Quantum $SU(2)$ and $E(2)$ groups.
Contraction procedure.

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Abstract

In [3] it was shown (in the framework of deformed enveloping algebras) that quantum $SU(2)$ and $E(2)$ groups are related by the contraction procedure. We consider the same problem on the C^*-level. As a result we find a number of formulae coupling the comultiplications in quantum $SU(2)$ and $E(2)$. In particular we show that the comultiplications in both groups are implemented by partial isometries. An unexpected feature of quantum $E(2)$ is discovered and the corresponding strange behavior of quantum $SU(2)$ is described.

0 Introduction.

We shall consider two three-dimensional matrix groups:

$$\begin{aligned} SU(2) &= \left\{ \begin{pmatrix} \alpha, & -\overline{\gamma} \\ \gamma, & \overline{\alpha} \end{pmatrix} \in M_{2\times 2}(\mathbf{C}) : \; |\alpha|^2 + |\gamma|^2 = 1 \right\} \\ E(2) &= \left\{ \begin{pmatrix} v, & n \\ 0, & \overline{v} \end{pmatrix} \in M_{2\times 2}(\mathbf{C}) : \; |v| = 1 \right\} \end{aligned}$$

They have the common subgroup S^1 consisting of all diagonal matrices. The corresponding homogeneous spaces are: the two-dimensional sphere in the case of $SU(2)$ and the two dimensional Euclidean plane in the case of $E(2)$. Since for small regions, the spherical geometry may be well approximated by the Euclidean one, we may expect that the two groups look very similar in a sufficiently small neighbourhood of S^1. To reveal this similarity we use the same coordinates to parametrize $SU(2)$ and $E(2)$.

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For $SU(2)$ we set

$$\begin{pmatrix} v \\ n \end{pmatrix} = \begin{pmatrix} v\sqrt{1-|n|^2} & , & n \\ -\overline{n} & , & \overline{v}\sqrt{1-|n|^2} \end{pmatrix} \in SU(2),$$

where $v, n \in \mathbf{C}$, $|v|= 1$ and $|n|< 1$. The coordinate system covers the dense open subset of $SU(2)$ consisting of all matrices with non-vanishing diagonal elements. For $E(2)$ we set

$$\begin{pmatrix} v \\ n \end{pmatrix} = \begin{pmatrix} v & , & n \\ 0 & , & \overline{v} \end{pmatrix} \in E(2),$$

where $v, n \in \mathbf{C}$ and $|v|= 1$. In this case the coordinate system covers the whole of $E(2)$. In both cases $\begin{pmatrix} v \\ n \end{pmatrix}$ belongs to a neighbourhood of S^1 if n is sufficiently small.

We have

$$\begin{aligned}
\begin{pmatrix} v_1 \\ n_1 \end{pmatrix} \cdot_{SU(2)} \begin{pmatrix} v_2 \\ n_2 \end{pmatrix} &= \begin{pmatrix} \text{Phase}\,(v_1 v_2 \sqrt{(1-|n_1|^2)(1-|n_2|^2)} - n_1\overline{n_2}) \\ v_1\sqrt{1-|n_1|^2}\, n_2 + n_1\overline{v_2}\sqrt{1-|n_2|^2} \end{pmatrix}, \\
\begin{pmatrix} v_1 \\ n_1 \end{pmatrix} \cdot_{E(2)} \begin{pmatrix} v_2 \\ n_2 \end{pmatrix} &= \begin{pmatrix} v_1 v_2 \\ v_1 n_2 + n_1\overline{v_2} \end{pmatrix}.
\end{aligned}$$

Comparing the right hand sides of these relations, when $n_1,\ n_2 \longrightarrow 0$ we see that the difference of the first coordinates is small of the second order in (n_1, n_2). The second coordinates are small of the first order in (n_1, n_2), whereas their difference is small of the third order in (n_1, n_2).

Let μ be a positive number smaller than 1. For any $l \in \mathbf{R}$ we set

$$\tau^l \begin{pmatrix} v \\ n \end{pmatrix} = \begin{pmatrix} v \\ \mu^l\, n \end{pmatrix}.$$

Clearly $(\tau^l)_{l\in\mathbf{R}}$ is a one-parameter group of automorphisms of $E(2)$. It retracts $E(2)$ to S^1 when $l \to \infty$. Using the above analysis of the behaviour of '$\cdot_{SU(2)}$' and '$\cdot_{E(2)}$' in a neighbourhood of S^1 we get:

$$\begin{pmatrix} v_1 \\ n_1 \end{pmatrix} \cdot_{E(2)} \begin{pmatrix} v_2 \\ n_2 \end{pmatrix} = \lim_{l\to\infty} \tau^{-l} \left[\tau^l \begin{pmatrix} v_1 \\ n_1 \end{pmatrix} \cdot_{SU(2)} \tau^l \begin{pmatrix} v_2 \\ n_2 \end{pmatrix} \right]. \tag{1}$$

The limit appearing in (1) is known as the contraction procedure. We say that $E(2)$ is the contraction of $SU(2)$. We refer to [7], [8] and [2], where the contraction procedure is considered (mainly on the level of the Lie algebra).

The idea to use the contraction procedure in the theory of quantum groups goes back to E. Celeghini et al.. In a series of papers [3] – [5] they applied this procedure to quantum deformations of simple Lie Groups producing new examples of quantum groups. The paper [6] contains the review of their results.

In all these papers the quantum groups are considered in the purely algebraic setting of q-deformed universal enveloping algebras.

In our paper we show how the contraction procedure works in the theory of topological quantum groups. The main aim is to prove the quantum analog of formula (1). We consider the simplest case: contraction of $S_\mu U(2)$ to $E_\mu(2)$. The deformation parameter is kept constant.

In Section 1 we recall the description of the quantum $SU(2)$ and $E(2)$ groups given in [15] and [18]. We work with concrete Hilbert space representations of the function algebras $A_{SU(2)}$ and $A_{E(2)}$. The Hilbert spaces are related by the inclusion $H_{SU(2)} \subset H_{E(2)}$. This corresponds to the use of the same coordinates on $SU(2)$ and $E(2)$ in the classical case.

The main results of the paper are listed in Section 2. They include a number of formulae relating the multiplications in $S_\mu U(2)$ and $E_\mu(2)$. One of them is the quantum version of (1). The proofs are given in Section 3.

Section 4 is devoted to some unexpected features of quantum $SU(2)$ and $E(2)$ that follow from our results. It turns out that in certain aspects they behave like locally compact semigroups. For example there exists a proper open subset in $S_\mu U(2)$ invariant under all right translations.

We shall freely use the concepts related to non-unital C^*-algebras such as multipliers, affiliated elements and morphisms (cf [10], [14], [13], [1] and [16]). The computations presented in this paper heavily depends on the results of [17]. We shall use the topology of almost uniform convergence on the multiplier algebra.

Let A be a C^*-algebra and (a_l) be a sequence of elements of $M(A)$. We recall that the sequence converges almost uniformly to an element $a_\infty \in M(A)$ if for any $x \in A$, $\| a_l x - a_\infty x \| \to 0$ and $\| {a_l}^* x - {a_\infty}^* x \| \to 0$. In such a case we write

$$\text{a.u.}\lim_{l\to\infty} a_l = a_\infty.$$

In what follows, $\lim_{l\to\infty} a_l$ always denotes the norm limit.

Combining the Lebesgue integral theory with the spectral theory of normal operators we obtain

Proposition 0.1 *Let n be a normal operator acting on a Hilbert space H and $(f_k)_{k=1,2,\dots}$ be a sequence of continuous bounded functions on* $\mathrm{Sp}\,(n)$ *such that $| f_k(\lambda) | \leq | \lambda |$ and $\lim_{k\to\infty} f_k(\lambda) = \lambda$ for all $\lambda \in \mathrm{Sp}\,(n)$. Then for any $\psi \in H$*

$$(\psi \in \mathcal{D}(n)) \Longleftrightarrow \begin{pmatrix} \textit{The sequence } f_k(n)\psi \\ \textit{is norm converging} \end{pmatrix}.$$

Moreover

$$n\psi = \lim_{k\to\infty} f_k(n)\psi$$

for any $\psi \in \mathcal{D}(n)$.

1 The function algebras $A_{SU(2)}$ and $A_{E(2)}$.

Let $\mu \in]0,1[$. To introduce the algebra $A_{SU(2)}$, which plays the role of "the algebra of all continuous functions on $S_\mu U(2)$" we shall not follow [15], where $A_{SU(2)}$ was defined via generators and relations. Instead we consider a Hilbert space $H_{SU(2)}$ equipped with an orthonormal basis $(e_{ij})_{i=0,1,2,\ldots;j\in\mathbf{Z}}$ and two operators $\alpha,\gamma\in B(H_{SU(2)})$ such that

$$\alpha e_{ij} = \sqrt{1-\mu^{2i}}\,e_{i-1,j} \tag{2}$$
$$\gamma e_{ij} = \mu^i e_{i,j-1} \tag{3}$$

The operators α and γ satisfy the commutation relations

$$\left.\begin{array}{cc} \alpha\gamma=\mu\gamma\alpha, & \\ \alpha\gamma^*=\mu\gamma^*\alpha, & \alpha^*\alpha+\gamma^*\gamma=I_{SU(2)}, \\ \gamma\gamma^*=\gamma^*\gamma, & \alpha\alpha^*+\mu^2\gamma^*\gamma=I_{SU(2)}, \end{array}\right\} \tag{4}$$

where $I_{SU(2)}$ denotes the identity operator acting on $H_{SU(2)}$. By definition $A_{SU(2)}$ is the smallest norm closed *-subalgebra of $B(H_{SU(2)})$ containing α and γ.

The two methods of introducing $A_{SU(2)}$ are obviously equivalent: one can easily verify that the representation of the commutation relations (4) given by (2) and (3) weakly contains any other representation.

The group structure on $S_\mu U(2)$ is described by the comultiplication $\Phi_{SU(2)}\in\mathrm{Mor}(A_{SU(2)},A_{SU(2)}\otimes A_{SU(2)})$. It acts on the distinguished elements in the following way:

$$\Phi_{SU(2)}(\alpha)=\alpha\otimes\alpha-\mu\gamma^*\otimes\gamma, \tag{5}$$

$$\Phi_{SU(2)}(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma. \tag{6}$$

In the following we shall find a partial isometry implementing $\Phi_{SU(2)}$ (cf (35)).

The algebra $A_{E(2)}$ "of all continuous functions on $E_\mu(2)$ vanishing at infinity" is introduced in [16] and [18]. We follow the description given in [18]. Let $H_{E(2)}$ be a Hilbert space equipped with an orthonormal basis $(e_{ij})_{i,j\in\mathbf{Z}}$ and v, n be operators acting on $H_{E(2)}$ defined by

$$v e_{ij} = e_{i-1,j} \tag{7}$$
$$n e_{ij} = \mu^i e_{i,j+1}. \tag{8}$$

It is understood that the set of all finite linear combinations of vectors e_{ij} ($i,j\in\mathbf{Z}$) is a core for n. The operator v is unitary, n is normal and the spectrum of n coincides with the set

$$\overline{\mathbf{C}}^\mu := \{\lambda\in\mathbf{C} : \lambda=0 \text{ or } |\lambda|\in\mu^{\mathbf{Z}}\}.$$

Moreover

$$vnv^*=\mu n. \tag{9}$$

By definition $A_{E(2)}$ is the smallest closed subspace of $B(H_{E(2)})$ contaning all elements of the form $v^k f(n)$, where $k\in\mathbf{Z}$ and $f\in C_\infty(\overline{\mathbf{C}}^\mu)$ ($C_\infty(\overline{\mathbf{C}}^\mu)$ denotes the set of all continuous functions on $\overline{\mathbf{C}}^\mu$ vanishing at infinity). Due to (9), $A_{E(2)}$ is closed under multiplication and hermitian conjugation, so it is a C^*-algebra. The operators v and n are affiliated with $A_{E(2)}$: $v, n\,\eta A_{E(2)}$. $I_{E(2)}$ will denote the identity operator acting on $H_{E(2)}$.

The group structure on $E_\mu(2)$ is described by the comultiplication $\Phi_{E(2)} \in \mathrm{Mor}(A_{E(2)}, A_{E(2)} \otimes A_{E(2)})$. This is the only morphism such that

$$\begin{aligned} \Phi_{E(2)}(v) &= v \otimes v, &(10)\\ \Phi_{E(2)}(n) &= v \otimes n \dot{+} n \otimes v^*. &(11) \end{aligned}$$

The action of $\Phi_{E(2)}$ is unitarily implemented. Let V be the unitary operator acting on $H_{E(2)} \otimes H_{E(2)}$ such that $V(e_{ij} \otimes e_{kl}) = e_{ij} \otimes e_{k+i+j,l}$ $(i,j,k,l \in \mathbf{Z})$ and W be the unitary operator acting on $H_{E(2)} \otimes H_{E(2)}$ defined by

$$W = F_\mu(n^{-1}v \otimes vn)V,$$

where F_μ is the continuous function on $\overline{\mathbf{C}}^\mu$ investigated in [17] (one can easily verify that $n^{-1}v \otimes vn$ is a normal operator and that $\mathrm{Sp}\,(n^{-1}v \otimes vn) = \overline{\mathbf{C}}^\mu$). Then

$$\Phi_{E(2)}(a) = W(a \otimes I_{E(2)})W^* \qquad (12)$$

for any $a \in A_{E(2)}$. It is sufficient to verify this formula for $a = v$ and $a = n$. The case $a = v$ is easy: $V(v \otimes I_{E(2)})V^* = v \otimes v$ and $v \otimes v$ commutes with $n^{-1}v \otimes vn$. To prove (12) for $a = n$ one has to verify that $V(n \otimes I_{E(2)})V^* = n \otimes v^*$ and then use formula (2.1) of [17] (with R and S replaced by $n \otimes v^*$ and $v \otimes n$ resp.).

Identifying the basis vectors of $H_{SU(2)}$ with the corresponding basis vectors of $H_{E(2)}$ we embed $H_{SU(2)} \hookrightarrow H_{E(2)}$. Consequently we have the embeding $B(H_{SU(2)}) \hookrightarrow B(H_{E(2)})$. By definition any element of $B(H_{SU(2)})$ kills the orthogonal complement of $H_{SU(2)}$. In particular $I_{SU(2)}$ is the orthogonal projection onto $H_{SU(2)}$.

The main result of this section is contained in the following two relations:

$$A_{SU(2)} \subset A_{E(2)}, \qquad (13)$$

$$A_{SU(2)} = I_{SU(2)} A_{E(2)} I_{SU(2)}. \qquad (14)$$

For any $\lambda \in \overline{\mathbf{C}}^\mu$ we set

$$\begin{aligned} f_\alpha(\lambda) &= \sqrt{1 - |\lambda|^2}\ \chi(|\lambda| \leq 1),\\ f_\gamma(\lambda) &= \overline{\lambda}\ \chi(|\lambda| \leq 1), \end{aligned}$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Clearly f_α, $f_\gamma \in C_\infty(\overline{\mathbf{C}}^\mu)$. Taking into account the definitions of α, γ, v and n one can easily check that

$$\begin{aligned} \alpha &= v f_\alpha(n), &(15)\\ \gamma &= f_\gamma(n), &(16) \end{aligned}$$

and (13) follows. In particular $I_{SU(2)} \in A_{E(2)}$ and ($I_{SU(2)}$ is the unit of $A_{SU(2)}$) $A_{SU(2)} \subset I_{SU(2)} A_{E(2)} I_{SU(2)}$.

To prove the converse inclusion, one has to show that

$$I_{SU(2)}v^k f(n) I_{SU(2)} \in A_{SU(2)}$$

for any $k \in \mathbf{Z}$ and $f \in C_\infty(\overline{\mathbf{C}}^\mu)$. To this end it is sufficient to notice that $f(n)I_{SU(2)} = I_{SU(2)}f(\gamma^*)$, $I_{SU(2)}v^k I_{SU(2)} = ((I-\mu^2\gamma^*\gamma)^{-\frac{1}{2}}\alpha)^k$ for $k > 0$ and $I_{SU(2)}v^k I_{SU(2)} = (\alpha^*(I-\mu^2\gamma^*\gamma)^{-\frac{1}{2}})^{-k}$ for $k < 0$. The proof of (14) is complete.

Remark: The use of the same letter 'e' to denote the basis vectors in $H_{SU(2)}$ and $H_{E(2)}$ precisely corresponds to the use of the same coordinates on classical $SU(2)$ and $E(2)$, as we did in Section 0. The result (14) is however much stronger than could be obtained in the classical case. For topological reasons there exists no continuous mapping of $SU(2)$ into $E(2)$ keeping the points of $S^1 \subset SU(2) \bigcap E(2)$ fixed. In the quantum case this argument does not work: $S_\mu U(2)$ and $E_\mu(2)$ exhibit some properties of a disconnected space.

For any $l \in \mathbf{Z}$ and $a \in A_{E(2)}$ we set:

$$\tau^l(a) = v^l a v^{-l}.$$

Clearly τ^l is an inner automorphism of $A_{E(2)}$. By virtue of (10), $\Phi_{E(2)}(\tau^l(a)) = (\tau^l \otimes \tau^l)\Phi_{E(2)}(a)$. It means that $(\tau^l)_{l\in\mathbf{Z}}$ is a group of automorphism of $E_\mu(2)$. Notice that

$$\tau^l(v) = v,$$
$$\tau^l(n) = \mu^l n.$$

Therefore the group $(\tau^l)_{l\in\mathbf{Z}}$ corresponds to the automorphism group of classical $E(2)$ considered in Section 0.

Let $f \in C_{\text{bounded}}(\overline{\mathbf{C}}^\mu)$. Then

$$\tau^l f(n) = f(\mu^l n). \tag{17}$$

If $l \longrightarrow \infty$, then the function $f(\mu^l \cdot)$ tends almost uniformly to the constant function with the value $f(0)$. Remembering that $n\eta A_{E(2)}$ we get

$$\text{a.u.}\lim_{l\to\infty} \tau^l f(n) = f(0) I_{E(2)}. \tag{18}$$

In particular (cf (15) and (16))

$$\text{a.u.}\lim_{l\to\infty} \tau^l(\gamma) = 0, \tag{19}$$

$$\text{a.u.}\lim_{l\to\infty} \tau^l(\alpha) = v. \tag{20}$$

Let us notice that $I_{SU(2)} = \chi(|n| \le 1)$. Therefore, by virtue of (18)

$$\text{a.u.}\lim_{l\to\infty} \tau^l(I_{SU(2)}) = I_{E(2)}. \tag{21}$$

The same result one obtains using (19), (20) and any of the relations in the second column of (4). By virtue of (17): $\tau^l(I_{SU(2)}) = \chi(|\,n\,| \le \mu^{-l})$. It shows that $(\tau^l(I_{SU(2)}))_{l\in\mathbf{Z}}$ is an increasing sequence of projections. Consequently $(\tau^l(A_{SU(2)}))_{l\in\mathbf{Z}}$ is an increasing sequence of norm-closed *-subalgebras of $A_{E(2)}$. Let

$$A_{E(2)}^{comp} = \bigcup_{l\in\mathbf{Z}} \tau^l(A_{SU(2)}). \tag{22}$$

By virtue of (21), $A_{E(2)}^{comp}$ is dense in $A_{E(2)}$. In what follows $A_{E(2)}^{comp}$ plays a very important role.

Proposition 1.1 *Let (Λ,ρ) be a measure space, f and g be non-negative ρ-integrable functions and, for any $R\geq 0$, $\Lambda_R=\{\lambda\in\Lambda : f(\lambda)\leq R$ and $g(\lambda)\leq R\}$. Then*

$$\lim_{R\to\infty}\frac{1}{R}\int_{\Lambda_R} f(\lambda)g(\lambda)d\rho(\lambda) \;=\; 0. \tag{23}$$

Proof:

Replacing f and g by $\max(f,g)$ we increase (23). Therefore it is sufficient to consider the case $f=g$. Assume that (23) does not hold. Then there exist $\delta>0$ and a sequence $(R_k)_{k=0,1,2,\ldots}$ such that $R_k>2R_{k-1}$ and

$$\frac{1}{R_k}\int_{\Lambda_{R_k}} f(\lambda)^2 d\rho(\lambda) \;\geq\; \delta. \tag{24}$$

Denote the left hand side of the above inequality by δ_k. Clearly $R_k f(\lambda)\geq f(\lambda)^2$ for $\lambda\in\Lambda_{R_k}$. Therefore

$$\int_{\Lambda_{R_k}-\Lambda_{R_{k-1}}} f(\lambda)d\rho(\lambda)\geq\frac{1}{R_k}(\delta_k R_k-\delta_{k-1}R_{k-1})\geq\delta_k-\frac{1}{2}\delta_{k-1}.$$

Remembering that $f\in L^1(\rho)$ we get

$$\sum_{k=1}^{\infty}(\delta_k-\frac{1}{2}\delta_{k-1})<\infty.$$

This statement is in clear contradiction with (24) saying that $\delta_k>\delta$ for all k.

Q.E.D.

We shall use this Proposition in the following context. Let $\Lambda=\overline{\mathbf{C}}^{\mu}\times\overline{\mathbf{C}}^{\mu}$, $f(\lambda_1,\lambda_2)=|\lambda_1|^2$, $g(\lambda_1,\lambda_2)=|\lambda_2|^2$ and $d\rho(\lambda)=(\psi\mid dE(\lambda_1)\otimes dE(\lambda_2)\mid\psi)$, where $dE(\cdot)$ is the spectral measure corresponding to the normal operator n and $\psi\in H_{E(2)}\otimes H_{E(2)}$. Then, for $R=\mu^{-2l}$ $(l\in\mathbf{Z})$, we have

$$\int_{\Lambda_R} f(\lambda)g(\lambda)d\rho(\lambda)=\mu^{-4l}||\,(\tau^l\gamma^*\otimes\tau^l\gamma)\psi\,||^2$$

Let us notice that $f\in L^1(\rho)$ ($g\in L^1(\rho)$ resp.) if and only if $\psi\in\mathcal{D}(n\otimes I_{E(2)})$ ($\psi\in\mathcal{D}(I_{E(2)}\otimes n)$ resp.). Therefore, using (23) we see that

$$\lim_{l\to\infty}\mu^{-l}||\,(\tau^l\gamma^*\otimes\tau^l\gamma)\psi\,||=0 \tag{25}$$

for any $\psi\in\mathcal{D}(n\otimes I_{E(2)})\bigcap\mathcal{D}(I_{E(2)}\otimes n)$.

2 The main results.

The contraction procedure leads to a number of formulae relating the quantum groups $S_\mu U(2)$ and $E_\mu(2)$. In this Section we just list these formulae, the proofs are contained in Section 3. We start with the following formula directly corresponding to (1):

For any $a \in A_{E(2)}^{comp}$

$$\Phi_{E(2)}(a) = \lim_{k\to\infty}(\tau^k \otimes \tau^k)\Phi_{SU(2)}(\tau^{-k}a). \tag{26}$$

Let us notice that $\tau^{-k}a \in A_{SU(2)}$ for sufficiently large k, so $\Phi_{SU(2)}(\tau^{-k}a)$ is a well defined element of $A_{SU(2)} \otimes A_{SU(2)} \subset A_{E(2)} \otimes A_{E(2)}$. For general $a \in A_{E(2)}$ this is not always the case and the above formula becomes slightly more complicated:

$$\Phi_{E(2)}(a) = \lim_{k\to\infty}(\tau^k \otimes \tau^k)\Phi_{SU(2)}(I_{SU(2)}(\tau^{-k}a)I_{SU(2)}). \tag{27}$$

To justify (27) it is sufficient to notice that it coincides with (26) for $a \in A_{E(2)}^{comp}$ (in this case $I_{SU(2)}(\tau^{-k}a) = (\tau^{-k}a)I_{SU(2)} = \tau^{-k}a$ for sufficiently large k) and that the right hand side of (27) depends continuously on a.

It turns out that the limit (26) may be computed explicitely and we shall obtain a closed formula relating the comultiplications in $S_\mu U(2)$ and $E_\mu(2)$. Let

$$t = \prod_{k=1}^{\infty}(I_{SU(2)} - \mu^{2k}\gamma^*\gamma), \tag{28}$$

$$X = \sum_{r=0}^{\infty} c_r(-\mu\gamma^* \otimes \gamma)^r(v \otimes v)^{-r}, \tag{29}$$

where

$$c_r = \prod_{i=1}^{r}\frac{1}{1-\mu^{2i}}.$$

By definition the first term in (29) (corresponding to $r = 0$) is equal to $I_{SU(2)} \otimes I_{SU(2)}$. Clearly the infinite product (28) is norm-converging and t is a positive element of $A_{SU(2)}$. Since all factors in (28) are invertible (in $A_{SU(2)}$), so is t. In what follows we denote by $t^{-\frac{1}{2}}$ the inverse of $t^{\frac{1}{2}}$: $t^{-\frac{1}{2}}t^{\frac{1}{2}} = t^{\frac{1}{2}}t^{-\frac{1}{2}} = I_{SU(2)}$. Similarly the series (29) is norm-converging and $X \in A_{E(2)} \otimes A_{E(2)}$.

Let $a \in A_{SU(2)}$. Computing the right hand side of (26) we shall obtain

$$\Phi_{E(2)}(a) = Z^*\Phi_{SU(2)}(a)Z \tag{30}$$

where

$$Z = \Phi_{SU(2)}(t^{-\frac{1}{2}})(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})X \tag{31}$$

is a partial isometry belonging to $A_{E(2)}$:

$$Z^*Z = \Phi_{E(2)}(I_{SU(2)}) \tag{32}$$

$$ZZ^* = I_{SU(2)} \otimes I_{SU(2)} \tag{33}$$

Multiplying both sides of (30) by Z from the left and Z^* from the right we obtain

$$\Phi_{SU(2)}(a) = Z\Phi_{E(2)}(a)Z^* \tag{34}$$

for any $a \in A_{SU(2)}$. Combining this formula with (12) we get

$$\Phi_{SU(2)}(a) = Q(a \otimes I_{E(2)})Q^*, \tag{35}$$

where $Q = ZW$ is a partial isometry acting on $H_{E(2)} \otimes H_{E(2)}$:

$$QQ^* = I_{SU(2)} \otimes I_{SU(2)}, \tag{36}$$

$$Q^*Q = I_{SU(2)} \otimes I_{E(2)}. \tag{37}$$

3 Proofs.

The proof of (26) consists in three steps.

At first (Step 1) we have to show that the limit

$$\Phi'(a) = \lim_{k\to\infty} (\tau^k \otimes \tau^k)\Phi_{SU(2)}(\tau^{-k}a) \tag{38}$$

exists for any $a \in A_{E(2)}^{comp}$. If this is the case then $\Phi' : A_{E(2)}^{comp} \longrightarrow A_{E(2)} \otimes A_{E(2)}$ is a norm-preserving *-algebra homomorphism. It admits the unique continuous extension (denoted by the same symbol) on $A_{E(2)}$.

Next (Step 2) we prove that $\Phi' \in \mathrm{Mor}(A_{E(2)}, A_{E(2)} \otimes A_{E(2)})$ To this end it is sufficient to find a bounded sequence (a_l) of elements of $A_{E(2)}$ such that

$$\text{a.u.}\lim_{l\to\infty} \Phi'(a_l) = I_{E(2)} \otimes I_{E(2)}. \tag{39}$$

Finally (Step 3) we have to prove that Φ' acts in the correct way on generators v and n:

$$\Phi'(v) = v \otimes v, \tag{40}$$

$$\Phi'(n) = v \otimes n \dot{+} n \otimes v^*. \tag{41}$$

Then $\Phi' = \Phi_{E(2)}$ and (26) follows.

Step 1: By virtue of (22) it is sufficient to show the existence of the limit (38) for $a \in A_{SU(2)}$.

We shall use the following formulae

$$\lim_{k\to\infty} \alpha^k v^{-k} = t^{\frac{1}{2}}, \tag{42}$$

$$\lim_{k\to\infty} \Phi_{SU(2)}(\alpha^k)(v^{-k} \otimes v^{-k}) = (t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})X. \tag{43}$$

To prove (42) we write $\alpha^k v^{-k} = \prod_{s=1}^{k}(v^{s-1}\alpha v^{-s})$. By virtue of (15) $\alpha^k v^{-k} = \prod_{s=1}^{k} v^s f_\alpha(n) v^{-s} = \prod_{s=1}^{k} f_\alpha(\mu^s n)$. One can easily verify that the latter operator kills all e_{ij} for $i<0$, whereas its action on e_{ij} $(i \geq 0)$ coincides with that of $\prod_{s=1}^{k}(I_{SU(2)} - \mu^{2s}\gamma^*\gamma)^{\frac{1}{2}}$. Now using (28) we get (42).

Remark: Formula (42) means that

$$\lim_{k\to\infty} \| t^{\frac{1}{2}} v^k - \alpha^k \| = 0. \tag{44}$$

The proof of (43) is more complicated. Let us notice that the terms in (5) satisfy the relation $(\alpha\otimes\alpha)(\gamma^*\otimes\gamma) = \mu^2(\gamma^*\otimes\gamma)(\alpha\otimes\alpha)$. Therefore using the binomial formula we have

$$\Phi_{SU(2)}(\alpha^k) = \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix}_{\mu} (-\mu\gamma^*\otimes\gamma)^r (\alpha\otimes\alpha)^{k-r}, \tag{45}$$

where $\begin{pmatrix} k \\ r \end{pmatrix}_{\mu}$ are deformed binomial coefficients:

$$\begin{pmatrix} k \\ r \end{pmatrix}_{\mu} = \frac{\prod_{i=1}^{k}(1-\mu^{2i})}{\prod_{i=1}^{k-r}(1-\mu^{2i})\prod_{i=1}^{r}(1-\mu^{2i})}.$$

One can easily check that the binomial coefficients are bounded: There exists a constant c (independent of k and r) such that

$$\begin{pmatrix} k \\ r \end{pmatrix}_{\mu} \leq c. \tag{46}$$

Moreover

$$\lim_{k\to\infty} \begin{pmatrix} k \\ r \end{pmatrix}_{\mu} = \prod_{i=1}^{r} \frac{1}{1-\mu^{2i}}. \tag{47}$$

Multiplying the both sides of (45) by $v^{-k}\otimes v^{-k}$ we get

$$\Phi_{SU(2)}(\alpha^k)(v\otimes v)^{-k} = \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix}_{\mu} (-\mu\gamma^*\otimes\gamma)^r (\alpha\otimes\alpha)^{k-r}(v\otimes v)^{r-k}(v\otimes v)^{-r}.$$

Due to (46), the norm of the r^{th} term on the right hand side is smaller than $c\mu^r$. Since this estimate does not depend on k, we can compute the limit for $k\to\infty$ term by term. Taking into account (47) and (42) we see that the r^{th} term converges to $c_r(-\mu\gamma^*\otimes\gamma)^r(t^{\frac{1}{2}}\otimes t^{\frac{1}{2}})(v\otimes v)^{-r}$ and (cf (29)) formula (43) follows.

Now we are ready to compute the limit (38). Let $a \in A_{SU(2)}$. Then $a = t^{\frac{1}{2}} t^{-\frac{1}{2}} a t^{-\frac{1}{2}} t^{\frac{1}{2}}$ and

$$\Phi'(a) = \lim_{k\to\infty} (v\otimes v)^k \Phi_{SU(2)}(v^{-k} t^{\frac{1}{2}} t^{-\frac{1}{2}} a t^{-\frac{1}{2}} t^{\frac{1}{2}} v^k)(v\otimes v)^{-k}.$$

Due to (44) we may replace $t^{\frac{1}{2}} v^k$ by α^k and $v^{-k} t^{\frac{1}{2}}$ by $(\alpha^*)^k$:

$$\Phi'(a) = \lim_{k\to\infty} (v\otimes v)^k \Phi_{SU(2)}((\alpha^*)^k t^{-\frac{1}{2}} a t^{-\frac{1}{2}} \alpha^k)(v\otimes v)^{-k}.$$

Using now (43) we obtain

$$\Phi'(a) = Z^* \Phi_{SU(2)}(a) Z, \tag{48}$$

where Z is an element of $A_{E(2)}$ defined by (31). In this way the existence of the limit (38) is proved.

Remark: For any $a \in A^{comp}_{E(2)}$ and $l \in \mathbf{N}$ we have:

$$\Phi'(\tau^l a) = (\tau^l \otimes \tau^l)\Phi'(a). \tag{49}$$

This formula follows immediately from (38).

Step 2: Inserting in (48) $a = t$ we obtain (cf (31))

$$\Phi'(t) = X^*(t \otimes t)X.$$

Let l be a natural number. By virtue of (49)

$$\Phi'(\tau^l(t)) = (\tau^l \otimes \tau^l)(X^*(t \otimes t)X). \tag{50}$$

We investigate the behavior of the right hand side when $l \longrightarrow \infty$. Using (29) and (19) one can easily show that

$$\text{a.u.}\lim_{l\to\infty}(\tau^l \otimes \tau^l)X = I_{E(2)} \otimes I_{E(2)}. \tag{51}$$

Indeed ($r = 0$)-term in (29) is the only one that survives. Due to (18)

$$\text{a.u.}\lim_{l\to\infty}\tau^l(t^{\frac{1}{2}}) = I_{E(2)}. \tag{52}$$

Inserting these data into (50) we see that the sequence $a_l = \tau^l(t)$ solves the problem (39).

Step 3: We already know that $\Phi' \in \mathrm{Mor}(A_{E(2)}, A_{E(2)} \otimes A_{E(2)})$. By the general theory [16], Φ' may be applied to any element $T \eta A_{E(2)}$ (η is the affiliation relation). The result $\Phi'(T)$ is affiliated with $A_{E(2)} \otimes A_{E(2)}$.

Inserting $a = t^{\frac{1}{2}}\alpha t^{\frac{1}{2}}$ in (48) and using (49) we obtain:

$$\begin{aligned}\Phi'(\tau^l(t^{\frac{1}{2}}\alpha t^{\frac{1}{2}})) &= (\tau^l \otimes \tau^l)(X^*(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})\Phi_{SU(2)}(\alpha)(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})X) \\ &= (\tau^l \otimes \tau^l)(X^*(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})(\alpha \otimes \alpha - \mu\gamma^* \otimes \gamma)(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})X).\end{aligned}$$

Let $l \longrightarrow \infty$. According to (51), (52), (20) and (19), the right hand side converges almost uniformly to $v \otimes v$. On the other hand a.u. $\lim_{l\to\infty}\tau^l(t^{\frac{1}{2}}\alpha t^{\frac{1}{2}}) = v$ and using almost uniform continuity of morphisms we obtain (40).

The proof of (41) is more complicated. In this case we have to deal with unbounded elements. We shall need precise information about the rate of convergence in (51). Rewriting (29) in the form

$$X = \sum_{r=0}^{\infty} c_r (v \otimes v)^{-r}(-\mu\tau^r(\gamma^*) \otimes \tau^r(\gamma))^r$$

and using (25) we see that

$$\lim_{l\to\infty} \mu^{-l} \mid\mid (\tau^l \otimes \tau^l) X\psi - \psi \mid\mid = 0 \tag{53}$$

for any $\psi \in \mathcal{D}(n \otimes I_{E(2)}) \bigcap \mathcal{D}(I_{E(2)} \otimes n)$.

Inserting $a = t^{\frac{1}{2}}\gamma^* t^{\frac{1}{2}}$ in (48) and using (49) we obtain:

$$\begin{aligned}\Phi'(\tau^l(t^{\frac{1}{2}}\gamma^* t^{\frac{1}{2}})) &= (\tau^l \otimes \tau^l)(X^*(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})\Phi_{SU(2)}(\gamma^*)(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})X) \\ &= (\tau^l \otimes \tau^l)(X^*(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})(\gamma^* \otimes \alpha^* + \alpha \otimes \gamma^*)(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}})X).\end{aligned}$$

For any $\lambda \in \overline{\mathbf{C}}^\mu$ and $l \in \mathbf{N}$ we set

$$f_l(\lambda) = \lambda \prod_{i=l+1}^{\infty} (1 - \mu^{2i}|\lambda|^2).$$

Then the sequence of functions (f_l) satisfies the assumptions of Proposition 0.1: notice that $f_l(\lambda) = 0$ for $|\lambda| > \mu^{-l}$, so

$$|f_l(\lambda)| \leq \mu^{-l} \tag{54}$$

for all $\lambda \in \overline{\mathbf{C}}^\mu$. Moreover one can easily verify that

$$f_l(n) = \mu^{-l}\tau^l(t^{\frac{1}{2}}\gamma^* t^{\frac{1}{2}}).$$

Therefore our formula may be rewritten in the following way

$$f_l(\Phi'(n)) = R_l + S_l, \tag{55}$$

where

$$R_l = (\tau^l \otimes \tau^l)(X)^*(I_{E(2)} \otimes \tau^l(t^{\frac{1}{2}}\alpha^* t^{\frac{1}{2}}))f_l(n \otimes I_{E(2)})(\tau^l \otimes \tau^l)(X),$$

$$S_l = (\tau^l \otimes \tau^l)(X)^*(\tau^l(t^{\frac{1}{2}}\alpha t^{\frac{1}{2}}) \otimes I_{E(2)})f_l(I_{E(2)} \otimes n)(\tau^l \otimes \tau^l)(X).$$

Let $\psi \in \mathcal{D}(v \otimes n + n \otimes v^*) = \mathcal{D}(n \otimes I_{E(2)}) \bigcap \mathcal{D}(I_{E(2)} \otimes n)$. By virtue of (53) and (54)

$$\lim_{l\to\infty} \mid\mid f_l(I_{E(2)} \otimes n)(\tau^l \otimes \tau^l)X\psi - f_l(I_{E(2)} \otimes n)\psi \mid\mid = 0$$

and (cf Proposition 0.1)

$$\lim_{l\to\infty} f_l(I_{E(2)} \otimes n)(\tau^l \otimes \tau^l)X\psi = (I_{E(2)} \otimes n)\psi.$$

Remembering that the almost uniform topology is stronger than the strong operator topology and using (51), (52) and (20) we obtain:

$$\lim_{l\to\infty} S_l\psi = (v \otimes n)\psi.$$

In the same way we get

$$\lim_{l\to\infty} R_l\psi = (n \otimes v^*)\psi.$$

Using now Proposition 0.1 we see that $\psi \in \mathcal{D}(\Phi'(n))$ and $\Phi'(n)\psi = (v \otimes n + n \otimes v^*)\psi$. It means that $\Phi'(n)$ is an extension of $(v \otimes n + n \otimes v^*)$. Remembering that normal operators have no proper normal extensions we get (41).

This completes the proof of our main formula (26). Formula (30) coincides now with (48). According to (42), $\lim_{k\to\infty} \alpha^k (\alpha^*)^k = t$. Using (43) we have

$$(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}}) X X^* (t^{\frac{1}{2}} \otimes t^{\frac{1}{2}}) = \lim_{k\to\infty} \Phi_{SU(2)}(\alpha^k (\alpha^*)^k) = \Phi_{SU(2)}(t)$$

and (33) follows. Inserting now $a = I_{SU(2)}$ in (30) we get (32). (36) follows immediately from (33). (37) may be proved by direct computation (cf (12)):

$$Q^* Q = W^* Z^* Z W = W^* \Phi_{E(2)}(I_{SU(2)}) W = I_{SU(2)} \otimes I_{E(2)}.$$

4 Paradoxes (Semigroup behaviour).

Let A and B be C^*-algebras. By definition (cf [14] and [13]) $\phi \in \mathrm{Mor}(A, B)$ if and only if ϕ is a *-algebra homomorphism from A into $M(B)$ such that $\phi(A)B$ is dense in B. In general $\phi(A)$ is not contained in B.

Let $\Phi \in \mathrm{Mor}(A, A \otimes A)$ be the comultiplication related to a classical (i.e. not quantum) locally compact, noncompact group G. Then $A = C_\infty(G)$, $A \otimes A = C_\infty(G \times G)$ and $\Phi(a)(g_1, g_2) = a(g_1 g_2)$ for any $a \in A$; $g_1, g_2 \in G$. Let us notice that the function $\Phi(a)$ is constant on sets of the form $\{(g_1, g_2) : g_1 g_2 = \text{const.}\}$. Since all these sets are non-compact, $\Phi(a)$ does not vanish at infinity (unless $a = 0$). It means that

$$\Phi(A) \bigcap (A \otimes A) = \{0\} \tag{56}$$

It is not difficult to show that the same relation holds for the Pontryagin duals of c.q.m.grps and for double groups built over c.q.m.grps (c.q.m.grp stays for compact quantum matrix group); these groups are described in [12]. One may think that the property (56) is characteristic for non-compact quantum groups. However it follows immediately from (27) that

$$\Phi_{E(2)}(A_{E(2)}) \subset (A_{E(2)} \otimes A_{E(2)}) \tag{57}$$

Since this result contradicts our intuition, we give an independent purely computational proof. Let $F_\mu \in C(\overline{\mathbf{C}}^\mu)$ be the special function considered in [17] and

$$f_k(z) = \frac{1}{2\pi} \int_0^{2\pi} F_\mu(e^{i\theta} z) e^{ik\theta} d\theta \tag{58}$$

be its Fourier transform. It is known (cf the proof of Proposition 5.2 in [17]) that $f_k \in C_\infty(\overline{\mathbf{C}}^\mu)$. Moreover

$$f_k(e^{i\theta} z) = e^{-ik\theta} f_k(z) \tag{59}$$

for any $\theta \in [0, 2\pi]$ and $z \in \overline{\mathbf{C}}^\mu$. According to the formula (3.1) of [17],

$$F_\mu(e^{i\theta} \Phi_{E(2)}(n)) = F_\mu(e^{i\theta} n \otimes v^*) F_\mu(e^{i\theta} v \otimes n).$$

Integrating over θ we get

$$f_0(\Phi_{E(2)}(n)) = \sum_{k=-\infty}^{+\infty} f_k(n \otimes v^*) f_{-k}(v \otimes n).$$

By virtue of (59), $f_k(n \otimes v^*) = f_k(n) \otimes v^k$ and $f_{-k}(v \otimes n) = v^k \otimes f_{-k}(n)$. Therefore

$$\Phi_{E(2)}(f_0(n)) = \sum_{k=-\infty}^{+\infty} f_k(n) v^k \otimes v^k f_{-k}(n). \tag{60}$$

Each term on the right hand side belongs to $A_{E(2)} \otimes A_{E(2)}$. Remembering that $|F_\mu(z)|= 1$ for all $z \in \overline{\mathbf{C}}^\mu$ we get the estimate $|f_k(z)|\leq 1$ for all $k \in \mathbf{Z}$ and $z \in \overline{\mathbf{C}}^\mu$.

It follows easily from the definition of $F_\mu(\cdot)$ that the function $F_\mu(e^{i\theta} z)$ admits the holomorphic extension into the strip $0 \geq \Im(\theta) > \log \mu$, bounded on each line $\Im(\theta) = \mathrm{const}$ with the bound independent of z. Shifting the integration contour in (58) down in the complex plane we get the inequality:

$$|f_k(z)|\leq C_a a^k$$

where $a \in [1, \mu^{-1}[$ and C_a is a constant independent of $k \in \mathbf{Z}$ and $z \in \overline{\mathbf{C}}^\mu$.

Due to the above inequalities, the norm of the k^{th} term in (60) is bounded by $C_a a^{-|k|}$; the series (60) is norm-converging and $\Phi_{E(2)}(f_0(n)) \in A_{E(2)} \otimes A_{E(2)}$. Therefore $\Phi_{E(2)}(\tau^l f_0(n)) = (\tau^l \otimes \tau^l)\Phi_{E(2)}(f_0(n)) \in A_{E(2)} \otimes A_{E(2)}$. Let $a \in A_{E(2)}$. By virtue of (18), $\lim_{l\to\infty} \tau^l(f_0(n))a = a$. Therefore $\Phi_{E(2)}(a) = \lim_{l\to\infty} \Phi_{E(2)}(\tau^l(f_0(n)))\Phi_{E(2)}(a) \in A_{E(2)} \otimes A_{E(2)}$ and (57) follows.

Inserting $a = (I_{E(2)} + n^*n)^{-1} \in A_{E(2)}$ in (27), we obtain

$$\lim_{l\to\infty} || \Phi_{SU(2)}((I_{SU(2)} + \mu^{-2l}\gamma^*\gamma)^{-1}) - (\tau^{-l} \otimes \tau^{-l})\Phi_{E(2)}((I_{E(2)} + n^*n)^{-1}) ||= 0$$

Let x be an element of $A_{SU(2)} \otimes A_{SU(2)}$ such that a.u. $\lim_{l\to\infty}(\tau^l \otimes \tau^l)x = 0$. This assumption is satisfied for $x = \gamma^\# \otimes I_{SU(2)}$, $I_{SU(2)} \otimes \gamma^\#$ and $\gamma^\# \otimes \gamma^\#$, where $\gamma^\#$ is either γ or γ^* (cf (19)). Remembering that $\Phi_{E(2)}((I_{E(2)} + n^*n)^{-1}) \in A_{E(2)} \otimes A_{E(2)}$ we get

$$|| x(\tau^{-l} \otimes \tau^{-l})\Phi_{E(2)}((I_{E(2)} + n^*n)^{-1}) ||=|| (\tau^l \otimes \tau^l)(x)\Phi_{E(2)}((I_{E(2)} + n^*n)^{-1}) ||\longrightarrow 0$$

when $l \longrightarrow \infty$. Comparing the last two relations we see that

$$\lim_{l\to\infty} x\Phi_{SU(2)}((I_{SU(2)} + \mu^{-2l}\gamma^*\gamma)^{-1}) = 0$$

and

$$\lim_{l\to\infty} x\Phi_{SU(2)}(a_l) = x, \tag{61}$$

where $a_l = \gamma^*\gamma(\mu^{2l} I_{SU(2)} + \gamma^*\gamma)^{-1} = I_{SU(2)} - (I_{SU(2)} + \mu^{-2l}\gamma^*\gamma)^{-1}$.

We know [11] that $S_\mu U(2)$contains the classical subgroup S^1. Removing this subgroup we obtain a locally compact (non-compact) quantum space $S_\mu U(2) - S^1$. Let B be the corresponding C^*-algebra. By definition B is the closed ideal in $A_{SU(2)}$ generated by γ and γ^*. One should notice that $a_l \in B$ for all l. According to (3), $\ker\gamma = \ker\gamma^* = \{0\}$. Therefore $a = 0$ is the only element of $A_{SU(2)}$ such that $ab = 0$ for all $b \in B$. Remembering that B is an ideal in $A_{SU(2)}$ we see that $A_{SU(2)} \subset M(B)$. Consequently

$$A_{SU(2)} \otimes A_{SU(2)} \subset M(B \otimes A_{SU(2)}), \tag{62}$$

$$A_{SU(2)} \otimes A_{SU(2)} \subset M(B \otimes B) \tag{63}$$

and so on.

Let $\tilde{\Phi} = \Phi_{SU(2)}|_B$. Using the commutation relations (4) one can easily see that $A_{SU(2)}\gamma + A_{SU(2)}\gamma^*$ is dense in B. Inserting $x = \gamma^\# \otimes I_{SU(2)}$ and $x = \gamma^\# \otimes \gamma^\#$ in (61) we see that $((\tilde{\Phi}(a_l))_{l=1,2,\ldots}$ is an approximate unit for $B \otimes A_{SU(2)}$ and $B \otimes B$. Keeping in mind the inclusions (62) and (63) we see that

$$\tilde{\Phi} \in \mathrm{Mor}(B, B \otimes A_{SU(2)}), \tag{64}$$

$$\tilde{\Phi} \in \mathrm{Mor}(B, B \otimes B). \tag{65}$$

Relation (64) means that $S_\mu U(2) - S^1$ is invariant under right shifts (so we have an example of a non-compact homogeneous space for a compact group). A similar result holds for the Podleś sphere $S_\mu U(2)/S^1$. Removing the classical point (there is only one) we obtain a non-compact 'quantum plane' with a transitive(?) action of $S_\mu U(2)$.

Relation (65) means that $S_\mu U(2) - S^1$ is a subsemigroup of $S_\mu U(2)$.

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