Gravity and the Shape of Turbulence

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Related works by S. Minwalla et.al.

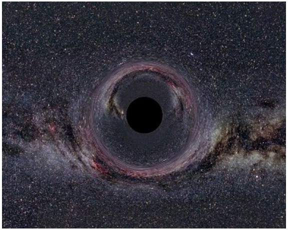
Outline

Turbulence

- The dynamics of fluids is a long standing challenge that remained as an unsolved problem for centuries.
- Understanding its main features, chaos and turbulence, is likely to provide an understanding of the principles and non-linear dynamics of a large class of systems far from equilibrium.

Black Hole Dynamics

- We consider a conceptually new viewpoint to study these features using the gravitational field variables.
- Since the gravitational field is characterized by a curved geometry, the gravity variables provide a geometrical framework for studying the dynamics of fluids: A *geometrization* of turbulence.



The Navier-Stokes Equations

- The fundamental formulation of the nonlinear dynamics of fluids is given by the incompressible Navier-Stokes (NS) equations

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i P + \nu \partial_{jj} v_i + f_i \tag{1}$$

- $v_i(x,t), i = 1, ..., d$, $(d \geq 2)$ obeying $\partial_i v_i = 0$ is the velocity vector field, $P(x,t)$ is the fluid pressure divided by the density, ν is the (kinematic) viscosity and $f_i(x,t)$ are the components of an externally applied force.
- The equations can be studied mathematically in any space dimensionality d, with two and three space dimensions having an experimental realization.

Reynolds Number and Turbulence

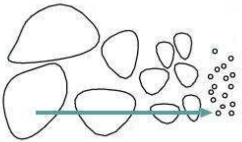
- An important dimensionless parameter is the Reynolds number
$$\mathcal{R}_e = \frac{LV}{\nu} \tag{2}$$
where L and V are, respectively, a characteristic scale and velocity of the flow.
- Experimental and numerical analysis data show that for $\mathcal{R}_e \ll 1$, the flows are regular (Laminar). For a Reynolds number in the range between 1 and 100 the flow exhibits a complicated (chaotic) structure, while for $\mathcal{R}_e \gg 100$ the flow is highly irregular (turbulent) with a complex spatio-temporal pattern formed by the turbulent velocity field.

Turbulence in Nature

- Most flows in nature are turbulent. This is simple to see by noting that the viscosity of water is $\nu \simeq 10^{-6} \frac{m^2}{sec}$, while that of air is $\nu \simeq 1.5 \times 10^{-5} \frac{m^2}{sec}$. Thus, a medium size river has a Reynolds number $\mathcal{R}_e \sim 10^7$.

The Statistical Approach

- Although the NS equation is deterministic (without a random force), it is useful to obtain a statistical description of the turbulent flows.
- In the energy cascade picture introduced by Kolmogorov in 1941, energy is transmitted to the fluid by large eddies at a scale L, that transmit the energy to smaller scales by breaking to smaller eddies due to instability, until the viscous scale l is reached, where the energy dissipates due to friction.



Anomalous Scaling

- There is experimental and numerical evidence that in the range of distance scales $l \ll r \ll L$, called the inertial range, the flows exhibit a universal behavior, e.g. the space-averaged equal-time correlators of velocity differences in the inertial range are characterized by critical exponents.
- For instance, the longitudinal n-point functions scale as ($\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$)

$$S_n(r) \equiv \langle \left((\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{r}}{r} \right)^n \rangle \sim r^{\xi_n} \tag{3}$$

  The 1941 exact scaling result of Kolmogorov $\xi_3 = 1$, agrees well with the experimental data, while the other exponents are measurable real numbers.

Anomalous Exponents from Experiment

- A major open problem is to calculate these exponents. While in two space dimensions the anomalous exponents ξ_n seem to follow the Kolmogorov linear scaling and are given by rational numbers, this is not the case in three space dimensions. Examples in three space dimensions taken from a wind tunnel data $\xi_2 = 0.7, \xi_4 = 1.28$ differ from the Kolmogorov linear scaling $\xi_2 = \frac{2}{3}, \xi_4 = \frac{4}{3}$.

The Third Millennium Problem

- The NS and Euler equations are nonlinear partial differential evolution equations. A major open problem posed by these equations is the understanding and control of their solutions.
- Of particular importance is the short distance behavior and existence of singularities in the solutions, i.e. starting from smooth initial data, with a bounded energy condition and a smooth external force, can the solutions develop a finite-time singularity.
- This is known not to be the case in two space dimensions, but is not known in three space dimensions ("The Third Millennium Problem" announced by the Clay Mathematics Institute).

Finite Time Singularities

- Numerical computations appear to exhibit blowup for solutions of the Euler equations, but the extreme numerical instability of the equations makes it very hard to draw reliable conclusions.
- From a physical viewpoint, such a finite-time blowup would mean a breakdown of the large distance effective theory that is supposed to describe fluid dynamics, and a need for an ultraviolet (short distance) information for a complete description of the dynamics.

Knudsen Number

- Hydrodynamics applies under the condition that the correlation length of the fluid l_{cor} is much smaller than the characteristic scale L of variations of the macroscopic fields.
- In order to characterize this, one introduces the dimensionless Knudsen number

$$Kn \equiv l_{cor}/L \tag{4}$$

- Since the only dimensionfull parameter is the characteristic temperature of the fluid T, one has by dimensional analysis,

$$l_{cor} = (\hbar c/k_B T)G(\lambda) \tag{5}$$

where λ denotes all the dimensionless parameters of the CFT.

Stress-Energy Tensor

- The stress-energy tensor of the CFT obeys

$$\partial_\nu T^{\mu\nu} = 0, \qquad T^\mu_\mu = 0 \tag{6}$$

- The equations of relativistic hydrodynamics are determined by the constitutive relation expressing $T^{\mu\nu}$ in terms of the temperature $T(x)$ and the four-velocity field $u^\mu(x)$ satisfying $u_\mu u^\mu = -1$.
- The constitutive relation has the form of a series in the small parameter $Kn \ll 1$,

$$T^{\mu\nu}(x) = \sum_{l=0}^{\infty} T_l^{\mu\nu}(x), \;\; T_l^{\mu\nu} \sim (Kn)^l, \tag{7}$$

where $T_l^{\mu\nu}(x)$ is determined by the local values of u^μ and T and their derivatives of a finite order.

Stress-Energy Tensor

- Keeping only the first term in the series gives ideal hydrodynamics and the stress-energy tensor reads

$$T_{\mu\nu} = T^4[\eta_{\mu\nu} + 4u_\mu u_\nu] \tag{8}$$

- The dissipative hydrodynamics is obtained by keeping $l = 1$ term in the series. The stress-energy tensor reads

$$T_{\mu\nu} = T^4\left[\eta_{\mu\nu} + 4u_\mu u_\nu\right] - c\eta\sigma_{\mu\nu} \tag{9}$$

where

$$\sigma_{\mu\nu} = (\partial_\mu u_\nu + \partial_\nu u_\mu + u_\nu u^\rho \partial_\rho u_\mu + u_\mu u^\rho \partial_\rho u_\nu - \frac{2}{3}\partial_\alpha u^\alpha [\eta_{\mu\nu} + u_\mu u_\nu] \tag{10}$$

- The dissipative hydrodynamics of a CFT is determined by only one kinetic coefficient - the shear viscosity η.

Non-Relativistic Limit

- The hydrodynamics of relativistic conformal field theories is intrinsically relativistic as is the microscopic dynamics.
- However, the limit of non-relativistic macroscopic motions of a CFT hydrodynamics leads to the non-relativistic incompressible Euler and Navier-Stokes equations for ideal and dissipative hydrodynamics of the CFT, respectively.
- The non-relativistic slow motions limit: $v \ll c$ where v^i is the three-velocity of the fluid, $u^\mu = (\gamma, \gamma v^i/c)$ and $\gamma = [1 - v^2/c^2]^{-1/2}$.

$$T = T_0 \left[1 + P/c^2 + o(1/c)\right], \qquad \nu \equiv \frac{\hbar c^2 F(\lambda)}{4k_B T_0} \tag{11}$$

- For example, for strongly coupled CFTs described by an AdS gravity dual $F = 1/\pi$.

Gravitational Dual Description

- Relativistic CFT hydrodynamics provides a universal description of the large scale dynamics of the CFT. The AdS/CFT correspondence suggests that the large-time dynamics of gravity provides a dual description of the CFT hydrodynamics.
- The four-dimensional CFT hydrodynamics equations are the same as the equations describing the evolution of large scale perturbations of the five-dimensional black brane.
- Since we can obtain the Euler and NS equations in the non-relativistic limit of CFT hydrodynamics, the AdS/CFT correspondence implies that Euler and NS equations have a dual gravitational description. The dual description is obtained by taking the non-relativistic limit of the geometry dual to the relativistic CFT hydrodynamics.

Gravitational Dual Description

- Consider the five-dimensional Einstein equations with negative cosmological constant

$$R_{mn} + 4g_{mn} = 0, \qquad R = -20 \tag{12}$$

- These equations have a particular "thermal equilibrium" solution - the boosted black brane

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f[br] u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

where

$$f(r) = 1 - \frac{1}{r^4}, \quad P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu} \tag{14}$$

and the constant $T = 1/\pi b$ is the temperature.

Gravitational Dual Description

- One looks for a solution of the Einstein equation by the method of variation of constants using the ansatz

$$g_{mn} = (g_0)_{mn} + \delta g_{mn} \tag{15}$$

$$(g_0)_{mn} dy^m dy^n = -2u_\mu(x^\alpha) dx^\mu dr - r^2 f[b(x^\alpha)r] u_\mu(x^\alpha) u_\nu(x^\alpha) dx^\mu dx^\nu + r^2 P_{\mu\nu}(x^\alpha) dx^\mu dx^\nu \tag{16}$$

- $y = (x^\mu, r)$. As in the Boltzmann equation, the condition of constructibility of the series solution produces equations for $u^\mu(x^\alpha)$ and $T(x^\alpha) = 1/\pi b(x^\alpha)$. The series for g_{mn} is the series in the Knudsen number of the boundary CFT hydrodynamics.

Membrane Dynamics

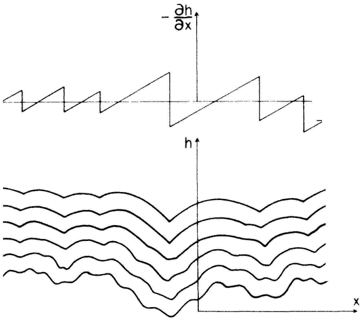
- The way the black brane horizon geometry encodes the boundary fluid dynamics is reminiscent of the *Membrane Paradigm* in classical general relativity, according to which any black hole has a fictitious fluid living on its horizon.
- The real fluid whose dynamics we wish to study is at the boundary of the space. It is natural to ask to what extent it can be identified under the duality map with the membrane paradigm fluid.
- Using the Membrane Paradigm approach as developed by Damour, we analyze the dynamics of a membrane defined by the event horizon of a black brane in asymptotically AdS space-time. We show that it is described by the incompressible NS equations.

Membrane Dynamics

- The analysis that we perform also holds for any non-singular null hypersurface when a large scale hydrodynamic limit exists. Thus, for instance, that the dynamics of the Rindler acceleration horizon is also described by the incompressible NS equations.
- The connection between the horizon hypersurface dynamics and the NS equation is analogous to the connection between the Burgers and the KPZ equations. The Burgers equation provides a simplified model for turbulence, while the KPZ equation describes a local growth of an interface using a height function. The height gradient obeys the Burgers equation.

Membrane Dynamics

- Our result shows that real turbulence may also be seen as resulting from a physically natural surface dynamics.



Definition of Coordinates

- We use the convention $8\pi G = c = \hbar = k_B = 1$.
- Consider a $(d+2)$-dimensional bulk space-time M with coordinates $X^A, A = 0, ..., d+1$ with a Lorentzian metric g_{AB}.
- Let H be a $(d+1)$-dimensional null hypersurface (notion akin to horizon) characterized by the null normal vector $\mathbf{n}$ which components n^A obey

$$\mathbf{n} \cdot \mathbf{n} = g_{AB} n^A n^B = 0 \tag{17}$$

Note this condition implies that for a null hypersurface the normal vector is also a tangent vector.

The Normal to the Horizon

- We define the hypersurface in the bulk space-time by $x^{d+1} \equiv r = const$, and denote the other coordinates as $x^\mu = (t, x^i), i = 1, ..., d$.
- The coordinate t parameterizes a slicing of space-time by spatial hypersurfaces and x^i are coordinates on sections of the horizon with constant t.
- In this coordinate system

$$n^r = 0, \quad n^t = 1, \quad n^i = v^i \tag{18}$$

Time Slices

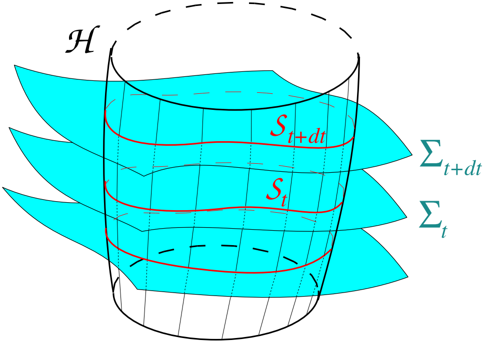


Figure: E. Gourgoulhon, J.L. Jaramillo, Phys.Rept. 423 (2006) 159

Slow Motion Limit

- In the case of black branes in AdS, the event horizon is located in the bulk space-time at $r = \pi T_0$, where T_0 corresponds to the Hawking temperature. The horizon coordinates (t, x^i) can be identified with time and space Eddington-Finkelstein coordinates in the AdS boundary.
- We consider the slow motion limit where v^i is a small perturbation. In order to keep track of the different terms we impose the scaling $\partial_t \sim \varepsilon^2, v^i \sim \partial_i \sim \varepsilon$, where ε is a small parameter corresponding to c^{-1}.

The Induced Metric

- The first fundamental form (induces metric) reads

$$ds_H^2 = h_{ij}(dx^i - v^i dt)(dx^j - v^j dt) \tag{19}$$

where h_{ij} is the metric on sections S_t of the horizon H at constant t.

- For the equilibrium black brane, at zeroth order

$$h_{ij}^{(0)} = (\pi T_0)^2 \delta_{ij} \tag{20}$$

The details of the subleading term, which is of order ε^2, will not be needed for our analysis.

The Extrinsic Curvature

- The second fundamental form of the horizon hypersurface is the extrinsic curvature $K_\mu^{\ \nu}$ defined by

$$\nabla_\mu n^A = K_\mu^{\ \nu} e_\nu^A \tag{21}$$

where we use the horizon basis e_μ^A

- Together, the first and second fundamental forms provide a complete description the embedding of the null hypersurface in the bulk space-time.

Expansion and Shear

- Consider Lie transport of h_{ij} along the null normal vector $\mathbf{n}$, which is given by the Lie derivative $\mathcal{L}_{\mathbf{n}}$

$$\mathcal{L}_{\mathbf{n}} h_{ij} = \frac{1}{2}\partial_t h_{ij} + \frac{1}{2}(\pi T_0)^2 (D_i v_j + D_j v_i) \tag{22}$$

  D_i is the covariant derivative with respect to the metric h_{ij}.
- This can be split into its trace part (the expansion θ) and trace free part (the shear σ_{ij})

$$\begin{aligned} \theta &= \frac{1}{2} h^{ij}\partial_t h_{ij} + D_j v^j && (23) \\ \sigma_{ij} &= \mathcal{L}_{\mathbf{n}} h_{ij} - \theta h_{ij}/d && (24) \end{aligned}$$

Expansion and Shear

- For the black brane we get to leading order the $O(\varepsilon^2)$ expressions

$$\theta = \partial_i v_i \quad (25)$$

$$\sigma_{ij} = \frac{1}{2}(\pi T_0)^2(\partial_i v_j + \partial_j v_i - 2\partial_k v_k \delta_{ij}/d) \quad (26)$$

The Pressure

- Using $\mathbf{n}$ and e_i^A as a tangent basis, the components of the horizon extrinsic curvature are

$$K_n^n = \kappa(x) \tag{27}$$
$$K_i^n = \Omega_i \tag{28}$$
$$K_j^i = \sigma_j^i + \theta\delta_j^i/d \tag{29}$$

- $\kappa(x)$ is the surface gravity defined by $n^B\nabla_B n^A = \kappa(x)n^A$. It can be parameterized as

$$\kappa(x) = 2\pi T_0(1 + P(x) - v^2/2) \tag{30}$$

where $P(x)$ and v^2 scale as ε^2. We will identify $P(x)$ as the fluid pressure.

The Momentum

- Ω_i is defined by

$$\Omega_i = m^A \nabla_i n_A \tag{31}$$

where $m^A n_A = 1$. For the black brane at leading order we have

$$\Omega_i = 2\pi T_0 v_i \tag{32}$$

The Focusing Equation

- The dynamics of the horizon geometry perturbations are governed by the Einstein equations. Consider first the contraction of the Einstein equations with $n^A n^B$. The black brane is a solution to the Einstein equations with negative cosmological constant

$$R_{AB} + (d+1) g_{AB} = T^{matt}_{AB} \tag{33}$$

- There is no contribution from the cosmological constant term proportional to the metric due to (17), and we get the focusing equation

$$-n^A \nabla_A \theta + \kappa(x)\theta - \theta^2/d - \sigma_{AB}\sigma^{AB} - T^{matt}_{AB} n^A n^B = 0 \tag{34}$$

The Incompressibility Condition

- Consider first the case $T_{AB}^{matt} = 0$. Plugging our previous results for the expansion, shear, and surface gravity into (34), we find at leading order the incompressibility condition

$$\partial_i v_i = 0 \tag{35}$$

The Navier-Stokes Equations

- Consider next the contraction of the Einstein equations with $n^A e_i^B$. Again the cosmological constant term does not contribute because by construction e_i^B and n^A are orthogonal. From the remaining terms one finds

$$\mathcal{L}_{\mathbf{n}}\Omega_i = -\partial_i \kappa(x) + D_j \sigma_i^j - \frac{1}{d}\partial_i \theta - n^A e_i^B T_{AB}^{matt} \qquad (36)$$

where

$$\mathcal{L}_{\mathbf{n}}\Omega_i = (\partial_t + \theta)\Omega_i + v^j D_j \Omega_i + \Omega_j D_i v^j \qquad (37)$$

- When $T_{AB}^{matt} = 0$ we find that the leading order terms are at order ε^3 and give the NS equation (1) without a force term and with a kinematic viscosity $\nu = (4\pi T_0)^{-1}$.

Forcing

- If there is a non-zero matter stress tensor it acts as a forcing term in the NS equations, the force being

$$f_i = T_{AB}^{matt} n^B e_i^A \tag{38}$$

- For instance, adding a dilaton ϕ to gravity, results in

$$f_i = \nabla_A \phi \nabla_B \phi n^B e_i^A / 2 \tag{39}$$

Energy Balance Equation

- In the following we will consider the NS equations without a forcing term. From the NS equations one can derive the energy balance equation

$$\int \frac{1}{2}\partial_t v^2 d^d x = -\int \nu \partial_i v_j \partial^i v^j d^d x \tag{40}$$

that relates the rate of change of the fluid energy to minus the energy dissipation per unit time due to fluid friction. To interpret this equation in terms of the horizon geometry, consider the focusing equation (34) expanded to order ε^4 with $T^{matt}_{AB} = 0$.

Area Theorem

- The expansion of the horizon is defined as the fractional rate of change in the cross-sectional area along the horizon generators

$$\theta = \mathcal{L}_{\mathbf{n}} \ln \sqrt{h} \ , \tag{41}$$

where $\mathcal{L}_{\mathbf{n}}$ is the Lie derivative and h is the determinant of h_{ij}.

- Integrating the focusing equation over a horizon cross-section one has

$$\partial_t A = \nu (\pi T_0)^d \int \partial_i v_j \partial^i v^j \ d^d x \ , \tag{42}$$

where A is the total horizon area.

Area Theorem

- Imposing the energy balance law (40), we find that

$$\partial_t \left(A/A_0\right) = -\int \partial_t v^2/2 \, d^d x \qquad (43)$$

where A_0 is the zeroth order area density $(\pi T_0)^d$. Thus, as the kinetic energy of the fluid on the boundary decreases in time due to viscous dissipation, the horizon area grows. This is consistent with the classical area increase theorem of General Relativity.

Comments

- The derivation of the NS equations required knowledge of the horizon embedding and employed a local analysis near this horizon. There was no need to know the asymptotic structure of the full bulk space-time.
- The results will apply to a general non-singular null hypersurface, as long as there is a separation between the characteristic scale L of the macroscopic perturbations and some intrinsic microscopic scale.
- The non-singularity requirement was used when contracting the Einstein equations in order to obtain the membrane equations.

Knudsen Number

- Consider for example black holes in asymptotically flat spaces with horizons of a spherical topology.
- By dimensional analysis the correlation length of a fluid will scale as $l_c \sim T_0^{-1}$, where T_0 is the Hawking temperature.
- In the asymptotically flat cases $T_0^{-1} \sim r_0$, where r_0 the horizon radius. Since the horizon is now also compact, the L can be no greater than $\sim r_0$. The dimensionless Knudsen number $Kn \equiv l_c/L$ is of order unity, implying that the derivative expansion used above is not valid and that hydrodynamics is not the appropriate effective description.

Rindler Horizon

- Rindler space is associated with accelerated observers in $d+2$ Minkowski spacetime

$$ds^2 = -\kappa^2\xi^2 d\tau^2 + d\xi^2 + \sum_{i=1}^{d} dx^i dx_i \tag{44}$$

where κ is a constant.

- To the uniformly accelerated observer with worldline $\xi = const.$, the surface $\xi = 0$ is a causal horizon, with intrinsic metric $h_{ij} = \delta_{ij}$, that prevents him from an access to the entire spacetime.

Rindler Horizon

- The constant κ can be identified with a temperature. Unruh showed that accelerated observers feel the quantum vacuum to be a thermal state at temperature $T = a/2\pi$, where a the observer's proper acceleration.
- This local temperature can be expressed as $T = \kappa/2\pi\chi$, where $\chi = \sqrt{-g_{\tau\tau}} = \kappa\xi$ is the redshift factor.
- We define $\kappa = 2\pi T_0$ as the location independent temperature of the system.

Rindler Horizon

- To perturb this horizon we allow, a slowly varying ($L \gg T_0^{-1}$) fluid velocity $n^i = v^i$ of $\sim \varepsilon$ and parameterize $\kappa(x)$ as in (30).
- Applying the membrane analysis to the Rindler horizon also shows that its dynamics is determined by the incompressible NS equations with kinematic viscosity $(4\pi T_0)^{-1}$.
- In this case though the result cannot be understood as mirroring the hydrodynamics of a field theory fluid living on an asymptotic boundary of spacetime.

Implications

- Exact solutions of the NS equations such as vortices and others are mapped immediately into geometrical horizons.
- Finite time singularities, where fluid velocity gradients diverge, are mapped into the naked curvature singularities in the gravity description. Thus, the extremely important problem of singularities in the NS equations in three space dimensions appears linked to a cosmic censorship principle in gravity.

Implications

- space-averaged equal-time correlators of velocity differences $S_n(r)$ in the inertial range are characterized by critical exponents. In the geometrical picture, $S_n(r)$ correspond to the space-averaged equal-time correlators of differences of normals to the horizon.
- Multi-fractality of the horizon surface may give for the first time a consistent dynamical basis for the multifractal model of turbulence, which expresses the anomalous exponents of turbulence in terms of the spectrum of fractal dimensions.