DUALITY OF LOCALLY COMPACT GROUPS

S.L. WORONOWICZ *

Centre de Physique Théorique
C.N.R.S. - Luminy - Case 907
13288 MARSEILLE CEDEX 9
France

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* Département de Physique Mathématique, Université de Provence

On leave from the Department of Mathematical Methods in Physics, University of Warsaw, Hoza 74, 00-682 Warszawa, Poland

This paper contains in the first part the new complete proof of the Tatsuuma duality of locally compact groups (cf. D. Kastler and M. Takesaki : Group duality and the Kubo-Martin-Schwinger Condition II, Comm. math. Phys. 85, 155-176 (1982)). The second part is devoted to the generalization in the spirit indicated in our paper : S.L. Woronowicz, Pseudospaces, pseudogroups and Pontriagin duality, Lecture Notes in Physics 116, p.405.

NOTATION :

G - locally compact group
dx - left invariant measure on G
$L^2(G)$ - the corresponding Hilbert space
L - left; R - right regular representation of G

(1) $$(L_x\varphi)(y) = \varphi(x^{-1}y)$$

(2) $$(R_x\varphi)(y) = \varphi(yx)$$

R_x is proportional to a unitary operator

W - unitary operator acting on $L^2(G)\otimes L^2(G) = L^2(G\times G)$ introduced by the formula

(3) $$(Wf)(x,y) \overset{df}{=} f(x,xy)$$

THEOREM 1 :

Let $Q \in B(L^2(G))$ and $Q \neq 0$. Then

$$\Big(W^*(Q\otimes I)W = Q\otimes Q\Big) \iff \left(\begin{array}{ll}\text{there exists} & x\in G\\ \text{such that} & Q = L_x\end{array}\right)$$

Proof : Let $f \in L^2(G\times G)$. Then using (1) and (3) we have

$$(W^*(L_x\otimes I)Wf)(z,y) = ((L_x\otimes I)Wf)(z,z^{-1}y) =$$
$$= (Wf)(x^{-1}z,z^{-1}y) = f(x^{-1}z,x^{-1}y) = ((L_x\otimes L_x)f)(z,y)$$

Therefore $W^*(L_x\otimes I)W = L_x\otimes L_x$. This ends the „ $\Leftarrow$ " part of the proof.

The second part of the proof we start with the similar computation :

$$(W^*(I\otimes R_x)Wf)(z,y) = ((I\otimes R_x)Wf)(z,z^{-1}y) =$$
$$= (Wf)(z,z^{-1}yx) = f(z,yx) = ((I\otimes R_x)f)(z,y)$$

Therefore

(4) $$W^*(I\otimes R_x)W = I\otimes R_x$$

Assume now that

(5) $$W^*(Q\otimes I)W = Q\otimes Q$$

It follows immediately that $I\otimes R_x$ and $Q\otimes Q$ commute. It means that

(6) $$QR_x = R_xQ$$

for any $x\in G$

Let $\delta_\alpha \in C_0(G)$ be a sequence of non-negative continuous functions with compact supports such that

$$\int_G \delta_\alpha(x)\,dx = 1$$

$$\operatorname{supp} \delta_\alpha \longrightarrow e$$

e denotes the neutral element of G

We shall use the sequence of operators

$$V_\alpha : L^2(G \times G) \longrightarrow L^2(G)$$

introduced by the formula

$$(7) \qquad (V_\alpha f)(x) = \int_G f(x,y)\, \delta_\alpha(y)\, dy$$

One can easily check that

$$V_\alpha (Q \otimes I) = Q V_\alpha$$

for any $Q \in B(L^2(G))$.

Let $\varphi_1, \varphi_2 \in L^2(G)$. Then

$$(8) \qquad \begin{aligned} V_\alpha W(Q\varphi_1 \otimes Q\varphi_2) &= V_\alpha W(Q \otimes Q)(\varphi_1 \otimes \varphi_2) = \\ &= V_\alpha (Q \otimes I) W(\varphi_1 \otimes \varphi_2) = Q V_\alpha W(\varphi_1 \otimes \varphi_2). \end{aligned}$$

We have :

$$(V_\alpha W(\varphi_1 \otimes \varphi_2))(x) = \int_G (W(\varphi_1 \otimes \varphi_2))(x,y)\, \delta_\alpha(y)\, dy =$$

$$= \int_G (\varphi_1 \otimes \varphi_2)(x, xy)\, \delta_\alpha(y)\, dy = \varphi_1(x) \int_G \varphi_2(xy)\, \delta_\alpha(y)\, dy$$

and similarly

$$(V_\alpha W(Q\varphi_1 \otimes Q\varphi_2))(x) = (Q\varphi_1)(x) \int_G (Q\varphi_2)(xy)\, \delta_\alpha(y)\, dy$$

Assume now that $\varphi_2, Q\varphi_2 \in C_\infty(G)$. ($C_\infty(G)$ denotes the C^*-algebra of all continuous, tending to 0 at infinity functions on G with values in $\mathbb{C}$). Then the RHS of the two above equations are converging in the L^2-sens to the pointwise products $\varphi_1 \cdot \varphi_2$ and $(Q\varphi_1)\cdot(Q\varphi_2)$ respectively. Therefore setting $\alpha \to \infty$ in (8) we get

$$(9) \qquad Q(\varphi_1 \cdot \varphi_2) = (Q\varphi_1)\cdot(Q\varphi_2)$$

for any $\varphi_1, \varphi_2 \in L^2(G)$ assuming that φ_2, $Q\varphi_2$ are continuous vanishing at infinity functions. Using this relation repeatedly we get

$$Q(\varphi_2^n) = (Q\varphi_2)^n$$

Taking the nth-root of the L^2-norm and setting $n \to \infty$ we finally obtain

$$(10) \qquad \| Q\varphi_2 \|_{L^\infty} \leq \| \varphi_2 \|_{L^\infty}$$

for any $\varphi_2 \in L^2(G) \cap C_\infty(G)$ such that $Q\varphi_2 \in C_\infty(G)$
We shall show that the last condition is automatically satisfied.

Let $\varphi \in L^\infty(G) \cap L^2(G)$. Then

$$\varphi_\alpha \stackrel{df}{=} \int_G R_x \varphi \; \delta_\alpha(x)\, dx \quad \in C_\infty(G)$$

and (cf. (6))

$$Q\varphi_\alpha = \int_G R_x Q\varphi \; \delta_\alpha(x)\, dx \in C_\infty(G)$$

and using (10) we have $\|Q\varphi_\alpha\|_{L^\infty} \leq \|\varphi_\alpha\|_{L^\infty}$. Setting $\alpha \to \infty$ and remembering that $\varphi \in L^\infty(G)$ we get $Q\varphi \in L^\infty(G)$ and

$$\| Q\varphi \|_{L^\infty} \leq \| \varphi \|_{L^\infty}$$

Now let again $\varphi \in L^2(G) \cap C_\infty(G)$. Then $\lim_{x\to e} \| R_x\varphi - \varphi \|_{L^\infty} = 0$. Therefore $\lim_{x\to\infty} \| R_x Q\varphi - Q\varphi \|_{L^\infty} = 0$. It shows that $Q\varphi$ is uniformly continuous function and since is square integrable we see that $Q\varphi \in C_\infty(G)$.
This way we proved the following result :

$$Q : L^2(G) \cap C_\infty(G) \longrightarrow L^2(G) \cap C_\infty(G)$$

$$\| Q \|_{L^\infty, L^\infty} \leq 1$$

The last relation shows that Q admits an extension $\tilde{Q} : C_\infty \to C_\infty$. Obviously $\tilde{Q}$ is multiplicative (cf.(9)) and commutes with right translations (cf. (6)). Therefore the map

$$C_\infty(G) \ni \varphi \xrightarrow{\chi} (\tilde{Q}\varphi)(e) \in \mathbb{C}$$

is linear and multiplicative and for any $y \in G$ we have

$$(\tilde{Q}\varphi)(y) = (\tilde{Q} R_y \varphi)(e) = \chi(R_y \varphi)$$

It shows that χ does not vanish identically (otherwise $Q = 0$ what contradicts our assumption saying that $Q \neq 0$). Therefore χ is a character of $C_\infty(G)$. According to the Gelfand theory there exists $x \in G$ such that $\chi(\varphi) = \varphi(x)$. Now the above relation can be rewritten in a more explicite way :

$$(\tilde{Q}\varphi)(y) = \chi(R_y\varphi) = (R_y\varphi)(x) = \varphi(xy) = (L_{x^{-1}}\varphi)(y)$$

It shows that $Q = L_{x^{-1}}$ and the theorem is proved.

Let $L^1(L^2(G))$ denote the space of all trace-class operators acting on $L^2(G)$ equipped with the well-known trace norm. For any $\rho \in L^1(L^2(G))$ we define a function $f_\rho \in C_\infty(G)$:

$$f_\rho(x) \stackrel{df}{=} \mathrm{Tr}\,(L_x \rho)$$

For any $\rho_1, \rho_2 \in L^1(L^2(G))$ the pointwise product

(11) $$f_{\rho_1} \cdot f_{\rho_2} = f_{\rho_1 * \rho_2}$$

where

(12) $$\rho_1 * \rho_2 \stackrel{df}{=} (id \otimes \mathrm{Tr})(W(\rho_1 \otimes \rho_2) W^*)$$

Indeed for any $x \in G$ we have

$$\begin{aligned} f_{\rho_1}(x) f_{\rho_2}(x) &= (\mathrm{Tr} \otimes \mathrm{Tr})((L_x \otimes L_x)(\rho_1 \otimes \rho_2)) = \\ &= (\mathrm{Tr} \otimes \mathrm{Tr})(W^*(L_x \otimes I) W(\rho_1 \otimes \rho_2)) = \\ &= (\mathrm{Tr} \otimes \mathrm{Tr})((L_x \otimes I) W(\rho_1 \otimes \rho_2) W^*) = \\ &= \mathrm{Tr}\ L_x(\rho_1 * \rho_2). \end{aligned}$$

Let S denote the set of all functions of the form f_ρ, where $\rho \in L^1(L^2(G))$. Clearly (cf. (11)) S is a subalgebra of . One can easily check that S is a Banach algebra with the norm

(13) $$\| f \|_S \stackrel{df}{=} \inf \{ \|\rho\|_{\mathrm{Tr}} : \rho \in L^1(L^2(G)),\ f_\rho = f \}$$

The Th.1 can be now rewritten in the following equivalent version :

<u>THEOREM 2</u>
Let $\chi : S \longrightarrow \mathbb{C}$ be a linear multiplicative continuous (with respect to the norm (13)) nonvanishing identically map. Then there exists $x \in G$ such that

$$\chi(f) = f(x)$$

for any $f \in S$.

<u>Proof</u> : The continuity of χ (together with the linearity) implies that there exists $Q \in B(L^2(G))$ such that

$$\chi(f_\varrho) = \mathrm{Tr}\,(Q\varrho) \tag{14}$$

for any $\varrho \in L^1(L^2(G))$.

Let $\varrho_1, \varrho_2 \in L^1(L^2(G))$. Using (12), (14), (11), the multiplicativity of χ and again (14), we obtain :

$$\begin{aligned}(\mathrm{Tr}\otimes\mathrm{Tr})(W^*(Q\otimes I)W(\varrho_1\otimes\varrho_2)) &= (\mathrm{Tr}\otimes\mathrm{Tr})((Q\otimes I)W(\varrho_1\otimes\varrho_2)W^*) = \\ &= \mathrm{Tr}\,(Q(\varrho_1 * \varrho_2)) = \chi(f_{\varrho_1 * \varrho_2}) = \chi(f_{\varrho_1} f_{\varrho_2}) = \chi(f_{\varrho_1})\chi(f_{\varrho_2}) = \\ &= (\mathrm{Tr}\otimes\mathrm{Tr})((Q\otimes Q)(\varrho_1\otimes\varrho_2))\end{aligned}$$

Therefore the operator Q satisfies the equality (5) and using Th.1 we find $x \in G$ such that $Q = L_x$. Now we have

$$\chi(f_\varrho) = \mathrm{Tr}(Q\varrho) = \mathrm{Tr}(L_x\varrho) = f_\varrho(x)$$

Q.E.D.

According to the Gelfand theory of maximal ideals in commutative Banach algebras a complex number λ belongs to the spectrum of an element f if and only if there exists a character χ such that $\chi(f) = \lambda$. Therefore in the Banach algebra S we have

$$\mathrm{Sp}\, f = \{ f(x) \ :\ x \in G \} = f(G).$$

for any $f \in S$. Now, using the holomorphic function calculus in Banach algebras, we get immediately

<u>THEOREM 3</u>

Let $f \in S$ and F be a holomorphic function defined on an open neighbourhood of $f(G)$ such that $G(0) = 0$ (if 0 belongs to the domain of F). Then $f \circ F \in S$.

We shall use this result to generalize the Th. 1.

Let π be a nondegenerate representation of the C^*-algebra $C_\infty(G)$ acting on a Hilbert space H. Then there exists a spectral measure dE_π on G such that

$$\pi(f) = \int_G f(x)\, dE_\pi(x) \tag{15}$$

for any $f \in C_\infty(G)$. Let

$$L_\pi \overset{df}{=} \int_G dE_\pi(x) \otimes L_x \tag{16}$$

Obivously L_π is an unitary operator acting on $H \otimes L^2(G)$. If H is one-dimensional, then π is a character of $C_\infty(G)$, dE_π is supported by a point $x \in G$ and $L_\pi = L_x$.

To formulate our final result we need the special notation. For any operator Q acting on $H \otimes L^2(G)$ we denote by Q_{01} and Q_{02} operators acting on $H \otimes L^2(G) \otimes L^2(G)$ introduced by the formulae :

$$Q_{01} \overset{df}{=} Q \otimes I$$

$$Q_{02} \overset{df}{=} (I\otimes T)(Q\otimes I)(I\otimes T)$$

where T is the twisting operating acting on $L^2(G)\otimes L^2(G)$: $T(f_1\otimes f_2) = f_2 \otimes f_1$ for all $f_1, f_2 \in L^2(G)$. For example if $Q = \alpha\otimes\beta$, where $\alpha \in B(H)$, $\beta \in B(L^2(G))$, then $Q_{01} = \alpha\otimes\beta\otimes I$, $Q_{02} = \alpha\otimes I\otimes\beta$.

<u>THEOREM 4</u>

Let Q be a unitary operator acting on $H\otimes L^2(G)$. Then

$$\left(\begin{array}{c}(I\otimes W^*)(Q\otimes I)(I\otimes W) = \\ = Q_{01}Q_{02}\end{array}\right) \Longleftrightarrow \left(\begin{array}{l}\text{there exists a nondegenerated re-}\\ \text{presentation } \pi \text{ of } C_\infty(G) \text{ acting}\\ \text{on } H \text{ such that}\\ \qquad Q = L_\pi\end{array}\right)$$

Proof :

$\Leftarrow$ Assume that $Q = L_\pi = \int_G dE_\pi(x) \otimes L_x$. Then

$$Q_{01} = \int_G dE_\pi(x) \otimes L_x \otimes I$$

$$Q_{02} = \int_G dE_\pi(x) \otimes I \otimes L_x$$

$$\begin{aligned} Q_{01} Q_{02} &= \int_G dE_\pi(x) \otimes L_x \otimes L_x = \\ &= \int_G dE_\pi(x) \otimes W^*(L_x \otimes I) W = \\ &= (I \otimes W^*)\Big(\int_G dE_\pi(x) \otimes L_x \otimes I\Big)(I \otimes W) = \\ &= (I \otimes W^*)(Q \otimes I)(I \otimes W) . \end{aligned}$$

$\Rightarrow$ Like in the proof of Th.1 one can check that Q commutes with right translations :

$$Q(I \otimes R_x) = (I \otimes R_x) Q$$

In other words $Q \in (I \bar{\otimes} \mathcal{R})' = B(H) \bar{\otimes} \mathcal{L}$, where $\mathcal{R}$ and $\mathcal{L}$ denote the von Neumann algebras generated by right and left regular representations respectively and $\bar{\otimes}$ denotes the tensor product followed by the weak closure. We used the well known Godeman result : $\mathcal{R}' = \mathcal{L}$.

Let $\rho_1, \rho_2 \in L^1(L^2(G))$. Assume that $f_{\rho_1} = f_{\rho_2}$. Then remembering that $Q \in B(H) \bar{\otimes} \mathcal{L}$ we see that

$$(id \otimes Tr)(Q(I \otimes \rho_1)) = (id \otimes Tr)(Q(I \otimes \rho_2))$$

It means that $(id \otimes Tr)(Q(I \otimes \rho))$ depends only on $f_\rho \in S$. In other words it means that there exists a linear map

$$\pi_0 : S \longrightarrow B(H)$$

such that

(17) $$\pi_0(f_\rho) = (id \otimes Tr)(Q(I \otimes \rho))$$

for any $\rho \in L^1(L^2(G))$. Clearly π_0 is continuous with respect to the $\|\cdot\|_S$ norm in S and the usual operator norm in B(H).

The following calculation shows that π_0 is multiplicative. Let $\rho_1, \rho_2 \in L^1(L^2(G))$. Then

$$\begin{aligned} \pi_0(f_{\rho_1}) \pi_0(f_{\rho_2}) &= [(id \otimes Tr)(Q(I \otimes \rho_1))][(id \otimes Tr)(Q(I \otimes \rho_2))] = \\ &= (id \otimes Tr \otimes Tr)(Q_{01} Q_{02}(I \otimes \rho_1 \otimes \rho_2)) = \\ &= (id \otimes Tr \otimes Tr)((I \otimes W^*)(Q \otimes I)(I \otimes W)(I \otimes \rho_1 \otimes \rho_2)) = \\ &= (id \otimes Tr \otimes Tr)((Q \otimes I)(I \otimes W(\rho_1 \otimes \rho_2) W^*)) = \\ &= (id \otimes Tr)(Q(I \otimes \rho_1 * \rho_2)) = \pi_0(f_{\rho_1 * \rho_2}) = \\ &= \pi_0(f_{\rho_1} f_{\rho_2}) . \end{aligned}$$

We shall use the following

LEMMA

For any $f \in S$, the complex conjugate function $\bar{f} \in S$ and

(18) $$\pi_0(\bar{f}) = \pi_0(f)^*$$

It can be easily check that the functions in S separate points in G . Therefore according to the Stone-Weierstrasse theorem (applied to the one-point compactification of G) S is dense in $C_\infty(G)$. We shall prove in a moment that

(19) $$\|\pi_0(f)\| \leqslant \|f\|_{L^\infty}$$

for any $f \in S$. It shows that there exists a continuous linear map

$$\pi : C_\infty(G) \longrightarrow B(H)$$

which extends π_0 : $\pi(f_\rho) = \pi_0(f_\rho)$ for any $\rho \in L^1(L^2(G))$. Clearly π is a representation of $C_\infty(G)$. Assume for the moment that there exists a projection $E \in B(H)$ such that $\pi(f)E = 0$ for any $f \in C_\infty(G)$. Then for any $\rho \in L^1(L^2(G))$ we have

$$0 = \pi_0(f_\rho)E = (id \otimes Tr)(Q(I \otimes \rho)) \cdot E =$$
(28)
$$= (id \otimes Tr)(Q(E \otimes I)(I \otimes \rho))$$

Therefore in this case $Q(E \otimes I) = 0$. Remembering that Q is unitary we see that $E = 0$. It shows that the representation π is nondegenerate.

Let $\rho \in L^1(L^2(G))$. Then we have

$$(id \otimes Tr)(L_\pi(I \otimes \rho)) = (id \otimes Tr)\left(\int_G dE_\pi(x) \otimes L_x \rho\right) = \int_G f_\rho(x)\, dE_\pi(x) =$$

$$= \pi(f_\rho) = \pi_0(f_\rho) = (id \otimes Tr)(Q(I \otimes \rho)).$$

It shows that

$$L_\pi = Q$$

To end the proof we have to show estimate (19). Let $f \in S$ and $\|f\|_{L^\infty} < 1$. Then using Th. 3 we see that $h = 1 - \sqrt{1 - |f|^2} \in S$. Therefore, using lemma we have :

$$\pi_0(f)^* \pi_0(f) = \pi_0(|f|^2) = \pi_0(2h - h^2) =$$

$$= 2\pi_0(h) - \pi_0(h)^2 = I - (I - \pi_0(h))^2$$

Since h is real, $\pi_0(h)$ is self-adjoint (cf Lemma) and $(I - \pi_0(h))^2$ is positive. Therefore $\pi_0(f)^*\pi_0(f) \leq I$ and $\|\pi_0(f)\| \leq 1$. The estimate (19) is proved.

Q.E.D.

Proof of the Lemma :

Let

$$S_0 \overset{df}{=} \{ f \in S : \text{The support of } f \text{ is compact} \}$$

$$L^1_0(L^2(G)) \overset{df}{=} \{ \rho \in L^1(L^2(G)) : f_\rho \in S_0 \}$$

For example if χ_K denotes the multiplication by the characteristic function of a compact subset $K \subset G$ then for any $\rho \in L^1(L^2(G))$ the function $f_{\chi_K \rho \chi_K} \in S_0$ and $\chi_K \rho \chi_K \in L^1_0(L^2(G))$. This example shows that $L^1_0(L^2(G))$ is dense in $L^1(L^2(G))$ and S_0 is dense in S (in the sense of $\|\cdot\|_S$-norm).

Let $b \in S_0$ and $\rho \in L^1_0(L^2(G))$. Then the map

$$G \ni x \longrightarrow b \cdot f_{\rho L_x^{-1}} \in S$$

is continuous and vanishes outside a compact set. Therefore we can compute the integral over G with respect to the Haar measure dx. We claim that this integral is proportional to b :

$$\int_G b \cdot f_{\rho L_x^{-1}}\, dx = \left(\int_G f_\rho(x^{-1})\, dx \right) \cdot b$$

Indeed for any $y \in G$ we have $f_{\rho L_x^{-1}}(y) = Tr(L_y \rho L_x^{-1}) = Tr(L_{x^{-1}y}\rho) = f_\rho(x^{-1}y)$ and the above formula follows immediately from the left invariance of the Haar measure.

Applying π_0 to the both sides we get

$$\int_G \pi_0(b)\pi_0(f_{\rho L_x^{-1}})\, dx = \left(\int_G f_\rho(x^{-1})\, dx \right) \pi_0(b)$$

where the integral is taken in the norm operator topology. According to the argumentation following eq.(20) the representation π_0 is non-degenerate. Therefore

$$\int_G \pi_0(f_{\rho L_x^{-1}})\, dx = \left(\int_G f_\rho(x^{-1})\, dx \right) \cdot I$$

where now the integral is taken in the weak operator topology.

Now we replace ρ by $\rho * \rho_1$, where $\rho, \rho_1 \in L^1_0(L^2(G))$. Using (12) and Th. 1 one can easily check that $(\rho * \rho_1) L_x^{-1} = \rho L_x^{-1} * \rho_1 L_x^{-1}$. Therefore

$$\int_G \pi_0(f_{\rho L_x^{-1}}) \pi_0(f_{\rho_1 L_x^{-1}})\, dx = \left(\int_G f_\rho(x^{-1}) f_{\rho_1}(x^{-1})\, dx \right) \cdot I$$

This equation holds for any $\rho_1 \in L^1_0(L^2(G))$. Remembering that $\pi_0(f_{\rho_1 L_x^{-1}}) = (id \otimes Tr)((I \otimes \rho_1 L_x^{-1})Q)$ and that $f_{\rho_1}(x^{-1}) = Tr(L_x^{-1}\rho_1)$ we get

$$\int_G \left(\pi_0(f_{\rho L_x^{-1}}) \otimes L_x^{-1}\right) dx \cdot Q =$$

$$= I \otimes \int_G f_\rho(x^{-1}) L_x^{-1}\, dx$$

At this moment the assumption that Q is unitary enters in the most essential way ; We rewrite this equation in the following form :

$$\int_G \left(\pi_0(f_{\rho L_x^{-1}}) \otimes L_x^{-1}\right) dx =$$

(*)

$$= \left(I \otimes \int_G f_\rho(x^{-1}) L_x^{-1}\, dx \right) Q^* .$$

Let J denotes the complex conjugation in $L^2(G)$: $(J\varphi)(x) = \overline{\varphi(x)}$. Then J is an antiunitary involution commuting with L_x. Moreover we have

$$\text{(21)} \qquad f_\rho(x^{-1}) = Tr(L_x^{-1}\rho) = \overline{Tr(L_x \rho^*)} =$$

$$= Tr\, J L_x \rho^* J = Tr\, L_x J\rho^* J = f_{J\rho^* J}(x).$$

$$\overline{f_\rho(x)} = \overline{Tr(L_x \rho)} = Tr\, J L_x \rho J = Tr\, L_x J \rho J = f_{J\rho J}(x)$$

The last equation shows that S is invariant under the complex conjugation.

Now let $\rho \in L^1_0(L^2(G))$, $\rho_1 \in L^1(L^2(G))$ and $y \in G$. Then using (21) we have

$$Tr\left(\rho_1 \int_G f_\rho(x^{-1}) L_x^{-1}\, dx \cdot L_y\right) = \int_G f_\rho(x^{-1}) f_{\rho_1}(x^{-1}y)\, dx =$$

$$= \int_G f_\rho(x^{-1}y^{-1}) f_{\rho_1}(x^{-1})\, dx = \int_G f_{J\rho^* J}(yx) f_{\rho_1}(x^{-1})\, dx =$$

$$= Tr \int_G L_y L_x J\rho^* J\, f_{\rho_1}(x^{-1})\, dx =$$

$$= Tr\left(\int_G L_x J \rho^* J\, f_{\rho_1}(x^{-1})\, dx \cdot L_y \right).$$

Remembering that Q and therefore Q^* belong to $B(H) \otimes \mathcal{L}$ we get

$$(id \otimes Tr)\left(\left(I \otimes \rho_1 \int_G f_\rho(x^{-1}) L_x^{-1}\, dx\right) Q^*\right) =$$

$$= (id \otimes Tr)\left(\left(I \otimes \int_G L_x J\rho^* J\, f_{\rho_1}(x^{-1})\, dx\right) Q^*\right)$$

On the other hand multiplying both sides of (*) by $(I \otimes \rho_1)$ and computing $(id \otimes Tr)$ we get

$$\int_G \pi_0(f_{\rho L_x^{-1}})\, f_{\rho_1}(x^{-1})\, dx =$$

$$= (id \otimes Tr)\left(\left(I \otimes \rho_1 \int_G f_\rho(x^{-1}) L_x^{-1}\, dx\right) Q^*\right)$$

Comparing the last two equations and remembering that f_{ρ_1} runs over a dense subset of $C_\infty(G)$ we get :

$$\pi_0(f_{\rho L_x^{-1}}) = (id \otimes Tr)\left((I \otimes L_x J\rho^* J)\, Q^*\right)$$

for almost all $x \in G$. Since both sides depend continuously on

x the equality holds for all $x \in G$. In particular for $x = e$ we get

$$\pi_0(f_\rho) = (id \otimes Tr)((I \otimes J\rho^* J)\,Q^*)$$

Now we have

$$\pi_0(f_\rho)^* = (id \otimes Tr)((I \otimes J\rho J)\,Q) =$$

$$= \pi_0(f_{J\rho J}) = \pi_0(\bar{f}_\rho)$$

Q.E.D.

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