Inhomogeneous quantum groups

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In this paper inhomogeneous group G is a semidirect product of a finite-dimensional vector space V by a group G_0 of linear transformations of V. Then V is a normal subgroup of G and $G_0 = G/V$ is the factor group. The Hopf algebra $\mathcal{A} = \mathrm{Poly}(G)$ of polynomial functions on G is generated by the sub-algebra $\mathcal{A}_0 = \mathrm{Poly}(G_0)$ and N distinguished linearly independent elements elements $x_1, x_2, \ldots x_N$ ($N = \dim V$) being the linear forms on V. The comultiplication Δ restricted to $\mathcal{A}_0$ coincides with the one describing the group structure of G_0 and for any $k = 1, 2, \ldots, N$

$$\Delta(x_k) = \sum_{l=1}^{N} \Lambda_{kl} \otimes x_l + x_k \otimes I, \tag{1}$$

where $\Lambda = (\Lambda_{kl})_{k,l=1,2,\ldots N}$ is the matrix corepresentation of $\mathcal{A}_0$ describing the action of G_0 on V.

In the classical case $\mathcal{A}_0$ is commutative and the elements $x_1, x_2, \ldots x_N$ commute with the elements of $\mathcal{A}_0$. Moreover $x_1, x_2, \ldots x_N$ mutually commute: we have $\frac{(N-1)N}{2}$ relations $x_k x_l = x_l x_k$ ($l = 2, 3, \ldots, N$; $k = 1, 2, \ldots, l-1$). In the quantum case $\mathcal{A}_0$ is no longer commutative (G_0 is a quantum group) and the relations describing the commutation properties of $x_1, x_2, \ldots x_N$ are more complicated. We shall assume that:

1. Elements of $\mathcal{A}_0$ may be dragged from the right hand side of x_k to the left: for any $a \in \mathcal{A}$ and any $k = 1, 2, \ldots, N$ there exist $a_0, b_1, b_2, \ldots, b_N \in \mathcal{A}_0$ such that

$$x_k a = a_0 + b_1 x_1 + b_2 x_2 + \ldots + b_N x_N. \tag{2}$$

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2. The elements x_k ($k = 1, 2, \ldots, N$) satisfy $\frac{(N-1)N}{2}$ independent quadratic relations of the form

$$\sum_{k,l=1}^{N} a_{kl} x_k x_l + \sum_{k=1}^{N} b_k x_k + c = 0, \tag{3}$$

where $a_{kl}, b_k, c \in \mathcal{A}_0$

The following problem seems to be interesting: for given Hopf algebra $\mathcal{A}_0$ and given matrix corepresentation $(\Lambda_{kl})_{k,l=1,2,\ldots N}$ of $\mathcal{A}_0$ one has to find the commutation relation of the form (2) and (3) compatible with the Hopf algebra structure. If there are many solutions (that seems to be the case in general), one has to classify them.

Let Γ be the set of all elements of the form

$$a = b_0 + b_1 x_1 + b_2 x_2 + \ldots + b_N x_N \tag{4}$$

where $b_k \in \mathcal{A}_0$, $k = 0, 1, 2, \ldots, N$. By virtue of the assumption 1 (cf (2), Γ is a bimodule over $\mathcal{A}_0$. So is $\dot{\Gamma} = \Gamma/\mathcal{A}_0$. According to the (1), $\Delta(\Gamma) \subset \Gamma \otimes \mathcal{A}_0 + \mathcal{A}_0 \otimes \Gamma$. Remembering that $\Delta(\mathcal{A}_0) \subset \mathcal{A}_0 \otimes \mathcal{A}_0 = \Gamma \otimes \mathcal{A}_0 \cap \mathcal{A}_0 \otimes \Gamma$ we see that Δ induces a linear mapping from $\dot{\Gamma}$ into $\dot{\Gamma} \otimes \mathcal{A}_0 \oplus \mathcal{A}_0 \otimes \dot{\Gamma}$. We denote by

$$\Delta_R : \dot{\Gamma} \longrightarrow \dot{\Gamma} \otimes \mathcal{A}_0,$$

$$\Delta_L : \dot{\Gamma} \longrightarrow \mathcal{A}_0 \otimes \dot{\Gamma}$$

the components of this mapping. The further investigations are based on the following simple observation:

Theorem 1 $(\dot{\Gamma}, \Delta_R, \Delta_L)$ *is a bicovariant bimodule over* $\mathcal{A}_0$ *in the sense of* [3, *Definition* 2.3]*.*

Let $\dot{x}_k \in \dot{\Gamma}$ be the element corresponding to $x_k \in \Gamma$ ($k = 1, 2, \ldots N$). According to (1),

$$\Delta_R \dot{x}_k = \dot{x}_k \otimes I, \tag{5}$$

$$\Delta_L \dot{x}_k = \sum_{l=1}^{N} \Lambda_{kl} \otimes \dot{x}_l \tag{6}$$

Relation (5) means that $\dot{x}_k \in \dot{\Gamma}_{\text{inv}}$, where $\dot{\Gamma}_{\text{inv}}$ is the set of all right-invariant elements of $\dot{\Gamma}$. Taking into account (4) and using [3, Theorem 2.3.1] we see that $\dot{\Gamma}$ is a free left $\mathcal{A}_0$-module with the basis $(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N)$. It shows that in the decomposition (4) the coefficients $b_k \in \mathcal{A}_0$, $k = 0, 1, 2, \ldots, N$ are determined uniquely by a.

Now we shall analyse the quadratic relations. Applying the comultiplication Δ on the both sides of (3) we get:

$$\sum_{k,l=1}^{N} \Delta(a_{kl}) \left[\sum_{s=1}^{N} \Lambda_{ks} \otimes x_s + x_k \otimes I \right] \left[\sum_{r=1}^{N} \Lambda_{lr} \otimes x_r + x_l \otimes I \right] + \ldots = 0$$

and

$$\sum_{k,l=1}^{N} \Delta(a_{kl}) \left[\sum_{s=1}^{N} \Lambda_{ks} x_l \otimes x_s + \sum_{r=1}^{N} x_k \Lambda_{lr} \otimes x_r \right] + \ldots = 0,$$

where in the last equation the dots represent terms belonging to $\mathcal{A}_0 \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{A}_0$. Passing to the factor bimodule $\dot{\Gamma} = \Gamma / \mathcal{A}_0$ we obtain

$$\sum_{k,l,r=1}^{N} \Delta(a_{kl}) \left[\Lambda_{kr} \dot{x}_l \otimes \dot{x}_r + \dot{x}_k \Lambda_{lr} \otimes \dot{x}_r \right] = 0.$$

Using the formula[1] 2.33 of [3] one can easily check that

$$\dot{x}_k \Lambda_{lr} = \sum_{n,s=1}^{N} c_{klns} \Lambda_{nr} \dot{x}_s. \tag{7}$$

Inserting this result into the previous formula we get

$$\sum_{k,l,r,s=1}^{N} \Delta(a_{kl}) \left[\left(\Lambda_{kr} \delta_{ls} + \sum_{n=1}^{N} c_{klns} \Lambda_{nr} \right) \otimes I \right] (\dot{x}_s \otimes \dot{x}_r) = 0.$$

Remembering that $(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N)$ is a basis of the free left $\mathcal{A}_0$-module $\dot{\Gamma}$ we obtain

$$\sum_{k,l=1}^{N} \Delta(a_{kl}) \left[\left(\Lambda_{kr} \delta_{ls} + \sum_{n=1}^{N} c_{klns} \Lambda_{nr} \right) \otimes I \right] = 0$$

for any $r, s = 1, 2, \ldots, N$. Multiplying the both sides by $(\Lambda^{-1})_{rn}$ and summing over r we have

$$\sum_{k,l=1}^{N} \Delta(a_{kl}) \left[\left(\delta_{kn} \delta_{ls} + c_{klns} \right) I \otimes I \right] = 0$$

and finally

$$\sum_{k,l=1}^{N} a_{kl} \left(\delta_{kn} \delta_{ls} + c_{klns} \right) = 0 \tag{8}$$

for any $n, s = 1, 2, \ldots, N$.

To understand better the last relation we recall the canonical twisted flip mapping $\sigma : \dot{\Gamma} \otimes_{\mathcal{A}_0} \dot{\Gamma} \to \dot{\Gamma} \otimes_{\mathcal{A}_0} \dot{\Gamma}$ introduced in [3, Proposition 3.1]. By definition this is the unique bicovariant bimodule homomorphism such that

$$\sigma(\dot{y} \otimes_{\mathcal{A}_0} \dot{x}) = \dot{x} \otimes_{\mathcal{A}_0} \dot{y} \tag{9}$$

for any left-invariant element $\dot{y}$ and any right-invariant element $\dot{x}$ of $\dot{\Gamma}$.

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[1] actually there is a misprint in this formula, to get the correct form one has to replace f_{ij} by $f_{ij} \circ \kappa^{-2}$

Let

$$\tau = \sum_{k,l=1}^{N} a_{kl} \dot{x}_k \otimes_{\mathcal{A}_0} \dot{x}_l. \tag{10}$$

We shall compute $\sigma^{-1}(\tau)$. To this end we consider elements $\dot{y}_k = \sum_{l=1}^{N} (\Lambda^{-1})_{kl} \dot{x}_l$ $(k = 1, 2, \ldots, N)$. Then

$$\dot{x}_l = \sum_{r=1}^{N} \Lambda_{lr} \dot{y}_r. \tag{11}$$

Inserting this formula into (10) and using (7) we see that

$$\tau = \sum_{k,l,r=1}^{N} a_{kl} \dot{x}_k \otimes_{\mathcal{A}_0} \Lambda_{lr} \dot{y}_r = \sum_{k,l,r=1}^{N} a_{kl} \dot{x}_k \Lambda_{lr} \otimes_{\mathcal{A}_0} \dot{y}_r = \sum_{k,l,r,n,s=1}^{N} c_{klns} a_{kl} \Lambda_{nr} \dot{x}_s \otimes_{\mathcal{A}_0} \dot{y}_r.$$

Using (6), one can easily check that elements $\dot{y}_r$ are left-invariant:

$$\Delta_L(\dot{y}_r) = I \otimes \dot{y}_r.$$

Now, taking into account (9) and using once more (11) we get

$$\sigma^{-1}(\tau) = \sum_{k,l,r,n,s=1}^{N} c_{klns} a_{kl} \Lambda_{nr} \dot{y}_r \otimes_{\mathcal{A}_0} \dot{x}_s = \sum_{k,l,n,s=1}^{N} c_{klns} a_{kl} \dot{x}_n \otimes_{\mathcal{A}_0} \dot{x}_s,$$

and one can easily see that (8) is equivalent to $\tau + \sigma^{-1}(\tau) = 0$ and to

$$\sigma(\tau) = -\tau. \tag{12}$$

This way we showed that any quadratic relation (3) leads to an eigenelement of σ corresponding to the eigenvalue -1. Taking into account the Condition 3, we obtain

Theorem 2 *The number -1 is an eigenvalue of σ with the multiplicity not smaller then $\frac{(N-1)N}{2}$.*

We applied the above theory to the classification of quantum Poincaré groups [2]. In this case G_0 is a quantum Lorentz group and Λ is the four-dimensional $(\frac{1}{2}, \frac{1}{2})$-representation of G_0 (like in the classical case, the smooth representations of the Lorentz group are labelled by a pair of spins). The quantum Lorentz groups are well classified [4]. Each quantum Lorentz group has assigned a numerical parameter $q \in \mathbf{C}$ that enters the commutation relations describing the algebra $\mathcal{A}_0$. For the classical Lorentz group the value $q = 1$. However this is also the case for many quantum Lorentz groups. For example the groups described in the Section 3.3 of [4] have by definition the value $q = 1$.

It turns out that for any quantum Lorentz group described in [4], there exist only two[2] bicovariant bimodules corresponding to the $(\frac{1}{2},\frac{1}{2})$-representation. One can compute the eigenvalues of σ for these bimodules. The results are the following:

for the first bimodule:

$$\begin{array}{lcccc} \text{Eigenvalues of } \sigma: & |q|^{-1}, & -q^2|q|^{-1}, & -\bar{q}^2|q|^{-1}, & |q|^3, \\ \text{Multiplicities:} & 9, & 3, & 3, & 1, \end{array}$$

for the second one:

$$\begin{array}{lcccc} \text{Eigenvalues of } \sigma: & |q|, & -q^{-2}|q|, & -\bar{q}^{-2}|q|, & |q|^{-3}. \\ \text{Multiplicities:} & 9, & 3, & 3, & 1. \end{array}$$

The reader should not be surprised with the appearence of the multiplicities 9, 3, 3, 1. They come from the decomposition

$$\begin{array}{ccccccccccc} (\frac{1}{2},\frac{1}{2}) & \otimes & (\frac{1}{2},\frac{1}{2}) & = & (1,1) & \oplus & (1,0) & \oplus & (0,1) & \oplus & (0,0) \\ 4 & \times & 4 & = & 9 & + & 3 & + & 3 & + & 1 \end{array}$$

In the considered case $\frac{(N-1)N}{2} = 6$. Taking into account the eigenvalues of σ given above, we see that only the Lorentz groups with $q = \pm 1$ satisfy the demand formulated in Theorem 2. It means that only these groups may be used to construct quantum Poincaré groups. For these groups the multiplicity of the eigenvalue -1 is precisely 6, so we have to take into account all quadratic relations coming from solutions of the eigenequation (12). It shows that the choice of the quantum Lorentz group (with $q = \pm 1$) and the choice of one of the two bimodules determine the main terms in (2) and (3). The lower order terms (a_0 in (2) and coefficients b_k and c in (3)) are not fixed yet. One possibility is to consider homogeneous relations only, assuming that the lower order terms vanish. This way we obtain a large class of quantum Poincaré groups: two for each quantum Lorentz group with $q = \pm 1$.

In general case the determination of the lower order terms is very painful. For the Poincaré groups the problem is simpler, because the tensor product of any irreducible representation v of the Lorentz group with the $(\frac{1}{2},\frac{1}{2})$-representation is disjoint with v. This makes it possible to determine the form of the lower order terms up to a numerical constants. The latter satisfy a complicated set of linear and quadratic equations, to solve them we used the computer *MATHEMATICA* program. With this help we found all possible forms of the lower order terms for the most of the quantum Lorentz groups. The only case that is not solved yet is the one when the corresponding Lorentz group is the classical one. One should notice however that this case contains the most celebrated κ-deformation of Poincaré group [1] (cf [5]).

[2] in particular cases they may coincide

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