# ON QUANTUM GROUP OF UNITARY OPERATORS
# QUANTUM '$az + b$' GROUP

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ABSTRACT. The concept of *a quantum group of unitary operators* is relevant for the theory of non-compact locally compact quantum groups. It plays similar role as the concept of *a quantum matrix group* in the compact case. To show the usefulness of this notion we present an approach to a construction of quantum '$az + b$' group based on this idea. A brief survey of the present status of quantum group theory is also included.

## 1. INTRODUCTION

The Conferences in Białowieża were unforgettable events due to a rare combination of a beautiful wild surroundings with its unique old forest and a specific atmosphere. Hopefully for us they started almost at the same time as quantum group theory was initiated and from the very beginning they were open for reports concerning a development of a general quantum group theory as well as to constructions of examples. Due to this we (especially the second author) participated in many of them. The jubilee occasion seems to be a good one to give also a brief account of the present status of the theory.

Roughly speaking 'a quantum group' is 'a quantum space' endowed with a group structure. One may consider a pure algebraic version of the theory or a topological one.

The first one is known as a Hopf $^*$-algebra approach. Let $\mathcal{A}$ be a unital $^*$-algebra and $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$ be unital $^*$-homomorphism such that

$$(1.1) \qquad (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

(coassociativity). In this approach $\mathcal{A}$ encodes a quantum space. Now $\Delta$ endows a group structure on it if $(\mathcal{A}, \Delta)$ is a Hopf $^*$-algebra.

We recall that $(\mathcal{A}, \Delta)$ is a Hopf $^*$-algebra if there exist linear mapping $e : \mathcal{A} \to \mathbb{C}$ and $\kappa : \mathcal{A} \to \mathcal{A}$ such that

$$(e \otimes \mathrm{id})\Delta(a) = a = (\mathrm{id} \otimes e)\Delta(a)$$

and

$$m(\kappa \otimes \mathrm{id})\Delta(a) = e(a)I = m(\mathrm{id} \otimes \kappa)\Delta(a)$$

for any $a \in \mathcal{A}$, where $m : \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A} \to \mathcal{A}$ denotes the multiplication map, i.e. $m(a \otimes b) = ab$ for any $a, b \in \mathcal{A}$. It is known that $e$ (called counit) and $\kappa$ (called coinverse or antipode) are uniquely determined. Moreover $e$ is a unital $^*$-algebra homomorphism, $\kappa$ is antimultiplicative, anticomultiplicative and

$$\kappa \left( \kappa(a^*)^* \right) = a$$

for any $a \in \mathcal{A}$.

The topological approach uses $C^*$-algebra language. Any (locally compact) quantum space is encoded by a $C^*$-algebra $A$. This is approved by Gelfand-Naimark theorem which says that any commutative $C^*$-algebra $A$ is isomorphic to the algebra $C_\infty(\Lambda)$ of all complex continuous and vanishing at infinity functions on some locally compact space $\Lambda$. The space $\Lambda$ is unique up to a homeomorphism, i.e. we have a correspondence

$$\Lambda \qquad \longleftrightarrow \qquad A = C_\infty(\Lambda).$$

Moreover $A$ is unital if and only if $\Lambda$ is a compact space. No theory of such type is known for noncommutative $C^*$-algebra $A$. Formally one can solve the problem (cf. e.g. [15]) by considering the category dual to the category of $C^*$-algebras. Objects of this dual category are called 'locally compact quantum spaces' and its morphisms - continuous mappings of (locally compact) quantum spaces. The notion of quantum spaces introduces a new language to the $C^*$-algebra theory. In particular, if $A$ is unital (or non-unital) we say that we deal with compact case (or noncompact, respectively).

Now we pass to the description of the (locally compact) quantum groups. It turned out that the compact case was relatively easy and starting from simple axioms a nice theory, parallel to that for classical compact groups was build [22] on early stage of the theory.

<div style="border:1px solid">Gcomp</div>

**Definition 1.1.** *Let $G = (A, \Delta)$, where $A$ is a separable $C^*$-algebra and $\Delta : A \longrightarrow A \otimes A$ is a unital $^*$-algebra homomorphism. We say that $G$ is a compact quantum group if*
*(1) $\Delta$ is coassociative (cf (1.1)) and*
*(2) The sets*

$$\{ (b \otimes I)\Delta(a) : \ a, b \in A \}, \qquad \{ (I \otimes b)\Delta(a) : \ a, b \in A \}$$

*are total subsets of $A \otimes A$.*

Using these axioms one shows that there are many finite-dimensional representations of $G$. We recall that a unitary matrix $V = (v_{kl}) \in M_N(A) = M_N(\mathbb{C}) \otimes A$ is a $N$-dimensional unitary representation of $G$ if

$$\Delta(v_{kl}) = \sum_{r=1}^{N} v_{kr} \otimes v_{rl}$$

or using leg-numbering notation:

<div style="border:1px solid">rep1</div>

(1.2)                                        $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}.$

Let $\mathcal{A}$ be the set of all linear combinations of matrix elements of all finite-dimensional unitary representations of $G$. Then we have the following result [22, Theorem 2.2 and Theorem 2.3]

<div style="border:1px solid">G1</div>

**Theorem 1.2.** *Let $G= (A, \Delta)$ be a compact quantum group. Then*
*1. $\mathcal{A}$ is dense $^*$-subalgebra of $A$ and $\Delta(\mathcal{A}) \subset \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$.*
*2. $(\mathcal{A}, \Delta|_{\mathcal{A}})$ is a Hopf $^*$-algebra.*
*3. There exists unique state (normalized positive linear functional) $h$ on $A$ such that*

$$(h \otimes \mathrm{id})\Delta(a) = h(a)I = (\mathrm{id} \otimes h)\Delta(a).$$

*for any $a \in \mathcal{A}$.*
*4. $h$ is faithful on $\mathcal{A}$, i.e. if $a \in \mathcal{A}$ and $h(a^*a) = 0$ then $a = 0$.*

Statement 1 and Statement 2 give a connection between $C^*$-algebra approach and Hopf $^*$-algebra one in the compact case. Clearly the functional $h$ in Statement 3 of the above theorem is a Haar state (measure). It is still an open problem how to guarantee the faithfulness of $h$ on the whole algebra $A$. If this would be the case then a deeper insight into the structure of $G$ could be obtained [22, Theorem 2.]. Let us note that left-invariant state and right-invariant state coincide. Therefore a compact quantum group is unimodular as in classical case. Nevertheless the Haar state needs not be central one. Therefore if $h$ is faithful there is modular structure coming from the Tomita-Takesaki theory.

Definition 1.1 is not very useful if one is looking for examples of compact quantum groups. In this aspect a concept of a compact matrix quantum group [17],[18] turns out to be very useful.

Let $A$ be unital $C^*$-algebra and $V = (v_{kl}) \in M_N(A) = M_N(\mathbb{C}) \otimes A$. We say that $V$ generates $A$ if the smallest $^*$-algebra containing all matrix elements $v_{kl}$, $k, l = 1, 2, ...N$ is dense in $A$.

<div style="border:1px solid">Gmatrix</div>

**Definition 1.3.** *Let $A$ be unital $C^*$-algebra and $V = (v_{kl}) \in M_N(A) = M_N(\mathbb{C}) \otimes A$. Assume that*
*1. $A$ is generated by $V$.*
*2. There exists a unital $^*$-algebra homomorphism $\Delta : A \longrightarrow A \otimes A$ such that*

<div style="border:1px solid">rep2</div>

(1.3)                                        $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}.$

*3. $V$ and $V^\top$ are invertible in $M_N(A)$, where $^\top$ denotes transposition.*
*Then $(A, V)$ is called a compact quantum matrix group.*

One can easily show that $\Delta$ is uniquely defined by (1.3) and when it exists it is automatically coassociative. Moreover, $G = (A, \Delta)$ is a compact group [22, Remark 2] in the sense of Definition 1.1. If $V$ is a unitary matrix than $V$ is a unitary representation of $G$ (cf.(1.2)).

The majority of known examples of compact quantum group is of this kind. In particular the first nontrivial example, namely the quantum $SU_q(2)$ introduced in [16] is a compact matrix quantum group.

Now we turn to the non-compact case. Then a $C^*$-algebra $A$ is non-unital. In the case of a non-compact locally compact space $\Lambda$, $A = C_\infty(\Lambda)$ consists of continuous functions vanishing at infinity on $\Lambda$. On the other hand one has to consider also other classes of continuous functions, such

as bounded or all ones. The counterparts of above notions for general noncommutative $C^*$-algebra are provided by concepts of a multiplier and an affiliated element. We recall these notions.

Let $A$ be a (separable) $C^*$-algebra. To simplify further considerations we assume that $A$ is a non-degenerate $C^*$-algebra of operators acting on the Hilbert space $H$, $A \subset B(H)$. The non-degeneracy means that $AH$ is dense in $H$. To abbreviate the notation we write $A \in C^*(H)$.

The multiplier algebra of $A$ is denoted by $M(A)$ and

$$M(A) = \left\{ a \in B(H) : \; aA \subset A \text{ and } Aa \subset A \right\}.$$

Clearly $M(A)$ is a unital $C^*$-subalgebra of $B(H)$. Now we define elements affiliated with $A$. These should be treated as "unbounded" multipliers. To be more precise, let $T$ be a closed densely defined linear operator acting on $H$. Its $z$-transform is by definition

`zT` (1.4)
$$z_T = T(I + T^*T)^{-\frac{1}{2}}.$$

Then $z_T$ is a bounded operator and $\|z_T\| \leq 1$. One should notice that $\|T\| < \infty$ if and only if $\|z_T\| < 1$. The affiliation relation, denoted by $\eta$, is introduced as follows.

`aff` (1.5)
$$\left( \; T \, \eta \, A \; \right) \Longleftrightarrow \left( \begin{array}{c} z_T \in M(A) \text{ and} \\[2mm] (I + T^*T)^{-\frac{1}{2}} A \text{ is dense in } \; A \end{array} \right).$$

The set of all affiliated elements is denoted by $A^\eta$. Let us remark that since $(I + T^*T)^{-\frac{1}{2}}$ is selfadjoint, the density of $(I + T^*T)^{-\frac{1}{2}} A$ is equivalent to the density of $A(I + T^*T)^{-\frac{1}{2}}$. Since $(z_T)^* = z_{T^*}$, $z_T \in M(A)$ if and only if both $z_T$ and $z_{T^*}$ are right multipliers of $A$.

Clearly $A \subset M(A) \subset A^\eta$. Moreover if $T \in A^\eta$ and $T$ is bounded then $T \in M(A)$. If $A$ is unital then $A = M(A) = A^\eta$. On the other hand for $A = C_\infty(\Lambda)$ we have $M(A) = C_{\text{bounded}}(\Lambda)$ and $A^\eta = C(\Lambda)$. For the algebra $A = \mathcal{K}(H)$ (of all compact operators on $H$) we get $M(A) = B(H)$ and $A^\eta$ is the set of all closed densely defined operators acting on $H$. Therefore in general $A^\eta$ is not an algebra or even a vector space. Nevertheless the set $A^\eta$ is closed under $^*$-operation: $T^* \, \eta \, A$ and $T^*T \, \eta \, A$ for any $T \in A^\eta$.

The natural topology of $M(A)$ is that of pointwise convergence on $A$. This is called a strict topology of $M(A)$. It is known that $M(A)$ equipped with this topology is a topological $*$-algebra. In turn the strict topology of $M(A)$ induces (by $z$-transform) a natural topology of $A^\eta$.

Let $A$ and $B$ be $C^*$-algebras and let $B \in C^*(K)$. We say that $\pi$ is a morphism from $A$ into $B$ if $\pi$ is a (non-degenerate) representation of $A$ on the Hilbert space $K$ and $\pi(A)B$ is dense subset of $B$. The set of all morphism from $A$ into $B$ is denoted by $\text{Mor}(A, B)$. In particular $\pi \in \text{Mor}(A, \mathcal{K}(K))$. Clearly $\pi$ maps $A$ into $M(B)$ and one can easily prove that any $\pi \in \text{Mor}(A, B)$ has unique extension to a $C^*$-algebra homomorphism from $M(A)$ into $M(B)$ (this allows to make a composition of morphisms possible) and then to a $*$-preserving map from $A^\eta$ into $B^\eta$.

Let us note that a bad algebraic structure of $A^\eta$ is the main source of a great discrepancy between Hopf $*$-algebra approach to non-compact quantum groups and $C^*$-algebra one. In this case elements of Hopf algebra have to be represented in general by unbounded (closed and densely defined) operators on a Hilbert space. Since the sum and the product of two such operators are badly defined, no representation of the whole Hopf algebra exists in general. There are also more subtle reasons of the discrepancy coming from the existence of symmetric operators without self-adjoint extensions or due to the phenomenon of commuting but not strongly commuting pairs of selfadjoint operators. These diversity of situations not apparent on the Hopf algebra level makes the constructions of topological quantum groups much more sophisticated. On the other hand the nice algebraic structure of $A^\eta = A$ for unital $C^*$-algebra $A$ explains why the Statement 1 and 2 of Theorem 1.2 holds for compact quantum groups and fails in non-compact case.

Now we are ready to describe briefly the present status of the theory of locally compact quantum groups. The theory of multiplicative unitary operators plays the central role in the approach. It was developed by S.Baaj and G.Skandalis [3]. Let $H$ be a Hilbert space. A unitary operator $W$ acting on $H \otimes H$ is called multiplicative unitary if it satisfies the pentagon equation:

`WU` (1.6)
$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

Such operators have appeared long ago in the theory of locally compact groups in the context of generalized Pontryagin duality. Let $G$ be a locally compact group and $H = L^2(G, dg)$, where $dg$ is

a right-invariant Haar measure. For any $x \in H$ and $g, g' \in G$ we set

$\boxed{\text{kac}}$ (1.7) $$(Wx)(g, g') = x(gg', g').$$

Then one can easily verify that $W$ is a unitary operator acting on $H \otimes H$. Moreover

$$\left(W_{23}W_{12}x\right)(g_1, g_2, g_3) = x(g_1(g_2g_3), g_1g_2, g_2), \quad \left(W_{12}W_{13}W_{23}x\right)(g_1, g_2, g_3) = x((g_1g_2)g_3, g_1g_2, g_2)$$

for any $x \in H \otimes H \otimes H$ and any $g_1, g_2, g_3 \in G$. Now the pentagon equation (1.6) is equivalent to the associativity of the group multiplication. The operator $W$ introduced by formula (1.7) is called Kac-Takesaki operator. It contains the full information about the group $G$. Following this idea to any multiplicative unitary operator $W$ one may try to assign a quantum group. It turns out [3] that it is possible if pentagon equation is supplemented by a *regularity condition*:

$\boxed{\text{reg}}$ (1.8) $$\left\{(\mathrm{id} \otimes \omega)(\Sigma W) : \omega \in B(H)_*\right\}^{\text{norm closure}} = \mathcal{K}(H)$$

where $\Sigma \in B(H \otimes H)$ is a flip automorphism, $\Sigma(x \otimes y) = y \otimes x$ for any $x, y \in H$ and $B(H)_*$ is the space of all normal functionals on $B(H)$.

Unfortunately this theory does not apply to all multiplicative unitaries related to quantum groups [1],[2]. To overcome this difficulty the regularity condition was replaced by a less restrictive condition of manageability [23]. It was shown [6] that any quantum group may be related to a manageable multiplicative unitary. On the other hand it is not easy to verify manageability condition in particular examples. Moreover the natural choice for the multiplicative unitaries in specific examples is not manageable. Let us stress that the correspondence between multiplicative unitaries and quantum groups is not one to one. Different multiplicative unitaries may describe the same quantum object. Using this non-uniqueness a weakened condition of modularity was introduced in [13].

A multiplicative unitary $W$ is called *modular* if there exist strictly positive selfadjoint operators $\hat{Q}$ and $Q$ acting on $H$ and a unitary operator $\tilde{W}$ acting on $\overline{H} \otimes H$ such that

$\boxed{\text{modular0}}$ (1.9) $$W(\hat{Q} \otimes Q) = (\hat{Q} \otimes Q)W$$

and

$\boxed{\text{modular}}$ (1.10) $$(x \otimes u | W | z \otimes y) = \left(\bar{z} \otimes Qu \,\Big|\, \tilde{W} \,\Big|\, \bar{x} \otimes Q^{-1}y\right)$$

for any $x, z \in H$, $u \in \mathcal{D}(Q)$ and $y \in \mathcal{D}(Q^{-1})$. In the above definition $\overline{H}$ is the complex conjugate Hilbert space related to $H$ by the antiunitary mapping $H \ni x \longrightarrow j(x) = \bar{x} \in \overline{H}$. In what follows $\top$ will denote a transposition map

$\boxed{\text{transp}}$ (1.11) $$B(H) \ni m \longrightarrow m^\top = j \circ m^* \circ j^{-1} \in B(\overline{H}).$$

Clearly it is antiisomorphisms of $C^*$-algebras.

Now multiplicative unitary is *manageable* whenever it is modular with $\hat{Q} = Q$. We have

$$\text{regularity} \implies \text{manageability} \implies \text{modularity}.$$

Let $A$ be a $C^*$-algebra and $\Delta \in \mathrm{Mor}(A, A \otimes A)$. We say that $(A, \Delta)$ is a $C^*$-bialgebra if $\Delta$ is coassociative. The following result [13, Theorem 2.3] is a structure theorem for bialgebra $(A, \Delta)$ which is related to a modular multiplicative unitary.

$\boxed{\text{GMU}}$ **Theorem 1.4.** *Let $W \in B(H \otimes H)$ be a modular multiplicative unitary. Define*

$\boxed{\text{AMU}}$ (1.12) $$A = \{(\omega \otimes \mathrm{id})W : \omega \in B(H)_*\}^{\text{norm closure}}$$
$$\hat{A} = \{(\mathrm{id} \otimes \omega)W^* : \omega \in B(H)_*\}^{\text{norm closure}}.$$

*Then*

(1) *$A$ and $\hat{A}$ are nondegenerate separable $C^*$-subalgebras in $B(H)$.*
(2) *$W \in M(\hat{A} \otimes A)$.*
(3) *There exists a unique $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that*

$$(\mathrm{id} \otimes \Delta)W = W_{12}W_{13}.$$

*Moreover $\Delta$ is coassociative and*

$$\{(b \otimes I)\Delta(a) : a, b \in A\}, \qquad \{(I \otimes b)\Delta(a) : a, b \in A\}$$

*are total subsets of $A \otimes A$.*

(4) *There exists a unique closed linear operator $\kappa$ on the Banach space $A$ such that $\{(\omega \otimes \mathrm{id})W : \omega \in B(H)_*\}$ is a core for $\kappa$ and*

$$\kappa\left((\omega \otimes \mathrm{id})W\right) = (\omega \otimes \mathrm{id})W^*$$

*for any $\omega \in B(H)_*$. Moreover*
   (i) *the domain of $\kappa$ is a subalgebra of $A$ and $\kappa$ is antimultiplicative: $\kappa(ab) = \kappa(b)\kappa(a)$ for any $a, b \in \mathcal{D}(\kappa)$.*
   (ii) *$\kappa\left((\mathcal{D}(\kappa)\right) = \mathcal{D}(\kappa)^*$ and $\kappa\left(\kappa(a^*)^*\right) = a$ for all $a \in \mathcal{D}(\kappa)$.*
   (iii) *the operator $\kappa$ admits the following polar decomposition*

$$\kappa = R \circ \tau_{i/2},$$

   *where $R$ is involutive (normal) antiautomorphism of $A$ and $\tau_{i/2}$ is the analytic generator of a one parameter group of $*$-automorphisms $\{\tau_t\}_{t \in \mathbb{R}}$ of the $C^*$-algebra $A$,*
   (iv) *$R$ commutes with $\tau_t$ for any $t \in \mathbb{R}$, in particular $\mathcal{D}(\tau) = \mathcal{D}(\tau_{i/2})$,*
   (v) *$R$ and $\{\tau_t\}_{t \in \mathbb{R}}$ are uniquely determined.*
(5) *We have*
   (i) *$\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$ for all $t \in \mathbb{R}$,*
   (ii) *$\Delta \circ R = \sigma(R \otimes R) \circ \Delta$, where $\sigma$ is a flip automorphisms of $A \otimes A$.*
(6) *Let $\tilde{W}$ and $Q$ be the operators related to $W$ by modularity condition. Then*
   (i) *$\tau_t(a) = Q^{2it}aQ^{-2it}$ for any $a \in A$ and $t \in \mathbb{R}$,*
   (ii) *$W^{\top \otimes R} = \tilde{W}^*$, where $a^R$ denotes $R(a)$ for any $a \in A$ and $\top$ is the transposition map (1.11).*

The antiautomorphism $R$ which appear in the polar decomposition of the antipode $\kappa$ is called the unitary antipode and $\{\tau_t\}_{t \in \mathbb{R}}$ is called a scaling group. It is clear that for any locally compact group $\kappa = R$ and the scaling group is trivial.

Now using above Theorem we can say that $C^*$-bialgebra $G = (A, \Delta)$ is a quantum group if it is related to some modular multiplicative unitary in the above sense ($A$ coincides with the $C^*$-algebra introduced by he first formula (1.12) and $\Delta$ as in Statement 3). To verify such definition one has to know a multiplicative unitary $W$ in advance. It is not easy. But if this is the case one has the rich theory of modular multiplicative operators in disposition. In particular the operator $\widehat{W} = \Sigma W^* \Sigma$ is a modular multiplicative unitary (operators $Q$ and $\hat{Q}$ exchanges their position) and a quantum group related to $\widehat{W}$ is a Pontryagin dual group $\hat{G} = (\hat{A}, \hat{\Delta})$.

As we noticed the above concept of quantum group is very involved and it would be nice to have much simpler set of axioms that will guarantee that a $C^*$-bialgebra $(A, \Delta)$ is a quantum group in the above sense. Now the situation is much better than a few years ago. There are two approaches in this direction. The first one, proposed by J.Kustermans and S.Vaes is very close to Definition 1.1 for the compact quantum group. A supplementary assumption is the existence of a left-invariant faithful weight and right-invariant one. It is very interesting that their theory anticipated non-invariance of Haar wights with respect to the scaling group. In [14] A.Van Daele shows that it really happens in the case of the quantum 'az + b' group for deformation parameter being a root of unity.

The second approach, presented in [6] assumes the existence only one faithful Haar weight but in addition one postulates the polar decomposition of antipode $\kappa$. Then clearly, if $\phi$ is a left-invariant weight then composing it with unitary antipode one gets a right-invariant one.

In both approaches the existence of a Haar weight is postulated. On the other hand when dealing with examples one is able to find such weight (cf [14], [26]). In particular when modular multiplicative unitary is known then a formula for an invariant weight can be derived [26, Theorem 1.1]. Let us recall that a (right) Haar weight on the quantum group $G = (A, \Delta)$ is by definition a lower semicontinuous faithful locally finite weight $h$ on $A$ such that

Haar      (1.13)                          $$h\left[(\mathrm{id} \otimes \phi)\Delta(a)\right] = \phi(I)h(a)$$

for any positive functional $\phi$ on $A$ and any positive $a \in A$ such that $h(a) < \infty$. A weight $h$ is a locally finite if the set $\{c \in A : h(c^*c) < \infty\}$ is dense in $A$.

fHaar     **Proposition 1.5.** *Let $G = (A, \Delta)$ be a quantum group related to a modular multiplicative $W$ and $Q$ and $\hat{Q}$ be strictly positive selfadjoint operators entering formulae (1.9) and (1.10). For any positive $c \in A$ we set*

fHaar1    (1.14)                          $$h(c) = \mathrm{Tr}(\hat{Q}c\hat{Q}).$$

*Then $h$ is faithful lower semicontinuous weight on $A$. Assume that $h$ is locally finite. Then $h$ is a (right) Haar weight on the quantum group $G$.*

At the moment it is not clear whether for any quantum group there exists a modular multiplicative unitary such that the weight (1.14) is locally finite.

Now we shall consider concepts of $C^*$-algebra generated by families of affiliated elements. For a non-unital $C^*$-algebra the problem of generating is more complicated. Even when we deal with classical non-compact groups the matrix elements of unitary representations are unbounded continuous functions in general. Therefore they are only affiliated with algebra of continuous functions vanishing at infinity. This means that for non-unital $C^*$-algebra $A$ one has to precise what does it mean that $A$ is generated by elements which does not belong to $A$. This problem was solved in [21]. It turns out that algebra $A$ should to be known in advance. At first we recall the concept of a $C^*$-algebra generated by finite set of affiliated elements [21, Definitin 3.1].

Let $A$ be a $C^*$-algebra and $T_j \in A^\eta$, $j = 1, 2, ...N$. We say that $A$ is generated by $T_1, T_2, ....T_N$ if for any Hilbert space $K$, any $B \in C^*(K)$ and any $\pi \in \mathrm{Rep}(A, K)$ we have

<div style="text-align:left">`genkl`</div> (1.15)
$$\begin{pmatrix} \pi(T_j)\,\eta\,B \text{ for any} \\ j = 1, 2, ...N \end{pmatrix} \Longrightarrow \Big(\ \pi \in \mathrm{Mor}(A, B).\ \Big)$$

This condition is not easy to verify but we have a nice criterion ([21, Theorem 3.3].

<div style="text-align:left">`kryt1`</div> **Theorem 1.6.** *Let $A$ be a $C^*$-algebra and $T_j \in A^\eta$ for any $j = 1, 2, ...N$ and let*
$$\mathcal{R} = \left\{ (I + T_j^* T_j)^{-1}, (I + T_j T_j^*)^{-1} : j = 1.2, ...N \right\}.$$
*Assume that*

*1. $T_1, T_2, ...T_N$ separates representations of $A$ : if $\varphi_1, \varphi_2$ are different elements of $\mathrm{Rep}(A, H)$ then $\varphi_1(T_j) \neq \varphi(T_j)$ for some $j = 1, 2, ...N$.*

*2. There exist elements $r_1, r_2, ...r_k \in \mathcal{R}$ such that the product $r_1 r_2 ... r_k \in A$.*

*Then $A$ is generated by $T_1, T_2, ....T_N$.*

For commutative $C^*$-algebra this criterion simplifies ([21, Exemple 2].

<div style="text-align:left">`kryt2`</div> **Proposition 1.7.** *Let $\Lambda$ be a locally compact space and $f_1, f_2, ...f_N \in C(\Lambda)$. Assume that $f_1, f_2, ...f_N$ separates points of $\Lambda$ and*
$$\lim_{\lambda \to \infty} \sum_{j=1}^{N} |f_j(\lambda)| = +\infty.$$
*Then $A$ is generated by $f_1, f_2, ....f_N$.*

To introduce a notion of a quantum group of unitary operators we shall use the concept of a C$^*$-algebra generated by a quantum family of affiliated elements [24, Definition 4.1].

Let $C$, $A$ be $C^*$-algebras and $V$ be an element affiliated with $C \otimes A$. We may regard $V$ as a family of elements of $C^\eta$ labelled by the "quantum space" $A$. We say that $A$ is generated by an element $V \eta (C \otimes A)$ if and only if for any representation $\pi$ of $A$ and any $B \in C^*(H_\pi)$ we have:

<div style="text-align:left">`genkw`</div> (1.16)
$$\Big((\mathrm{id} \otimes \pi)V \eta (C \otimes B)\Big) \Longrightarrow \Big(\pi \in \mathrm{Mor}(A, B)\Big)$$

Let us note that if $V$ generates $A$ and $B$ is a $C^*$-algebra then any morphism $\phi \in \mathrm{Mor}(A, B)$ is completely determined by its value on $V$. To be more precise, let $\phi_1, \phi_2 \in \mathrm{Mor}(A, B)$ and $B \in C^*(K)$. Then

<div style="text-align:left">`phijed`</div> (1.17)
$$\Big((\mathrm{id} \otimes \phi_1)V = (\mathrm{id} \otimes \phi_2)V\Big) \Longrightarrow \Big(\phi_1 = \phi_2\Big)$$

Indeed, let $\tilde{\phi} = \phi_1 \oplus \phi_2$. Then $\tilde{\phi} \in \mathrm{Mor}(A, B \oplus B)$. Let $\tilde{B} = \{b \oplus b : b \in B\}$. Clearly $\tilde{B} \in C^*(K \oplus K)$ and one can easily verify that $\tilde{B}^\eta = \{b \oplus b : b \in B^\eta\}$. Now our assumption means that $(\mathrm{id} \otimes \tilde{\phi})V \eta C \otimes \tilde{B}$. Therefore $\tilde{\phi} \in \mathrm{Mor}(A, \tilde{B})$ and by definition of $\tilde{B}$
$$\tilde{\phi}(c) = \phi_1(c) \oplus \phi_2(c) \in M(\tilde{B}).$$
for any $c \in A$. Since $M(\tilde{B}) \subset \tilde{B}^\eta$, the statement is proven.

In this more general situation we also have a useful criterion (cf. [24, Example 10, page 507]):

**Proposition 1.8.** *Let $C$, $A$ be $C^*$-algebras and $V$ be a unitary element of $\mathrm{M}(C \otimes A)$. Assume that there exists a faithful representation $\phi$ of $C$ such that:*

    *1. For any $\phi$-normal linear functional $\omega$ on $C$ we have: $(\omega \otimes \mathrm{id})V \in A$*

    *2. The smallest $^*$-subalgebra of $A$ containing $\{(\omega \otimes \mathrm{id})V : \omega \text{ is } \phi\text{-normal}\}$ is dense in $A$.*

*Then $A$ is generated by $V \in \mathrm{M}(C \otimes A)$.*

Let us remind that a linear functional $\omega$ on $C$ is said to be $\phi$-normal if there exists a trace-class operator $\rho$ acting on $H_\phi$ such that $\omega(c) = \mathrm{Tr}(\rho\phi(c))$ for all $c \in C$.

A unitary element $V \in M(\mathcal{K}(K) \otimes A)$ may be treated as a "strongly continuous family" (labelled by the quantum space $A$) of unitary operators acting on the Hilbert space $K$. To precise when such a family is a (quantum) group we shall accept for the purpose of our paper the following definition.

**Definition 1.9.** *Let $A$ be a $C^*$-algebra, $K$ be a Hilbert space and let $V$ be a unitary element of $M(\mathcal{K}(K) \otimes A)$. Assume that*
*1. $A$ is generated by $V$.*
*2. There exists a morphism $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that*

$$(1.18) \qquad\qquad (\mathrm{id} \otimes \Delta)V = V_{12}V_{13}.$$

*Then we say that $(A, V)$ is a quantum group of unitary operators.*

**Remark 1.10.** By previous considerations there is at most one $\Delta \in \mathrm{Mor}(A, A \otimes A)$ satisfying (1.18). On the other hand if $\Delta$ exists then it is co-associative. Indeed, $\Phi_1 = (\mathrm{id} \otimes \Delta)\Delta$ and $\Phi_2 = (\Delta \otimes \mathrm{id})\Delta$ are both elements of $\mathrm{Mor}(A, A \otimes A \otimes A)$ and

$$(\mathrm{id} \otimes \Phi_1)V = V_{12}V_{13}V_{14} = (\mathrm{id} \otimes \Phi_2)V$$

Since they coincide on $V$, they coincide. Therefore $(A, \Delta)$ is a bi-algebra and $V$ is a co-representation. Now one can study whether $G = (A, \Delta)$ is a quantum group in the sense described in Section 1. If this is the case then $V$ may be treated as the fundamental unitary representation of $G$.

Recently (cf [11]) an approach basing on above concepts was used for construction of new deformations of quantum '$ax + b$' group. In this approach a role of generating aspects is more transparent. To demonstrate these ideas we consider the construction of quantum '$az + b$' group introduced in [20] from this point of view. This is a content of next sections. There are no new results concerning the theory of quantum '$az + b$' group. Nevertheless, with respect to the methodology and to the tools involved in the approach this presentation may be interesting.

## 2. Group $\Gamma$, related special functions and generating algebras

In this section we recall the basic facts concerning the construction of quantum '$az + b$' group for real values of deformation parameter (cf [24], [20], [9]). To this end for a fixed value of a real parameter $q$, $0 < q < 1$ we consider a multiplicative subgroup $\Gamma$ of nonzero complex numbers,

$$\Gamma = \left\{ z \in \mathbb{C} : |z| \in q^{\mathbb{Z}} \right\}.$$

Then $\Gamma$ is an abelian locally compact group. Denote by $d\gamma$ the Haar measure:

$$\int_\Gamma x(\gamma)d\gamma = \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \int_0^{2\pi} x(q^n e^{i\varphi})d\varphi.$$

Clearly any $\gamma \in \Gamma$ is of the form $\gamma = q^{i\varphi+n}$ for unique $n \in \mathbb{Z}$ and $\varphi \in \left[0, -\frac{2\pi}{\log q}\right[$. For any $\gamma, \gamma' \in \Gamma$ we set

$$(2.1) \qquad\qquad \chi(\gamma, \gamma') = \chi(q^{i\varphi+n}, q^{i\varphi'+n'}) = q^{i(\varphi n' + \varphi' n)}.$$

Then $\chi \colon \Gamma \times \Gamma \to S^1$ and $\chi$ is a symmetric function. One can easily check that

$$(2.2) \qquad\qquad \chi(\gamma, q) = \mathrm{Phase}\,\gamma, \qquad \chi(\gamma, q^{it}) = |\gamma|^{it}$$

for any $\gamma \in \Gamma$. Moreover $\chi$ is a nondegenerate bicharacter on $\Gamma$. Therefore we may identify $\Gamma$ with its Pontryagin dual $\widehat{\Gamma}$.

Let $\overline{\Gamma}$ denote the closure of $\Gamma$. Clearly $\overline{\Gamma} = \Gamma \cup \{0\}$. The $C^*$- algebras $C_\infty(\Gamma)$ and $C_\infty(\overline{\Gamma})$ play a key role in further consideration. We consider $C^*$-algebra $C_\infty(\Gamma)$ first.

Let

gengamma  (2.3) $$f_1(\gamma) = \gamma, \qquad f_2(\gamma) = \gamma^{-1}$$

for any $\gamma \in \Gamma$. Then $f_1, f_2 \in C(\Gamma) = C_\infty(\Gamma)^\eta$. Moreover $f_1$ and $f_2$ separates points of $\Gamma$ and $|f_1(\gamma)| + |f_1(\gamma| \longrightarrow +\infty$ whenever $\gamma$ tends to infinity in $\Gamma$, i.e. $|\gamma| \longrightarrow 0$ or $|\gamma| \longrightarrow +\infty$. Therefore by Proposition 1.7, $f_1, f_2$ generates $C_\infty(\Gamma)$.

Let $X$ be a normal operator acting on the Hilbert space $K$. Assume that $X$ is invertible and $\mathrm{Sp}\, X \subset \overline{\Gamma}$. Then the mapping

repres1  (2.4) $$C_\infty(\Gamma) \ni f \longrightarrow \pi(f) = f(X) \in B(K)$$

is a representation of $C_\infty(\Gamma)$ acting on $K$. Operators $X$ and $X^{-1}$ are determined by $\pi$. Indeed, $X = \pi(f_1)$ and $X^{-1} = \pi(f_2)$, where $f_1, f_2$ are given by (2.3). Recall that $f_1, f_2$ generate $C_\infty(\Gamma)$. Therefore for any representation $\pi$ of $C_\infty(\Gamma)$ and any $B \in C^*(H_\pi)$ we have:

$$\Big(\pi(f_1), \pi(f_2)\, \eta\, B\Big) \Longrightarrow \Big(\pi \in \mathrm{Mor}(C_\infty(\Gamma), B)\Big) \Longrightarrow \Big(\pi(f)\, \eta\, B \text{ for any } f \in C(\Gamma)\Big)$$

In particular for $\pi$ introduced by (2.4) and $B \in C^*(K)$ we get:

tauT  (2.5) $$\begin{pmatrix} X, X^{-1}\, \eta\, B \\ f \in C(\Gamma) \end{pmatrix} \Longrightarrow \Big(f(X)\, \eta\, B\Big).$$

Let $f_\gamma(\gamma') = \chi(\gamma', \gamma)$ for any $\gamma, \gamma' \in \Gamma$. Then $f_\gamma \in C(\Gamma)$ and $f_\gamma(X) = \chi(X, \gamma)$ is a unitary element of $B(K)$. Let us note that $X$ is completely determined by $\chi(X, \gamma)$. Indeed, using (2.2) one can easily show

chirownosc  **Proposition 2.1.** *Let $X_k$ ($k = 1, 2$) be a normal invertible operator acting on a Hilbert space $K$ and such that $\mathrm{Sp}\, X_k \subset \overline{\Gamma}$. Then*

$$\begin{pmatrix} \chi(X_1, \gamma) = \chi(X_2, \gamma) \\ \text{for all } \gamma \in \Gamma \end{pmatrix} \Longleftrightarrow \Big( X_1 = X_2 \Big).$$

Assume that $X$ is normal invertible operator and $\mathrm{Sp}\, X \subset \Gamma$. Then the mapping

chiX  (2.6) $$\Gamma \ni \gamma \longrightarrow \chi(X, \gamma) \in B(K)$$

is strongly continuous. By the general theory strongly continuous mappings from $\Gamma$ into the set of unitary operators acting on $K$ correspond to unitary multipliers of $\mathcal{K}(K) \otimes C_\infty(\Gamma)$.

chiXgen  **Proposition 2.2.** *Let $X$ be a normal invertible operator acting on a Hilbert space $K$ and $\mathfrak{X} \in M(\mathcal{K}(K) \otimes C_\infty(\Gamma))$ be the unitary corresponding to the mapping (2.6). Assume that the spectral measure of $X$ is absolutely continuous with respect to the Haar measure on $\Gamma$.*
*Then $\mathfrak{X}$ generates $C_\infty(\Gamma)$.*

*Proof.* We use Proposition 1.8. For any normal linear functional $\omega$ on $B(K)$ we set $f_\omega = (\omega \otimes \mathrm{id})\mathfrak{X}$. Then $f_\omega \in M(C_\infty(\Gamma)) = C_{\mathrm{bounded}}(\Gamma)$. Clearly

$$f_\omega(\gamma) = \omega\Big(\chi(X, \gamma)\Big)$$

for any $\gamma \in \Gamma$. Since the spectral measure of $X$ is absolutely continuous with respect to the Haar measure, $f_\omega \in C_\infty(\Gamma)$ by the Riemann-Lebesgue lemma.

We shall show that $f_\omega$ separates points of $\Gamma$. To this end let $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$. Suppose that $f_\omega(\gamma) = f_\omega(\gamma')$ for all $\omega$. Then $\chi(X, \gamma) = \chi(X, \gamma')$ and $\chi(X, \gamma_0) = I$ where $\gamma_0 = \gamma'\gamma^{-1}$. This means that the spectral measure of $X$ is supported by the set $\{z \in \Gamma : \chi(z, \gamma_0) = 1\}$. Inspecting formula (2.1) we find that this is a discrete subset of $\Gamma$. This is in contradiction with the assumption of absolute continuity with respect to the Haar measure. Therefore $f_\omega$ separates points of $\Gamma$. Now by the Stone-Weierstrass theorem the smallest *-subalgebra of $C_\infty(\Gamma)$ containing all $f_\omega$ is dense in $C_\infty(\Gamma)$. This ends the proof.

$\square$

As a conclusion we formulate the following Proposition which will be very useful in further considerations.

chigen  **Proposition 2.3.** *Let $X$ be a normal invertible operator acting on a Hilbert spaces $K$. Assume that $\mathrm{Sp}\, X \subset \overline{\Gamma}$ and the spectral measure of $X$ is absolutely continuous with respect to the Haar measure.*

*Let $Z$ be a normal invertible operator acting on a Hilbert space $H$. Assume that $\operatorname{Sp} Z \subset \overline{\Gamma}$. Then for any $A \in C^*(H)$ we have:*

$$\left( \chi(X \otimes I, I \otimes Z) \in M(\mathcal{K}(K) \otimes A) \right) \implies \left( Z, Z^{-1} \, \eta \, A \right)$$

*Proof.* For any $f \in C_\infty(\Gamma)$ we set $\pi(f) = f(Z)$. Then $\pi$ is a representation of $C_\infty(\Gamma)$ acting on the Hilbert space $H_\pi = H$. Let $\mathfrak{X} \in \mathrm{M}(\mathcal{K}(K) \otimes C_\infty(\Gamma))$ be unitary introduced in Proposition 2.2. A moment of reflection shows that $(\mathrm{id} \otimes \pi)\mathfrak{X} = \chi(X \otimes I, I \otimes Z)$. If $\chi(X \otimes I, I \otimes Z)$ is affiliated with $\mathcal{K}(K) \otimes A$ then $\pi \in \mathrm{Mor}(C_\infty(\Gamma), A)$ and $\pi$ maps continuous functions on $\Gamma$ into elements affiliated with $A$. Applying this rule to the functions $f_1$, $f_2$ introduced by (2.3) we obtain $Z = \pi(f_1) \, \eta \, A$, $Z^{-1} = \pi(f_2) \, \eta \, A$. $\qquad\square$

We shall need an operator version of Proposition 2.1.

**Proposition 2.4.** *Let $X$ be a normal invertible operator acting on a Hilbert spaces $K$. Assume that $\operatorname{Sp} X \subset \overline{\Gamma}$ and the spectral measure of $X$ is absolutely continuous with respect to the Haar measure. Let $Z_k$ $(k = 1, 2)$ be a normal invertible operator acting on a Hilbert space $H$. Assume that $\operatorname{Sp} Z_k \subset \overline{\Gamma}$. Then*

$$\left( \chi(X \otimes I, I \otimes Z_1) = \chi(X \otimes I, I \otimes Z_2) \right) \implies \left( Z_1 = Z_2 \right).$$

*Proof.* Let $A = \{m \oplus m : m \in \mathcal{K}(H)\}$ and $Z = Z_1 \oplus Z_2$. Then $\chi(X \otimes I, I \otimes Z) \in M(\mathcal{K}(K) \otimes A)$ due to the assumption. Therefore $Z \in A^\eta$ by Proposition 2.3. This means that $Z_1 = Z_2$. $\qquad\square$

Now we pass to the set $\overline{\Gamma}$ and the $C^*$-algebra $C_\infty(\overline{\Gamma})$.

Let

(2.7) $$f_0(\gamma) = \gamma$$

for any $\gamma \in \overline{\Gamma}$. Then $f_0 \in C(\overline{\Gamma}) = C_\infty(\overline{\Gamma})^\eta$. Using Proposition 1.7 one easily verifies that $C_\infty(\overline{\Gamma})$ is generated by $f_0$.

Let $Y$ be a normal operator acting on a Hilbert space $K$ and $\operatorname{Sp} Y \subset \overline{\Gamma}$. Then the mapping

(2.8) $$C_\infty(\overline{\Gamma}) \ni f \longrightarrow \pi(f) = f(Y) \in B(K)$$

is a representation of $C_\infty(\overline{\Gamma})$ acting on $K$. The operator $Y$ is determined by $\pi$, $Y = \pi(f_0)$ (cf (2.7)). Since $f_0$ generate $C_\infty(\overline{\Gamma})$,

$$\left( \pi(f_0) \, \eta \, B \right) \implies \left( \pi \in \mathrm{Mor}(C_\infty(\overline{\Gamma}), B) \right) \implies \left( \pi(f) \, \eta \, B \text{ for any } f \in C(\overline{\Gamma}) \right)$$

for any representation $\pi$ of $C_\infty(\overline{\Gamma})$ and any $B \in C^*(H_\pi)$. In particular for $\pi$ introduced by (2.8) and $B \in C^*(K)$ we get

(2.9) $$\left( \begin{array}{c} Y \, \eta \, B \\ f \in C(\overline{\Gamma}) \end{array} \right) \implies \left( f(Y) \, \eta \, B \right).$$

Now consider a special function $F_q \colon \overline{\Gamma} \to \mathbb{C}$. This is a *quantum exponential function* introduced in [24] by the formula

$$F_q(\gamma) = \prod_{k=0}^{\infty} \frac{1 + q^{2k}\overline{\gamma}}{1 + q^{2k}\gamma}$$

for $\gamma \in \overline{\Gamma} \setminus \{-1, -q^{-2}, -q^{-4}, \ldots\}$. Setting $F_q(\gamma) = -1$ for $\gamma \in \{-1, -q^{-2}, -q^{-4}, \ldots\}$ one gets a continuous function on $\overline{\Gamma}$. In addition $F_q(0) = 1$.

Let $z, \gamma \in \Gamma$. Due to [24, p. 427] the asymptotic behavior of $F_q(z\gamma)$ for large $|z\gamma|$ is described by the formula

(2.10) $$F_q(z\gamma) \sim \alpha(z)\alpha(\gamma)\chi(z, \gamma)$$

where $\alpha(z) = (\operatorname{Phase} z)^{\log_q |z|^{-1}}$ and '$\sim$' means that the difference goes to 0 when $|z\gamma| \longrightarrow \infty$.

We know that $F_q \in C(\overline{\Gamma})$ and assumes values of modulus one. Therefore if $Y$ is a normal operator acting on a Hilbert space $H$ and $\operatorname{Sp} Y \subset \overline{\Gamma}$ then $F_q(Y)$ is unitary. Moreover we have

Fqrownosc **Proposition 2.5.** *Let $Y_k$ $(k = 1, 2)$ be normal operator acting on a Hilbert space $H$ and such that* $\operatorname{Sp} Y_k \subset \overline{\Gamma}$. *Then*

$$\left( \begin{array}{c} F_q(zY_1) = F_q(zY_2) \\[2mm] \text{for all } z \in \Gamma \end{array} \right) \Longleftrightarrow \left( \; Y_1 = Y_2 \; \right).$$

*Proof.* One may proceed as in the proof of [10, Lemma 3.1] but here we present another proof. It is known (cf [24, p.425]) that asymptotic behavior of $F_q(\gamma)$ for small $\gamma$ is described by the formula

asF (2.11)
$$F_q(\gamma) = 1 - \frac{\gamma}{1 - q^2} + \frac{\overline{\gamma}}{1 - q^2} + o(|\gamma|).$$

Let

fnF (2.12)
$$f_n(\gamma) = \frac{1}{2\pi i} \int_{|z| = q^n} \frac{F_q(z\gamma)}{z} \frac{dz}{z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{F_q(q^n e^{i\varphi} \gamma)}{q^n e^{i\varphi}} d\varphi.$$

where $n$ is an integer. Then $f_n \in C_{\text{bounded}}(\overline{\Gamma})$. Set $n \longrightarrow +\infty$. Then $f_n(\gamma) \longrightarrow -\frac{1}{1-q^2}\gamma$ for all $\gamma \in \overline{\Gamma}$ due to (2.11). The convergence is almost uniform. Therefore if $Y$ is a normal operators acting on a Hilbert space $H$ and $\operatorname{Sp} Y \subset \overline{\Gamma}$ we have

$$\lim_{n \to +\infty} f_n(Y) = -\frac{1}{1 - q^2} Y$$

in a natural topology (cf. [21]) on the set of affiliated elements $\mathcal{K}(H)^\eta$. We know that $F_q(zY_1) = F_q(zY_2)$. Therefore $f_n(Y_1) = f_n(Y_2)$ and $Y_1 = Y_2$ (the limit is unique). $\qquad \square$

To reveal the usefulness of $F_q$ we need a notion of a $G$-pair. This notion involves a pair $(X, Y)$ of normal operators and assigns a precise meaning to the relations of the form

Gpair (2.13)
$$XY = q^2 Y X, \qquad XY^* = Y^* X.$$

They were investigated in [24],[20].

q2 **Definition 2.6.** *Let $X$ and $Y$ are closed densely defined operators acting on a Hilbert space $H$. We say that $(X, Y)$ is a $G$-pair on $H$ if $X$ and $Y$ are normal, $\operatorname{Sp} X$, $\operatorname{Sp} Y \subset \overline{\Gamma}$, $\ker X = \{0\}$ and*

q2pair (2.14)
$$\chi(X, \gamma) Y \chi(X, \gamma)^* = \gamma Y$$

*for all $\gamma \in \Gamma$.*

Setting $\gamma = q$ and $\gamma = q^{it}$ in the above formula we have (cf (2.2))

$$(\text{Phase } X) Y (\text{Phase } X)^* = qY \quad \text{and} \quad |X|^{it} Y |X|^{-it} = q^{it} Y.$$

respectively. In particular $|X|$ and $|Y|$ strongly commute and $(\text{Phase } X) |Y| = q |Y| (\text{Phase } X)$.

Weyl **Remark 2.7.** It is known that if $(X, Y)$ is a $G$-pair on $H$ then $(Y^*, X^*)$ and $(XY, Y)$ are $G$-pairs on $H$ as well. If in addition $Y$ is an invertible operator then formula (2.14) takes the form of Weyl relation:

q2pairp (2.15)
$$\chi(X, \gamma) \chi(Y, \gamma') = \chi(\gamma, \gamma') \chi(Y, \gamma') \chi(X, \gamma)$$

for any $\gamma, \gamma' \in \Gamma$. Then one can show that $(Y, X^{-1})$, $(Y^{-1}, X)$ and $(Y^{-1}, Y^{-1}X)$ are $G$-pairs on $H$.

We shall need the following result [24, Theorem 2.1, Theorem 2.2 and Theorem 3.1].

Qexp **Theorem 2.8.** *Let $(X, Y)$ be a $G$-pair on a Hilbert space $H$. Then the sum $Y + X$ is a densely defined closeable operator and its closure $Y \dotplus X$ is a normal operator and $\operatorname{Sp}(Y \dotplus X) \subset \overline{\Gamma}$. Moreover*

EXP (2.16)
$$F_q(Y \dotplus X) = F_q(Y) F_q(X).$$

*If in addition $\ker Y = \{0\}$ then*

Fqsum (2.17)
$$Y \dotplus X = F_q(Y^{-1}X) Y F_q(Y^{-1}X)^*.$$

The reader should notice that the last formula combined with (2.16) leads to

FXFY (2.18)
$$F_q(Y)F_q(X) = F_q(Y^{-1}X)F_q(Y)F_q(Y^{-1}X)^*.$$

Formula (2.16) supports the name "quantum exponential function" assigned to the function $F_q$.

Now we shall introduce a generating element for $C_\infty(\overline{\Gamma})$ associated with $F_q$. To this end for any $z \in \Gamma$ and $\gamma \in \overline{\Gamma}$ we set:

unitarny (2.19)
$$\Phi(z, \gamma) = \overline{F_q(\gamma)}F_q(z\gamma).$$

Then $|\Phi(z, \gamma)| = 1$ and $\Phi$ is a continuous function on $\Gamma \times \overline{\Gamma}$. Therefore it may be treated as a unitary element of $\mathrm{M}\left(C_\infty(\Gamma) \otimes C_\infty(\overline{\Gamma})\right)$. We have

Gene1 **Proposition 2.9.** *The $C^*$-algebra $C_\infty(\overline{\Gamma})$ is generated by $\Phi \in \mathrm{M}\left(C_\infty(\Gamma) \otimes C_\infty(\overline{\Gamma})\right)$.*

*Proof.* We shall use Proposition 1.8 setting $C = C_\infty(\Gamma)$, $A = C_\infty(\overline{\Gamma})$ and $V = \Phi$. Denote by $dz$ the Haar measure on $\Gamma$ and let $\phi$ be the natural representation of $C_\infty(\Gamma)$ acting on $L^2(\Gamma, dz)$: $\phi(h)$ is the multiplication by $h$ for any $h \in C_\infty(\Gamma)$. Then $\phi$ is faithful representation and a linear functional $\omega$ on $C_\infty(\Gamma)$ is $\phi$-normal if and only if it is of the form

$$\omega(h) = \int_\Gamma h(z)\varphi_\omega(z)\, dz,$$

where $\varphi_\omega \in L^1(\Gamma, dz)$.

Let $f_\omega = (\omega \otimes \mathrm{id})\Phi$. Then $f_\omega \in \mathrm{M}\left(C_\infty(\overline{\Gamma})\right)$ i.e. $f_\omega$ is a bounded continuous function on $\overline{\Gamma}$. Clearly for any $\gamma \in \overline{\Gamma}$ we have

funkcje (2.20)
$$f_\omega(\gamma) = \int_\Gamma \Phi(z, \gamma)\varphi_\omega(z)\, dz = \overline{F_q(\gamma)} \int_\Gamma F_q(z\gamma)\varphi_\omega(z)\, dz.$$

Now using the asymptotic behavior (2.10) and the Riemann–Lebesgue lemma one can show that the integral on the right hand side vanish when $|\gamma| \to +\infty$. This means that $f_\omega \in C_\infty(\overline{\Gamma})$.

To prove that the smallest $^*$-algebra containing all functions of the form (2.20) is dense in $C_\infty(\overline{\Gamma})$ we apply the Stone-Weierstrass theorem to the one point compactification of $\overline{\Gamma}$. Clearly for any $\gamma \in \overline{\Gamma}$ one can find a functional $\omega$ such that $f_\omega(\gamma) \neq 0$. It remains to show that functions $f_\omega$ separate points of $\overline{\Gamma}$. Let $\gamma, \gamma' \in \overline{\Gamma}$ and assume that $f_\omega(\gamma) = f_\omega(\gamma')$ for all $\phi$-normal functionals $\omega$. Then $\overline{F_q(\gamma)}F_q(z\gamma) = \overline{F_q(\gamma')}F_q(z\gamma')$ for all $z \in \Gamma$. Recall that $F_q$ is a continuous function and $F_q(0) = 1$. Therefore taking the limit $z \to 0$ we get $\overline{F_q(\gamma)} = \overline{F_q(\gamma')}$. This formula combined with the previous one imply that $F_q(z\gamma) = F_q(z\gamma')$ for all $z \in \Gamma$. Then for any integer $n$ the function $f_n$ introduced by formula (2.12) attains the same value on $\gamma$ and $\gamma'$, $f_n(\gamma) = f_n(\gamma')$. Remembering that $\lim_{n\to\infty} f_n(\gamma) = -\frac{1}{1-q^2}\gamma$ for any $\gamma \in \overline{\Gamma}$ we conclude that $\gamma = \gamma'$. The statement is proved. $\qquad\square$

To solve some technical problems which appear in further considerations we need the following result.

FToz **Proposition 2.10.** *Let $Y$, $U$ and $X$ be operators acting on a Hilbert space $H$ and $C \in C^*(H)$. Assume that:*

*1. $X$ and $Y$ are normal and $(X, Y)$ is a G-pair on $H$,*

*2. $U$ is unitary and commutes with $X$,*

*3. Operators $X$, $X^{-1}$, $Y$ and $U$ are affiliated with $C$.*

*Then $F_q(Y) \in \mathrm{M}(C)$ and*

*1. For any representation $\rho$ of $C$ and any $B \in C^*(H_\rho)$ we have:*

$$\begin{pmatrix} \rho(X),\ \rho(X^{-1}),\ \rho\left(F_q(Y)U\right) \\ \text{are affiliated with } B \end{pmatrix} \implies \begin{pmatrix} \rho(Y),\ \rho(U) \\ \text{are affiliated with } B \end{pmatrix}$$

*2. For any representations $\rho_1$ and $\rho_2$ of $C$ acting on the same Hilbert space $H_{\rho_1} = H_{\rho_2}$ we have:*

$$\begin{pmatrix} \rho_1(X) = \rho_2(X), \\ \rho_1\left(F_q(Y)U\right) = \rho_2\left(F_q(Y)U\right) \end{pmatrix} \implies \begin{pmatrix} \rho_1(Y) = \rho_2(Y), \\ \rho_1(U) = \rho_2(U) \end{pmatrix}$$

*Proof.* Relation $F_q(Y) \in \mathrm{M}(C)$ follows immediately from (2.9).

Ad 1. Let $z \in \Gamma$. Using the commutation relations satisfied by operators $X, Y$ and $U$ we have:

$$\chi(X, z)\, F_q(Y)U\, \chi(X, z)^* = F_q(zY)U.$$

Passing to a representation $\rho$ of $C$ we get

$$\chi\left(\rho(X), z\right) \rho\left(F_q(Y)U\right) \chi\left(\rho(X), z\right)^* = \rho\left(F_q(zY)U\right).$$

If $\rho(X)$, $\rho(X^{-1})$, $\rho\left(F_q(Y)U\right) \eta\, B$, then all factors on the left hand side of the above equation belong to $\mathrm{M}(B)$ and depend continuously on $z$ in the strict topology of $\mathrm{M}(B)$ (cf [20, Theorem 5.2]). Therefore $\rho\left(F_q(zY)U\right) \in \mathrm{M}(B)$ for any $\gamma \in \Gamma$ and the mapping

$$\Gamma \ni z \longmapsto \rho\left(F_q(zY)U\right) \in \mathrm{M}(B)$$

is strictly continuous. Multiplying from the right by the hermitian conjugation of $\rho\left(F_q(Y)U\right)$ we get

$$\rho\left(F_q(zY)F_q(Y)^*\right) = \rho\left(F_q(Y)^*F_q(zY)\right) = \rho\left(\Phi(z, Y)\right) \in \mathrm{M}(B)$$

where $\Phi$ is the function introduced by formula (2.19). Moreover the mapping

<div style="float:left">21</div>

(2.21)
$$\Gamma \ni z \longmapsto \rho\left(\Phi(z, Y)\right) \in \mathrm{M}(B)$$

is strictly continuous. By general theory (cf [21]) such mappings from $\Gamma$ into $\mathrm{M}(B)$ correspond to elements of $\mathrm{M}(C_\infty(\Gamma) \otimes B)$. A moment of reflection shows that the mapping (2.21) corresponds to the element $(\mathrm{id} \otimes \rho \circ \pi)\Phi$, where $\pi$ is the representation of $C_\infty(\overline{\Gamma})$ introduced by (2.8). Therefore $(\mathrm{id} \otimes \rho \circ \pi)\Phi \in \mathrm{M}(C_\infty(\Gamma) \otimes B)$. Now using Proposition 2.9 we conclude that $\rho \circ \pi \in \mathrm{Mor}(C_\infty(\overline{\Gamma}), B)$. In consequence $\rho \circ \pi$ maps continuous functions on $\overline{\Gamma}$ into elements affiliated with $B$. Applying this rule to function $f_0$ (cf. (2.7)) and $F_q$ we obtain that $\rho(Y)$ is affiliated with $B$ and $\rho(F_q(Y)) \in \mathrm{M}(B)$. By passing to adjoint $\rho(F_q(Y)^*) \in \mathrm{M}(B)$. We have assumed that $\rho\left(F_q(Y)U\right) \in \mathrm{M}(B)$. Therefore $\rho(U) \in \mathrm{M}(B)$ and Statement 1 is proved.

Ad 2. Let $\rho = \rho_1 \oplus \rho_2$. Then $H_\rho = H_{\rho_1} \oplus H_{\rho_2}$ and $\rho(c) = \rho_1(c) \oplus \rho_2(c)$. In our case $H_{\rho_1} = H_{\rho_2}$. We set: $B = \{m \oplus m : m \in \mathcal{K}(H_{\rho_1})\}$. Then $B \in C^*(H_\rho)$. One can easily verify that for any $c\,\eta\,C$ we have:

$$\left(\rho(c)\,\eta\,B\right) \iff \left(\rho_1(c) = \rho_2(c)\right).$$

Now Statement 2 follows immediately from Statement 1.

$\square$

We shall use slightly different version of Statement 2 of the above Proposition.

<div style="float:left">FToz1</div>

**Proposition 2.11.** *Let $Y_1$, $U_1$, $Y_2$, $U_2$, $X$ be operators acting on a Hilbert space $H$. Assume that for each $k = 1, 2$ the operators $Y_k$, $U_k$, $X$ satisfy the assumptions 1-3 of the previous Proposition. Then*

<div style="float:left">22.50</div>

(2.22)
$$\left(F_q(Y_1)U_1 = F_q(Y_2)U_2\right) \implies \begin{pmatrix} Y_1 = Y_2, \\ U_1 = U_2. \end{pmatrix}$$

<div style="float:left">UF</div>

**Remark 2.12.** *Since $F_q(Y)U = UF_q(U^*YU)$ the same result holds under assumption that we have $U_1 F_q(Y_1) = U_2 F_q(Y_2)$.*

*Proof.* Let $C = \mathcal{K}(H) \oplus \mathcal{K}(H)$ and for any $m_1, m_2 \in \mathcal{K}(H)$ we set $\rho_k(m_1 \oplus m_2) = m_k$ ($k = 1, 2$). We use Proposition 2.10 with $Y$, $U$ and $X$ replaced by $Y_1 \oplus Y_2$, $U_1 \oplus U_2$ and $X \oplus X$, Now (2.22) follows immediately from Statement 2 of Proposition 2.10. $\square$

## 3. Construction of quantum '$az + b$' group.

The quantum '$az + b$' group considered in this paper was introduced in [20, Appendix A]. Following the idea of [11] in this section we shall present it as a quantum group of unitary operators. In this approach one considers a C*-algebra $A$ and a Hilbert space $K$ endowed with a certain additional structure. The main object is a pair $(A, V)$ where $V$ is a unitary element of $M(\mathcal{K}(K) \otimes A)$. It may be treated as a quantum family of unitary operators acting on $K$ 'labeled by elements' of quantum space related to the C*-algebra $A$.

At first we define $A$. To this end we consider two operators $a$ and $b$ acting on the Hilbert space $H = L^2(\Gamma, d\gamma)$. For any $\gamma \in \Gamma$ let $u_\gamma$ denote the shift operator:

$$(u_\gamma x)(\gamma') = x(\gamma \gamma')$$

for any $x \in H$. Clearly $\Gamma \ni \gamma \longrightarrow u_\gamma \in B(H)$ is a unitary representation of $\Gamma$. Therefore by SNAG theorem [4, Chap. 6, §2, Theorem 1] there exists a spectral measure $dE(\gamma)$ on $\widehat{\Gamma} = \Gamma$ such that

$$u_\gamma = \int_\Gamma \chi(\gamma', \gamma) dE(\gamma')$$

for all $\gamma \in \Gamma$. Let

$$a = \int_\Gamma \gamma' dE(\gamma').$$

Then $a$ is a normal operator, $\ker a = \{0\}$ and $\operatorname{Sp} a \subset \overline{\Gamma}$. Moreover $u_\gamma = \chi(a, \gamma)$. By $b$ we denote the multiplication operator:

$$(bx)(\gamma') = \gamma' x(\gamma').$$

By definition a domain $\mathcal{D}(b)$ consists of all $x \in H$ such that the right hand side is square integrable. Clearly $b$ is normal and $\operatorname{Sp} b \subset \overline{\Gamma}$. Moreover $\ker b = \{0\}$. Now one can easily check that

(3.1)          $$\chi(a, \gamma) b \chi(a, \gamma)^* = u_\gamma b u_\gamma^* = \gamma b.$$

This means (cf (2.14)) that $(a, b)$ is a $G$-pair on $H$. We refer to it as a Schrödinger pair.

**Theorem 3.1.** *Let*

(3.2)          $$A = \left\{ f(b)g(a) : f \in C_\infty(\overline{\Gamma}), g \in C_\infty(\Gamma) \right\}^{\substack{\text{norm closed} \\ \text{linear envelope}}}.$$

*Then*: 1. *$A$ is a nondegenerate C*-algebra of operators acting on $L^2(\Gamma, d\gamma)$,*

2. *$a$, $a^{-1}$ and $b$ are affiliated with $A$: $a$, $a^{-1}$, $b \, \eta \, A$,*

3. *$a$, $a^{-1}$ and $b$ generate $A$.*

*Proof.* Ad 1. Operator $b$ is normal and $\operatorname{Sp} b \subset \overline{\Gamma}$. Therefore the mapping $C_\infty(\overline{\Gamma}) \ni f \longrightarrow f(b) \in B(H)$ is a representation of the C*-algebra $C_\infty(\overline{\Gamma})$ on the Hilbert space $H$. Let

(3.3)          $$B = \left\{ f(b) : f \in C_\infty(\overline{\Gamma}) \right\}.$$

Then $B$ is a non-degenerate C*-subalgebra of $B(H)$. Let $C_0(\Gamma, B)$ denote the set of all continuous mappings from $\Gamma$ into $B$ with compact support. Then

(3.4)          $$A = \left\{ \int_\Gamma h(\gamma) \chi(a, \gamma) d\gamma : h \in C_0(\Gamma, B) \right\}^{\text{norm closure}}.$$

Indeed, for $h(\gamma) = f(b)\hat{g}(\gamma)$, where $\gamma \in \Gamma$ and $\hat{g} \in C_0(\Gamma)$ we have

$$\int_\Gamma h(\gamma) \chi(a, \gamma) d\gamma = f(b) g(a),$$

where $g(\gamma') = \int_\Gamma \hat{g}(\gamma) \chi(\gamma', \gamma) d\gamma$ $(\gamma' \in \Gamma)$. By the Riemann-Lebesque lemma (e.g. [12, Theorem 1.2.4]), $g \in C_\infty(\Gamma)$ and the set consisting of functions of such form is dense in $C_\infty(\Gamma)$. This proves formula (3.4). Now (3.1) shows that the unitaries $\chi(a, \gamma)$ $(\gamma \in \Gamma)$ implement a one parameter group of automorphisms of $B$. Using the standard technique of the theory of crossed products (cf. [8, Section 7.6]) one can show that (3.4) is a (non-degenerate) C*-algebra of operators acting on $L^2(\Gamma, d\gamma)$.

Ad 2. The affiliation relation was introduced in (1.5). We consider the operator $a$ first. We know that $a$ is a normal invertible operator and $\mathrm{Sp}\, a \subset \Gamma$. Let

$$\boxed{\texttt{g1g2}} \quad (3.5) \qquad\qquad g_1(\gamma) = \frac{1}{\sqrt{1 + |\gamma|^2}}, \qquad g_2(\gamma) = \frac{\gamma}{\sqrt{1 + |\gamma|^2}},$$

for any $\gamma \in \Gamma$. For $T = a$ and $T = a^*$ we have $z_a = g_2(a)$ and $z_{a^*} = \bar{g}_2(a)$. In both cases $(I + T^*T)^{-\frac{1}{2}} = g_1(a)$. Clearly $g_1, g_2 \in M(C_\infty(\Gamma))$. Now inspecting definition (3.2) one can easily show that $Ag_1(a)$ is dense in $A$ and $z_a, z_{a^*}$ are right multipliers of $A$. This means (cf the comment after (1.5)) that $z_a$ is a multiplier of $A$ and $a$ is affiliated with $A$. In the same manner we prove that $a^{-1}\, \eta\, A$.

Now consider the operator $b$. Let $g_1$ and $g_2$ be given by the expression (3.5) again but now for any $\gamma \in \overline{\Gamma}$. Then $z_b = g_2(b)$, $z_{b^*} = \bar{g}_2(b)$ and in both cases $(I + T^*T)^{-\frac{1}{2}} = g_1(b)$. Now $g_1, g_2 \in M(C_\infty(\overline{\Gamma}))$. Therefore $g_1(b)A$ is dense in $A$ and $z_b, z_{b^*}$ are left multipliers. In consequence $z_b$ is a multiplier of $A$ and $b\, \eta\, A$.

Ad 3. Let $c \in A$ be of the form $c = f(b)g(a)$, where $f \in C_\infty(\overline{\Gamma})$ and $g \in C_\infty(\Gamma)$. By definition (3.2) the set of such elements is total in $A$. Let $\pi$ be a non-degenerate representation of $A$. Then $\pi(a)$ is invertible and $\pi(c) = f(\pi(b))g(\pi(a))$. Therefore $\pi$ is completely determined by $\pi(a)$ and $\pi(b)$. This means that $a, a^{-1}$ and $b$ separate representations of $A$.

Let $r_1 = (I + b^*b)^{-1}$, $r_2 = (I + a^*a)^{-1}$ and $r_3 = [I + (a^{-1})^*a^{-1}]^{-1}$. To end the proof it is sufficient (cf Theorem 1.6) to show that $r_1 r_2 r_3 \in A$. Since $r_1 r_2 r_3 = f(b)g(a)$ where $f(\gamma) = (1 + |\gamma|^2)^{-1}$, and $g(\gamma) = |\gamma|^2\,(1 + |\gamma|^2)^{-2}$, the result follows from (3.2). $\qquad\qquad \square$

Now we describe the Hilbert space $K$. The structure of $K$ is determined by two normal operators $\widehat{a}$ and $\widehat{b}$ such that

$$(\widehat{a}, \widehat{b}) \text{ is a } G\text{-pair on } K \text{ and } \ker \widehat{b} = \{0\}.$$

It is known that any such pair is unitary equivalent to the direct sum of copies of the Schrödinger pair. In particular spectral measures of $\widehat{a}$ and $\widehat{b}$ are absolutely continuous with respect to the Haar measure on $\Gamma$.

Let

$$\boxed{\texttt{V1}} \quad (3.6) \qquad\qquad V = F_q(\widehat{b} \otimes b)\, \chi(\widehat{a} \otimes I, I \otimes a).$$

It is the basic object considered in this Section. We shall prove

$\boxed{\texttt{AV}}$ **Theorem 3.2.**

    *1. $V$ is a unitary operator and $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$*

    *2. $A$ is generated by $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$.*

*Proof.* Let $Y = \widehat{b} \otimes b$, $U = \chi(\widehat{a} \otimes I, I \otimes a)$, $X = \widehat{a} \otimes I$ and $C = \mathcal{K}(K) \otimes A$. Then all the assumptions of Proposition 2.10 are satisfied. Therefore $V = F_q(Y)U \in \mathrm{M}(C)$ and Statement 1 is proved.

Let $\pi$ be a representation of $A$ and $B \in C^*(H_\pi)$. Then $\mathrm{id} \otimes \pi$ is a representation of $C$ acting on $K \otimes H_\pi$. Let us note that $(\mathrm{id} \otimes \pi)X = \widehat{a} \otimes I$ is affiliated with $\mathcal{K}(K) \otimes B$. Assume that $(\mathrm{id} \otimes \pi)V \in \mathrm{M}(\mathcal{K}(K) \otimes B)$. By Proposition 2.10, operators: $(\mathrm{id} \otimes \pi)Y = \widehat{b} \otimes \pi(b)$ and $(\mathrm{id} \otimes \pi)U = \chi(\widehat{a} \otimes I, I \otimes \pi(a))$ are affiliated with $\mathcal{K}(K) \otimes B$. By Proposition A.1 of [25] operator $\pi(b)$ is affiliated with $B$. On the other hand operators $\widehat{a}$ and $\pi(a)$ satisfy the assumptions of Proposition 2.3. Therefore $\pi(a)$ and $\pi(a)^{-1}$ are affiliated with $B$.

According to Statement 3 of Theorem 3.1, $a, a^{-1}$ and $b$ generate $A$. Therefore $\pi \in \mathrm{Mor}(A, B)$. This way we showed that $(\mathrm{id} \otimes \pi)V \in \mathrm{M}(\mathcal{K}(K) \otimes B)$ implies $\pi \in \mathrm{Mor}(A, B)$. It means that $A$ is generated by $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$. $\qquad\qquad \square$

Now we formulate the main result of this Section:

$\boxed{\texttt{mnoz}}$ **Theorem 3.3.** *There exists $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that*

$$\boxed{\texttt{delta}} \quad (3.7) \qquad\qquad (\mathrm{id} \otimes \Delta)V = V_{12}V_{13}$$

*Proof.* Let us recall that $b$ is an invertible operator. Therefore $(b^{-1}, a)$ and $(b^{-1}, b^{-1}a)$ are $G$-pairs on $H$ by Remark 2.7. In particular $b^{-1}a$ is normal and $\mathrm{Sp}\,(b^{-1}a) \subset \overline{\Gamma}$. Let

<span style="float:left">`W'def`</span> (3.8)
$$W = F_q(b^{-1}a \otimes b)\,\chi(b^{-1} \otimes I, I \otimes a).$$

Clearly $W$ is a unitary operator acting on $H \otimes H$. We shall prove that

<span style="float:left">`W'V`</span> (3.9)
$$V_{12}V_{13} = W_{23}V_{12}W_{23}^*.$$

To deal with shorter formulae we set

$$U = \chi(\widehat{a} \otimes I, I \otimes a), \qquad Z = \chi(b^{-1} \otimes I, I \otimes a).$$

Applying formula (2.14) for the $G$-pairs $(\widehat{a}, \widehat{b})$ and $(b^{-1}, a)$ one can easily verify that

<span style="float:left">`up`</span> (3.10)
$$U(\widehat{b} \otimes I)U^* = \widehat{b} \otimes a$$

and

<span style="float:left">`upZ`</span> (3.11)
$$Z(a \otimes I)Z^* = a \otimes a.$$

With the above notation $V = F_q(\widehat{b} \otimes b)U$ and

<span style="float:left">`W12W13`</span> (3.12)
$$V_{12}V_{13} = F_q(\widehat{b} \otimes b \otimes I)\,U_{12}\,F_q(\widehat{b} \otimes I \otimes b)\,U_{13}.$$

By the relation (3.10) we get

<span style="float:left">`eG1`</span> (3.13)
$$U_{12}\,F_q(\widehat{b} \otimes I \otimes b) = F_q(\widehat{b} \otimes a \otimes b)U_{12}$$

and

<span style="float:left">`fV12V13a`</span> (3.14)
$$V_{12}V_{13} = F_q(\widehat{b} \otimes b \otimes I)F_q(\widehat{b} \otimes a \otimes b)\,U_{12}U_{13}.$$

Let us consider the first factor in (3.14). We apply formula (2.18) with $X = \widehat{b} \otimes a \otimes b$ and $Y = \widehat{b} \otimes b \otimes I$. Then

$$F_q(Y^{-1}X) = F_q(I \otimes b^{-1}a \otimes b).$$

Now (3.14) takes the form

<span style="float:left">`fnV12V13`</span> (3.15)
$$V_{12}V_{13} = F_q(I \otimes b^{-1}a \otimes b)F_q(\widehat{b} \otimes b \otimes I)F_q(I \otimes b^{-1}a \otimes b)^*\,U_{12}U_{13}.$$

Since $\chi$ is a bicharacter, $U_{12}U_{13} = \chi(\widehat{a} \otimes I \otimes I, I \otimes a \otimes a)$. Since $a \otimes a$ commutes with $b^{-1}a \otimes b$,

$$F_q(I \otimes b^{-1}a \otimes b)^*\,U_{12}U_{13} = U_{12}U_{13}\,F_q(I \otimes b^{-1}a \otimes b)^*.$$

The relation (3.11) implies that $Z_{23}U_{12}Z_{23}^* = U_{12}U_{13}$ and the formula (3.15) takes now the form

<span style="float:left">`fnV12V13b`</span> (3.16)
$$V_{12}V_{13} = F_q(I \otimes b^{-1}a \otimes b)F_q(\widehat{b} \otimes b \otimes I)Z_{23}U_{12}Z_{23}^*F_q(I \otimes b^{-1}a \otimes b)^*$$

Finally $b \otimes I$ commutes with $Z$. Therefore $F_q(\widehat{b} \otimes b \otimes I)$ commutes with $Z_{23}$. Clearly $F_q(\widehat{b} \otimes b \otimes I)U_{12} = [F_q(\widehat{b} \otimes b) \otimes I]U_{12} = V_{12}$ and $F_q(I \otimes b^{-1}a \otimes b)Z_{23} = [I \otimes F_q(b^{-1}a \otimes b)]Z_{23} = W_{23}$. Now (3.9) follows immediately from (3.16).

Now we prove the main statement. For any $c \in A$ we set

<span style="float:left">`DeltaW`</span> (3.17)
$$\Delta(c) = W(c \otimes I)W^*.$$

Then $\Delta$ is a representation of $A$ acting on $L^2(\Gamma, d\gamma) \otimes L^2(\Gamma, d\gamma)$. We know that $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$. Formula (3.9) shows that

$$(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}.$$

Clearly $V_{12}, V_{13} \in \mathrm{M}(\mathcal{K}(K) \otimes A \otimes A)$. Therefore $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13} \in \mathrm{M}(\mathcal{K}(K) \otimes A \otimes A)$. Remembering that $A$ is generated by $V$ we conclude that $\Delta \in \mathrm{Mor}(A, A \otimes A)$. $\qquad\square$

We conclude the section by discussion to what extent $C^*$-bialgebra $A, \Delta)$ is a quantum group.

Using formula (3.17) one can calculate $\Delta(c)$ for any $c \in A$. The same is true for any $c$ affiliated with $A$. We shall show that

<span style="float:left">`Deltaab`</span> (3.18)
$$\Delta(a) = a \otimes a,$$
$$\Delta(b) = a \otimes b \dotplus b \otimes I.$$

Since $b^{-1}a \otimes b$ commutes with $a \otimes a$, formula for $\Delta(a)$ follows immediately from (3.11). To prove formula for $\Delta(b)$ we notice that $Z$ and $b \otimes I$ commute. Therefore

<span style="float:left">`db`</span> (3.19)
$$W(b \otimes I)W^* = F_q(b^{-1}a \otimes b)(b \otimes I)F_q(b^{-1}a \otimes b)^*$$

Now we use formula (2.17) with $X = a \otimes b$ and $Y = b \otimes I$. Then $Y^{-1}X = b^{-1}a \otimes b$ and the right hand side of (3.19) coincides with $X \dotplus Y$. The formula for $\Delta(b)$ is proved.

Formula (3.18) shows that $(A, \Delta)$ does not depend on the particular choice of a Hilbert space $K$ nor on operators $\widehat{a}$ and $\widehat{b}$. One can choose $K = L^2(\Gamma, d\gamma)$ and $(\widehat{a}, \widehat{b}) = (a, b)$. However it turns out that $K = L^2(\Gamma, d\gamma)$ and

`dbwyb`     (3.20)                           $(\widehat{a}, \widehat{b}) = (b^{-1}, b^{-1}a)$.

is a more interesting choice. If this is the case then operator (3.6) coincides with (3.8): $V = W$. Relation (3.9) takes the form:

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

This is a pentagon equation (1.6). It means that $W$ is a multiplicative unitary. It turns out [20],[13] that $W$ is modular with (cf formulae (1.9) and (1.10)

$$\hat{Q} = |b|, \qquad Q = |a|$$

and

`tW`     (3.21)          $\tilde{W} = F_q \left( (b^{-1}a)^\top \otimes (-qa^{-1}b) \right)^* \chi \left( (b^{-1})^\top \otimes I, I \otimes a \right).$

One can easily verify that $(A, \Delta)$ is related to $W$ in the sense explained after Theorem 1.4. Therefore $(A, \Delta)$ is a quantum group. Its structure is described by Theorem 1.4. In particular there exists an antipode admitting a polar decomposition. We shall show that in this case

`abR`     (3.22)                    $a^R = a^{-1}, \qquad b^R = -qa^{-1}b$

where $R$ is a unitary antipode (cf Statement 4(iii) of Theorem 1.4). We use Statement 6(ii) of this theorem to prove these formulae. Since $\top \otimes R$ is an antiisomorphism of $B(H) \otimes A$ into $B(\overline{H}) \otimes A$, it is antimultiplicative. We get

$$W^{\top \otimes R} = \chi \left( (b^{-1})^\top \otimes I, I \otimes a^R \right) F_q \left( (b^{-1}a)^\top \otimes b^R \right).$$

On the other hand (cf (3.21))

$$\tilde{W}^* = \chi \left( (b^{-1})^\top \otimes I, I \otimes a \right)^* F_q \left( (b^{-1}a)^\top \otimes (-qa^{-1}b) \right).$$

Clearly $\overline{\chi(\gamma', \gamma)} = \chi(\gamma', \gamma^{-1})$ for any $\gamma', \gamma \in \Gamma$. Therefore

$$\chi \left( (b^{-1})^\top \otimes I, I \otimes a \right)^* = \chi \left( (b^{-1})^\top \otimes I, I \otimes a^{-1} \right).$$

Now formula $W^{\top \otimes R} = \tilde{W}^*$ may be written as

`Rtoz`     (3.23)
$$\chi \left( (b^{-1})^\top \otimes I, I \otimes a^R \right) F_q \left( (b^{-1}a)^\top \otimes b^R \right)$$
$$= \chi \left( (b^{-1})^\top \otimes I, I \otimes a^{-1} \right) F_q \left( (b^{-1}a)^\top \otimes (-qa^{-1}b) \right).$$

We apply Proposition 2.11 with

$$Y_1 = (b^{-1}a)^\top \otimes b^R, \qquad U_1 = \chi \left( (b^{-1})^\top \otimes I, I \otimes a^R \right),$$
$$Y_2 = (b^{-1}a)^\top \otimes (-qa^{-1}b), \quad U_2 = \chi \left( (b^{-1})^\top \otimes I, I \otimes a^{-1} \right),$$

$X = b^\top \otimes I$ and $C = \mathcal{K}(\overline{H}) \otimes A$. Taking into account that $(b^\top, a^\top)$ is a $G$-pair on $\overline{H}$ one can easily check that all assumptions of Proposition 2.11 are satisfied. Then (cf Remark 2.12) we get $Y_1 = Y_2$ (this proves the second formula in (3.22)) and $U_1 = U_2$. Now the first formula in (3.22) follows by Proposition 2.4.

Now consider formula (1.14). In this case it takes the form

`fHaarn`     (3.24)                           $h(c) = \mathrm{Tr}(|b| \, c \, |b|)$.

We shall show that $h$ is locally finite (cf [26]), i.e. the set $\{c \in A : h(c^*c) < +\infty\}$ is dense in $A$. By Statement 1 of Theorem 3.1 we know that a set of elements of the form $c = g(a)f(b)$, where $g \in C_\infty(\Gamma)$ and $f \in C_\infty(\overline{\Gamma})$ is total in $A$. The same is true if $g$ is of the form $g(\gamma) = \int_\Gamma \hat{g}(\gamma')\chi(\gamma, \gamma')d\gamma'$, where $\hat{g} \in C_0(\Gamma)$. Clearly $h(c^*c) = \mathrm{Tr}((c\,|b|)^*c\,|b|)$ and since $[\chi(a, \gamma')x](\gamma) = x(\gamma\gamma')$, $c\,|b|$ is an integral operator

$$[c\,|b|\,x](\gamma) = \int_\Gamma \hat{g}(\gamma')f(\gamma\gamma')\,|\gamma\gamma'|\,x(\gamma\gamma')d\gamma' = \int_\Gamma \hat{g}(\gamma^{-1}\gamma')f(\gamma')\,|\gamma'|\,x(\gamma')d\gamma'$$

with a kernel $K_c(\gamma, \gamma') = \hat{g}(\gamma^{-1}\gamma')f(\gamma')\,|\gamma'|$. Therefore

$$h(c^*c) = \int_{\Gamma \times \Gamma} |K_c(\gamma, \gamma')|^2 \, d\gamma d\gamma' = \left( \int_\Gamma |f(\gamma)|^2 \, |\gamma|^2 \, d\gamma \right) \left( \int_\Gamma |\hat{g}(\gamma)|^2 \, d\gamma \right)$$

where we used the invariance of Haar measure on $\Gamma$. Now by Plancherel formula

$$h(c^*c) = \left( \int_{\overline{\Gamma}} |f(\gamma)|^2 \, |\gamma|^2 \, d\gamma \right) \left( \int_\Gamma |g(\gamma)|^2 \, d\gamma \right).$$

Clearly $L^2(\Gamma, d\gamma) \cap C_\infty(\Gamma)$ is dense in $C_\infty(\Gamma)$ and since the measure $d\mu(\gamma) = |\gamma|^2 \, d\gamma$ is locally finite on $\overline{\Gamma}$, $L^2(\overline{\Gamma}, d\mu) \cap C_\infty(\overline{\Gamma})$ is dense in $C_\infty(\overline{\Gamma})$. This means that $h$ is finite on a dense subset of $A$. Therefore $h$ is a (right) Haar weight by Proposition 1.5.

We recall that (cf Statement 6, (ii) of Theorem 1.4) the scaling group acts in the following way

$$\tau_t(c) = Q^{2it}cQ^{-2it} = |a|^{2it}\,c\,|a|^{-2it}.$$

Remembering that $|a|$ and $|b|$ commute we conclude (cf (3.24)) that in this case (in contrast to the quantum '$az + b$' group at roots of unity) the Haar weight is scaling invariant, $h\circ\tau_t = h$.

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