strong continuity means that if $\mathrm{SU}(2) \ni U_n \to U \in \mathrm{SU}(2)$, then $\|\Pi(U_n)\psi - \Pi(U)\psi\| \to 0$ for all $\psi \in \mathcal{H}$.

where $a_i \in \mathbb{C}$ and hence $\dim_{\mathbb{C}}(V_m) = m + 1$. For $p, q \in V_m$, the following inner product

$$\langle p|q\rangle = \int_{|z_1|^2+|z_2|^2=1} \overline{p(z_1, z_2)} q(z_1, z_2) d\Omega(z_1, z_2), \tag{2.17}$$

where Ω is the Lebesgue measure on a unit sfere in $\mathbb{C}^2$, provides the Hilbert space structure on V_m.

Pick $U \in \mathrm{SU}(2)$ and consider the transformation on V_m given by the formula

$$\Pi_m(U)f(z) = f(U^{-1}z), \tag{2.18}$$

where, on the right-hand side of the above equation, U^{-1} acts on a column vector $z = (z_1, z_2)^T \in \mathbb{C}^2$. $\Pi_m(U)$ is clearly $\mathbb{C}$-linear for all $U \in \mathrm{SU}(2)$. Moreover, since

$$U^{-1}z = \left(\begin{array}{cc} (U^{-1})_{11} & (U^{-1})_{12} \\ (U^{-1})_{21} & (U^{-1})_{22} \end{array}\right)\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right) = \left(\begin{array}{c} (U^{-1})_{11}z_1 + (U^{-1})_{12}z_2 \\ (U^{-1})_{21}z_1 + (U^{-1})_{22}z_2 \end{array}\right)$$

we have

$$\begin{aligned}
\Pi_m(U)f(z_1, z_2) &= \sum_{k=0}^{m} a_k [(U^{-1})_{11}z_1 + (U^{-1})_{12}z_2]^{m-k}[(U^{-1})_{21}z_1 + (U^{-1})_{22}z_2]^k \\
&= \sum_{k=0}^{m}\sum_{i=0}^{m-k}\sum_{j=0}^{k} a_k \binom{m-k}{i}\binom{k}{j}[(U^{-1})_{11}z_1]^{m-k-i}[(U^{-1})_{21}z_1]^{k-j} \times \\
&\times [(U^{-1})_{12}z_2]^i[(U^{-1})_{22}z_2]^j,
\end{aligned}$$

and it is easy to see that $i + j$ does never exceed m, thus the obtained object is again a homogeneous polynomial of the form (2.16). Now, for $U_1, U_2 \in \mathrm{SU}(2)$, we have

$$\begin{aligned}
\Pi_m(U_1)[\Pi_m(U_2)f](z) &= [\Pi_m(U_2)f](U_1^{-1}z) = f(U_2^{-1}U_1^{-1}z) = f((U_1U_2)^{-1}z) \\
&= \Pi_m(U_1U_2)f(z),
\end{aligned}$$

thus Π_m is a finite-dimensional, complex representation: $\Pi_m : \mathrm{SU}(2) \to \mathrm{GL}(V_m)$. Moreover, since the Lebesgue measure Ω in (2.17) coincides with the so-called Haar measure of SU(2) (see Chapter 3), it turns out that Π_m is unitary with respect to the inner product (2.17).

Theorem 2.5.1. *Representations Π_m described above are irreducible for any integer $m \geq 0$.*

The following lemma, which is a special case of more general theorem, will help us prove Theorem 2.5.1:

Lemma 2.5.2. ***(Shur's lemma)*** *A unitary representation Φ of a Lie group G, acting on a finite-dimensional, complex vector space V is irreducible if and only if the only linear operators on V, which commute with $\Phi(g)$ for all $g \in G$ are operators of the form $c\mathbb{1}$ with $c \in \mathbb{C}$.*

See [8] for general case and proof.

Proof of Theorem 2.5.1 In the proof, we follow [7]. Pick a constant $a \in \mathbb{C}$ from the unit circle: $|a|= 1$. Then define and element $U_a \in \mathrm{SU}(2)$ by

$$U_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Set $f_k(z_1, z_2) = z_1^k z_2^{m-k}$ for $k = 0, 1..., m$. This is a basis of V_m. Pick A such that it commutes with all the $\Pi_m(U)$'s. We have

$$\Pi_m(U_a)f_k(z) = f_k(U_a^{-1}z) = (a^{-1}z_1)^k(az_2)^{m-k} = a^{m-2k}z_1^k z_2^{m-k} = a^{m-2k}f_k(z),$$

so f_k is an eigenvector for $\Pi_m(U_a)$ for all $0 \leq k \leq m$. On the other hand

$$\Pi_m(U_a)Af_k(z) = A\Pi_m(U_a)f_k(z) = Aa^{m-2k}f_k(z) = a^{m-2k}Af_k(z)$$

by definition of A. We can choose a so that the constants a^{m-2k} are distinct. If so, the eigenspaces of $\Pi_m(U_a)$ must all have dimension one and are spanned by f_k's. But from the above we see that Af_k is also the eigenvector of $\Pi_m(U_a)$ with the same eigenvalue as f_k, so we must have $Af_k \in \mathrm{span}\{f_k\}$, that is, $Af_k = c_k f_k$ for all k and some $c_k \in \mathbb{C}$. Now, for $t \in [-\pi, \pi[\subset \mathbb{R}$, consider $U_t \in \mathrm{SU}(2)$ given by:

$$U_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Clearly

$$U_t^{-1} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Let us now compute $A\Pi_m(U_t)f_m(z)$. Since $f_m(z_1, z_2) = z_1^m$:

$$\begin{aligned} A\Pi_m(U_t)f_m(z) &= Af_m(U_t^{-1}z) = A(\cos(t)z_1 + \sin(t)z_2)^m \\ &= A\sum_{k=0}^{m}\binom{m}{k}\cos(t)^k z_1^k \sin(t)^{m-k} z_2^{m-k} \\ &= A\sum_{k=0}^{m}\binom{m}{k}\cos(t)^k \sin(t)^{m-k} z_1^k z_2^{m-k} \\ &= \sum_{k=0}^{m}\binom{m}{k}\cos(t)^k \sin(t)^{m-k} Af_k(z_1, z_2) \\ &= \sum_{k=0}^{m}\binom{m}{k}\cos(t)^k \sin(t)^{m-k} c_k f_k(z_1, z_2). \end{aligned} \tag{2.19}$$

On the other hand,

$$\begin{aligned} \Pi_m(U_t)Af_m(z) &= c_m f_m(U_t^{-1}z) = c_m(\cos(t)z_1 + \sin(t)z_2)^m \\ &= \sum_{k=0}^{m}\binom{m}{k}\cos(t)^k \sin(t)^{m-k} c_m f_k(z_1, z_2). \end{aligned} \tag{2.20}$$

But (2.19) and (2.20) should be equal because A commutes with $\Pi_m(U_t)$. It can be shown that functions $t \mapsto \cos(t)^k \sin(t)^{m-k}$ are linearly independent, so we can compare coefficients term by term. We conclude that, for all $0 \le k \le m$, we have $c_k = c_m$ and $A = c_m \mathbb{1}$.

$\square$

Theorem 2.5.3. *Every irreducible representation of* $\mathrm{SU}(2)$ *is equivalent to one and only one of the* Π_m*'s.*

The proof requires a deeper insight into the theory of characters, see [7].

The above theorems say that we are essentially done, as far as theory of representations of $\mathrm{SU}(2)$ is concerned, because we know that every irreducible representation of $\mathrm{SU}(2)$ must be finite-dimensional, and we know that every finite-dimensional representation of $\mathrm{SU}(2)$, and every infinite-dimensional representation of $\mathrm{SU}(2)$ acting on separable Hilbert space can be decomposed as a direct sums (infinite direct sum in infinite-dimensional case) of irreducible representations, and that this decomposition is unique up to equivalence.

Let us now turn to $\mathfrak{su}(2)$, the Lie algebra of $\mathrm{SU}(2)$. The following theorem is crucial.

Theorem 2.5.4. *Let* $(\Pi, \mathcal{H})$ *be the finite-dimensional representation of* $\mathrm{SU}(2)$*. Then there exists a unique finite-dimensional representation* $\pi : \mathfrak{su}(2) \to \mathfrak{gl}(\mathcal{H})$ *corresponding to* Π*, such that the following identity holds for all* $X \in \mathfrak{su}(2)$*:*

$$\Pi(e^X) = e^{\pi(X)}. \tag{2.21}$$

Moreover, $\pi(gXg^{-1}) = \Pi(g)\pi(X)\Pi(g)^{-1}$ *for all* $X \in \mathfrak{su}(2)$ *and for all* $g \in \mathrm{SU}(2)$*, and* π *can be computed explicitly by the formula:*

$$\pi(X) = \left.\frac{d}{dt}\right|_{t=0} \Pi(e^{tX}) \tag{2.22}$$

for all $X \in \mathfrak{su}(2)$*.*

Theorem 2.5.4 says that every finite-dimensional representation of $\mathrm{SU}(2)$ gives rise a representation of $\mathfrak{su}(2)$. In fact, this result can be extended to any matrix Lie group homomorphism between any matrix Lie group, which, on the other hand, is itself a special case of more general theorem from theory of Lie groups. For a proof in matrix Lie group case, see [5], but the proof is based on the notion of the so-called one parameter group, which we have not defined.

In the case of SU(2), the converse of Theorem 2.5.4 is also true:

Theorem 2.5.5. *If $\pi : \mathfrak{su}(2) \to \mathfrak{gl}(\mathcal{H})$ is a representation of $\mathfrak{su}(2)$, then there exists a representation Π of $\mathrm{SU}(2)$ acting on the same space, such that Π and π are related as in Theorem 2.5.4. Moreover, while passing from π to Π (or from Π to π) in this way, equivalence and irreducibility are preserved, namely, Π is irreducible if and only if π is irreducible, and representations Π_1, Π_2 of $\mathrm{SU}(2)$ are equivalent if and only if the corresponding representations of $\mathfrak{su}(2)$ are equivalent.*

The above theorem is true due to the fact that SU(2) is connected and simply connected.

We can now use Theorem 2.5.4 to obtain the corresponding representations π_m of the Lie algebra $\mathfrak{su}(2)$. Following (2.22), for any $X \in \mathfrak{su}(2)$ we have

$$\pi_m(X)f(z) = \frac{d}{dt}\bigg|_{t=0} \Pi_m(e^{tX})f(z) = \frac{d}{dt}\bigg|_{t=0} f(e^{-tX}z).$$

For $z = (z_1, z_2)$, let $z(t) = (z_1(t), z_2(t))$ be the curve in $\mathbb{C}^2$ given by $z(t) = e^{-tX}(z)$. Using the chain rule and the fact that $dz/dt(0) = -Xz$, we see that

$$\pi_m(X)f(z_1, z_2) = -\frac{\partial}{\partial z_1} f(z_1, z_2)(X_{11}z_1 + X_{12}z_2) - \frac{\partial}{\partial z_2}(X_{21}z_1 + X_{22}z_2). \quad (2.23)$$

It is a simple matter to check that the right-hand side of (2.23) is again an element of V_m. Now, we know from Proposition 2.4.1 that every finite-dimensional representation of $\mathfrak{su}(2)$ extends uniquely to a $\mathbb{C}$-linear representation of $\mathfrak{su}^{\mathbb{C}}(2) \simeq \mathfrak{sl}(2, \mathbb{C})$. If we extend formula (2.23) to $\mathfrak{sl}(2, \mathbb{C})$, then it is clearly $\mathbb{C}$-linear representation, and this extension is unique. Thus, from now on, we regard π_m's as representations of $\mathfrak{sl}(2, \mathbb{C})$. Now, pick the following basis of $\mathfrak{sl}(2, \mathbb{C})$ (as a complex vector space):

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.24)$$

From (2.23) we obtain

$$\pi_m(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}, \quad \pi_m(X) = -z_2\frac{\partial}{\partial z_1}, \quad \pi_m(Y) = -z_1\frac{\partial}{\partial z_2}. \quad (2.25)$$

Theorem 2.5.6. *The representation π_m of $\mathfrak{sl}(2, \mathbb{C})$ is irreducible.*

Since π_m is irreducible as a representation of $\mathfrak{su}(2)$ (see Theorem 2.5.5), the proof of theorem 2.5.6 is actually a corollary from the following proposition

Proposition 2.5.1. *Let π be a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. Then π is irreducible as a representation of $\mathfrak{su}(2)$ if and only if it is irreducible as a representation of $\mathfrak{sl}(2, \mathbb{C})$.*

Proof. If π is irreducible as a representation of $\mathfrak{su}(2)$, and W is invariant under the action of $\mathfrak{sl}(2,\mathbb{C})$, then W must be invariant also under the action of $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$, so W must be trivial invariant subspace. Thus π is irreducible as a representation of $\mathfrak{sl}(2,\mathbb{C})$. Conversely, suppose that π is irreducible as a representation of $\mathfrak{sl}(2,\mathbb{C})$, and W is invariant under the action of $\mathfrak{su}(2)$. Then W will be invariant under the action of $\pi(X+iY) = \pi(X) + i\pi(Y)$ for all $X, Y \in \mathfrak{su}(2)$, but every element of $\mathfrak{sl}(2,\mathbb{C})$ is of this form, so W is again trivial. This means that π is irreducible as a representation of $\mathfrak{su}(2)$.

□

For educational reasons, we present alternative, purely algebraic proof of Theorem 2.5.6.

Proof. Applying (2.25) to basis vector $f_k(z_1, z_2) = z_1^k z_2^{m-k}$, we easily obtain

$$\begin{aligned}
\pi_m(H)f_k &= -kz_1^k z_2^{m-k} + (m-k)z_1^k z_2^{m-k} = (m-2k)z_1^k z_2^{m-k} = (m-2k)f_k,\\
\pi_m(X)f_k &= -kz_1^{k-1} z_2^{m-k+1} = -kf_{k-1},\\
\pi_m(Y)f_k &= (k-m)z_1^{k+1} z_2^{m-k-1} = (k-m)f_{k+1}.
\end{aligned} \tag{2.26}$$

Let $V \subset V_m$ be non-zero invariant subspace. There is at least one non-zero element $v = a_0 z_2^m + a_1 z_1 z_2^{m-1} + ... + a_m z_1^m$. Let k_0 be such that $a_{k_0} \neq 0$ but $a_k = 0$ for $k > k_0$. Thus $v = a_0 z_2^m + a_1 z_1 z_2^{m-1} + ... + a_{k_0} z_1^{k_0} z_2^{m-k_0}$. By (2.25) and (2.26), we see that only the last term, $a_{k_0} z_1^{k_0} z_2^{m-k_0}$, will survive the application of $\pi_m(X)^{k_0}$ to v. But $\pi_m(X)^{k_0} = k_0!\,(-1)^{k_0} a_{k_0} z_2^m$, and since V is invariant, $z_2^m \in V$. But now from (2.26) we see that we can obtain multiple of any basis vector by applying $\pi_m(Y)$ to z_2^m many times. Thus $z_1^k z_2^{m-k} \in V$ for all $0 \leq k \leq m$, and hence V is in fact the whole V_m.

□

The standard result from linear algebra tells us that every $U \in \mathrm{SU}(2)$ can be written as $U = U_0 D_\theta U_0^{-1}$, with $U_0 \in \mathrm{SU}(2)$[5] and D_θ is of the form:

$$D_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

with $\theta \in [0, 2\pi[$. Simple computation shows that $\Pi_n(D_\theta)f_k = e^{i\theta(2k-n)}f_k$ for all $0 \leq k \leq n$. Thus, since $U_0 \in \mathrm{SU}(2)$ and Π_n is unitary:

$$\begin{aligned}
\chi_{\Pi_n}(U) &= \mathrm{Tr}(\Pi_n(U)) = \mathrm{Tr}(\Pi_n(U_0)\Pi_n(D_\theta)\Pi_n(U_0)^{-1}) = \mathrm{Tr}(\Pi_n(D_\theta))\\
&= \sum_{k=0}^{n} e^{i\theta(2k-n)}.
\end{aligned} \tag{2.27}$$

[5] In general, we have $U = U_1 D_\theta U_1^{-1}$ with $U_1 \in \mathrm{U}(2)$, but if so, then $\det(U_1) = e^{i\phi}$ for some $\phi \in [0, 2\pi[$, then we can put $U_0 = e^{-i\phi/2}U_1$, and it can be checked trivially that $U_0 \in \mathrm{SU}(2)$ and $U = U_0 D_\theta U_0^{-1}$.

We know so far that V_n is irreducible as a representation, but if we consider $V_n \otimes V_k$, there is no reason to expect it to be irreducible, and if it is not irreducible, it can be decomposed into direct sum of irreducible representations. Finding this decomposition is a part of **Clebsh-Gordan theory**, and in the case of SU(2), it has significant applications in quantum mechanics.

Theorem 2.5.7. *The following decomposition holds:*

$$V_n \otimes V_k \cong \bigoplus_{i=0}^{\min\{n,k\}} V_{n+k-2i}. \tag{2.28}$$

See [7] for a proof of this fact. It is actually based on analysis of the analogous formula for characters of the form (2.27) (the character of direct sum is the sum of characters and the character of tensor product is a product of characters, and the representation is determined uniquely, up to equivalence, by its character, see Chapter 3). Now let us introduce the following notation: $s = V_{2s}$ (s is a *number and also a symbol* that labels spaces). Then, by using (2.28), one can easily check that:

$$\frac{1}{2} \otimes \frac{1}{2} \cong 1 \oplus 0. \tag{2.29}$$

The *number* s is called a **spin**, and what we have just done is called **addition of angular momentum** in quantum mechanics.

Now the time has come to deal with the converse problem. Proposition 2.5.1 tells us that, in order to determine all finite-dimensional irreducible representations of $\mathfrak{su}(2)$, we can pass to $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{su}^{\mathbb{C}}(2)$ without risk that we will lose any information. For the basis of $\mathfrak{sl}(2,\mathbb{C})$ as in (2.24), we have the following commutation relations:

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H. \tag{2.30}$$

Warning. Definitions 2.5.1 and 2.5.2 below are temporary. They are adjusted to the case of $\mathfrak{sl}(2,\mathbb{C})$ and are formulated only for the sake of this section. They are the very special case of more general definitions where roots and weights are linear functionals (See Chapter 4).

Definition 2.5.1. *A complex number $\alpha \in \mathbb{C}$ is a* ***root*** *if $\alpha \neq 0$, and there exists $Z \in \mathfrak{sl}(2,\mathbb{C}$ such that $[H,Z] = \alpha Z$. Z is called a* ***root vector*** *corresponding to α.*

The above definition says that a root is simply a non-zero eigenvalue of the linear operator $\mathrm{ad}(H)$ defined by the formula: $\mathrm{ad}(H)Z := [H,Z]$. One can easily see that the map $\mathrm{ad} : \mathfrak{sl}(2,\mathbb{C}) \ni A \mapsto \mathrm{ad}(A) \in \mathfrak{gl}(\mathfrak{sl}(2,\mathbb{C}))$ is a representation, and this representation is called **adjoint representation**. The commutation relations (2.30) tell us that we have two roots: $\alpha_1 = 2$ and $\alpha_2 = -2$, where corresponding root vectors are X and Y. In fact, it is not a coincidence that $\alpha_1 = -\alpha_2$. In particular, $\alpha_2 \in \text{span}\{\alpha_1\}$. We will call α_1 a **simple root** to emphasize this fact (of course, this choice is arbitrary). Let π be an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, acting on a finite-dimensional complex vector space V. Since π is a representation, we must have

$$[\pi(H),\pi(X)] = 2\pi(X), \quad [\pi(H),\pi(Y)] = -2\pi(Y), \quad [\pi(X),\pi(Y)] = \pi(H). \tag{2.31}$$

Now, because we are working over an algebraically close field of complex numbers, $\pi(H)$ must have at least one eigenvalue.

Definition 2.5.2. *A complex number $\mu \in \mathbb{C}$ is a **weight** for a representation π if there exists a non-zero vector u in V such that $\pi(H)u = \mu u$. u is called a **weight vector** corresponding to μ, and the space of all weight vectors of μ is called a **weight space** of μ.*

Thus a weight for π is just an eigenvalue of $\pi(H)$. Note that a root is a non-zero weight for the adjoint representation. As noted above, π has at least one weight. Denote it by μ and the corresponding weight vector by u. From the commutation relations (2.31), we obtain

$$\begin{aligned} \pi(H)\pi(X)u &= (\pi(X)\pi(H) + 2\pi(X))u = \pi(X)\pi(H)u + 2\pi(X)u = (\mu+2)\pi(X)u, \\ \pi(H)\pi(Y)u &= (\pi(Y)\pi(H) - 2\pi(Y))u = \pi(Y)\pi(H)u - 2\pi(Y)u = (\mu-2)\pi(Y)u, \end{aligned} \tag{2.32}$$

thus either $\pi(X)u$ is zero, or it is a weight vector for weight $\mu+2$, and similarly, either $\pi(Y)u$ is zero, or it is a weight vector for weight $\mu-2$. This means, for example, that $\pi(H)\pi(X)^n u = (\mu+2n)\pi(X)^n u$, so again, either $\pi(X)^n u$ is zero, or it is a weight vector for weight $\mu+2n$. We see that we can build new weights from old ones in this process. New weights are of the form $\mu+\alpha_1 n$ or $\mu+\alpha_2 n$. We introduce the following partial ordering in the set of weights:

Definition 2.5.3. *Let $\alpha_1 = 2$ be a simple root as described above, and let μ_1, μ_2 be two weights. We say that μ_1 is **higher** than μ_2 if*

$$\mu_1 - \mu_2 = t\alpha_1$$

*with $t \geq 0$. We also say that μ_2 is **lower** than μ_1, and denote this relation by $\mu_1 \succeq \mu_2$ or $\mu_2 \preceq \mu_1$. If the weight μ_0 satisfies $\mu_0 \succeq \mu$ for all weights μ of π, then μ_0 is called the **highest weight**.*

Let us go back to our analysis. There is some $n_0 \geq 0$ such that $\pi(X)^{n_0}u \neq 0$, but $\pi(X)^{n_0+i}u = 0$ for all $i > 0$. This follows from the fact that weights $\mu+2n$ are all distinct, so their weight vectors must be linearly independent, and V is assumed to be finite-dimensional. Take $u_0 = \pi(X)^{n_0}u$ and $b = a + 2n_0$. Define $u_k = \pi(Y)^k u_0$ for $k \geq 0$. By the second identity of (2.32) we have $\pi(H)u_k = (b-2k)u_k$.

Lemma 3.2.1. For $k > 0$, we have

$$\pi(X)u_k = (bk - k(k-1))u_{k-1}.$$

Proof. Using $[\pi(X), \pi(Y)] = \pi(H)$ and definition of u_1, we have $\pi(X)u_1 = \pi(X)\pi(Y)u_0 = (\pi(Y)\pi(X) + \pi(H))u_0 = bu_0$, and this is the proof in the case $k = 1$. Now, let us proceed by induction:

$$\begin{aligned}
\pi(X)u_{k+1} &= \pi(X)\pi(Y)u_k \\
&= (\pi(Y)\pi(X) + \pi(H))u_k \\
&= \pi(Y)(kb - k(k-1))u_{k-1} + (b - 2k)u_k \\
&= (kb - k(k-1) + (b - 2k))u_k \\
&= ((k+1)b - (k+1)k)u_k.
\end{aligned}$$

□

Recall that $\pi(H)u_k = (b - 2k)u_k$. But again the $(b - 2k)$'s are distinct, so u_k's cannot be all non-zero. There exists $m \geq 0$ such that $u_k = \pi(Y)^k u_0 \neq 0$ for $k \leq m$ but $u_{m+1} = 0$. Then $\pi(X)u_{m+1} = ((m+1)b - m(m+1))u_m = (m+1)(b-m)u_m$ must give 0, but $m+1 \neq 0$ and $u_m \neq 0$ by definition of m, so we must have $b = m$. Let us summarize our efforts:

Given π, a finite-dimensional, $\mathbb{C}$-linear, irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ acting on a complex space V, there exists an integer $m \geq 0$ and non-zero vectors $u_0, ..., u_m$ such that

$$\begin{aligned}
\pi(H)u_k &= (m - 2k)u_k, \\
\pi(Y)u_k &= u_{k+1}, \qquad k < m, \\
\pi(Y)u_m &= 0, \\
\pi(X)u_k &= (km - k(k-1))u_{k-1}, \qquad k > 0, \\
\pi(X)u_0 &= 0.
\end{aligned} \tag{2.33}$$

Of course, u_k's are linearly independent, and $\text{span}\{u_0, ..., u_m\}$ is invariant under $\pi(A)$ for all $A \in \mathfrak{sl}(2,\mathbb{C})$. This span must be the whole space, since π is irreducible by assumption. Moreover, one can show that (2.33) actually defines a representation of $\mathfrak{sl}(2,\mathbb{C})$, and it is irreducible. We see that there exists an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, acting on the space of dimension $m+1$, and every irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ must be of the form (2.33). It is clear that, if ψ and θ are representations of the form (2.33), acting on the spaces V and W, respectively, then V has a basis $v_0, ..., v_m$ and W has basis a $w_0, ..., w_m$, and both bases satisfy (2.33), moreover, the unitary map $\phi : V \to W$ with $\phi(v_i) = w_i$ is the (unitary) isomorphism of representations. It follows that two irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$, which have the same dimension, are equivalent. In particular, the representation π_m obtained before, and given by the formulas (2.23) and (2.25), must be equivalent to π from (2.33). By introducing the basis $u_k(z_1, z_2) = \pi(Y)^k(z_2^m) = (-1)^k \frac{m!}{(m-k)!} z_1^k z_2^{m-k} = (-1)^k \frac{m!}{(m-k)!} f_k$, one can easily show, by straightforward computations, that this is indeed the case. In other words, we have proven the following lemma, which is a special case of the so-called classification theorem:

Lemma 2.5.8.

1. *Every ($\mathbb{C}$-linear) finite-dimensional irreducible representation of* $\mathfrak{sl}(2,\mathbb{C})$ *has a unique highest weight* μ_0*, and two equivalent irreducible representations have the same highest weight. The highets weight of* π *described by (2.33) is equal to* m*.*

2. *Two irreducible representations of* $\mathfrak{sl}(2,\mathbb{C})$ *with the same highest weight are equivalent.*

3. *The highest weight of an irreducible representation of* $\mathfrak{sl}(2,\mathbb{C})$ *is non-negative integer.*

4. *If* m *is non-negative integer, then there exists a unique, up to isomorphism, finite-dimensional irreducible representation of* $\mathfrak{sl}(2,\mathbb{C})$ *with the highest weight* $\mu_0 = m$*.*

□

In light of Propositions 2.4.1 and 2.5.1, we have found all finite-dimensional irreducible representations of $\mathfrak{su}(2)$. Moreover, Theorem 2.5.5 tells us that these are in one-to one correspondence with the complex irreducible representations of SU(2), because SU(2) is connected and simply connected. This is another proof of the fact that Π_m's are all irreducible representations of SU(2), and every finite-dimensional representation of SU(2) is equivalent to one of the Π_m's.

Chapter 3

Basic representation theory of compact groups

In this chapter we focus on compact groups. We begin with giving basic definitions of topological group, locally compact group and compact group. We will see that, among topological groups, compact groups are very special from the point of view of the theory of representations due to the fact that the Haar measure is finite on such a group. Intuitively, as far as representations are concerned, a compact group behaves a bit like a finite group (with the natural counting measure defined on it). We explain why restriction to unitary representations is justified in the case of compact groups and we formulate theorems concerning decomposition into direct sums of irreducible representations and orthogonality relations for characters and matrix elements of representations. We end this chapter with some comments and definitions on Lie groups, Lie algebras and compact Lie groups. In our presentation we mainly follow [6].

3.1 Compact groups and their representations

We start with the definition of a topological group.

Definition 3.1.1. *A **topological group** G is a Hausdorff topological space which is also a group, such that the product operation $G \times G \to G$ and taking an inverse $G \to G$ are continuous.*

In the above definition, $G \times G$ is the topological space with the so-called **product topology**, that is, the weakest topology for which the projections $G \times G \ni (g_1, g_2) \mapsto g_1 \in G$ and $G \times G \ni (g_1, g_2) \mapsto g_1 \in G$ are continuous. In this topology, open sets are unions of the sets of the form $O_1 \times O_2$, where O_1, O_2 are open subsets of G.

Let us now give a temporary definition of a finite-dimensional representation of a topological group. Later we will see that, if the topological group under interest is compact, it is desirable to restrict attention to unitary representations, but at this very moment by **finite-dimensional representation** $(\Pi, \mathcal{H})$ of a topological group G we mean the continuous group homomorphism $\Pi : G \to \mathrm{GL}(\mathcal{H})$, where $\mathrm{GL}(\mathcal{H})$ is a group of invertible linear transformations on a finite-dimensional complex Hilbert space $\mathcal{H}$.

Although the following will not be used directly, it is a deep and useful result. Recall that a subgroup N of a group G is called **normal** if and only if $gng^{-1} \in N$ for all $n \in N$ and all $g \in G$.

Theorem 3.1.1. *If H is a closed, normal subgroup of a topological group G then G/H has a natural structure of a topological group. Moreover, any open subgroup of topological group is also a closed subgroup.*

Definition 3.1.2. *H is called a **discrete subgroup** of G if there is an open cover $\mathcal{O}$ of G such that every $O \in \mathcal{O}$ contains exactly one element of H.*

Definition 3.1.3. *A **locally compact group** G is a topological group G for which the underlying topology is locally compact.*

Locally compact topology means that every point $x \in G$ has compact neighbourhood, i.e. there exists an open set U and a compact set K such that $x \in U \subseteq K$.

Definition 3.1.4. *A **compact group** G is a topological group which is compact as a topological space, that is, for every open cover of G there exists a finite subcover.*

The following theorem deals with the most important (after a compact group) object in our presentation in this chapter.

Theorem 3.1.2. *Let G be a locally compact group. Then there exists a non-zero, regular, Borel measure μ_G on the Borel $\sigma-$algebra in G, called the **Haar measure**, which is left-translation invariant, that is, for all $g \in G$ and all Borel subset $E \subset G$, we have $\mu_G(gE) = \mu_G(E)$, and which is finite on every compact subset of G.*

The Haar measure defined above always exists for locally compact groups. But in the case of compact groups (which are locally compact by definition), the Haar measure has additional properties:

Theorem 3.1.3. *Let G be a compact group. Then the Haar measure on G, μ_G, is also right-translation invariant, that is, for all $g \in G$ and all Borel subset $E \subset G$, we have $\mu_G(Eg) = \mu_G(E)$, and it is unique up to multiplication by a constant, that is, if ν is another Haar measure on G, then for all Borel sets $E \subset G$ we have $\mu(E) = a\nu(E)$ for some $a > 0$.*

Theorem 3.1.2 implies that, if G is the compact group, then $\mu(G) < \infty$, and in particular it can be normalized so that $\mu(G) = 1$. Since Theorem 3.1.3 implies that Haar measure is unique up to multiplication by a constant for a compact group, it follows that the normalized Haar measure μ is unique in this case.

Example. The Haar measure on SU(2). Recall that SU(2) is homeomorphic to 3-dimensional sphere $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ and if $U \in \mathrm{SU}(2)$, then U may be written in the form:

$$U = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

We can choose the following parametrization of S^3:

$$\begin{aligned} x_1 &= \cos(\theta), \\ x_2 &= \sin(\theta)\cos(\phi), \\ x_3 &= \sin(\theta)\sin(\phi)\cos(\psi), \\ x_4 &= \sin(\theta)\sin(\phi)\sin(\psi), \end{aligned} \tag{3.1}$$

where $\theta, \phi \in [0, \pi]$ and $\psi \in [0, 2\pi[$. One can easily check that the Jacobian of the reparametrization (3.1) is $\sin^2(\theta)\sin(\psi)$, and we can define

$$\int_{\mathrm{SU}(2)} f(U) d\mu_{\mathrm{SU}(2)}(U) = \frac{1}{2\pi^2} \int_0^{\pi} d\theta \int_0^{\pi} d\phi \int_0^{2\pi} d\psi f(\theta, \phi, \psi) \sin^2(\theta)\sin(\psi) \tag{3.2}$$

for a function $f : \mathrm{SU}(2) \to \mathbb{C}$. One can show that $\mu_{\mathrm{SU}(2)}$ has all the properties of the Haar measure and $\mu_{\mathrm{SU}(2)}(\mathrm{SU}(2)) = 1$.

If G is a compact group, we have

$$\int_G f(U_0 U) d\mu_G(U) = \int_G f(U) d\mu_G(U) \tag{3.3}$$

and also

$$\int_G f(U U_0) d\mu_G(U) = \int_G f(U) d\mu_G(U) \tag{3.4}$$

for any $U_0 \in G$ and we can consider $\mathrm{L}^p(G, d\mu_G)$ spaces for $p \in [1, \infty[$.

Now, let $(\Pi, \mathcal{H})$ be a finite-dimensional representation of a compact group G. Pick any inner product on $\mathcal{H}$ and denote it by $\langle \cdot | \cdot \rangle_{\mathcal{H}}$. For $v_1, v_2 \in \mathcal{H}$ define

$$\langle v_1 | v_2 \rangle_G := \int_G \langle \Pi(U) v_1 | \Pi(U) v_2 \rangle_{\mathcal{H}} d\mu_G(U). \tag{3.5}$$

One can easily see that (3.5) is well defined because μ_G is finite, and that it is actually the inner product on $\mathcal{H}$. Now, take $U_0 \in G$. We have:

$$\begin{aligned} \langle \Pi(U_0) v_1 | \Pi(U_0) v_2 \rangle_G &= \int_G \langle \Pi(U)\Pi(U_0) v_1 | \Pi(U)\Pi(U_0) v_2 \rangle_{\mathcal{H}} d\mu_G(U) \\ &= \int_G \langle \Pi(U U_0) v_1 | \Pi(U U_0) v_2 \rangle_{\mathcal{H}} d\mu_G(U) \\ &= \int_G \langle \Pi(U) v_1 | \Pi(U) v_2 \rangle_{\mathcal{H}} d\mu_G(U) \\ &= \langle v_1 | v_2 \rangle_G \end{aligned} \tag{3.6}$$

since Π is a representation and μ_G is right-invariant. We see that $\Pi(U_0)$ is unitary with respect to the inner product $\langle\cdot|\cdot\rangle_G$ for any $U_0 \in G$. That is, every representation of a topological group G can be made into unitary representation if G is compact group. This motivates:

Definition 3.1.5. *Let G be a compact group. A* ***finite-dimensional representation*** *of G is a continuous group homomorphism Π of G to the group of unitary maps $U(\mathcal{H})$ on some finite-dimensional, complex Hilbert space $\mathcal{H}$. Two representations Π and Ψ acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, are said to be* ***equivalent*** *if and only if there is a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for all $g \in G$ we have $\Psi(g) = U\Pi(g)U^{-1}$. U is called* ***isomorphism*** *of representations Π and Ψ, and we write $\Pi \cong \Psi$. The definition of irreducibility of representation is the same as in Definition 2.4.3.*

In Chapter 2, we restricted ourselves to unitary representations of SU(2) because it is compact as a topological group, and the first half of analysis of representations of SU(2) in Chapter 2 was actually a special case of analysis we are doing here. In particular, we have the general definition of complete reducibility of representations and the theorem concerning complete reducibility of general compact groups:

Definition 3.1.6. *A finite-dimensional representation of a topological group G acting on $\mathcal{H}$ is called* ***completely reducible****, if given a invariant subspace W of $\mathcal{H}$, there is an invariant subspace $U \subset \mathcal{H}$ such that $\mathcal{H} = W \oplus U$.*

Theorem 3.1.4.

1. *Let G be a compact group and let Π be a finite-dimensional representation of G acting on $\mathcal{H}$. Then Π is completely reducible.*

2. *A finite-dimensional completely reducible representation of a topological group is equivalent to the direct sum of irreducible representations.*

The proof of 1. is the same as the proof of Theorem 2.4.1, because the only property that is used is that Π being finite-dimensional unitary representation. The above theorem implies that every finite-dimensional representation of a compact group is equivalent to the direct sum of irreducible representations.

Definition 3.1.7. *Given a finite-dimensional representation Π of a compact group G acting on $\mathcal{H}$, we define the* ***character*** *$\chi_\Pi : G \to \mathbb{C}$ of Π by the formula*

$$\chi_\Pi(g) := \mathrm{Tr}(\Pi(g)) \tag{3.7}$$

for $g \in G$.

It can be shown that, for two representations Π and Ψ, we have $\chi_{\Pi\oplus\Psi}(g) = \mathrm{Tr}(\Pi(g)) + \mathrm{Tr}(\Psi(g))$ and $\chi_{\Pi\otimes\Psi}(g) = \mathrm{Tr}(\Pi(g))\mathrm{Tr}(\Psi(g))$. If we denote the entries of matrix $\Pi(g)$ in some orthonormal basis of $\mathcal{H}$ (that is, $D^\Pi_{ij}(g) = \langle e_i|\Pi(g)e_j\rangle$ where

$\{e_i\}$ is orthonormal basis of $\mathcal{H}$) by $D^{\Pi}_{ij}(g)$, where $1 \leq i, j \leq \dim(\Pi) = \dim_{\mathbb{C}}(\mathcal{H})$, we see that

$$\chi_{\Pi}(g) = \mathrm{Tr}(\Pi(g)) = \sum_{i=1}^{\dim(\Pi)} D^{\Pi}_{ii}(g). \tag{3.8}$$

Now, take two equivalent representations of G, say, Π and Ψ, acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. By definition, there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for all $g \in G$ we have $\Psi(g) = U\Pi(g)U^{-1}$. But then

$$\chi_{\Psi}(g) = \mathrm{Tr}(\Psi(g)) = \mathrm{Tr}(U\Pi(g)U^{-1}) = \mathrm{Tr}(\Pi(g)) = \chi_{\Pi}(g) \tag{3.9}$$

since the trace is invariant under unitary transformation. We see that two equivalent representations have the same character. Moreover, since we are dealing with a compact group, we know that every finite-dimensional representation is equivalent to the direct sum of irreducible representations. Thus, from now on, we can pay attention only on finite-dimensional irreducible representations.

Suppose Irreps(G) is a set of irreducible representations of a compact group G. We do not know yet if these even exist, but let us forget about technicalities for a moment. Since we have the notion of equivalence of representations, we can consider classes of equivalent, irreducible representations, namely, for $\theta \in \mathrm{Irreps}(G)$ define $[\theta] = \{\pi \in \mathrm{Irreps}(G) \mid \theta \cong \pi\}$. It is easy to see that "$\cong$" is an equivalence relation. For any $\Pi, \Psi \in \mathrm{Irreps}(G)$ we thus have $[\Pi] = [\Psi]$ or $[\Pi] \cap [\Psi] = \emptyset$.

Definition 3.1.8. *Define* $\widehat{G} := \mathrm{Irreps}(G)/\cong$.

Thus, $\widehat{G}$ is a set of classes of equivalent irreducible representations. We can consider $\chi_{[\Pi]}$ to be the character of any irreducible representation $\Phi \in [\Pi]$, since in light of (3.9) the character depends only on $[\Pi]$, not on the choice of representatives.

Theorem 3.1.5. *For all* $[\Pi], [\Psi] \in \widehat{G}$ *we have*

$$\int_G \overline{\chi_{[\Pi]}}(U)\chi_{[\Psi]}(U)d\mu_G(U) = \delta_{[\Pi],[\Psi]}, \tag{3.10}$$

where $\delta_{[\Pi],[\Psi]} = 1$ *if* $[\Pi] = [\Psi]$ *(or equivalently, if* $\Pi \simeq \Psi$*), and* $\delta_{[\Pi],[\Psi]} = 1$ *if* $[\Pi] \cap [\Psi] = \emptyset$ *(that is, if* $\Pi \ncong \Psi$*).*

We can take $D^{[\Pi]}_{ij}(g)$ to be any matrix realization of any irreducible representation $\Phi \in [\Pi]$. Of course, equivalent representations must have the same dimensions, so define $\dim([\Pi]) = \dim(\Psi)$ for any $\Psi \in [\Pi]$. Thus, in the case of a matrix realization $D^{[\Pi]}_{ij}(g)$ we have $1 \leq i, j \leq \dim([\Pi])$. From (3.9) we obtain

$$\chi_{[\Pi]}(g) = \sum_{i=1}^{\dim([\Pi])} D^{[\Pi]}_{ii}(g). \tag{3.11}$$

Moreover, we have the following theorem

Theorem 3.1.6. *For all $[\Pi], [\Psi] \in \widehat{G}$ we have*

$$\int_G \overline{D_{ij}^{[\Pi]}}(U) D_{kl}^{[\Psi]}(U) d\mu_G(U) = \frac{\delta_{[\Pi],[\Psi]}\delta_{i,k}\delta_{j,l}}{\dim([\Pi])}. \tag{3.12}$$

Relations (3.10) and (3.12) are called **orthogonality relations** for characters and matrix elements of representations.

Let Ψ be any finite-dimensional representation of a compact group G. We know that Ψ is equivalent to the direct sum of irreducible representations. We write

$$\Psi \cong \bigoplus_{[\Pi]\in\widehat{G}} \bigoplus^{N_{[\Pi]}} \Pi. \tag{3.13}$$

It may happen that $N_{[\Pi_0]} = 0$ for some irreducible Π_0, that means there is no representation equivalent to Π_0 in decomposition (3.13). The sum (3.13) runs over all possible classes from $\widehat{G}$. We have the following, very important and useful theorem.

Theorem 3.1.7.

$$N_{[\Pi]} = \int_G \overline{\chi_{[\Pi]}}(U)\chi_\Psi(U) d\mu_G(U), \tag{3.14}$$

where χ_Ψ is the character of Ψ given by (3.13).

It turns out that decomposition (3.13) is unique up to equivalence, and

Corollary 3.1.1. *Any finite-dimensional representation Ψ of a compact group G is uniquely determined, up to equivalence, by its character.*

At the end of this section, let us give more result from abstract theory of representations of compact groups, namely, the Peter-Weyl theorem and three additional theorems which proofs are based on Peter-Weyl theorem, see [6].

Theorem 3.1.8. ***(Peter-Weyl)*** *Let G be a compact group. The set of finite linear combinations*

$$\{D_{ij}^{[\Pi]}(U)\}_{[\Pi]\in\widehat{G}, 1\leq i,j\leq \dim([\Pi])}$$

is dense in $\|\cdot\|_\infty$ norm in $C(G)$ (the space of continuous functions on G).

Just as in the case of SU(2), which is an example of compact group, we have

Definition 3.1.9. *The **infinite-dimensional representation** of a compact group G is the strongly continuous (see footnote attached to Definition 2.4.9 in Chapter 1) group homomorphism $\Pi : G \to U(\mathcal{H})$, where $\mathcal{H}$ is infinite-dimensional, complex, separable Hilbert space and $U(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$.*

The definitions of invariant subspaces and irreducibility are almost the same as in the finite case, but with the exception that we require an invariant subspace to be *closed* subspace of $\mathcal{H}$.

Theorem 3.1.9. *Let G be a compact group.*

1. *The set*
$$\{\sqrt{\dim([\Pi])}D^{[\Pi]}_{ij}(U)\}_{[\Pi]\in\widehat{G},1\leq i,j\leq\dim([\Pi])}$$
is an orthonormal basis of $\mathrm{L}^2(G, d\mu_G)$*;*

2. *The set*
$$\{\chi_{[\Pi]}\}_{[\Pi]\in\widehat{G}}$$
is an orthonormal basis of the subspace $\{f \in \mathrm{L}^2(G, d\mu_G) \mid f(U_1U_2U_1^{-1}) = f(U_2)$ *for all* U_1 *and almost every* U_2 *with respect to* $\mu_G\}$*;*

3. *Let $\mathcal{H}$ be a separable, complex Hilbert space, and Ψ be a unitary representation of G, acting on $\mathcal{H}$. Then Ψ is equivalent to direct sum of finite-dimensional irreducible representations. If $\mathcal{H}$ is infinite-dimensional, then the direct sum is infinite.*

In particular, 3. implies that if G is a compact group, then every irreducible representation of G must be finite-dimensional: there simply does not exists infinite-dimensional irreducible representation of G. It follows that the Π_m's, the finite-dimensional irreducible representations of SU(2) found in the previous chapter, are all irreducible representations of SU(2), up to equivalence.

3.2 Lie groups

We now turn to the specific class of topological groups, namely Lie groups.

Definition 3.2.1. *A **Lie group** is a topological group G which is also a C^∞-manifold and such that the product operation and taking an inverse are smooth (C^∞) maps.*

We use the terms "C^∞" and "smooth" interchangeably.

We hope that the reader is familiar with basics of differential geometry and theory of differential manifolds, and we will not give any recap of these topics here. Nevertheless, it is important to mention that, for any Lie group G, there exists a real vector space $\mathfrak{g}$ consisting of left invariant vector fields on G (a vector field X on G is called left-invariant if and only if for all $g \in G$ we have $X(gh) = (L_g)_*(X(h))$ with $L_g(h) = gh$). This real vector space is closed under the vector field commutator, and its dimension (as a real vector space) is equal to the dimension of G (as a C^∞-manifold). Moreover, we have

$$\mathfrak{g} \cong T_e(G), \tag{3.15}$$

where $T_e(G)$ is a tangent space of G at the point e - the group identity of G.

Definition 3.2.2. $\mathfrak{g}$ *defined above is called the* ***Lie algebra*** *of a Lie group* G.

Definition 3.2.3. *If* G, H *are two Lie groups with Lie algebras* $\mathfrak{g}, \mathfrak{h}$*, respectively, then we define a* ***Lie group homomorphism*** $\Phi : G \to H$ *to be a continuous group homomorphism. A Lie group homomorphism which is a bijection is called a* ***Lie group isomorphism****. If there exists an isomorphism between Lie groups* G *and* H*, then we say that* G *and* H *are isomorphic and this property is denoted by* $G \cong H$*. We define Lie algebra homomorphism for Lie algebras of general Lie groups in the same way as in Definition 2.3.3.*

It turns out that, if a group homomorphism between two Lie groups is continuous (that is, it is a Lie group homomorphism), then it is also smooth. Thus, we have only two classes of homomorphisms between Lie groups: discontinuous and smooth ones.

We will now give the definition of a general Lie algebra, without explicit reference to a Lie group.

Definition 3.2.4. *A* ***finite dimensional real (complex) Lie algebra*** *is a finite dimensional, real (complex) vector space, denoted* $\mathfrak{g}$*, with an additional product operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *(one often calls* $[\cdot,\cdot]$ *a bracket), which has the following properties:*

1. $[\cdot,\cdot]$ *is bilinear (*$\mathbb{R}$ *-bilinear if* $\mathfrak{g}$ *is real and* $\mathbb{C}$ *-bilinear if* $\mathfrak{g}$ *is complex);*
2. $[X, Y] = -[Y, X]$ *for all* $X, Y \in \mathfrak{g}$.
3. *the Jacobi identity holds:* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

We define the real or complex **subalgebra** of $\mathfrak{g}$ as a real or complex subspace of $\mathfrak{g}$ which is closed under the bracket. Of course, a subalgebra of a Lie algebra is itself a Lie algebra. There is a result which says that every finite-dimensional Lie algebra is isomorphic to a Lie algebra of a Lie group.

The following proposition is extremely important.

Proposition 3.2.1. *Every matrix Lie group is a Lie group. Every matrix Lie group homomorphism is a Lie group homomorphism. A Lie algebra of a Lie group (Definition 3.2.2) is a real Lie algebra in the sense of Definition 3.2.4, and a Lie algebra of a matrix Lie group is (isomorphic to) a Lie algebra in the sense of Definition 3.2.2.*

We introduced the notion of "matrix Lie group" in Chapter 1 to emphasize the fact that every matrix Lie group is a Lie group, as stated in the above proposition, although this is not obvious and requires proof. We will not prove this here, but notice that a matrix Lie group is defined to be a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, and $\mathrm{GL}(n, \mathbb{C})$ is an open subset of $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. It turns out that a matrix Lie group is a manifold embedded in some $\mathbb{R}^m$. Note also that $\mathfrak{su}^{\mathbb{C}}(2)$ is a complex Lie algebra.

Another reason for introducing notions of matrix Lie group and its Lie algebra, without explicit reference to manifold theory, is that at the level of matrix Lie groups everything is much easier, especially when it comes to the Lie algebra, which is defined simply via the exponential mapping being standard power series.

For a compact Lie group G, it is known that the Haar measure is induced from left-invariant volume form on G. This volume form is unique up to multiplication by a constant.

One of the most important objects in the study of Lie groups are tori, the groups which are (isomorphic to) products of circle groups.

Definition 3.2.5. *Let G be a compact Lie group. A torus T is an abelian subgroup of G which is connected and compact. A maximal torus is a maximal (in the sense of inclusion) subgroup with these properties.*

It turns out that a compact abelian (the product is commutative) connected Lie group is necessarily a torus. It is a consequence of classification of compact Lie groups obtained in the works of E. Cartan [4] and H. Weyl [3].

Example. Let $G = \mathrm{GL}(n, \mathbb{K})$ or $G = \mathrm{SL}(n, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let B^+ and B^- be groups of upper- and lower-triangular matrices with entries in $\mathbb{K}$, respectively. Their intersection, $B^+ \cap B^-$, is a maximal torus - the subgroup of diagonal matrices.

Once the notion of a Lie group was established, we can now introduce a slightly more advanced concept of a complex Lie group.

Definition 3.2.6. *The **complex Lie group** G is a group which is also a complex-analytic manifold, such that the maps $G \ni g \mapsto g^{-1} \in G$ and $G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$ are holomorphic.*

Example. The group $\mathrm{SL}(2, \mathbb{C})$, viewed as a subset of $\mathbb{C}^4$, is a complex Lie group.

Chapter 4

Basic theory of representations of semisimple Lie groups and Borel-Weil theorem

In this chapter we present Borel-Weil theorem, which gives a geometric construction of irreducible representations of compact connected Lie groups. However, in order to formulate and prove it, one needs a significant background in representation theory. Thus Borel-Weil theorem will be presented in the last section of this chapter and in the first three sections we will recall the necessary definitions and facts concerning representation theory. In the first section we present the most basic notions such as that of an ideal or complexification. These notions will be used throughout the rest of the paper, although the more experienced reader may skip reading this section. In the second and third sections we present some of the most important properties of semisimple and compact Lie groups and semisimple Lie algebras. In particular, we present definitions of important decompositions, such as Cartan decomposition, and the definitions of weights and roots. Finally, in the last section we formulate Borel-Weil theorem and present one of its proofs. We also mention some of the generalizations of this theorem.

4.1 Basic terminology

The only purpose of this section is to recall some definitions which will be widely used in this chapter. They are absolutely elementary, and if the reader is familiar with notions of simplicity, semisimplicity, ideals, etc., they can skip it and go to the next section.

Definition 4.1.1. *Let $\mathfrak{g}$ be a Lie algebra of a Lie group G. An* ***ideal*** *of $\mathfrak{g}$, denoted $\mathfrak{i}$, is a subalgebra of $\mathfrak{g}$ with $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$. The* ***commutator series*** *(or* ***central series****) of $\mathfrak{g}$ is the non-increasing (in the sense of inclusion) sequence of ideals $\mathfrak{g}^i$ with $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]$. The* ***lower central series*** *of $\mathfrak{g}$ is the non-increasing sequence of ideals $\mathfrak{g}_j$ with $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j]$. We say $\mathfrak{g}$ is* ***solvable*** *if its cummutator series ends in 0,* ***nilpotent*** *if its lower central series ends in 0, and* ***abelian*** *if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$. A Lie algebra $\mathfrak{g}$ is called* ***simple*** *if it is* *nonabelian and has no proper non-zero ideals, and* ***semisimple*** *if it is a direct sum of simple Lie algebras. We say that Lie group G is* ***semisimple*** *if its Lie algebra $\mathfrak{g}$ is semisimple, and G is said to be* ***simple*** *if its Lie algebra $\mathfrak{g}$ is simple.*

In particular, $\mathrm{SU}(2)$ and $\mathrm{SL}(2, \mathbb{C})$ are semisimple Lie groups.

Definition 4.1.2. *We say that a Lie group G is* ***linear*** *if it is (isomorphic to) a subgroup of* $\mathrm{GL}(n, \mathbb{C})$.

Definition 4.1.3. *Let G be a semisimple Lie group, and suppose that $G^{\mathbb{C}}$ is a complex semisimple Lie group such that G is a Lie subgroup of $G^{\mathbb{C}}$ and the Lie algebra of $G^{\mathbb{C}}$ is the complexification of the Lie algebra of G. Then we say that $G^{\mathbb{C}}$ is a complexification of G.*

For example, $\mathrm{SU}(n)$ and $\mathrm{SL}(n, \mathbb{R})$ both have $\mathrm{SL}(n, \mathbb{C})$ as a complexification.

It is worth noting that not every real Lie group has a complexification. A connected semisimple Lie group (defined in the following definition) has a complexification if and only if it is linear. On the other hand every compact Lie group has a complexification.

Proposition 4.1.1. *Let $\mathfrak{g}$ be a Lie algebra. There exists a unique maximal solvable ideal, called* ***radical****.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two solvable ideals of $\mathfrak{g}$. Then $\mathfrak{a} + \mathfrak{b}$ is again an ideal of $\mathfrak{g}$, and it is solvable because it is an extension of $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ by $\mathfrak{a}$. Now consider the sum of all the solvable ideals of $\mathfrak{g}$. It is nonempty since $\{0\}$ is a solvable ideal, and it is solvable ideal by the sum property just derived. Clearly it is unique maximal solvable ideal. □

There is an equivalent definition of semisimple Lie algebra; a Lie algebra $\mathfrak{g}$ is semisimple if the radical of $\mathfrak{g}$ is zero.

Definition 4.1.4. *A* ***linear connected reductive group*** *is a closed connected group of matrices that is closed under hermitian conjugation. A* ***linear connected semisimple group*** *is a linear connected reductive group with finite center. We call a Lie group* ***reductive*** *if its Lie algebra $\mathfrak{g}$ is reductive, i.e. it is a direct sum of a semisimple Lie algebra $\mathfrak{s}$ and an abelian Lie algebra $\mathfrak{a}$, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$.*

Every group mentioned in this paper is reductive, but for the sake of completeness, we will sometimes stress that the groups are reductive. We will later show that any compact connected Lie group can be realized as a linear connected reductive Lie group.

4.2 Some of the most important notions

Definition 4.2.1. *The* ***Killing form*** *on* $\mathfrak{g}$ *is the symmetric bilinear form* $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ *given by*

$$B(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)), \tag{4.1}$$

for $(x, y) \in \mathfrak{g} \times \mathfrak{g}$*. Here and in the rest of this paper,* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is the adjoint representation,* $\mathrm{ad}(x)(y) = [x, y]$ *for* $x, y \in \mathfrak{g}$*.*

The Killing form appears often in the theory of Lie algebras, for example in the following useful theorems:

Theorem 4.2.1. *Let* $\mathfrak{g}$ *be a Lie algebra and* $B(x, y)$ *a Killing form. Then we have:*

1. ***Cartan's criterion for solvability:***
   $\mathfrak{g}$ *is solvable if and only if* $B(x, y) = 0$*, for all* $x \in \mathfrak{g}$ *and* $y \in [\mathfrak{g}, \mathfrak{g}]$*.*

2. ***Cartan's criterion for semisimplicity:***
   $\mathfrak{g}$ *is semisimple if and only if the Killing form on* $\mathfrak{g}$ *is nondegenerate.*

There are many ways to decompose Lie groups into products of its subgroups and such decompositions are useful tool in the study of Lie groups. One of them, namely the Iwasawa decomposition, is explicitly used in a proof of Borel-Weil theorem - the central theorem of this chapter. We begin, however, with Cartan decomposition of semisimple Lie algebras.

Definition 4.2.2. *An involution (i.e. an automorphism of the Lie algebra with square equal to identity)* $\tilde{\theta}$ *of a real semisimple Lie algebra* $\mathfrak{g}$ *such that the symmetric bilinear form*

$$B_{\tilde{\theta}}(X, Y) = -B(X, \tilde{\theta}Y) \tag{4.2}$$

is positive definite is called a Cartan involution.

The map $\theta(X) = -X^*$ is the Cartan involution. To see that it respects brackets, we can simply write

$$\theta[X, Y] = -[X, Y]^* = -[Y, X]^* = [-X^*, -Y^*] = [\theta(X), \theta(Y)].$$

One can check that any real semisimple Lie algebra has a Cartan involution and that the Cartan involution is unique up to inner automorphism. As a consequence of the proof, one can also obtain a converse: Every real semisimple Lie algebra can be realized as a Lie algebra of real matrices closed under transpose.
The Cartan involution θ of a real semisimple Lie algebra $\mathfrak{g}$ yields an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{4.3}$$

of $\mathfrak{g}$ into $+1$ and -1 eigenspaces. Since θ is an automorphism, these space must bracket according to the following rules

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Definition 4.2.3. *The decomposition (4.3) is called the Cartan decomposition.*

We now have the following important lemma:

Lemma 4.2.2. *If $\mathfrak{g}$ is a real semisimple Lie algebra and θ is a Cartan involution, then*

$$(\mathrm{ad}X)^* = -\mathrm{ad}(\theta X) \quad \text{for all } X \in \mathfrak{g},$$

where $(\cdot)^*$ is defined relative to the inner product B_θ.

Proof. We have

$$\begin{aligned} B_\theta((\mathrm{ad}(\theta X))Y, Z) &= -B([\theta X, Y], \theta Z) \\ &= B(Y, [\theta X, \theta Z]) = B(Y, \theta[X, Z]) \\ &= -B_\theta(Y, (\mathrm{ad}X)Z) = -B_\theta((\mathrm{ad}X)^*Y, Z). \end{aligned}$$

□

Now we will present the theorem, which collects basic properties of semisimple Lie groups. The proof of this theorem is not difficult but we shall omit it, because it is fairly long.

Theorem 4.2.3 ([2])**.** *Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, and let K be the Lie subgroup of G with Lie algebra $\mathfrak{k}$. Then*

1. *there exists a Lie group automorphism Θ of G with differential θ, and $\Theta^2 = \mathrm{id}_G$.*
2. *the subgroup of G fixed by Θ is K.*
3. *the mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp(X)$ is a diffeomorphism onto.*
4. *K is closed.*
5. *K contains the center Z of G.*
6. *K is compact if and only if Z is finite.*
7. *when Z is finite, K is a maximal compact subgroup of G.*

We now let B be any nondegenerate symmetric invariant bilinear form on semisimple Lie algebra $\mathfrak{g}$ such that $B(X, Y) = B(\theta X, \theta Y)$, for all X and Y in $\mathfrak{g}$, and such that B_θ defined in terms of 4.2 is positive definite. The form is invariant in the sense that

$$B((\mathrm{ad}X)Y, Z) = -B(Y, (\mathrm{ad}X)Z) \tag{4.4}$$

for all X,Y and Z in $\mathfrak{g}$. The alternative way of writing 4.4 is

$$B([X, Y], Z) = B(X, [Y, Z]).$$

One of the possible choices for B is the Killing form.

Proposition 4.2.1 ([1]). ***(Iwasawa decomposition of Lie algebra)*** *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition. Then $\mathfrak{g}$ is a vector-space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is solvable, and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.*

This result can be lifted to the Lie groups and so we have:

Theorem 4.2.4 ([1]). ***(Iwasawa decomposition)*** *Let G be a semisimple Lie group, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G, and let A and N be the Lie subgroups of G with corresponding Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. Then the map $K \times A \times N \to G$ given by $(k, a, n) \to kan$ is a diffeomorphism onto. The groups A and N are simply connected.*

Example. Let $G = \mathrm{SL}(m, \mathbb{C})$. The group K from the Cartan decomposition is $\mathrm{SU}(m)$. Let A be the subgroup of G of diagonal matrices with positive diagonal entries, and let N be the group of upper-triangular matrices with 1 in each diagonal entry. The Iwasawa decomposition is $G = KAN$ in the sense that multiplication $K \times A \times N \to G$ is a diffeomorphism onto. One can think of the Gram-Schmidt orthogonalization process in linear algebra as a prototype of Iwasawa decomposition. To see that Iwasawa decompostion of $\mathrm{SL}(m, \mathbb{C})$ amounts to the Gram-Schmidt orthogonalization process, let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{C}^m$. Form the basis $\{ge_1, \ldots, ge_m\}$, for fixed $g \in G$. The Gram-Schmidt process yields the orthonormal basis $\{v_1, \ldots, v_m\}$ such that

$$\begin{aligned} \mathrm{span}\{ge_1, \ldots, ge_j\} &= \mathrm{span}\{v_1, \ldots, v_j\} \\ v_j &\in \mathbb{R}^+(ge_j) + \{v_1, \ldots, v_{j-1}\} \end{aligned}$$

for $1 \leq j \leq m$. Define a matrix $k \in \mathrm{U}(n)$ by $k^{-1}v_j = e_j$. Then $k^{-1}g$ is upper-triangular with positive diagonal entries. Since g has determinant 1 and k has determinant of modulus 1, k must have a determinant 1. Then k is in $K = \mathrm{SU}(m)$, $k^{-1}g$ is in AN and $g = k(k^{-1}g)$ exhibits g as in $K(AN)$. This proves that $K \times A \times N \to G$ is onto. It is $1 - 1$ since $K \cap AN = \{1\}$, and the inverse is smooth because of the explicit formulas for the Gram-Schmidt process.

Definition 4.2.4. *A* ***Borel subgroup*** *of a Lie group G is a maximal connected solvable closed subgroup of G.*

Example. Let $G = \mathrm{SL}(n, \mathbb{C})$. Then the Borel subgroup can be taken to be the subgroup B^+ or B^- of the upper- and lower-triangular matrices in G, respectively.

Definition 4.2.5. ***Cartan subalgebra*** *$\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra of Lie algebra $\mathfrak{g}$. Such subalgebra exists and contains $Z(\mathfrak{g})$ - the center of $\mathfrak{g}$. A* ***Cartan subgroup*** *T of G is a maximal connected abelian subgroup. Such subgroups are closed and are exactly the Lie subgroups of G which correspond to Cartan subalgebras.*

Example.

1. $T \hookrightarrow \mathrm{U}(n)$ is the group $\mathrm{U}(1)^n \hookrightarrow \mathrm{U}(n)$ of diagonal matrices with entries of absolute value one.
2. $T \hookrightarrow \mathrm{SU}(n+1)$ is $\mathrm{U}(1)^n \hookrightarrow \mathrm{SU}(n+1)$.

The notions of the Cartan subalgebra and Cartan subgroup play major role in the classification of compact Lie groups.

Recall that the **centralizer** of a subset S of group G is defined as

$$Z_G(S) = \{g \in G : gs = sg \text{ for all } s \in S\},$$

and the **normalizer** of S in group G is defined as

$$N_G(S) = \{g \in G : gS = Sg\}.$$

The **centralizer subalgebra** of a subset of Lie algebra $S \subset \mathfrak{g}$ is the set of elements commuting with S, i.e.

$$\mathfrak{z}_{\mathfrak{g}}(S) = \{X \in \mathfrak{g} \mid [X, s] = 0 \text{ for all } s \in S\},$$

and the **normalizer subalgebra** of a subset of Lie algebra $S \subset \mathfrak{g}$ is the set

$$\mathfrak{n}_{\mathfrak{g}}(S) = \{X \in \mathfrak{g} \mid [X, s] \in S \text{ for all } s \in S\}.$$

Definition 4.2.6. *For maximal torus T and connected Lie group G, we define the* ***Weyl group****, denoted $W(G,T)$, by*

$$W(G,T) = N_G(T)/Z_G(T).$$

The Weyl group defined above is often called **analytically defined Weyl group**. Later in the text we introduce the algebraically defined Weyl group.

We finish this section with a very important theorem.

Theorem 4.2.5. *Any compact connected Lie group G can be realized as a linear connected reductive Lie group.*

Proof. Since a finite dimensional representation of a compact group is unitary, it is enough to produce a $1-1$ finite dimensional representation of G. It follows from Peter-Weyl theorem that for each $x \neq \mathbb{1}$ in G there is a finite dimensional representation Φ_x of G such that $\Phi_x \neq \mathbb{1}$. If the identity component G_0 is not $\{\mathbb{1}\}$, pick $x_1 \neq \mathbb{1}$ in the identity component G_0. Then $G_1 = \ker \Phi_{x_1}$ is a closed subgroup of G, and its identity component is a proper subgroup of G_0. If $(G_1)_0 \neq \{\mathbb{1}\}$, pick $x_2 \neq \mathbb{1}$ in $(G_1)_0$. Then $G_2 = \ker(\Phi_{x_1} \oplus \Phi_{x_2})$ is a closed subgroup of G_1, and its identity component is a proper subgroup of $(G_1)_0$. Continuing in this way and using finite-dimensionality of G, and the fact that proper subgroup of connected Lie group has strictly lower dimension, we can find a finite dimensional representation Φ_0 of G such that $\ker \Phi_0$ is 0-dimensional. Then $\ker \Phi_0$ is finite, being a comapact 0-dimensional Lie group. Let $\ker \Phi_0 = \{y_1, \ldots, y_n\}$. Then

$$\Phi = \Phi_0 \oplus \bigoplus_{j=1}^{n} \Phi_{y_j}$$

is a $1-1$ finite dimensional representation of G. $\square$

4.3 More on the structure of Lie algebras and Lie groups

In this section we introduce the most important notions of this paper, namely, we present the fundamental concepts and properties of weights and roots, and finish with the theorem of the highest weight.

Definition 4.3.1. *Let V be a representation of a Lie group G. A form $(\cdot,\cdot) : V \times V \to \mathbb{C}$ is called **G-invariant** if $(gv, gw) = (v, w)$ for $g \in G$ and $v, w \in V$.*

Lemma 4.3.1. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and (Φ, V) a finite dimensional representation of G. Let ϕ be the differential of Φ.*

1. *There exists a G-invariant inner product $(\cdot,\cdot)$ on V and for any such G-invariant inner product on V, $\phi(X)$ is skew-Hermitian, i.e. $(\phi(X)v, w) = -(v, \phi(X)w)$ for $X \in \mathfrak{g}$ and $v, w \in V$.*

2. *There exists an Ad-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$, i.e. $(\mathrm{Ad}(g)Y_1, \mathrm{Ad}(g)Y_2) = (Y_1, Y_2)$ for $g \in G$ and $Y_i \in \mathfrak{g}$. For any such inner product on $\mathfrak{g}$, ad is skew-symmetric.*

Now let G be a compact semisimple Lie group and (Φ, V) a finite dimensional representation of G. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. By lemma 4.3.1, there exists an inner product on V that is G-invariant and for which ϕ is skew-Hermitian on $\mathfrak{g}$ and is Hermitian on $i\mathfrak{g}$. Thus $\mathfrak{h}^{\mathbb{C}}$ acts on V as a family of commuting normal operators and so V is simultaneously diagonalizable under the action of $\mathfrak{h}^{\mathbb{C}}$. We have the following definition

Definition 4.3.2. *Let G be a compact semisimple Lie group, (Φ, V) a finite dimensional representation of G, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. There is a finite set $\Delta(V, \mathfrak{h}^{\mathbb{C}}) \subseteq (\mathfrak{h}^{\mathbb{C}})^*$, called the **weights** of V, so that*

$$V = \bigoplus_{\lambda \in \Delta(V, \mathfrak{h}^{\mathbb{C}})} V_\lambda, \tag{4.5}$$

where

$$V_\lambda = \{v \in V | \phi(H)v = \lambda(H)v, \quad H \in \mathfrak{h}^{\mathbb{C}}\}$$

*is nonzero. The equation (4.5) is called **weight space decomposition**.*

Theorem 4.3.2. *Let G be a compact Lie group, (Φ, V) a finite dimensional representation G, T a maximal torus of G, and $V = \bigoplus_{\lambda \in \Delta(V, \mathfrak{h}^{\mathbb{C}})} V_\lambda$ the weight space decomposition.*

1. *For each weight $\lambda \in \Delta(V, \mathfrak{h}^{\mathbb{C}})$, λ is purely imaginary on $\mathfrak{h}$ and is real valued on $\mathfrak{h}_{\mathbb{R}}$.*

2. *For $t \in T$, choose $H \in \mathfrak{h}$ so that $e^H = t$. Then $tv_\lambda = e^{\lambda(H)}v_\lambda$ for $v_\lambda \in V_\lambda$.*

Let G be a compact semisimple Lie group. For $g \in G$, we extend the domain of $\mathrm{Ad}(g)$ from $\mathfrak{g}$ to $\mathfrak{g}^{\mathbb{C}}$ by $\mathbb{C}$-linearity. Then $(\mathrm{Ad}, \mathfrak{g}^{\mathbb{C}})$ is a representation of G with differential given by ad (again extended by $\mathbb{C}$-linearity). It has a weight space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\lambda \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})} \mathfrak{g}_{\lambda}.$$

Note that the zero weight space is $\mathfrak{g}_0 = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = 0, \quad H \in \mathfrak{h}^{\mathbb{C}}\}$. Thus $\mathfrak{g}_0 = \mathfrak{h}^{\mathbb{C}}$ since $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$. This gives us the following definition

Definition 4.3.3. *Let G be a compact semisimple Lie group and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. There is a finite set of nonzero elements $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) \subseteq (\mathfrak{h}^{\mathbb{C}})^*$ called the roots of $\mathfrak{g}^{\mathbb{C}}$, so that*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})} \mathfrak{g}_{\alpha}, \tag{4.6}$$

*where $\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \alpha(H)X, \quad H \in \mathfrak{h}^{\mathbb{C}}\}$ is nonzero. The equation (4.6) is called the **root space decomposition** of $\mathfrak{g}^{\mathbb{C}}$.*

Theorem 4.3.3. *Let G be a compact semisimple Lie group, (Φ, V) a finite dimensional representation G, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Denote $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) = \Delta(\mathfrak{g}^{\mathbb{C}})$ and $\Delta(V, \mathfrak{h}^{\mathbb{C}}) = \Delta(V)$.*

1. *For $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}})$ and $\lambda \in \Delta(V)$, $\phi(\mathfrak{g}_{\alpha})V_{\lambda} \subseteq V_{\alpha+\lambda}$.*
2. *In particular, for $\alpha, \beta \in \Delta(\mathfrak{g}^{\mathbb{C}}) \cup \{0\}$, $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$.*
3. *Let $(\cdot, \cdot)$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}^{\mathbb{C}}$. For $\alpha, \beta \in \Delta(\mathfrak{g}^{\mathbb{C}}) \cup \{0\}$, $(\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}) = 0$ when $\alpha + \beta \neq 0$.*
4. *$\Delta(\mathfrak{g}^{\mathbb{C}})$ spans $(\mathfrak{h}^{\mathbb{C}})^*$.*

Recall that for a finite dimensional representation Φ of a Lie group G, acting on a space with an inner product $\langle \cdot | \cdot \rangle$, its character is the function

$$\chi_{\Phi}(x) = \mathrm{Tr}(\Phi(x)) = \sum_i \langle u_i | \Phi(x) u_i \rangle,$$

where $\{u_i\}$ is an orthonormal basis with respect to the inner product $\langle \cdot | \cdot \rangle$ and $x \in G$.

Definition 4.3.4. *Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\lambda \in (\mathfrak{h}_{\mathbb{R}})^*$. Let $H_{\lambda} \in \mathfrak{h}_{\mathbb{R}}$ be uniquely determined by the equation $\lambda(H) = B(H, H_{\lambda})$ for all $H \in \mathfrak{h}_{\mathbb{R}}$. When $\lambda \neq 0$, let*

$$u_{\lambda} = \frac{2H_{\lambda}}{B(H_{\lambda}, H_{\lambda})}.$$

Definition 4.3.5. *An **abstract root system** in a finite dimensional real inner product space V is a finite set Δ of nonzero elements of V such that*

- *Δ spans V.*
- *The orthogonal transformations $s_\alpha(\phi) = \phi - \frac{2\langle\phi,\alpha\rangle}{\|\alpha\|^2}\alpha$, for $\alpha \in \Delta$, leave Δ invariant.*
- *$2\langle\beta,\alpha\rangle/\|\alpha\|^2$ is in $\mathbb{Z}$ for all $\alpha, \beta \in \Delta$.*

*An abstract root system is **reduced** if $\alpha \in \Delta$ implies $2\alpha \notin \Delta$. An abstract root system is **irreducible** if Δ admits no nontrivial disjoint decomposition $\Delta = \Delta' \cup \Delta''$ with every member of Δ' orthogonal to every member of Δ''.*

In the above definition we also introduced the root reflection

$$s_\alpha(\phi) = \phi - \frac{2\langle\phi,\alpha\rangle}{\|\alpha\|^2}\alpha,$$

for $\alpha \in \Delta$ and $\phi \in (\mathfrak{h}_0)^*$, where $\mathfrak{h}_0$ denotes the $\mathbb{R}-$linear span of all H_α, and the inner product $\langle\cdot,\cdot\rangle$ is defined as

$$\langle\phi,\phi'\rangle = B(H_\phi, H_{\phi'}).$$

This is -1 on α and 1 on orthogonal complement of α. It can be shown that if $\alpha \in \Delta$, then the root reflection s_α carries Δ into itself.

Among all the structure of roots and weights, the notion of positivity will play a crucial role.

Definition 4.3.6. *A subset $\delta = \{\alpha_1, \dots, \alpha_n\}$ of Δ is called a **base** for the root system if*

1. *δ is a basis for V.*
2. *Every element $\beta \in \Delta$ can be written as $\beta = \sum k_i\alpha_i$ where all k_i are integers with the same sign (or zero).*

*Given a choice of base δ, the roots in Δ are called **simple roots**, and roots which are a positive linear combination of simple roots are called **positive roots** and denoted Δ^+.*

We have the following proposition, which relates roots defined in 4.3.3 and abstract root systems.

Proposition 4.3.1. *The root system for a compact connected linear semisimple Lie group G with respect to a Cartan subalgebra $\mathfrak{h}$ forms an abstract reduced root system in $(\mathfrak{h}_\mathbb{R})^*$. The system is irreducible if and only if the Lie algebra $\mathfrak{g}$ of G is simple, if and only if $\mathfrak{g}^\mathbb{C}$ is simple.*

Definition 4.3.7. *Let G be a compact semisimple Lie group and T a maximal torus of G with corresponding Cartan subalgebra $\mathfrak{h}$.*
*The set of **algebraically integral weights** is the set $P(\mathfrak{h})$ in $(\mathfrak{h}_{\mathbb{R}})^*$ given by*

$$P(\mathfrak{h}) = \{\lambda \in (\mathfrak{h}_{\mathbb{R}})^* \mid \lambda(u_\alpha) \in \mathbb{Z} \text{ for } \alpha \in \Delta(\mathfrak{g}^{\mathbb{C}})\}$$

where $(\mathfrak{h}_{\mathbb{R}})^*$ is extended to an element of $(\mathfrak{h}^{\mathbb{C}})^*$ by $\mathbb{C}$-linearity.
*The set of **analytically integral weights** is the set $A(T)$ in $(\mathfrak{h}_{\mathbb{R}})^*$ given by*

$$A(T) = \{\lambda \in (\mathfrak{h}_{\mathbb{R}})^* \mid \lambda(H) \in 2\pi i\mathbb{Z} \text{ whenever } \exp(H) = I \text{ for } H \in \mathfrak{h}\}.$$

Lemma 4.3.4. *Let G be a compact connected semisimple Lie group with Cartan subalgebra $\mathfrak{h}$. For $H \in \mathfrak{h}$, $\exp(H) \in Z(G)$ if and only if $\alpha(H) \in 2\pi i\mathbb{Z}$ for all $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}})$.*

Definition 4.3.8. *Let G be a compact semisimple Lie group and T a maximal torus. Write $\chi(T)$ for the **character group** on T, i.e. $\chi(T)$ is the set of all Lie homomorphisms $\zeta : T \to \mathbb{C}^\times$.*

Theorem 4.3.5. *Given $\lambda \in (\mathfrak{h}_{\mathbb{R}})^*$, $\lambda \in A(T)$ if and only if there exists $\zeta_\lambda \in \chi(T)$ satisfying*

$$\zeta_\lambda(\exp(H)) = e^{\lambda(H)}$$

for $H \in \mathfrak{h}$, where $\lambda \in (\mathfrak{h}_{\mathbb{R}})^*$ is extended to an element of $(\mathfrak{h}^{\mathbb{C}})^*$ by $\mathbb{C}$-linearity. The map $\lambda \to \zeta_\lambda$ establishes a bijection

$$A(T) \to \chi(T).$$

Definition 4.3.9. *A linear functional $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$, which is real on $\mathfrak{h}_{\mathbb{R}}$, is said to be **dominant** if*

$$\frac{2\langle \lambda, \alpha\rangle}{\|\alpha\|^2} \geq 0 \text{ for all } \alpha \in \Delta^+.$$

Having the notion of positivity of roots, we can introduce partial ordering on the set of weights.

Definition 4.3.10. *If $\lambda, \mu \in (\mathfrak{h}^{\mathbb{C}})^*$, then we say that λ is **higher** than μ if $\lambda - \mu$ is a linear combination of positive roots with non-negative real coefficients. We denote this relation by $\lambda \succeq \mu$). The weight which is the highest with respect to this partial ordering is called the **highest weight**.*

Now we can introduce a fundamental theorem which characterizes irreducible representations of compact connected Lie groups.

Theorem 4.3.6. ***(Theorem of the highest weight)*** *Let G be a compact connected semisimple Lie group and V an irreducible representation of G.*

1. *The highest weight λ_0 is dominant and analytically integral.*

2. *Up to isomorphism, representation V is uniquely determined by λ_0.*

3. *The weight space V_λ of the highest weight λ is 1-dimensional.*

4. *V has a unique highest weight, λ_0.*

5. *Every dominant, analytically integral weight is the highest weight of an irreducible representation.*

Theorem of the highest weight gives a parametrization of the irreducible representations of a compact Lie group. Lacking is an explicit realization of these representations, and Borel-Weil theorem repairs this gap. Now we algebraically define the Weyl group.

Definition 4.3.11. *Let Δ be a reduced abstract root system in a finite dimensional real inner product space V. The group generated by the s_α for $\alpha \in \Delta$ is called the* ***algebraically defined Weyl group*** *of Δ.*

One can immediately see two properties of an algebraically defined Weyl group:

- W is a finite group of orthogonal transformations of V. For example, for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $W = \{\text{permutations of } \{e_1, \ldots, e_n\}\}$ and $|W| = n!$.

- Straightforward computation shows that

$$s_{r\alpha}(r\phi) = r\phi - \frac{2\langle r\phi, r\alpha\rangle}{\|r\alpha\|^2} r\alpha = r\phi - \frac{2\langle \phi, \alpha\rangle}{\|\alpha\|^2} r\alpha = r(s_\alpha \phi),$$

  for any orthogonal transformation $r \in W$ of V, and we see that $s_{r\alpha} = r s_\alpha r^{-1}$.

It is also worth noting that for compact connected Lie group G, the analytically defined Weyl group $W(G, T)$ (considered as acting on $(\mathfrak{h}_\mathbb{R})^*$) coincides with the algebraically defined Weyl group of the root system Δ [8].

4.4 Borel-Weil theorem

In what follows, we will formulate and prove Borel-Weil theorem. We assume that the reader is familiar with basic notions of differential geometry such as line bundle and its sections. Borel-Weil theorem asserts that if B the Borel subgroup of the complex semisimple group $G^\mathbb{C}$, then all unitary irreducible representations can be obtained as the space of holomorphic line bundles associated to the fiber bundle over $G^\mathbb{C}/B \cong G/T$ with the fiber $\mathbb{C}_\chi$, which is the one dimensional representation corresponding to a dominant integral character χ; and vice versa. With use of structure theory we will embed our compact connected Lie group in a complexification and then we will interpret Iwasawa decomposition of complexification.
Before formulating and proving Borel-Weil theorem, we will state two very useful theorems which we will use in a proof of Borel-Weil theorem itself.

Theorem 4.4.1. ***(Weyl's unitary trick)*** *Let G be a linear connected semisimple group, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of its Lie algebra, and suppose $\mathfrak{k} \cap i\mathfrak{p} = 0$. Let U and $G^{\mathbb{C}}$ be the Lie groups of matrices with Lie algebras $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and $\mathfrak{g}^{\mathbb{C}} = (\mathfrak{k} \oplus \mathfrak{p})^{\mathbb{C}}$, and suppose U is simply connected. If V is any finite dimensional complex vector space, then a representation of any of the following kinds on V leads, via the formula*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u},$$

to a representation of each of the other kinds. Under this correspondence, invariant subspaces and equivalences are preserved:

1. *a representation of G on V,*
2. *a representation of U on V,*
3. *a holomorphic representation of $G^{\mathbb{C}}$ on V,*
4. *a representation of $\mathfrak{g}$ on V,*
5. *a representation of $\mathfrak{u}$ on V,*
6. *a complex-linear representation of $\mathfrak{g}^{\mathbb{C}}$ on V.*

Theorem 4.4.2. ***(Weyl's theorem)*** *If G is a compact linear connected semisimple Lie group, then the universal covering group of G is compact.*

In the following notation we follow [8]. We will denote our compact connected Lie group by K, and its Lie algebra by $\mathfrak{k}$. Complex Lie group $K^{\mathbb{C}}$ will be denoted by G with Lie algebra $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$. One can show that G in this case is linear connected reductive and $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ is the Cartan decomposition of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition with corresponding group decomposition $G = KAN$. We have $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = i\mathfrak{a}$, and $\mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{k}$, where $\mathfrak{m}^{\mathbb{C}} = \mathfrak{a} \oplus \mathfrak{m}$. From the fact that the roots in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}})$ are just complex-linear extensions to $\mathfrak{m}^{\mathbb{C}}$ of their restrictions to $\mathfrak{a}$ it follows that the root space decomposition of $(\mathfrak{g}, \mathfrak{a})$ coincides with the root space decomposition of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}})$:

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{k}_{\alpha} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} = \mathfrak{g}. \tag{4.7}$$

Thus the choice of $\mathfrak{n}$ determines a positive root system Δ^+ of Δ. With $M = Z_K(A)$, the group $B = MA\overline{N}$, where $\overline{N} = \Theta N$, is a complex subgroup of G because its Lie algebra

$$\mathfrak{b} = \mathfrak{m} \oplus \mathfrak{a} \oplus \theta\mathfrak{n} = \mathfrak{m}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{k}_{-\alpha}$$

is complex. Let λ be analytically integral on $(\mathfrak{m}^{\mathbb{C}})^*$ and let ζ_λ be the corresponding character of subgroup M. We extend it to a holomorphic one dimensional representation $\zeta_\lambda : B \to \mathbb{C}^\times$ by defining

$$\zeta_\lambda = \begin{cases} \exp(\lambda \log x) & \text{for } x \in A \\ 1 & \text{for } x \in \overline{N}. \end{cases}$$

Theorem 4.4.3. ***(Borel-Weil theorem)*** *Let K be a compact connected Lie group, and let $G = K^{\mathbb{C}}$ and B be as above. If λ is dominant and analytically integral and if ζ_λ denotes the corresponding holomorphic one dimensional representation of B, then a realization of an irreducible representation of K with the highest weight λ is as follows; we have a space*

$$\Gamma(\lambda) = \left\{ F : G \to \mathbb{C} \mid F(xb) = \zeta_\lambda(b)^{-1} F(x) \textit{ for } x \in G, b \in B \right\},$$

where F is holomorphic and K operates by the left regular representation.

For an outline of the proof from the perspective of complex analysis, see Chapter 14 of [10]. This proof follows [8] and therefore utilizes highest-weight theorem and proceeds in several steps, each in form of a lemma.
We introduce a structure of inner product space on $\Gamma(\lambda)$ by defining

$$(F_1, F_2) = \int_K F_1(k) \overline{F_2(k)} dk.$$

The left regular representation L of K operates isometrically.

Lemma 4.4.4. *Let τ be a finte dimensional representation of K on a complex vector space V. Then τ extends to a holomorphic representation of G on V.*

Proof. We will give a proof for K semisimple. By Theorem 4.4.2 and the fact that any compact connected Lie group may be taken to be a subgroup of a unitary group, we may assume K is simply connected. The result then follows from Theorem 4.4.1.

□

We apply now the theorem of the highest weight to obtain an irreducible unitary representation Φ_λ of K with highest weight λ. Let Φ_λ act in V and v_λ be a highest weight vector (a weight vector v with $(\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha)v = 0$) of norm one. By Lemma 4.4.4, we can extend Φ_λ to a holomorphic representation of G. For $v \in V$ we define Ψ_v on G by

$$\Psi_v(x) = (\Phi_\lambda(x)^{-1} v, v_\lambda).$$

Lemma 4.4.5. *For each v in V, Ψ_v is in $\Gamma(\lambda)$. Moreover, $L(k)\Psi_v = \Psi_{\Phi_\lambda(k)v}$, so that $\{\Psi_v | v \in V\}$ is an irreducible subspace of $\Gamma(\lambda)$ under K equivalent with Φ_λ.*

Proof. Let ϕ_v be the differential of Φ_λ. Since Φ_λ is unitary on K, ϕ_λ is skew-Hermitian on $\mathfrak{k}$. Since ϕ_λ is complex-linear on $\mathfrak{g}$, $\phi_v(\theta X) = -\phi_v(X)^*$ for all X in $\mathfrak{g}$. Thus $\Phi_\lambda(\Theta x) = \Phi_\lambda(x^{-1})^*$ for x in G. Hence $b = ma\overline{n}$ in $MA\overline{N}$ implies

$$\begin{aligned}
\Psi_v(xma\overline{n}) &= (\Phi_\lambda(ma\overline{n})^{-1}\Phi_\lambda(x)^{-1}v, v_\lambda)\\
&= (\Phi_\lambda(x)^{-1}v, \Phi_\lambda(ma^{-1}n)v_\lambda)\\
&= (\Phi_\lambda(x)^{-1}v, \Phi_\lambda(ma^{-1})v_\lambda)\\
&= (\Phi_\lambda(x)^{-1}v, \zeta_\lambda(m)\zeta_\lambda(a)^{-1}v_\lambda)\\
&= \overline{\zeta_\lambda(m)}\zeta_\lambda(a)^{-1}(\Phi_\lambda(x)^{-1}v, v_\lambda)\\
&= \zeta_\lambda(b)^{-1}\Psi_v(x).
\end{aligned}$$

The second equality comes from $n = \Theta\overline{n}$, the third from the fact that v_λ is the highest weight vector and the fourth - from the fact that v_λ has weight λ.
We can see that Ψ_v is holomorphic, and thus Ψ_v is in $\Gamma(\lambda)$. The lemma finally follows from

$$\begin{aligned}
\Psi_{\Phi_\lambda(k)v}(x) &= (\Phi_\lambda(x)^{-1}\Phi_\lambda(k)v, v_\lambda)\\
&= (\Phi_\lambda(k^{-1}x)^{-1}v, v_\lambda)\\
&= \Psi_v(k^{-1}x) = L(k)\Psi_v(x).
\end{aligned}$$

□

Now we want to prove that the mapping $v \mapsto \Psi_v$ from Lemma 4.4.5 carries V onto $\Gamma(\lambda)$; this will allow us to end the proof of the theorem. Let us denote $\Psi_{v_\lambda} = \Psi_\lambda$.

Lemma 4.4.6. *If F is in $\Gamma(\lambda)$, then*

$$\int_M F(mxm^{-1})dm = F(1)\Psi_\lambda(x),$$

for all $x \in G$. In this case dm *denotes normalized Haar measure on M.*

Proof. The main idea is to show near $x = 1$ that the left side is $F(1)$ times a power series in x that is independent of F. The power series is evaluated as the series for $\Psi_\lambda(x)$ by putting $F = \Psi_\lambda$. Since both sides are holomorphic on G and equal in a neighbourhood of 1, they are equal everywhere. We omit the rest of this proof, because it utilizes Birkhoff-Witt theorem and the notion of enveloping algebras which are out of the scope of this article.

□

We now can proceed to the end of the proof of the theorem in the following steps:

1.

$$\begin{aligned}
\|F\|^2 &= \int_K |F(k)|^2 dk = \int_K |F(mkm^{-1})|^2 dk \quad \text{for all } m \in M \\
&= \int_K \int_M |F(mkm^{-1})|^2 dm dk \\
&\geq \int_K |\int_M F(mkm^{-1}) dm|^2 dk \quad \text{by the Schwarz inequality} \\
&= |F(1)|^2 \int_K |\Psi_\lambda(k)|^2 dk \quad \text{by Lemma 4.4.6} \\
&= |F(1)|^2 \|\Psi_\lambda\|^2.
\end{aligned}$$

2. To each compact set $E \subseteq G$ corresponds a constant $C_E < \infty$ such that

$$|F(x)| \leq C_E \|F\|,$$

for all F in $\Gamma(\lambda)$ and x in E.

3. $\Gamma(\lambda)$ is complete. By 2., we have that Cauchy sequences converge uniformly on compact sets in G; moreover, the limit function is holomorphic and satisfies the correct transformation law under B.

4. $\Gamma(\lambda)$ is irreducible. Thus the map $v \mapsto \Psi_v$ is onto $\Gamma(\lambda)$. Let $U \subseteq \Gamma(\lambda)$ be nonzero closed invariant subspace and let $F \neq 0$ be in U. We can apply $L(k)$ and thus assume that $F(1) \neq 0$. Then, by 3.,

$$\int_M \overline{\zeta_\lambda(m)} L(m) F dm$$

is in U. But from Lemma 4.4.6 we have that this is $F(1)\Psi_v$, therefore Ψ_v is in U. Now assume that $U^\perp \neq 0$. Then Ψ_v is also in U and this gives us a contradiction. Hence we have that $U = 0$ or $U^\perp = 0$.

This concludes the proof of Borel-Weil theorem.

□

There are many generalizations of Borel-Weil theorem, one of them being Bott-Borel-Weil theorem, which extends it to higher cohomologies. In the language of sheaf cohomology, we have been looking at the zero degree cohomology of the sheaf of sections of a line bundle

$$\Gamma(L_\lambda) = H^0(K^{\mathbb{C}}/B, \mathcal{O}(L_\lambda)),$$

where $\mathcal{O}(L_\lambda)$ is a space of all sections of a line bundle L_λ, but there are higher degree cohomology groups which also provide irreducible representation of K. There is also an extension to quantum groups or to Harish-Chandra sheaves to construct the infinite dimensional representations, but these are out of the scope of this article.

4.5 Borel-Weil theorem for SU(2)

In the case of $K = \mathrm{SU}(2)$ we have explicit construction of irreducible representations in terms of homogeneous polynomials of two variables.
We can express that in the language of Borel-Weil theorem by identifying holomorphic sections in terms of homogeneous polynomials. In this case we have $K = \mathrm{SU}(2)$, $T = U(1)$, $K^{\mathbb{C}} = G = \mathrm{SL}(2,\mathbb{C})$, $L_\lambda = \mathrm{SU}(2) \times_{U(1)} \mathbb{C}$, so

$$K/T = \mathrm{SU}(2)/U(1) = \mathrm{SL}(2,\mathbb{C})/B = \mathbb{C}\mathrm{P}^1.$$

Elements of $\mathrm{SL}(2,\mathbb{C})$ are of the form

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \ \alpha, \beta, \gamma, \delta \in \mathbb{C},\ \alpha\delta - \beta\gamma = 1 \right\}$$

and

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{C} \right\}.$$

One can check that for $b \in B$ we have

$$b \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Subgroup N of B (corresponding to $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} g_\alpha$) consists of matrices of the form

$$\left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \middle| \ \alpha \in \mathbb{C} \right\}.$$

We also have subgroup $T_{\mathbb{C}}$ corresponding to an algebra $\mathfrak{h}^{\mathbb{C}}$ which appears in decomposition of $\mathfrak{g}$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. $T_{\mathbb{C}}$ consists of matrices of the following form

$$T_{\mathbb{C}} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \middle| \ \alpha \in \mathbb{C} \right\}.$$

Space of holomorphic sections $\Gamma(L_{-k})$ is the space of functions on $\mathrm{SL}(2,\mathbb{C})$ on which N acts trivially from the right and $T_{\mathbb{C}}$ acts via character of T, which corresponds to an integer k. Explicitly we have

$$\Gamma(L_{-k}) = \{ f : \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C} | f(gb) = \alpha^k f(g) \forall b \in B \}.$$

Let $b \in N$ and we have

$$gb = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta'\alpha + \beta \\ \gamma & \beta'\gamma + \delta \end{pmatrix}$$

The condition $f(gb) = f(g)$ implies that f acts only on the first column of the matrix. Choosing $b \in T_{\mathbb{C}}$ gives

$$gb = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & (\alpha')^{-1} \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & (\alpha')^{-1}\beta \\ \gamma\alpha' & (\alpha')^{-1}\delta \end{pmatrix}$$

and condition $f(gb) = (\alpha')^k f(g)$ implies that

$$f(\alpha' \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}) = (\alpha')^k f(\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}).$$

It follows that our homogeneous polynomials of degree k in two variables $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ are elements of $\Gamma(L_{-k})$. These polynomials are all such sections, so we have that, up to equivalence, homogeneous polynomials are only finite-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ (and so of $\mathrm{SU}(2)$).

Bibliography

[1] A.W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, 1996

[2] M.R. Sepanski, *Compact Lie Groups*, Springer, 2007

[3] H. Weyl, Uspekhi Mat. Nauk **4**, pp. 201–246, 1938

[4] E. Cartan, *La topologie des groupes de Lie*, Hermann, 1936

[5] B.C. Hall, *An elementary introduction to groups and representations*

[6] B. Simon, *Representations of finite and compact groups*

[7] T. Brocker, T. tom Dieck, *Representations of compact Lie groups*

[8] A.W. Knapp, *Representation Theory of Semisimple Groups*, Princeton University Press, 1986

[9] P.T. Eendebak, *The Borel - Weil Theorem*, Utrecht University, 2001

[10] G. Segal, *Lectures on Lie groups and Lie algebras*, Cambridge University Press, 1995

[11] M. Reed, B. Simon Methods of Modern Mathematical Physics, vol I: Functional Analysis, Academic Press, Inc., 1980

[12] Andrew Baker Matrix Groups: An introduction to Lie group theory Springer 2002