The Thursday Colloquium
"The Algebra & Geometry of Modern Physics"

Lecture Notes

ASYMPTOTIC BEHAVIOR OF β ENSEMBLES

KAROL K. KOZŁOWSKI

(CNRS, INSTITUT DE MATHÉMATIQUES DE BOURGOGNE,
UNIVERSITÉ DE BOURGOGNE)

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K. K. Kozlowski[1].

Abstract

In these lecture notes we present large-deviation based techniques that allow one to prove the topological expansions in β ensembles

1 A not so short introduction

1.1 Integrals

Integral representations play an important role in physics and mathematics. On the very fundamental level they can be seen as efficient tools allowing one to construct explicit solutions to numerous problems be it differential or finite difference equiations, enumeration and other combinatorial issues, or compact resummations of sums, so as to name a few. For instance, the hypergeometric functions, which solve the differential equation

$$z(1-z)\cdot\frac{\mathrm{d}^2u}{\mathrm{d}^2z} \;+\; (c-(a+b+1)z)\cdot\frac{\mathrm{d}u}{\mathrm{d}z} - ab\cdot u \;=0 \tag{1.1}$$

have been shown to admit various types of one-fold integral representations such as the Gauss one

$$u(z) \;=\; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\cdot\mathrm{d}t \tag{1.2}$$

which is valid for $\Re(c)>\Re(b)>0$ and $|\arg(1-z)|<\pi$ or the Mellin-Barnes one

$$u(z) \;=\; \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\int\limits_{i\mathbb{R}} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s\cdot\frac{\mathrm{d}s}{2i\pi} \tag{1.3}$$

which is valid for $|\arg(-z)|<\pi$, $a,b\notin-\mathbb{N}$ with a path of integration that separates $s\in\mathbb{N}$ from $\{-a-\mathbb{N}\}\cup\{-b-\mathbb{N}\}$.

In itself, taking the example of the hypergeometric function, an integral representation, apart from providing a closed well-defined representation cannot be considered as the final answer. Without tools for analysing it,

[1]Université de Bourgogne, Institut de Mathématiques de Bourgogne, UMR 5584 du CNRS, France, karol.kozlowski@u-bourgogne.fr

ie extracting all the desired information on the object being represented, they would be merely another hardly useful formal object. Still, the development of complex analysis gave birth to the saddle-point method and to techniques of analysis based on contour deformations. It is these techniques that turned one-dimensional integral representations into extremely powerful tools. In the case of the hypergeometric function, they allow one for an easy access to

- determining the regions of analyticity in its parameters a, b, c and in its principal variable z;
- determining the small z expansions;
- determining the value of the hypergeometric function at special points (*eg* for $z = 1$ (1.2) simply reduces to the Euler integral, whereas computing the explicit value of the associated series demands a much tougher analysis);
- extracting asymptotic behaviours in $z \to \infty$ or in the auxiliary parameters a, b, c.

It is to be expected that when a problem becomes too complex, then obtaining some closed expression for its solution might demand to recourse to higher dimensional integrals. Although, in principle, this might seem fine and acceptable, this is not such an ideal situation in that on the one hand the structure of domains of integration may be extremely complicated as soon as one moves into higher dimensions then 1, and, on the other hand, there is no *per se* method of steepest descent for integrals over many variables. True, in some cases, one can save the day by applying, repeatedly, the one dimensional steepest descent method.

This is however, by far, not a generic case. A typical example where such a procedure would fail corresponds to a multiple integral in which the number of integration N is the large parameter. These integrals, and especially their large-N behaviour, play important roles in physics. They can be thought of as baby models for finite lattice approximations of path integrals. Furthermore, they arise naturally in the study of models of classical statistical physics and especially in the calculation of their partition functions. They are also intimately related with matrix models, the latter having a large domain of applications: counting of various types of graphs that can be drawn on a Riemann Surface, analysis of statistics of noise in signal processing, 0-dimensional quantum field theories, statistics of eigenvalues of heavy nuclei... so as to name a few.

These integrals step so much out of the "well-understood" scheme for "classical" single or many-fold integrals that, as I shall argue further in these notes, they deserve to be call "semi-classical" integrals. Techniques for their analysis are still not fully developed and solely certain specific cases or families could have been treated so far. Nonetheless, these examples already led to the development of a new kind of mathematics. In particular, there has been observed to exist an intimate connection between these and sequences of probability measures on certain Polish spaces. The study of such sequences through so-called large deviation principles brought, on the one hand, a certain impetus to the theory of probabilities and on the other numerous deep results on the objects being represented by these multiple integrals.

The purpose of these lecture notes is to discuss particular examples of theses "semi-classical" integrals and introduce techniques that allow one to extract informations on their large N behaviour, with N being the number of integrals.

We shall begin by introducing a few examples of such integrals and outlining the type of questions and problems one would like to resolve in such a context.

1.2 Classical statistical mechanics

Consider a system of N classical particles on the line in an external confining potential $N\beta^{-1}V(\lambda)$ interacting through a two-body interaction $W(\lambda, \mu)$. Then, the "spacial" part of the model's partition function at temperature

$T = \beta^{-1}$ takes the form

$$Z_N[V, W] \; = \; \int_{\mathbb{R}^N} \prod_{a,b=1}^{N} \mathrm{e}^{-\frac{\beta}{2} W(\lambda_a, \lambda_b)} \prod_{a=1}^{N} \mathrm{e}^{-N V(\lambda_a)} \cdot \mathrm{d}^N \lambda \; . \tag{1.4}$$

The potential V is supposed to be confining meaning that

$$V(\lambda) \underset{\lambda \to \infty}{\longrightarrow} +\infty \; , \tag{1.5}$$

this sufficiently fast so as to ensure the convergence of the integral (1.4). The two-body interaction may or may not present singularities. However, typically for realistic models they present a divergence on the diagonal, *ie* $W(\lambda, \mu) \to +\infty$ when $\lambda \to \mu$. The latter merely translate a sort of impenetrability condition between the various particles. As follows from numerous considerations of statistical mechanics, the partition function -or slight modifications thereof- allow one to access to many observables associated with the system under investigation. In fact, from the perspective of studying observables of the model, it is convenient to introduce a generating function of observables

$$\mathcal{G}_N[h] \; = \; \int_{\mathbb{R}^N} \prod_{a=1}^{N} \mathrm{e}^{-h(\lambda_a)} \cdot \prod_{a,b=1}^{N} \mathrm{e}^{-\frac{\beta}{2} W(\lambda_a, \lambda_b)} \prod_{a=1}^{N} \mathrm{e}^{-N V(\lambda_a)} \cdot \frac{\mathrm{d}^N \lambda}{Z_N[V, W]} \; . \tag{1.6}$$

For instance, the average position $\langle x \rangle_N$ of a particle is obtained through

$$\langle x \rangle_N \; = \; \frac{1}{N} \frac{\partial}{\partial \alpha} \mathcal{G}_N[\alpha h]_{|\alpha=0} \qquad \text{with} \qquad \mathrm{h}(\lambda) = \lambda \; . \tag{1.7}$$

Clearly, for fixed N the multiple integral representation for $Z_N[W, V]$, without even mentioning more involved objects such as $\mathcal{G}_N[h]$, can only be considered as a formal object. Indeed, unless the external potential V and the two-body interaction W are chosen both to take an utterly specific form, the integral cannot be computed in a closed form. However, from the perspective of statistical mechanics, one is usually interested in the behaviour of these quantities in the case of a large number N of interacting particles. In this respect, one can address the following questions

- What is the large-N behaviour of Z_N? In particular, does it admit a large-N asymptotic expansion

$$\ln Z_N \; = \; -N^2 \cdot F_2[V, W] \; + \; N \cdot F_1[V, W] \; + \; \cdots \tag{1.8}$$

- Once can think of

$$\mathrm{d}P_N^{W;V}(\boldsymbol{\lambda}_N) \; = \; \prod_{a,b=1}^{N} \mathrm{e}^{-\frac{\beta}{2} W(\lambda_a, \lambda_b)} \prod_{a=1}^{N} \mathrm{e}^{-N V(\lambda_a)} \cdot \frac{\mathrm{d}^N \lambda}{Z_N[V, W]} \tag{1.9}$$

  as a probability measure on the configuration space. Then $X_N(\boldsymbol{\lambda}_N) \; = \; \sum_{a=1}^{N} h(\lambda_a)$ is a sum of random variables. Does it converge to some random variable? What is a typicall distribution in this case.

- A somehow related (but stronger) questions relates to the existence of the large-N limit directly for the generating function $\mathcal{G}_N[h]$?

In these notes we are going to introduce an analogue of the saddle-point technique that allow one to treat two cases of interest

- W is a bounded $\mathscr{C}^2(\mathbb{R}^2)$ function ;
- $W(\lambda, \mu) = -2 \ln |\lambda - \mu|$.

The first case will be rather easy whereas the second will emphasize the difficulty of taking into account "singular" interactions which so-often appear in physics. The second case corresponds to log-gases and is referred to as β-ensembles. In fact, this class of integrals at $\beta = 1/2, 1$ and 2 is intimately related with various classical ensembles of random matrices.

1.3 The classical random matrix ensembles

Random matrices have been first introduced by Wishart in the late 1920's (1928) as a tool for studying statistics of noise in the measurement of samples. Then, in 1958 Wigner proposed to use certain ensembles of random matrices so as to model excitation spectra for heavy nuclei. In a nutshell, his ideas were the following. The very details of the interactions in a heavy atomic nucleus are hardly accessible. Due to the large number of interacting particles and the possible change of the precise and explicit form of the interactions due to fine tuning in the system, one may, in fact, treat the model's Hamiltonian as a random variable solely satisfying to overall explicit symmetries of the model under investigation. Furthermore, the nature of the spectrum, *ie* typical statistical features should not be altered whether one considers some random operator or a sufficiently large random matrix. Considerations about the invariance of the system under time reversal gave rise to three "classical" ensembles of random matrices:

- the unitary ensemble $\mathcal{E}_N$ consisting of $N \times N$ hermitian matrices $M = M^{\dagger}$ with a probability distribution that is invariant under unitary transformations $M \mapsto UMU^{\dagger}$, $U^{\dagger}U = I_N$;
- the orthogonal ensemble $\mathcal{S}_N$ consisting of $N \times N$ symmetric matrices $M = M^t$ with a probability distribution that is invariant under orthogonal transformations $M \mapsto OMO^t$, $O^tO = I_N$;
- the symplectic ensemble $\mathcal{J}_N$ consisting of $2N \times 2N$ hermitian self-dual matrices $M = M^{\dagger} = J_N M^t J_N^t$ with

$$J_N = \operatorname{diag}\big(\sigma, \dots, \sigma\big) \quad , \qquad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.10}$$

  with a probability distribution that is invariant under unitary-symplectic conjugation transformations $M \mapsto UMU^{\dagger}$, (where $UU^{\dagger} = I_N$, $UJ_NU^t = J_N$);

The fact that one imposes the probability distribution to be invariant under specific conjugation is a mere restatement of the fact that a hermitian matrix M and $U^{\dagger}MU$ will lead to an exactly identical description of the quantum system. Hence, all equivalent realisation ought to be treated on the same ground and thus be associated with an equal probability. The reasoning, in the case of orthogonal and symplectic ensembles, is much similar. The difference on the class of transformations leaving the probability measure invariant solely arises from the requirement that the type of conjugate transformations ought to respect the overall symmetry of the class of Hamiltonians (which is imposed by additional physical symmetries enjoyed on the system such a time invariance of etc...).

For all of these three ensemble, the probability distribution function takes the form

$$\mathrm{d}\mathbb{P}_N(M) = \frac{1}{\mathcal{N}_N} \mathrm{e}^{-\mathrm{tr}[Q(M)]} \cdot \mathrm{d}M \qquad \text{with} \qquad \mathcal{N}_N = \int_{\mathcal{E}} \mathrm{e}^{-\mathrm{tr}[Q(M)]} \cdot \mathrm{d}M \ , \tag{1.11}$$

in which dM is the Lebesgue measure on the algebraically independent entries. Furthermore, Q is some polynomial of even degree -so as to ensure the convergence of the integral-, hence making tr$[Q(M)]$ well defined. In fact, one could consider much more general confining potentials Q subject to the sole condition of growing sufficiently fast at infinity so as to ensure the convergence of the integral. In such a more general case, the quantity tr$[Q(M)]$ ought to be understood in the sense of matrix functional calculus.

Note that unless $\deg(Q) = 2$ (Gaussian distribution), the algebraically independent entries are *not* independent random variables; they are correlated.

In the following, we are going to provide a thorough analysis of the orthogonal ensemble. In particular, we are going to establish its connection with the log-gas at temperature $\beta = 1/2$. The analysis that we shall present can be repeated for the ensembles $\mathcal{E}_N$ and $\mathcal{J}_N$ as well but bears some additional technical complications that we shall not discuss in the present notes.

1.3.1 A case study : the orthogonal ensemble

An $N \times N$ symmetrix matrix M depends on $N(N+1)/2$ free parameters:

- the N diagonal entries M_{kk}, $k = 1, \dots, N$;
- the $N(N-1)/2$ upper-off diagonal entries.

Hence, in this case, the probability distribution takes the form

$$\mathrm{d}\mathbb{P}_N(M) \;=\; \frac{1}{\mathcal{N}_N} \mathrm{e}^{-\mathrm{tr}[Q(M)]} \cdot \prod_{a=1}^{N} \mathrm{d}M_{aa} \cdot \prod_{a<b}^{N} \mathrm{d}M_{ab} \;. \tag{1.12}$$

The purpose of this section will be to establish the

Theorem 1.1 *Let $f \in L^1(\mathcal{S}_N, \mathrm{d}\mathbb{P}_N)$ be orthogonal invariant $f(OMO^t) = f(M)$. Then, f is a symmetric function F of the eigenvalues $\lambda_1, \dots, \lambda_N$ of $M \in \mathcal{S}_N$: $f(M) = F(\lambda_1, \dots, \lambda_N)$ and the ensemble average reduces to an integration over the eigenvalues*

$$\int_{\mathcal{S}_N} f(M) \cdot \mathrm{d}\mathbb{P}_N(M) \;=\; \frac{1}{\mathcal{Z}_N^{(1/2)}[Q]} \int_{\mathbb{R}^N} F(\lambda_1, \dots, \lambda_N) \cdot \prod_{a<b}^{N} |\lambda_a - \lambda_b| \cdot \prod_{a=1}^{N} \mathrm{e}^{-Q(\lambda_a)} \;, \tag{1.13}$$

in which

$$\mathcal{Z}_N^{(1/2)}[Q] \;=\; \int_{\mathbb{R}^N} \prod_{a<b}^{N} |\lambda_a - \lambda_b| \cdot \prod_{a=1}^{N} \mathrm{e}^{-Q(\lambda_a)} \cdot \mathrm{d}^N \lambda \tag{1.14}$$

At this point, one can already thing of several directions to investigate.

- The most natural being: what are the typical freatures of the distribution of the eigenvalues of $M \in \mathcal{S}_N$? A good insight on this issue can be obtained by investigation the large-N limit of the density of probability of finding an eigenvalue at λ_1:

$$p_{1;N}^{(1/2)}(\lambda_1) \;=\; \int_{\mathbb{R}^{N-1}} \prod_{a<b}^{N} |\lambda_a - \lambda_b| \cdot \prod_{a=1}^{N} \mathrm{e}^{-Q(\lambda_a)} \cdot \prod_{a=2}^{N} \mathrm{d}\lambda_a \;. \tag{1.15}$$

  Does $p_{1;N}^{(1/2)}(\lambda)$ has a good $N \to +\infty$ limit? In what sense such a limit exists? Is it possible to obtain an explicit control on the speed of convergence towards this limit? An explicit expression for the corrections?

- Do the eigenvalues have some "average" position $\gamma_1 < \cdots < \gamma_N$, assuming that they are ordered increasingly $\lambda_1 < \ldots < \lambda_N$? Are the fluctuations around these average positions strong?
- Are the eigenvalues correlated in the large-N limit? A good insight on this question can be given by studying the large-N behavior of

$$p_{2;N}^{(1/2)}(\lambda_1, \lambda_2) \; = \; \int\limits_{\mathbb{R}^{N-2}} \prod_{a<b}^{N} |\lambda_a - \lambda_b| \cdot \prod_{a=1}^{N} \mathrm{e}^{-Q(\lambda_a)} \cdot \prod_{a=3}^{N} \mathrm{d}\lambda_a \; . \tag{1.16}$$

- Provided a good scaling is chosen, are large gaps in the spectrum possible? What is their probability?

We shall start by checking that, indeed, the probability measure $\mathrm{d}\mathbb{P}_N$ is invariant under orthogonal transformations.

Lemma 1.1 *The probability measure* (1.12) *defined on the space of symmetric matrices is invariant under orthogonal transformations* $M \mapsto OMO^t$.

Proof —

It is clear that $\mathrm{tr}[Q(M)]$ is invariant in respect to the orthognal conjugations. Hence, it remains solely to check the invariance of the measure.

For a symmetric matrix, let

$$\boldsymbol{M} \; = \; \Big(M_{11}, \ldots, M_{NN}, M_{12}, \ldots, M_{N-1N} \Big) \tag{1.17}$$

denote its $N(N+1)/2$ vector column representation. It is clear that the transformation $\mathcal{T} : M \mapsto O^t M O$ defines a linear operator $\boldsymbol{T}$ on $\mathbb{R}^{\frac{N(N+1)}{2}}$ $\boldsymbol{M} \mapsto T\boldsymbol{M}$. Thus, the Jacobian of the transformation $M \mapsto O^t M O$ is given by $\det[T]$. Now, one has that

$$\mathrm{tr}[M^2] \; = \; \mathrm{tr}[(\mathcal{T} \cdot M)^2] \; = \; \sum_{a=1}^{N} M_{aa}^2 \; + \; 2 \sum_{j<k}^{N} M_{jk}^2 \; = \; \sum_{a=1}^{N} [\mathcal{T} M]_{aa}^2 \; + \; 2 \sum_{j<k}^{N} [\mathcal{T} M]_{jk}^2 \; . \tag{1.18}$$

Thus, setting

$$D \; = \; \mathrm{diag}\Big(\underbrace{1, \ldots, 1}_{N}, \underbrace{2, \ldots, 2}_{\frac{N(N-1)}{2}} \Big) \, , \tag{1.19}$$

we get that

$$\Big(\boldsymbol{M}, D\boldsymbol{M} \Big) \; = \; \Big(T\boldsymbol{M}, DT\boldsymbol{M} \Big) \qquad \text{with} \;\; \Big(\cdot, \cdot \Big) \; \text{the canonical scalar product on } \mathbb{R}^{\frac{N(N+1)}{2}} \tag{1.20}$$

so that T is orthogonal in respect to the scalar prodcut on $\mathbb{R}^{\frac{N(N+1)}{2}}$ induced by D, *ie* $T^t D T \; = \; D$. As a consequence, $(\det[T])^2 = 1$. ■

Every symmetric matrix M can be diagonalized by some orthogonal similarity transformation:

$$M = O \cdot \Lambda(\lambda_N) \cdot O^t \qquad \text{with} \qquad \Lambda(\lambda_N) \; = \; \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \qquad \text{and} \quad O \in O(N). \tag{1.21}$$

It thus appears reasonable to trade the integration over the space of symmetric matrices into one that would be compatible with the parametrization of a symmetric matrix by a diagonal and an orthogonal one. The main issue here is that such a parametrization is not unique. Indeed, let $\mathcal{H}_N$ be the closed subgroup of $O(N)$ consisting of diagonal matrices with entries $+1$ or -1. Then, for a fixed $\Lambda(\lambda_N)$, the matrices O and $O \cdot H$, $H \in \mathcal{H}_N$ will lead to the same matrix M. Furthermore, should two eigenvalues coincide, then an even greater choice of matrices O (arbitrary block form associated with the blocks of $\Lambda(\lambda_N) \propto$ the identity) will still lead to the same matrix M. Finally, even if the eigenvalues were all different, there would still remain a permutational freedom associated with the various ways of ordering them.

Hence, in order to carry out the change of variables, some more care is necessary. We are first going to show that, in fact, we can restrict the integration to a nice class of matrices, namely to

$$\mathcal{A}_S = \{M \in \mathcal{S}_N : M \text{ has simple spectrum}\} . \tag{1.22}$$

Lemma 1.2 *$\mathcal{A}_S$ is open and dense in $\mathcal{S}_N$. Furthermore, $\mathcal{S}_N \setminus \mathcal{A}_S$ is of $\mathbb{P}_N$-measure zero.*

Proof —

The first two statements are clear by standard perturbation theory. Further let

$$\Delta(M) = \prod_{a \neq b}^{N} (\lambda_a - \lambda_b) . \tag{1.23}$$

$\Delta(M)$ is a symmetric polynomial in the variables λ_a. Hence, it is a polynomial in the symmetric polynomials, and thus a polynomial in the coefficient of the matrix[1] M. We proceed by contradiction. Assume that

$$\mathbb{P}_N[\mathcal{S}_N \setminus \mathcal{A}_S] > 0 . \tag{1.24}$$

Since $\mathbb{P}_N$ is absolutely continuous in respect to Lebesgue's measure on $\mathbb{R}^{\frac{N(N+1)}{2}}$, it follows that $\Delta(M)$ is a polynomial on $\mathbb{R}^{\frac{N(N+1)}{2}}$ that vanishes on a set of non-zero Lebesgue measure. By the lemma to come, $\Delta(M) = 0$. Yet, it is readily seen that for any $M = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ such that $\lambda_a \neq \lambda_b$ for $a \neq b$, $\Delta(M) \neq 0$, a contradiction. ■

Lemma 1.3 *Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ be such that there exists a Lebesgue measurable set $E \subset \mathbb{R}^n$ such that*

$$P_{|E} = 0 \qquad \mathcal{L}_n[E] > 0 , \tag{1.25}$$

in which $\mathcal{L}_n$ is the Lebesgue measure on $\mathbb{R}^n$. Then $P = 0$

Proof —

The proof goes by induction. For $n = 1$, assume that $P \in \mathbb{C}[X]$

$$P_{|E} = 0 \tag{1.26}$$

for some $\mathcal{L}_1$-measurable set $E \subset \mathbb{R}$ with $\mathcal{L}_1[E] > 0$. Clearly, $\#E = +\infty$. Thus, P vanishes on a set of cardinality greater then $d^\circ P + 1$ and hence $P = 0$. Now, assume the statement holds up to some n. Let $P \in \mathbb{C}[X_1, \ldots, X_{n+1}]$ and E be as in the hypothesis of the lemma. Represent

$$P(X_1, \ldots, X_{n+1}) = \sum_{a=0}^{m} X_{n+1}^a Q_a(X_1, \ldots, X_n) . \tag{1.27}$$

[1]One can also obtain an explicit representation in terms of the resolvent $R(P, P')$, with $P = \det[\lambda - M]$, thus yielding the polinomiality in the coefficients of M explicitly.

Let $f_{\boldsymbol{x}_n} : x \mapsto (\boldsymbol{x}_n, x)$ and define $E_n = \{\boldsymbol{x}_n \in \mathbb{R}^n \ : \ f_{\boldsymbol{x}_n}(\mathbb{R}) \cap E \neq \emptyset\}$. Then, by Fubbini's theorem,

$$0 \ < \ \int \mathbf{1}_E \cdot \mathrm{d}\mathcal{L}_{n+1} \ = \ \int \mathbf{1}_{E_n}(\boldsymbol{x}_n) \cdot \mathcal{L}_1[\,f_{\boldsymbol{x}_n}^{-1}(E)] \cdot \mathrm{d}\mathcal{L}_n \ . \tag{1.28}$$

Thus, there exists a $\mathcal{L}_n$ measurable set $\mathcal{E} \subset \mathbb{R}^n$, $\mathcal{L}_n[\mathcal{E}] > 0$ such that $\mathcal{L}_1[\,f_{\boldsymbol{x}_n}^{-1}(E)] > 0$-$\mathcal{L}_n$ *a.e.* on $\mathcal{E}$. Removing from $\mathcal{E}$ a set of measure zero if necessary, we may assume that $\mathcal{L}_1[\,f_{\boldsymbol{x}_n}^{-1}(E)] > 0$ on $\mathcal{E}$. Hence, for any $\boldsymbol{x}_n \in \mathcal{E}$ the polynomial $P(x_1, \dots, x_n, X)$ vanishes on the set $f_{\boldsymbol{x}_n}^{-1}(\mathcal{E})$ of positive Lebesgue measure. It is thus zero. Hence, the polynomials $Q_0, \dots, Q_m$ in n variables vanish on $\mathcal{E}$ with $\mathcal{L}_n[\mathcal{E}] > 0$. By the induction hypothesis, $Q_k = 0$ for any k. ■

Hence, when integrating versus $\mathbb{P}_N$ we may restrict the integration to $\mathcal{A}_S$. We now build a diffeomorphism between $\mathcal{A}_S$ and

$$\mathbb{R}^N_{\uparrow} \times O(N)/\mathcal{H}_N \qquad \text{with} \quad \mathbb{R}^N_{\uparrow} \ = \ \{\boldsymbol{\lambda}_N \in \mathbb{R}^N \ : \ \lambda_1 < \cdots < \lambda_N\} \tag{1.29}$$

that will allow us to change the coordinates and integrate out the orthogonal group part. Note that we have imposed an ordering of the coordinates on the first space.

Lemma 1.4 *Let Φ be the map*

$$\Phi : \mathcal{A}_S \to \mathbb{R}^N_{\uparrow} \times O(N)/\mathcal{H}_N \qquad \Phi(M = O\Lambda(\boldsymbol{\lambda}_N)O^t) = (\boldsymbol{\lambda}_N, O \cdot \mathcal{H}_N) \ . \tag{1.30}$$

Then, Φ is a smooth diffeomorphism from $\mathcal{A}_S$ onto $\mathbb{R}^N_{\uparrow} \times O(N)/\mathcal{H}_N$ with inverse $\Psi(\boldsymbol{\lambda}_N, U) = O \cdot \Lambda(\boldsymbol{\lambda}_N) \cdot O^t$, in which O is any representative of the coset U. Furthermore,

$$\det\left[D_{(\boldsymbol{\lambda}_N, U)}\Psi\right] \ = \ \prod_{a<b}^{N} |\lambda_a - \lambda_b| \cdot g(U) \tag{1.31}$$

for some smooth function $g : O(N)/\mathcal{H}_N \to \mathbb{R}^+$.

Proof —

We first observe that Ψ is well defined. For if O and $O' = O \cdot H$ with $H \in \mathcal{H}_N$ are any two representatives of the coset U, we get that

$$O'\Lambda(\boldsymbol{\lambda}_n)(O')^t \ = \ O \underbrace{H\Lambda(\boldsymbol{\lambda}_n)H^t}_{=\Lambda(\boldsymbol{\lambda}_n)} O^t \tag{1.32}$$

since H and $\Lambda(\boldsymbol{\lambda}_n)$ are both diagonal and $H^2 = I_N$.

We are now in position to prove that

$$\Phi \circ \Psi \ = \ \mathrm{id}_{\mathbb{R}^N_{\uparrow} \times O(N)/\mathcal{H}_N} \qquad \text{and} \qquad \Psi \circ \Phi \ = \ \mathrm{id}_{\mathcal{A}_S} \ . \tag{1.33}$$

Indeed, one has

$$\Phi \circ \Psi(\boldsymbol{\lambda}_N, U) \ = \ \Phi(O\Lambda(\boldsymbol{\lambda}_N)O^t) \ = \ (\boldsymbol{\lambda}_N, \underbrace{O \cdot \mathcal{H}_N}_{U}) \ . \tag{1.34}$$

Also, for $M \in \mathcal{A}_S$ there exists $\boldsymbol{\lambda}_N \in \mathbb{R}^N_{\uparrow}$ and $O \in O(N)$ such that $M = O\Lambda(\boldsymbol{\lambda}_N)O^t$. Then,

$$\Psi \circ \Phi(M) \ = \ \Psi(\boldsymbol{\lambda}_N, O \cdot \mathcal{H}_N) = O\Lambda(\boldsymbol{\lambda}_N)O^t \ = \ M, \tag{1.35}$$

since O is a representative of the coset $O \cdot \mathcal{H}_N$.

• **Smoothness of Ψ**

We observe that $\mathcal{H}_N$ is a close subgroup of the Lie group $O(N)$. As a consequence, $O(N)/\mathcal{H}_N$ admits a unique structure of a $\mathscr{C}^\infty$ manifold such that the canonical projection $\pi : O(N) \mapsto O(N)/\mathcal{H}_N$ is a smooth submersion. Furthermore, relative to this manifold structure π is a principal fiber bundle with structure group $\mathcal{H}_N$ meaning that

- for any $U_0 \in O(N)/\mathcal{H}_N$ there exists an open neighborhood $\mathcal{U}_0$ of U_0 in $O(N)/\mathcal{H}_N$ and a smooth section $\tau : \pi^{-1}(\mathcal{U}_0) \to \mathcal{U}_0 \times \mathcal{H}_N$, such that $\pi = \mathrm{pr}_1 \circ \tau_{|W}$, that intertwines the $\mathcal{H}_N$ action on $\pi^{-1}(\mathcal{U}_0)$.

We introduce

$$g : \mathcal{U}_0 \to \pi^{-1}(\mathcal{U}_0) \quad \text{such that} \quad g(U) = \tau^{-1}(U, I_N) \,. \tag{1.36}$$

Then, g is a smooth local section on $O(N)/\mathcal{H}_N$.

For any $U \in \mathcal{U}_0$ and $\boldsymbol{\lambda}_N \in \mathbb{R}^N_{\uparrow}$, we have

$$\Psi(\boldsymbol{\lambda}_N, U) \;=\; g(U) \cdot \Lambda(\boldsymbol{\lambda}_N) \cdot (g(U))^t \,, \tag{1.37}$$

and the map is clearly smooth.

• **Smoothness of Φ**

Let $M_0 \in \mathcal{A}_S$. The matrix M_0 has distinct eigenvalues $\lambda_1(M_0) < \cdots < \lambda_N(M_0)$ and admits an orthonormal basis of eigenvectors $u_j(M_0)$,

$$M_0 u_j(M_0) \;=\; \lambda_j u_j(M_0) \,. \tag{1.38}$$

Since $\mathcal{A}_S$ is open there exists an open neighborhood $\mathcal{M}_0$ of M_0 in $\mathcal{A}_S$. We shrink the neighborhood if necessary so that, for any $M \in \mathcal{M}_0$

$$|\lambda_j(M) \,-\, \lambda_j(M_0)| \leq \frac{\delta}{10} \quad \text{with} \quad \delta \;=\; \min\Big\{|\lambda_a(M_0) - \lambda_b(M_0)| \;:\; a \neq b,\; a, b \in [\![\, 1\,;\, N \,]\!]\Big\} \,. \tag{1.39}$$

Then we introduce the spectral projections onto M's orthonormal of eigenvectors $u_j(M)$:

$$P_j(M) \;=\; \oint\limits_{|z-\lambda_j(M_0)|<\delta/3} \frac{1}{z-M} \cdot \frac{\mathrm{d}z}{2i\pi} \,. \tag{1.40}$$

It is readily seen that $u_j(M)$ are given by

$$u_j(M) \;=\; \frac{P_j(M) \cdot u_j(M_0)}{||P_j(M) \cdot u_j(M_0)||} \,. \tag{1.41}$$

The eigenvalues of M are expressed as

$$\lambda_j(M) \;=\; (u_j(M), M u_j(M)) \,. \tag{1.42}$$

As a consequence, both $M \mapsto \boldsymbol{\lambda}_N(M) \equiv (\lambda_1(M), \ldots, \lambda_N(M))$ and $M \mapsto O(M) = \mathrm{Mat}(u_1(M), \ldots, u_N(M)) \in O(N)$ are smooth on $\mathcal{M}_0$. Hence,

$$\Phi(M) \;=\; \Phi\Big(O(M) \cdot \Lambda(\boldsymbol{\lambda}_N(M)) \cdot O(M)^t\Big) \;=\; \Big(\Lambda(\boldsymbol{\lambda}_N(M)), \pi(O(M))\Big) \tag{1.43}$$

is also smooth.

• **Calculation of the Jacobian**

We now calculate the Jacobian of Ψ. For this purpose, fix a point $(\Lambda_0, U_0) \in \mathbb{R}^N_{\uparrow} \times O(N)/H$. Let $\boldsymbol{x}_\ell = (x_1, \ldots, x_\ell)$, $\ell = N(N-1)/2$ be a system of local coordinates in some neighborhood $\mathcal{U}_0$ of U_0:

$$\boldsymbol{x}_\ell \mapsto U(\boldsymbol{x}_\ell) \qquad \text{with} \qquad \sum_{a=1}^{\ell} x_a^2 \ < \ \epsilon \ . \tag{1.44}$$

Then, we denote $O(\boldsymbol{x}_\ell) = (g \circ U)(\boldsymbol{x}_\ell)$ with g defined as (1.36). In this way, we get an open neighbourhood $\mathcal{M}_0$ of $M_0 = \Psi(\Lambda_0, U_0)$ in $\mathcal{A}_S$:

$$\mathcal{M}_0 \ = \ \left\{ M \in \mathcal{A}_S \ : \ M = O(\boldsymbol{x}_\ell)\Lambda(\boldsymbol{\lambda}_N)O^t(\boldsymbol{x}_\ell) \ \text{ with } \ \boldsymbol{\lambda}_N \in \mathbb{R}^N_{\uparrow} \ \text{ and } \ \sum_{a=1}^{\ell} x_a^2 < \epsilon^2 \right\} . \tag{1.45}$$

Given $M \in \mathcal{A}_S$, we get

$$\partial_{x_k} M \ = \ \Big(\partial_{x_k} O(\boldsymbol{x}_\ell)\Big)\Lambda(\boldsymbol{\lambda}_N)O^t(\boldsymbol{x}_\ell) \ + \ O(\boldsymbol{x}_\ell)\Lambda(\boldsymbol{\lambda}_N)\Big(\partial_{x_k} O^t(\boldsymbol{x}_\ell)\Big) \quad \text{and} \quad \partial_{\lambda_k} M \ = \ O(\boldsymbol{x}_\ell)\cdot\partial_{\lambda_k}\Lambda(\boldsymbol{\lambda}_N)\cdot O^t(\boldsymbol{x}_\ell) \ . \tag{1.46}$$

Hence, due to $\Big(\partial_{x_a} O^t(\boldsymbol{x}_k)\Big)\cdot O(\boldsymbol{x}_\ell) \ = - \ O^t(\boldsymbol{x}_k)\Big(\partial_{x_a} O(\boldsymbol{x}_\ell)\Big)$ which is a consequence of $O^t(\boldsymbol{x}_\ell)O(\boldsymbol{x}_\ell) = I_N$,

$$O^t(\boldsymbol{x}_k)\cdot\Big(\partial_{x_k} M\Big)\cdot O(\boldsymbol{x}_\ell) \ = \ [S_j(\boldsymbol{x}_\ell), \Lambda(\boldsymbol{\lambda}_N)] \qquad \text{with} \qquad S_j(\boldsymbol{x}_\ell) \ = \ O^t(\boldsymbol{x}_\ell)\cdot\Big(\partial_{x_k} O(\boldsymbol{x}_\ell)\Big) \ . \tag{1.47}$$

We introduce the map

$$V_O A \ : \mapsto V_O(A) \ = \ O^t \cdot A \cdot O \ . \tag{1.48}$$

We have already established that V_O induces an orthogonal transformation on the vector representation $\boldsymbol{M} \in \mathbb{R}^{\frac{N(N+1)}{2}}$ of $M \in \mathcal{A}_S$, this in respect to the canonical scalar product induced by $\mathrm{tr}[M^2]$ on $\mathbb{R}^{\frac{N(N+1)}{2}}$. In particular, let $\mathcal{V}_O$ correspond to the linear transformation induced by V_O on $\mathbb{R}^{\frac{N(N+1)}{2}}$.

$$\mathcal{V}_O : \boldsymbol{M} \mapsto \mathcal{V}_O(\boldsymbol{M}) \equiv \boldsymbol{V_O(M)} \quad \text{then} \quad \det\left[\mathcal{V}_O\right] \ = \ \pm 1 \ . \tag{1.49}$$

Thence,

$$\mathcal{V}_O(\partial_{\lambda_1}\boldsymbol{M}, \ldots, \partial_{\lambda_N}\boldsymbol{M}, \partial_{x_1}\boldsymbol{M}, \ldots, \partial_{x_\ell}\boldsymbol{M}) \ = \ \Big(\boldsymbol{\partial_{\lambda_1}\Lambda}(\lambda_N), \ldots, \boldsymbol{\partial_{\lambda_N}\Lambda}(\lambda_N), \boldsymbol{[S_1(x_\ell), \Lambda(\lambda_N)]}, \cdots, \boldsymbol{[S_\ell(x_\ell), \Lambda(\lambda_N)]}\Big) \tag{1.50}$$

Where we do stress that $\boldsymbol{[S_k(x_\ell), \Lambda(\lambda_N)]}$ is the $\dfrac{N(N+1)}{2}$ dimensional vector that is canonically associated with the symmetric matrix $[S_k(\boldsymbol{x}_\ell), \Lambda(\boldsymbol{\lambda}_N)]$.

It is readily seen, since $\Lambda(\boldsymbol{\lambda}_N)$ is a diagonal matrix, that

$$\Big(\boldsymbol{\partial_{\lambda_1}\Lambda}(\lambda_N), \ldots, \boldsymbol{\partial_{\lambda_N}\Lambda}(\lambda_N), \boldsymbol{[S_1(x_\ell), \Lambda(\lambda_N)]}, \cdots, \boldsymbol{[S_\ell(x_\ell), \Lambda(\lambda_N)]}\Big) \ = \ \begin{pmatrix} I_N & 0 \\ 0 & X_N \end{pmatrix} \tag{1.51}$$

with

$$X_N \ = \ \begin{pmatrix} (\lambda_2-\lambda_1)(S_1)_{12} & (\lambda_2-\lambda_1)(S_2)_{12} & \ldots & (\lambda_2-\lambda_1)(S_\ell)_{12} \\ (\lambda_3-\lambda_1)(S_1)_{13} & (\lambda_3-\lambda_1)(S_2)_{13} & \ldots & (\lambda_3-\lambda_1)(S_\ell)_{13} \\ & & \vdots & \\ (\lambda_N-\lambda_{N-1})(S_1)_{NN-1} & (\lambda_N-\lambda_{N-1})(S_2)_{NN-1} & \ldots & (\lambda_N-\lambda_{N-1})(S_\ell)_{NN-1} \end{pmatrix} . \tag{1.52}$$

Thus,

$$\det_{\frac{N(N+1)}{2}}\Big(\partial_{\lambda_1}\boldsymbol{M},\ldots,\partial_{\lambda_N}\boldsymbol{M},\partial_{x_1}\boldsymbol{M},\ldots,\partial_{x_\ell}\boldsymbol{M}\Big) \;=\; (\pm 1)\cdot\det[X_N]\;. \tag{1.53}$$

Hence, all in all, in a neighbourhood of $\boldsymbol{M}_0 \in \mathbb{R}^{\frac{N(N+1)}{2}}$ the Jacobian of the map $\boldsymbol{M} \mapsto (\Lambda(\boldsymbol{\lambda}_N), U)$ is

$$\Big|\det\big[\frac{\partial \boldsymbol{M}}{\partial(\Lambda(\boldsymbol{\lambda}_N),U)}\big]\Big| \;=\; \prod_{a<b}^{N}|\lambda_b-\lambda_b|\cdot f(U)\;, \tag{1.54}$$

in which $f(U)>0$ is a smooth function on $O(N)/\mathcal{H}_N$. Note that $f(U)>0$ is a consequence of the maximalily of the rank of the differential of Ψ, as ensured by $\Phi\circ\Psi \;=\; \mathrm{id}_{\mathcal{A}_S}$. ■

We are now in position to establish the

Theorem 1.2 *Let $f\in L^1(\mathcal{S}_N,\mathrm{d}\mathbb{P}_N)$ be orthogonal invariant $f(OMO^t) = f(M)$. Then, f is a symmetric function F of the eigenvalues $f(M) \;=\; F(\lambda_1,\ldots,\lambda_N)$ and*

$$\int_{\mathcal{S}_N} f(M)\cdot\mathrm{d}\mathbb{P}_N(M) \;=\; \frac{1}{\mathcal{Z}_N^{(1/2)}[Q]}\int_{\mathbb{R}^N} F(\lambda_1,\ldots,\lambda_N)\cdot\prod_{a<b}^{N}|\lambda_a-\lambda_b|\cdot\prod_{a=1}^{N}\mathrm{e}^{-Q(\lambda_a)}\;, \tag{1.55}$$

in which

$$\mathcal{Z}_N^{(\beta)}[Q]\int_{\mathbb{R}^N}\prod_{a<b}^{N}|\lambda_a-\lambda_b|^{2\beta}\cdot\prod_{a=1}^{N}\mathrm{e}^{-Q(\lambda_a)}\;, \tag{1.56}$$

Proof —

As it has been already discussed, each $U_0\in O(N)/\mathcal{H}_N$ admits an open neighbourhood $\mathcal{U}_0$ and a smooth lift $g_0 : \mathcal{U}_0 \to \pi^{-1}(\mathcal{U}_0)\subset O(N)$. Furthermore, the neighbourhood $\mathcal{U}_0$ can be endowed with a system of local coordinates:

$$\mathcal{U}_0 \;=\; \Big\{U^{(0)}(\boldsymbol{x}_\ell)\;:\;\sum_{a=1}^{\ell}x_a^2\;<\;\epsilon\;\;\text{and}\;\;U^{(0)}(\boldsymbol{0})\;=;U_0\Big\} \tag{1.57}$$

$\cup_{U_0}\mathcal{U}_0$ is an open covering of $O(N)/\mathcal{H}_N$. Hence, by compactness, there exists a finite subcover $\mathcal{U}_1,\ldots,\mathcal{U}_m$ with associated lifts g_k. Let h_k, $k=1,\ldots,m$ be a smooth partition of the identity associated with $\cup_{k=1}^m\mathcal{U}_k$:

$$h_i\;\geq\;0\quad\sum_{a=1}^{m}h_a(U)\;=\;1\quad\text{for any}\quad U\in O(N)/\mathcal{H}_N\quad\text{and}\quad\mathrm{supp}(h_p)\subset\mathcal{U}_p\;. \tag{1.58}$$

Then, since $\mathcal{A}_S$ has full measure

$$\int f(M)\mathrm{e}^{-\mathrm{tr}\left[Q(M)\right]}\cdot\mathrm{d}M \;=\; \sum_{p=1}^{m}\int f(M)\cdot h_p\Big(\mathrm{pr}_2(\Phi(M))\Big)\cdot\mathrm{e}^{-\mathrm{tr}\left[Q(M)\right]}\cdot\mathrm{d}M \tag{1.59}$$

where pr_2 denotes the projection onto $O(N)/\mathcal{H}_N$.

Observe that the map Ψ restricts to a diffeomorphism from

$$\mathcal{W}_k \;=\; \mathbb{R}^N_{\uparrow}\times\Big\{U\in O(N)/\mathcal{H}_N\;:\;h_k(U)>0\Big\} \tag{1.60}$$

onto

$$\mathcal{V}_k \;=\; \Big\{ M \in \mathcal{A}_S \;\;:\;\; h_k\Big(\mathrm{pr}_2(\Phi(M))\Big) > 0 \Big\}\,. \tag{1.61}$$

Thus, the change of variables $M \mapsto \Phi(M)$, according to the previous results, leads to

$$\int f(M)\mathrm{e}^{-\mathrm{tr}\left[Q(M)\right]}\cdot \mathrm{d}M \;=\; \int\limits_{\mathbb{R}^N_{\uparrow}} F(\lambda_1,\ldots,\lambda_N)\prod_{a<b}^{N}|\lambda_b-\lambda_a|\prod_{a=1}^{N}\mathrm{e}^{-Q(\lambda_a)}\mathrm{d}^N\lambda$$
$$\times\sum_{p=1}^{m}\;\int\limits_{\sum_{a=1}^{\ell}x_a^2<\epsilon^2} h_p\Big(U^{(p)}(\boldsymbol{x}_\ell)\Big)\cdot f(U^{(p)}(\boldsymbol{x}_\ell))\cdot \mathrm{d}^\ell x\;, \tag{1.62}$$

where we remind that $U^{(p)}(\boldsymbol{x}_\ell)$ is a parametrization of elements of $\mathcal{U}_k$ by a system of local coordinates. The claim then follows since the second line of (1.62) does not depend on F and thus cancels out between the average of F and the partition function $\mathcal{N}_N$. ■

1.3.2 Other matrix ensembles and eigenvalue distributions

Although we shall not establish these properties here, one can establish similar properties for the unitary and symplectic ensembles. Namely

$$\int\limits_{\mathcal{E}_N} f(M)\cdot \mathrm{d}\mathbb{P}_N(M) \;=\; \frac{1}{\mathcal{Z}_N^{(1)}[Q]}\int\limits_{\mathbb{R}^N} F(\lambda_1,\ldots,\lambda_N)\cdot\prod_{a<b}^{N}|\lambda_a-\lambda_b|^2\cdot\prod_{a=1}^{N}\mathrm{e}^{-Q(\lambda_a)}\cdot\mathrm{d}^N\lambda \tag{1.63}$$

$$\int\limits_{\mathcal{J}_N} f(M)\cdot \mathrm{d}\mathbb{P}_N(M) \;=\; \frac{1}{\mathcal{Z}_N^{(2)}[2Q]}\int\limits_{\mathbb{R}^N} F(\lambda_1,\ldots,\lambda_N)\cdot\prod_{a<b}^{N}|\lambda_a-\lambda_b|^4\cdot\prod_{a=1}^{N}\mathrm{e}^{-2Q(\lambda_a)}\cdot\mathrm{d}^N\lambda\,. \tag{1.64}$$

In the case of the unitary ensemble, there is a "doubling" of the exponent in the repulsive two-body interaction. This effect takes its origin in the doubling of the "freedom" of choice for the off-diagonal entries of $M \in \mathcal{E}_N$. Indeed, both imaginary and real parts in the upper off-diagonal entries are now free from constraints. The additional doubling which is observed in the case of the symplectic ensemble, (*ie* an interaction of the form $|\lambda_a - \lambda_b|^4$) stems from the occurence of additional freedom in the construction of hermitian self-dual matrices. Also, note that the factor 2 present in front of the confining potential Q in the case of the symplectic ensemble issued multiple integral stems from the fact that eigenvalues of Hermitian self-dual matrices always appear in pairs: $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_N, \lambda_N)$.

Furthermore, the log-gas interpretation of the probability distribution function for the eigenvalues of the hermitian, orthogonal and symplectic ensembles is clear. As a matter of fact, one can reduce several other integrals over various one-matrix ensembles also to solely integrations over the spectrum.

For instance, consider an $L \times N$, $L \geq N$, random matrix X whose entries are real ($\beta = 1/2$), complex ($\beta = 1$) and real quaternion ($\beta = 2$) independent random variables distributed with a Gaussian density given resp. by

$$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x_{jk}^2} \quad , \quad \frac{1}{\pi}\mathrm{e}^{-|z_{jk}|^2} \quad \text{or} \quad \frac{2}{\pi}\mathrm{e}^{-2|z_{jk}|^2} \quad \text{and} \quad \frac{2}{\pi}\mathrm{e}^{-2|w_{jk}|^2} \tag{1.65}$$

where the occurence of two distributions in the real quaternion cases is due to the fact that a quaternion can be parameterized by 2 complex numbers z and w. The Wishart ensembles (at $\beta \in \{1/2, 1, 2\}$) are then defined as consisting of random matrices of the type $XX^{\dagger}$. Such a matrix X admits a singular value decomposition

$$X \;=\; U \cdot \Lambda \cdot V^{\dagger} \tag{1.66}$$

in which U, resp. V, is an $L \times L$, , resp. $N \times N$, unitary matrix and Λ is a $L \times N$ matrix of the form

$$\Lambda \;=\; \begin{pmatrix} \mathrm{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_N}) \\ \mathbf{0}_{(L-N)\times N} \end{pmatrix} . \tag{1.67}$$

There $\mu_1, \ldots, \mu_N$ are the N eigenvalues of $X^{\dagger}X$. The positive numbers $\sqrt{\mu_1}, \ldots, \sqrt{\mu_N}$ are called the singular values of X. By generalizing the handlings relative to the orthogonal ensemble, one shows that one can reduce the integration in the partition function to solely the singular value part leading to the so-called Laguerre ensemble based partition function

$$Z_N^{(Lag)} \;=\; \int_0^{+\infty} \prod_{a<b}^{N} |\lambda_a - \lambda_b|^{2\beta} \prod_{a=1}^{N} \Big\{ \lambda_a^{2\beta\alpha} \cdot \mathrm{e}^{-\beta\lambda_a} \Big\} \cdot \mathrm{d}^N\lambda \qquad \text{with} \qquad \alpha = L - N + 1 - \frac{1}{\beta} \tag{1.68}$$

in which, depending on the type of matrices considered, one should set $\beta = 1/2, 1, 2$.

Hence, one sees that changing certain overall characteristics of the "base" matrix ensemble may lead to qualitatively different forms of the pdf for the eigenvalues. In the case of Whishart matrices $X^{\dagger}X$, the main difference with the previous cases lies in the fact that the integration runs through a semi-axis and that one allows for a power-law singularity at the origin. This has rather important consequences on the universality properties associated with the model. For instance, the scaling limit for the distribution of eigenvalues "near" the edge $\lambda = 0$ takes a completely different form from the ones that can arise in the three "classical" ensembles of random matrices introduced so-far.

In fact, one can even construct ensembles of random matrices whose eigenvalues will be supported on some segment of $\mathbb{R}$, *ie* be bounded from below *and* above. Consider matrices $A = X^{\dagger}X$ and $B = Y^{\dagger}Y$ with $X \in \mathcal{M}_{L_1\times N}(\mathbb{K})$ and $Y \in \mathcal{M}_{L_1\times N}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}\,,\,\mathbb{C}\,,\,\mathbb{H}$ depending on $\beta = 1/2\,,1\,,2$, random matrices distributed according to the Gaussian laws introduced previously. Then the eigenvalues $x_1, \ldots, x_N$ of the matrix $(A+B)^{-\frac{1}{2}} \cdot A \cdot (A+B)^{-\frac{1}{2}}$ belong to $[\,0\,;1\,]$ and, upon setting $\lambda_a = 1 - 2x_a$, have a probability density function giving rise to the so-called Jacobi ensemble:

$$Z_N^{(Jac)} \;=\; \int_{-1}^{1} \prod_{a<b}^{N} |\lambda_a - \lambda_b|^{2\beta} \cdot \prod_{a=1}^{N} \Big\{ (1-\lambda_a)^{\beta\alpha} \cdot (1+\lambda_a)^{\beta\gamma} \Big\} \cdot \mathrm{d}^N\lambda \qquad \text{with} \qquad \left\{ \begin{array}{l} \alpha = N_1 - L + 1 - \dfrac{1}{\beta} \\ \gamma = N_2 - L + 1 - \dfrac{1}{\beta} \end{array} \right. \tag{1.69}$$

in which, again, depending on the type of matrices considered, $\beta = 1/2, 1, 2$.

1.4 A first hint towards building a relation with probabilities

We have introduced enough concepts so as to establish a connection between the problem of extracting the large-N behavior of N-fold integrals of interest and sequences of probability measures on Polsih spaces. Consider the partition function on $\mathbb{R}^N$:

$$Z_N[V, W] \;=\; \int_{\mathbb{R}^N} \prod_{a,b=1}^{N} \mathrm{e}^{-\frac{\beta}{2}W(\lambda_a,\lambda_b)} \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)} \cdot \mathrm{d}^N\lambda \,, \tag{1.70}$$

with V and W sufficiently regular so that the integral is defined for any N.

For a given $\boldsymbol{\lambda}_N \in \mathbb{R}^N$, one associates the so-called empirical measure

$$L_N^{(\lambda_N)} = \frac{1}{N}\sum_{a=1}^{N} \delta_{\lambda_a} \in \mathcal{P}(\mathbb{R}) , \tag{1.71}$$

in which δ_x is the Dirac mass at x and $\mathcal{P}(S)$ refers to the space of probability measures on the space S. Then, the partition function can be recast as

$$Z_N[V,W] = \int_{\mathbb{R}^N} \exp\Big\{-N^2 \mathcal{W}[L_N^{(\lambda_N)}]\Big\}\cdot \mathrm{d}^N\lambda \quad \text{where} \quad \mathcal{W}[\mu] = \frac{1}{2}\int_{\mathbb{R}^2} \Big[V(s)+V(t)+\beta W(s,t)\Big]\cdot \mathrm{d}\mu(s)\otimes \mathrm{d}\mu(t) . \tag{1.72}$$

Assume that $\mathcal{W}$ admits a unique minimum on $\mathcal{P}(\mathbb{R})$ and that it is "sufficiently-well" behaved as a function on $\mathcal{P}(\mathbb{R})$.

- The integrand $\exp\Big\{-N^2\mathcal{W}[L_N^{(\lambda_N)}]\Big\}$ behaves as $\mathrm{e}^{\mathrm{O}(N^2)}$, meaning that, it should produce an analogous behaviour of the partition function;
- The Lebesgue measure should generate, at most, an $\mathrm{e}^{\mathrm{O}(N)}$ behaviour. A heuristic argument in favor of this statement is that the volume of $[-M\,;M]$, $M>0$ goes as $(2M)^N$, *ie* grows exponentially fast.
- Atomic measures on $\mathbb{R}$ are dense in $\mathcal{P}(\mathbb{R})$, hence, given a measure $\mu \in \mathcal{P}(\mathbb{R})$, provided that the sequence $\boldsymbol{x}_N \in \mathbb{R}^N$ is chosen properly, $L_N^{(\boldsymbol{x}_N)} \to \mu$ in the $N\to+\infty$, where, for the time being, we do not give more specifications on the symbol $\to$.
- Pick a $\mu \in \mathcal{P}(\mathbb{R})$ such that $\delta = \mathcal{W}[\mu] - \inf_{\mu\in\mathcal{P}(\mathbb{R})}\mathcal{W}[\mu] > 0$. Then all points $\boldsymbol{\lambda}_N \in \mathbb{R}^N$ such that $L_N^{(\lambda_N)}$ "is close to" μ will have roughly a relative contribution $\mathrm{O}(\mathrm{e}^{-\delta N^2})$ in magnitude as compared to those configurations of points $\boldsymbol{\lambda}_N \in \mathbb{R}^N$ such that $L_N^{(\lambda_N)}$ "is close to" minimizing $\mathcal{W}$, *ie* $\mathcal{W}[L_N^{(\lambda_N)}] - \inf_{\mu\in\mathcal{P}(\mathbb{R})}\mathcal{W}[\mu]$.

Thus, on the basis of the above arguments, one can expect that the integration variables will localize -in the $N\to+\infty$ limit- in such a way that $L_N^{(\lambda_N)}$ will be "close" to minimizing $\mathcal{W}$. Hence, one may expect that

$$\lim_{N\to+\infty} \frac{1}{N^2}\cdot \ln Z_N[V,W] = \inf_{\mu\in\mathcal{P}(\mathbb{R})} \mathcal{W}[\mu] . \tag{1.73}$$

We are, indeed, going to establish this result, once upon we have specified more thoroughly the structures that we will be working with.

1.5 The need for scaling

The concentrated reader has probably noticed that the random matrix issued partition functions did not have a prefactor of N in front of the "potential" part $\mathrm{e}^{-Q(\lambda)}$, where we remind that Q is a polynomial of even degree:

$$Q(X) = \sum_{p=0}^{2m} a_p X^p \quad a_{2m} > 0 . \tag{1.74}$$

We are now going to explain the origin of the scaling with N and show how it can be "reinstalled by hand". Consider thus

$$\mathcal{Z}_N^{(\beta)}[Q] = \int_{\mathbb{R}^N} \prod_{a<b}^{N} |\lambda_a-\lambda_b|^{2\beta}\cdot \prod_{a=1}^{N} \mathrm{e}^{-Q(\lambda_a)}\cdot \mathrm{d}^N\lambda = \int_{\mathbb{R}^N} \exp\Big\{N^2\beta\cdot\frac{1}{N^2}\sum_{a\neq b}^{N}\ln|\lambda_a-\lambda_b| - N\cdot\frac{1}{N}\sum_{a=1}^{N} Q(\lambda_a)\Big\} . \tag{1.75}$$

As already argued in the previous section, in order to access to the leading large-N behavior of $\ln \mathcal{Z}_N^{(\beta)}[Q]$, one needs to maximize the argument of the exponential. In fact, in this respect, there will be two competing effects:

- The logarithmic potential part is a repelling interaction. Configurations of integration variables which are "as far apart as possible" will maximize its value.
- The confining potential part $-Q$ ensures the convergence of the integrals. It will tend to "keep" the integration variables localized in some finite (possibly growing with N) region.

However, there are different prefactors of N in the *rhs* of (1.75) in front of each term. Hence, their effect will not appear on the same scales.

Due to the "large" (of the order N^2) number of terms in the interaction potential part, for sufficiently "mild" separations between the variables λ_a and λ_b, $a \neq b$, this term will completely dominate the "confining" potential part, which only has N terms. Thus, the latter will start to compensate for the repelling logarithmic interaction solely when the variables will become "spaced" by an average distance scaling with some power of N. The aim of this scaling is to bring the scale of the variables in such a form that the "logarithmic" interaction and the "confining" potential are of the same order of magnitude already on a region of finite (in respect to N) lenght.

We are going to argue the correct power of the scaling on the basis of the assumption (that will be further justified rigorously by the analysis to come) that

$$\lambda_a = \mathrm{O}(1) \;\; a = 1, \dots, N \qquad \Rightarrow \qquad \frac{1}{N}\sum_{a=1}^{N} f(\lambda_a) \sim \mathrm{O}(1) \quad \text{and} \quad \frac{1}{N^2}\sum_{a=1}^{N} g(\lambda_a, \lambda_b) \sim \mathrm{O}(1) \; . \tag{1.76}$$

Thus, in the integral, we change variables $\lambda_a = N^{\alpha}\mu_a$, leading to

$$\mathcal{Z}_N^{(\beta)}[Q] \;=\; N^{\alpha\cdot N} N^{\alpha\cdot\beta N(N-1)} \cdot \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} |\lambda_a - \lambda_b|^{2\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-N^{\alpha 2m} V_N^{(\alpha)}(\lambda_a)} \cdot \mathrm{d}^N\lambda \qquad \text{with} \qquad V_N^{(\alpha)}(\lambda) \;=\; a_{2m}\lambda^{2m} + \sum_{p=0}^{2m-1} \frac{a_p \lambda^p}{N^{(2m-p)\cdot\alpha}} \; . \tag{1.77}$$

Here, again, the logarithmic interactions are of the order of $\mathrm{O}(N^2)$, on a finite with N size region whereas the "confining" potential interaction are of the order of $\mathrm{O}(N^{2m\alpha+1})$. Hence, the rescaling of variables by $N^{\frac{1}{2m}}$ seems to be the "good" scaling which immediately, in the new integration variables, allows one to tune the contributions of the "two-body interaction" part and of the "confining potential" to the same level of magnitude. Under such a scaling, the partition function is recast as

$$\mathcal{Z}_N^{(\beta)}[Q] \;=\; N^{\frac{N}{2m}} N^{\frac{\beta}{2m}N(N-1)} \cdot \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} |\lambda_a - \lambda_b|^{2\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-NV_N(\lambda_a)} \cdot \mathrm{d}^N\lambda \qquad \text{with} \qquad V_N^{(\frac{1}{2m})}(\lambda) \;=\; a_{2m}\lambda^{2m} + \sum_{p=0}^{2m-1} \frac{a_p \lambda^p}{N^{1-\frac{p}{2m}}} \; . \tag{1.78}$$

Hence, we basically recover the case of a partition function having an explicit N in front of the confining potential. True, the potential itself depends on N. However, for a fixed compact set K, setting $V_\infty(\lambda) = a_{2m}\lambda^{2m}$, we get that,

$$||V_N^{(\frac{1}{2m})} - V_\infty||_{L^\infty(K)} \underset{N\to+\infty}{\longrightarrow} 0 \; . \tag{1.79}$$

Furthermore, when going to infinity, *ie* moving "far" away from compacts,

$$\frac{V_N(\lambda)}{V_\infty(\lambda)} \underset{\lambda\to\pm\infty}{\longrightarrow} 0 \; . \tag{1.80}$$

Thus it seems plausible, and this fact will become apparent from our analysis, that the remainder $V_N - V_\infty$ will merely produce sub-leading corrections to the large-N behaviour of $\ln \mathcal{Z}_N^{(\beta)}[Q]$.

1.6 Occurrence of similar multiple integrals in other contexts

1.6.1 The quantum separation of variables

The so-called quantum separation of variables is one of the exact methods allowing one to fully characterize and compute -in terms of solutions to 1-dimensional spectral problems- the eigenvalues of numerous partial differential operators O_N in N variables. The operator O_N in naturally defined on some dense space of $L^2(\mathbb{R}^N)$. The method consists in constructing a unitary transform

$$\mathcal{U} \; : \; L^2(\mathbb{R}^N, \mathrm{d}^N x) \; \mapsto \; L^2(\mathbb{R}^N, \mu(\boldsymbol{y}_N)\mathrm{d}^N y) \qquad \text{with} \quad \mu(\boldsymbol{y}_N) \; = \; \prod_{a<b}^{N} \Big\{ \sinh\left[\pi\omega_1(y_a - y_b)\right] \cdot \sinh\left[\pi\omega_2(y_a - y_b)\right] \Big\} \; . \tag{1.81}$$

The unitray map allows one to solely work on the space $L^2(\mathbb{R}^N, \mu(\boldsymbol{y}_N)\mathrm{d}^N y)$ where the separation of variables occur, namely, if $\Psi(\boldsymbol{x}_N)$ is an eigenvector of O_N, then

$$\mathcal{U}[\Psi](\boldsymbol{y}_N) \; = \; \prod_{a=1}^{N} q_\Psi(y_a) \tag{1.82}$$

in which q_Ψ solves an auxiliary *one*-dimensional spectral problem. Therefore, the scalar product takes the form

$$\Big(\Psi, \Psi\Big)_{L^2(\mathbb{R}^N, \mathrm{d}^N x)} \; = \; \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh\left[\pi\omega_1(y_a - y_b)\right] \cdot \sinh\left[\pi\omega_2(y_a - y_b)\right] \Big\} \cdot \prod_{a=1}^{N} q^2(y_a) \cdot \mathrm{d}^N y \; . \tag{1.83}$$

Thus, the norm of the states is given by a N-fold integral. For various reasons, one is interested in extracting the large-N behaviour out of such multiple integrals.

In fact, in physics, one is mostly interested in computing the so-called form factor of local operators, *ie* acting on a "reduced number of variables", *ie* $p_1 = \frac{i}{\hbar}\partial_{x_1}$ is a kind of local operator of interest for physics. For numerous local operators O, one can find an expression for $\mathcal{U}^\dagger O \mathcal{U}$, *ie* describe explicitly how the operator acts on functions living on the space where the quantum separation of variables occurs. Then, form factors of certain such operators O take the form

$$\Big(\Phi, O \cdot \Psi\Big)_{L^2(\mathbb{R}^N, \mathrm{d}^N x)} \; = \; \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh\left[\pi\omega_1(y_a - y_b)\right] \cdot \sinh\left[\pi\omega_2(y_a - y_b)\right] \Big\} \cdot \prod_{a=1}^{N} q_\Phi(y_a) q_\Psi(y_a) \cdot \prod_{a=1}^{N} o(y_a) \mathrm{d}^N y \; . \tag{1.84}$$

There $o(y)$ are functions that represent a sort of perturbation of the integrand, much in the spirit of (1.6).

Studying the large-N behaviour of such integrals is still an (hard) open problem. Important complications arise due to the lack of factorization for the measure $\mu(\boldsymbol{y}_N)$ under a rescaling of the variables. Furthermore, the complicated formulation for the functions q is also a problems. Basically, most tools developed for log-gases issued multiple integrals break down and one has to invent more sophisticated techniques.

In fact, just as the log-gases issued multiple integrals did deserve to be called semi-classical multiple integrals, the former class of integrals should already be refered to as a quantum integrals, so large is the gap in the technical arsenal that is necessary for its analysis. Another instance of such integrals, in fact in a much more complex form, stems from Bethe Ansatz solvable one-dimensional spin chains.

1.6.2 The emptiness formation probability

The emptiness formation probability is a specific correlator that arizes in the so-called XXZ spin-$1/2$ chain. The Hamiltonian of this model acts on the Hilbert space

$$\mathfrak{h} \;=\; \otimes_{a=1}^{L} V_a \quad V_a \simeq \mathbb{C}^2 \;, \tag{1.85}$$

and takes the form

$$H_{XXZ} \;=\; \sum_{a=1}^{L} \Big\{ \sigma_a^x \sigma_{a+1}^x \;+\; \sigma_a^y \sigma_{a+1}^y \;+\; \cos(\zeta)\cdot \sigma_a^z \sigma_{a+1}^z \;-\; h \sum_{a=1}^{L} \sigma_a^z \Big\} \tag{1.86}$$

where $\sigma^{x,y,z}$ are standard Pauli matrices

$$\sigma^x \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{1.87}$$

and we have used tensor notations so as to write down the Hamiltonian in a compact form; namely for some operator $O \in \mathcal{L}(\mathbb{C})$

$$O_a \;=\; \underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{a-1 \text{ terms}} \otimes O \otimes \mathrm{id}\otimes\cdots\otimes\mathrm{id} \tag{1.88}$$

The emptiness formation probability corresponds to the below correlator

$$\tau_L(m) \;=\; \Big(\Psi_g^{(L)}, \prod_{a=1}^{m} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{[a]} \cdot \Psi_g^{(L)} \Big) \tag{1.89}$$

where $\Psi_g^{(L)}$ correspond to the ground state of the Hamiltonian H_{XXZ}. One can show that, in the so-called thermodynamic limit of the model $L \to +\infty$, $\lim_{L\to+\infty} \tau_L(m) = \tau(m)$ admits an m-fold multiple integral based representation:

$$\tau(m) \;=\; \frac{1}{m!} \int_{-q}^{q} \mathrm{d}^m\lambda \prod_{a,b=1}^{m} \left\{ \frac{\sinh(\lambda_a + i\zeta/2)\sinh(\lambda_a - i\zeta/2)}{\sinh(\lambda_a - \lambda_b - i\zeta)} \right\} \cdot F_m(\lambda_1, \ldots, \lambda_m) \;, \tag{1.90}$$

with

$$F_m(\lambda_1, \ldots, \lambda_m) \;=\; \lim_{\xi_k \to -i\zeta/2} \frac{\det_m \left[t\left(\lambda_j, \xi_k\right) \right] \det_m \left[\rho\left(\lambda_j, \xi_k\right) \right]}{\prod_{j<k}^{m} \sinh^2\left(\xi_j - \xi_k\right)} \tag{1.91}$$

The function t is explicit

$$t\left(\lambda, \mu\right) = \frac{-i\sin\zeta}{\sinh(\lambda-\mu)\sinh(\lambda-\mu-i\zeta)} \;. \tag{1.92}$$

ρ is interpreted as the density of certain parameters that parametrize the model's ground state. It is defined as the solution to integral equation

$$\rho\left(\lambda, \xi\right) + \int_{-q}^{q} K\left(\lambda-\mu\right)\rho\left(\mu, \xi\right) \cdot \frac{\mathrm{d}\mu}{2\pi} \;=\; \frac{t\left(\lambda, \xi\right)}{-2i\pi} \qquad \text{with} \qquad K\left(\lambda\right) = \frac{\sin 2\zeta}{\sinh(\lambda+\zeta)\sinh(\lambda-i\zeta)} \;. \tag{1.93}$$

For ξ close to $-i\zeta/2$ and $q = +\infty$, the solution is explicit

$$\rho\left(\zeta, \xi\right) = \frac{i}{2\zeta \sinh \dfrac{\pi}{\zeta}\left(\lambda - \xi\right)} \;, \tag{1.94}$$

and then the determinant involving ρ becomes computable in that it corresonds to a Cauchy matrix.

2 Some facts about probability measures on Polish spaces

In all of the setting, (S, d) will be a Polish space (complete, separable metric space). We shall endow it with its Borel σ-algebra $\mathcal{B}$ generated by open sets. We shall also denote by $\mathcal{P}(S)$ the space of Borel probability measures on S.

We remind two facts about finite measures on S.

Lemma 2.1 *Let μ be a finite measure on S and $\mathcal{A}$ a collection of disjoint Borel subsets of S. Then at most countably many elements of $\mathcal{A}$ has non-zero μ-measure.*

Proof —
Given $\ell \geq 1$ set $\mathcal{A}_\ell = \{A \in \mathcal{A} : \mu[A] > 1/\ell\}$. Thus, for any distinct $A_1, \dots, A_k$ in $\mathcal{A}_\ell$, one has

$$\mu[S] \geq \mu[\cup_{p=1}^{k} A_p] = \sum_{p=1}^{k} \mu[A_p] > \frac{k}{\ell}, \tag{2.1}$$

ie $\mathcal{A}_\ell$ has at most $[\ell \cdot \mu[S]]$ elements. ■

Proposition 2.1 *Any finite measure on S is regular in the sense that for any Borel subset $B \in \mathcal{B}$*

$$\mu[B] = \sup\{\mu[F] : F \subset B , F \text{ closed}\} = \inf\{\mu[O] : O \subset B , O \text{ open}\}. \tag{2.2}$$

The first equality is referred to as inner regularity whereas the second as outer regularity.

Prior to discussing the convergence of measures and its metrizability, we shall introduce a few concepts that will be useful in the course of our handlings. In general, when studying measures, it is often useful to know that the latter, basically, concentrates on compacts. This property is called tightness. As we shall establish right away, probability measures are always tight.

Definition 2.1 *A Borel measure μ on S is tight if given any $\epsilon > 0$ there exists a compact $K \Subset S$ such that*

$$\mu[S \setminus K] < \epsilon \tag{2.3}$$

Recall the convenient characterization of compacts in complete metric spaces.

Lemma 2.2 *Any totally bounded (for any $\epsilon > 0$ the set is covered by finitely many balls of radius ϵ) and closed subset K of S is compact.*

It is clear that the converse is true.

Proof — Let x_n be a sequence of elements in K. Since K is totally bounded, it can be covered by finitely many balls of radius $1/p$, this for any p. Hence, for any p, at least one of these balls contains infinitely many x_n's. Consider the following construction. For $p = 1$ take a ball B_1 of radius 1 such that

$$\mathcal{N}_1 = \Big\{n : x_n \in B_1\Big\} \tag{2.4}$$

is infinite and pick $n_1 \in \mathcal{N}_1$. Then, for $p = 2$ take a ball B_2 of radius $1/2$ such that

$$\mathcal{N}_2 = \Big\{n > n_1 : x_n \in B_1 \cap B_2\Big\} \tag{2.5}$$

is infinite and pick $n_2 \in \mathcal{N}_2$. Continue so-on, for $p+1$ taking a ball B_{p+1} or radius $1/(p+1)$ such that

$$\mathcal{N}_{p+1} \;=\; \left\{n > n_p \;:\; x_n \in B_1 \cap B_2 \cdots \cap B_p\right\} \tag{2.6}$$

is infinite and pick $n_{p+1} \in \mathcal{N}_{p+1}$. Thence the sequence (x_{n_p}) is a subsequence of (x_n) such that $x_{n_\ell} \in B_k$ for any $\ell \geq k$. It is thus a Cauchy sequence. As such it converges to some $x \in S$ in virtue of the latter's set completeness. K being closed, it follows that $x \in K$. As a consequence, K is compact. ■

Theorem 2.1 Ulam
Every probability measure on (S,d) is tight.

Proof —
Pick $\epsilon > 0$ and consider a sequence (x_n) that is dense in S. As a consequence, for any $\ell > 0$

$$S \;=\; \cup_{n=1}^{+\infty} \overline{B}(x_n, \frac{1}{\ell})\,. \tag{2.7}$$

Because S is of finite measure and μ is continuous, there exists n_ℓ such that

$$\mu\Big[S \setminus \cup_{n=1}^{n_\ell} \overline{B}(x_n, \frac{1}{\ell})\Big] \;\leq\; \frac{\epsilon}{2^\ell}\,. \tag{2.8}$$

The set

$$K \;=\; \cap_{\ell\geq 1} \cup_{n=1}^{n_\ell} \overline{B}(x_n, \frac{1}{\ell}) \tag{2.9}$$

is closed and totally bounded, hence compact. Furthermore,

$$\mu[S \setminus K] \;\leq\; \sum_{\ell\geq 1} \mu\Big[S \setminus \cup_{n=1}^{n_\ell} \overline{B}(x_n, \frac{1}{\ell})\Big] \;\leq\; \sum_{\ell\geq 1} \frac{\epsilon}{2^\ell} \;=\; \epsilon. \qquad ■ \tag{2.10}$$

2.1 Convergence of measures

Definition 2.2 *A sequence $\mu_N \in \mathcal{P}(S)$ converges weakly to $\mu \in \mathcal{P}(S)$, $\mu_N \rightharpoonup \mu$ if*

$$\int f \mathrm{d}\mu_N \to \int f \mathrm{d}\mu \tag{2.11}$$

for any $f \in C_b(S)$, the space of real-valued bounded continuous functions on S.

The notion of weak convergence can be, in fact, rephrased in terms of a convergence on closed (or open) sets

Theorem 2.2 Portmanteau
Let $\mu_N \in \mathcal{P}(S)$ be a sequence of probability measures on S. Then, the following statements are equivalent:

i) $\mu_N \rightharpoonup \mu \in \mathcal{P}(S)$ *;*

ii) for any open set U, $\limsup_{N\to+\infty} \mu_N[U] \;\geq\; \mu[U]$ *;*

iii) for any closed set F, $\limsup_{N\to+\infty} \mu_N[F] \;\leq\; \mu[F]$ *;*

iv) for any continuity set A of μ, ie $\mu[\partial A] = 0$, $\lim_{N\to+\infty} \mu_N[A] \ = \ \mu[A]$.

Proof —
i) ⇒ *ii*). Let U be open. Then define

$$f_m(x) \ = \ \min\{m \cdot d(x, U^{\mathrm{c}})\,,\, 1\}\ . \tag{2.12}$$

Since U^{c} is closed, $f_m \uparrow \mathbf{1}_U$. Furthermore, clearly, f_m is bounded and continuous.

$$\mu_N[U] \ \geq \ \int f_m(s)\cdot \mathrm{d}\mu_N(s) \ \to \ \int f_m(s)\cdot \mathrm{d}\mu(s) \qquad \Rightarrow \qquad \liminf_{N\to+\infty} \mu_N[U] \ \geq \ \int f_m(s)\cdot \mathrm{d}\mu(s)\ . \tag{2.13}$$

Since, by the monotone convergence theorem,

$$\int f_m(s)\cdot \mathrm{d}\mu(s) \ \to \ \mu[U]\ , \tag{2.14}$$

ii) follows.

ii) ⇔ *iii*) by taking complements. Namely, given F closed,

$$\liminf_{N\to+\infty} \mu_N[F^{\mathrm{c}}] \ = \ 1 - \limsup_{N\to+\infty} \mu_N[F] \ \geq \ \mu[F^{\mathrm{c}}] \ = \ 1 - \mu[F] \tag{2.15}$$

and given O closed,

$$\limsup_{N\to+\infty} \mu_N[O^{\mathrm{c}}] \ = \ 1 - \liminf_{N\to+\infty} \mu_N[O] \ \leq \ \mu[O^{\mathrm{c}}] \ = \ 1 - \mu[O] \tag{2.16}$$

ii)&*iii*) ⇒ *iv*)

Is A is a continuity set for μ, then

$$\mu[A] \ = \ \mu[\mathring{A}] \ \leq \ \liminf_{N\to+\infty} \mu_N[\mathring{A}] \leq \limsup_{N\to+\infty} \mu_N[\overline{A}] \ \leq \ \mu[\overline{A}] \ = \ \mu[A]\ . \tag{2.17}$$

iv) ⇒ *i*)

Let $f \in \mathscr{C}_b(S)$. Since the measure μ is finite, there exists at most countably many x's such that $\mu[f^{-1}(\{x\})] > 0$. Hence, for any ϵ, one can find a sequence $a_1 < a_2 < \cdots < a_M$ such that

$$\max(a_{k+1} - a_k) \ \leq \ \epsilon \quad \mu[f^{-1}(\{a_k\})] = 0 \quad \text{and} \quad \mathrm{range}(f) \subset [\,a_1\,;a_M\,]\ . \tag{2.18}$$

Then introduce the below approximation f_ϵ for f

$$F_k \ = \ \Big\{s \in \ S \ : \ a_k \leq f(s) < a_{k+1}\Big\} \qquad \text{and} \qquad f_\epsilon \ = \ \sum_{k=1}^{M} a_k \mathbf{1}_{F_k}\ . \tag{2.19}$$

One has that, by continuity of f, $\partial F_k \ = \ f^{-1}(\{a_k\}) \cup f^{-1}(\{a_{k+1}\})$. Thus, $\mu[f^{-1}(\{a_k\})] \ = \ 0$. As a consequence, by *iv*),

$$\int f_\epsilon \cdot \mathrm{d}\mu_N \ = \ \sum_{k=1}^{M} a_k \mu_N[F_k] \ \to \ \sum_{k=1}^{M} a_k \mu[F_k] \ = \ \int f_\epsilon \cdot \mathrm{d}\mu\ . \tag{2.20}$$

Since, by construction $|f - f_\epsilon| < \epsilon$, the claim follows upon relaxing $\epsilon \to 0$. ■

2.2 Metrizability

It so happens that one can metrize $\mathcal{P}(S)$. There will, in fact, arise two equivalent distances in our handlings. Working with the two provides one with a convenient way of proving Prokhorov's theorem which is the key result of this section. We start with the Levy-Prokhorov metric

Definition 2.3 *The ϵ-neighborhood A_ϵ of a set $A \subset S$ is defined by*

$$A_\epsilon \ = \ \Big\{ y \in S \ : \ \exists y \in A \ , \ d(x,y) < \epsilon \Big\} \tag{2.21}$$

Definition 2.4 *The quantity*

$$d_{LP}(\mu,\nu) \ = \ \inf \Big\{ \epsilon > 0 \ : \ \mu[A] \ \leq \ \nu[A_\epsilon] + \epsilon \quad \text{for all} \ \ A \in \mathcal{B} \Big\} \tag{2.22}$$

is called the Levy-Prokhorov distance between the probability measure μ and ν.

Proposition 2.2 *d_{LP} is a metric on $\mathcal{P}(S)$*

Proof —

We start by showing that d_{LP} is symmetric.

Hence, let $\mu, \nu \in \mathcal{P}(S)$. Assume that $d_{LP}(\mu,\nu) > \eta$. Then, by definition, there exists a set $A \in \mathcal{B}$ such that

$$\mu[A] \ > \ \nu[A_\eta] + \eta \ . \tag{2.23}$$

Note the inclusion

$$\Big((A_\eta)^{\mathrm{c}} \Big)_\eta \subset A^{\mathrm{c}} \ . \tag{2.24}$$

Indeed, if $x \in \Big((A_\eta)^{\mathrm{c}} \Big)_\eta$, then there exists $y \in (A_\eta)^{\mathrm{c}}$ such that $d(x,y) < \eta$. Furthermore, $d(y,A) \geq \epsilon$. Hence, $d(x,A) > 0$ and thus $x \in A^{\mathrm{c}}$. This inclusion implies, upon taking the complement of (2.23), that

$$\nu[(A_\eta)^{\mathrm{c}}] \ > \ \mu[A^{\mathrm{c}}] \ + \ \eta \ \geq \ \mu\Big[\Big((A_\eta)^{\mathrm{c}} \Big)_\eta \Big] \ + \ \eta \ . \tag{2.25}$$

In other words, the set $B = (A_\eta)^{\mathrm{c}}$ verifies that

$$\nu[B] \ > \ \mu[B_\eta] \ + \ \eta \qquad \Rightarrow \qquad d_{LP}(\nu,\mu) \ > \ \eta \ . \tag{2.26}$$

By sending $\eta \uparrow d_{LP}(\mu,\nu)$, one gets that $d_{LP}(\nu,\mu) \geq d_{LP}(\mu,\nu)$ so that, by symmetry, $d_{LP}(\nu,\mu) = d_{LP}(\mu,\nu)$.

We now establish the triangle inequality. Assume that

$$d_{LP}(\mu,\nu) \ \leq \ \eta \qquad \text{and} \qquad d_{LP}(\nu,\rho) \ \leq \ \epsilon. \tag{2.27}$$

Thus, for any $A \in \mathcal{B}$,

$$\mu[A] \ \leq \ \nu[A_\eta] + \eta \qquad \text{and} \qquad \nu[A_\eta] \ \leq \ \rho\Big[(A_\eta)_\epsilon \Big] \ + \ \epsilon \ . \tag{2.28}$$

As a consequence, since $(A_\eta)_\epsilon \subset A_{\eta+\epsilon}$, one has that

$$\mu[A] \ \leq \ \rho[A_{\eta+\epsilon}] \ + \ \epsilon \ + \ \eta \ . \tag{2.29}$$

Thus, $d_{LP}(\mu,\rho) \leq \eta + \epsilon$. The claim then follows by taking the infimum over ϵ and then over η.

Finally, we establish that d_{LP} fulfils the identity of indiscernibles. Hence, assume that $d_{LP}(\mu, \nu) = 0$. Then, for any closed set F, one has that

$$\mu[F] \;\leq\; \nu[F_{\frac{1}{n}}] \;+\; \frac{1}{n}\,. \tag{2.30}$$

$F_{\frac{1}{n}}$ is a decreasing sequence of measurable set and $\nu[F_1] < +\infty$. Hence, by continuity of the measure ν,

$$\lim_{n\to+\infty} \nu[F_{\frac{1}{n}}] \;=\; \nu\Big[\cap_{n\in\mathbb{N}} F_{\frac{1}{n}}\Big] \;=\; \nu[F] \tag{2.31}$$

Thence, $\mu[F] \;\leq\; \nu[F]$ and by symmetry, $\mu[F] \;=\; \nu[F]$ and by inner regularity $\mu = \nu$. ■

The second distance of interest in the study of probability measures is the bounded-Lipschitz distance.

Proposition 2.3 *Let*

$$d_{BL}(\mu, \nu) \;=\; \sup\Big\{\Big|\int f \mathrm{d}\mu \;-\; \int g \mathrm{d}\nu\Big| \;:\; ||f||_{BL} \leq 1\Big\} \tag{2.32}$$

where $||\cdot||_{BL}$ *is the bounded-Lipschitz norm*

$$||f||_{BL} \;=\; \sup_{\substack{x\neq y \\ x,y\in S}} \Big|\frac{f(x)-f(y)}{d(x,y)}\Big| \;+\; \sup_{x\in S}\big|f(x)\big|\,. \tag{2.33}$$

Then, d_{BL} *is a distance on* $\mathcal{P}(S)$ *.*

Proof —
It is clear that d_{BL} is symmetric and that is satisfies the triangle inequality. It hence solely remains to prove that $d_{BL}(\mu, \nu) = 0 \;\Rightarrow \mu = \nu$. Let $F \subset S$ be closed. Introduce

$$f_m(x) \;=\; \min\{m \cdot d(x, F)\,,\; 1\}\,. \tag{2.34}$$

Then, since F is closed, $f_m \uparrow \mathbf{1}_U$ where $U = F^{\mathrm{c}}$. Furthermore, $||f||_{BL} \;\leq\; m + 1$. As a consequence

$$\int f_m \cdot \mathrm{d}\mu \;=\; \int f_m \cdot \mathrm{d}\nu\,. \tag{2.35}$$

Hence, by the monotone convergence theorem, $\mu[U] = \nu[U]$. Recall that every finite Borel measure on S is outer regular,

$$\forall B \in \mathcal{B} \qquad \mu[B] \;=\; \inf\{\mu[U] \;:\; U \supset B,\; U \text{ open}\}\,. \tag{2.36}$$

This implies that $\mu = \nu$. ■

Prior to establishing the equivalence of the Levy-Prohorov and bounded-Lipschitz metrics as well as their compatibility with the weak convergence of probability measure we recall Arzela-Ascoli's theorem:

Theorem 2.3 *Any equicontinuous sequence* $f_p : E \to F$ *of continuous functions the compact metric spaces* E, F *admits a convergent subsequence* f_{p_k} *in respect to the sup-norm topology on* $C^0(E, F)$.

Theorem 2.4 *The four statements are equivalent:*

i) $\mu_N \rightharpoonup \mu$ *;*

ii) *for any bounded Lipschitz function* f

$$\int f(s) \cdot \mathrm{d}\mu_N(s) \underset{N\to+\infty}{\longrightarrow} \int f(s) \cdot \mathrm{d}\mu(s) \; ; \tag{2.37}$$

iii) $d_{BL}(\mu_N, \mu) \to 0$ *;*

iv) $d_{LP}(\mu_N, \mu) \to 0$ *;*

Proof —
i) $\Rightarrow$ *ii)* is obvious.
ii) $\Rightarrow$ *iii)* The idea is to approximate, for a given $\epsilon > 0$ *any* bounded Lipschitz function by elements from a *finite* set. There is no chance in doing so on the whole space S. However, Arzela-Ascoli theorem allows one to do so on compacts. Then Ulam's theorem allows one to conclude.

Hence, given $\epsilon > 0$, by Ulam's theorem, there exists $K \Subset S$ such that $\mu[K^{\mathrm{c}}] < \epsilon$. Introducing the function

$$g_\epsilon(x) \; = \; \max\{0, 1 - \frac{1}{\epsilon} d(x, K)\} \tag{2.38}$$

which satisfies $||g_\epsilon||_{BL} \; \leq \; 1 + \epsilon^{-1}$ and $\mathbf{1}_K \leq g_\epsilon \leq \mathbf{1}_{K_\epsilon}$, one gets that

$$\mu_N\Big[K_\epsilon\Big] \; \geq \; \int g_\epsilon(s) \cdot \mathrm{d}\mu_N(s) \; \to \; \int \int g_\epsilon(s) \cdot \mathrm{d}\mu(s) \; \geq \; \mu[K] \; \geq \; 1 - \epsilon \, . \tag{2.39}$$

Therefore, for any $N \geq N_0$ large enough, $\mu_N\Big[K_\epsilon\Big] \; \geq \; 1 - 2\epsilon$. In other words, μ_N is essentially concentrated on K_ϵ provided that N is large enough.

Further set

$$B \; = \; \Big\{ f \; : \; ||f||_{BL} \leq 1 \Big\} \qquad \text{and} \qquad B_K \; = \; \Big\{ f_{|K} \; : \; f \in B \Big\} \, . \tag{2.40}$$

By Arzela-Ascoli's theorem, B_K is compact hence totally bounded in respect to the $L^\infty(K)$ norm. Thus, given $\epsilon > 0$ there exists $f_1, \dots, f_m \in B_K$ such that, for any $f \in B_K$, there exists a $j \in [\![\, 1 \, ; \, m \,]\!]$

$$\sup_{x \in K} \Big| f(x) - f_j(x) \Big| \; < \; \epsilon \, . \tag{2.41}$$

In fact, such a uniform approximation can be extended to K_ϵ for if $x \in K_\epsilon$, pick $y \in K$ such that $d(x, y) < \epsilon$. Then,

$$\begin{aligned} \Big| f(x) - f_j(x) \Big| \; &\leq \; \Big| f(x) - f(y) \Big| \; + \; \Big| f(y) - f_j(y) \Big| \; + \; \Big| f_j(y) - f_j(x) \Big| \\ &\leq \; ||f||_{BL} d(x, y) \; + \; \epsilon \; + \; ||f_j||_{BL} d(x, y) \; \leq \; 3\epsilon . \end{aligned} \tag{2.42}$$

We are now in position to estimate

$$\Big| \int f(s)\mathrm{d}\mu(s) \; - \; \int f(s)\mathrm{d}\mu_N(s) \Big| \; \leq \; \Big| \int_{K_\epsilon} f(s)\mathrm{d}\mu(s) \; - \; \int_{K_\epsilon} f(s)\mathrm{d}\mu_N(s) \Big| \; + \; ||f||_{BL} \cdot \Big(\mu[K_\epsilon^{\mathrm{c}}] \; + \; \mu_N[K_\epsilon^{\mathrm{c}}] \Big)$$

$$\leq \; \Big| \int_{K_\epsilon} f_j(s)\mathrm{d}\mu(s) \; - \; \int_{K_\epsilon} f_j(s)\mathrm{d}\mu_N(s) \Big| + 2 \sup_{x \in K_\epsilon} \big| f(x) - f_j(x) \big| \; + \; 3\epsilon$$

$$\leq \; \Big| \int f_j(s)\mathrm{d}\mu(s) \; - \; \int f_j(s)\mathrm{d}\mu_N(s) \Big| + ||f_j||_{BL} \cdot \Big(\mu[K_\epsilon^{\mathrm{c}}] \; + \; \mu_N[K_\epsilon^{\mathrm{c}}] \Big) \; + \; 9\epsilon$$

$$\leq \; \max_{j \in [\![1 \, ; \, m]\!]} \Big| \int f_j(s)\mathrm{d}\mu(s) \; - \; \int f_j(s)\mathrm{d}\mu_N(s) \Big| \; + \; 12\epsilon \tag{2.43}$$

Then optimizing in respect to $f \in B$ yields,

$$d_{BL}(\mu, \mu_N) \; = \; \sup_{f \in B} \Big| \int f(s)\mathrm{d}\mu(s) - \int f(s)\mathrm{d}\mu_N(s) \Big| \; \leq \; \max_{j \in [\![1 \, ; \, m]\!]} \Big| \int f_j(s)\mathrm{d}\mu(s) - \int f_j(s)\mathrm{d}\mu_N(s) \Big| \; + \; 12\epsilon \tag{2.44}$$

Hence,

$$\limsup_{N \to +\infty} d_{BL}(\mu, \mu_N) \; \leq \; 12\epsilon \; , \tag{2.45}$$

so that $\epsilon \to 0^+$ allows one to conclude.

iii) $\Rightarrow$ *iv*)

For any Borel set $A \in \mathscr{B}$ introduce

$$g_\epsilon(x) \; = \; \max\Big\{ 0, 1 - \frac{1}{\epsilon} d(x, A) \Big\} \tag{2.46}$$

which satisfies $||g_\epsilon||_{BL} \; \leq \; 1 + \epsilon^{-1}$ and $\mathbf{1}_A \leq g_\epsilon \leq \mathbf{1}_{A_\epsilon}$. Then,

$$\mu_N[A] \; \leq \; \int g_\epsilon(s)\mathrm{d}\mu_N(s) \; + \; \int g_\epsilon(s)\mathrm{d}\mu(s) \; + \; (1 + \epsilon^{-1}) \cdot d_{BL}(\mu_N, \mu) \; \leq \mu[A_\epsilon] \; + \; (1 + \epsilon^{-1}) \cdot d_{BL}(\mu_N, \mu) \tag{2.47}$$

so that setting $\delta = \max\{\epsilon, (1 + \epsilon^{-1}) \cdot d_{BL}(\mu_N, \mu)\}$, one gets

$$\mu_N[A] \; \leq \; \mu[A_\delta] \; + \; \delta \; \Rightarrow \; d_{LP}(\mu_N, \mu) \; \leq \; \delta \; . \tag{2.48}$$

One can optimise δ in respect to ϵ. Taking $\epsilon = \sqrt{d_{BL}(\mu_N, \mu)}$ yields,

$$\text{if} \quad d_{BL}(\mu_N, \mu) \; \leq \; 1 \qquad \text{then} \qquad \delta \leq 2\sqrt{d_{BL}(\mu_N, \mu)} \tag{2.49}$$

and if

$$d_{BL}(\mu_N, \mu) \; \geq \; 1 \quad \text{since by definition} \quad d_{LP}(\mu_N, \mu) \leq 1 \; \leq 2\sqrt{d_{BL}(\mu_N, \mu)} \; . \tag{2.50}$$

In other words, the bound $d_{LP}(\mu_N, \mu) \; \leq 2\sqrt{d_{BL}(\mu_N, \mu)}$ always holds.

$iv) \Rightarrow i)$

By definition, if $d_{LP}(\mu_N,\mu) \to 0$ means that there exists a sequence $\epsilon_N \to 0^+$ such that

$$\mu_N[A] \ \leq \ \mu[A_{\epsilon_N}] \ + \ \epsilon_N \qquad \text{for any} \quad A \in \mathcal{B} \ . \tag{2.51}$$

By Portmanteau theorem, it is enough to show that for any set of continuity A of μ one has $\lim_{N\to+\infty}\mu_N[A] = \mu[A]$. Let A be such a set. Then

$$\begin{aligned} \mu_N[A] \ &\leq \ \mu[A_{\epsilon_N}] \ + \ \epsilon_N \ = \ \mu[A] \ + \ \mu[A_{\epsilon_N} \setminus A] \ + \ \epsilon_N & (2.52)\\ \mu_N[A^{\mathrm{c}}] \ &\leq \ \mu[A^{\mathrm{c}}_{\epsilon_N}] \ + \ \epsilon_N \ = \ \mu[A^{\mathrm{c}}] \ + \ \mu[A^{\mathrm{c}}_{\epsilon_N} \setminus A^{\mathrm{c}}] \ + \ \epsilon_N \ . & (2.53)\end{aligned}$$

In other words

$$\begin{aligned} \mu_N[A] \ - \ \mu[A] \ &\leq \ \mu[A_{\epsilon_N} \setminus A] \ + \ \epsilon_N & (2.54)\\ \mu[A] \ - \ \mu_N[A] \ &\leq \ \mu[A^{\mathrm{c}}_{\epsilon_N} \setminus A^{\mathrm{c}}] \ + \ \epsilon_N \ . & (2.55)\end{aligned}$$

Hence,

$$\big|\mu[A] \ - \ \mu_N[A]\big| \ \leq \ \mu[A^{\mathrm{c}}_{\epsilon_N} \setminus A^{\mathrm{c}}] \ + \ \mu[A_{\epsilon_N} \setminus A] \ + \ \epsilon_N \ . \tag{2.56}$$

Since μ is finite as a probability measure on S, by continuity

$$\begin{aligned} \lim_{N\to+\infty} \mu[A^{\mathrm{c}}_{\epsilon_N} \setminus A^{\mathrm{c}}] \ &= \ \mu\Big[\cap_{N\in\mathbb{N}} \{A^{\mathrm{c}}_{\epsilon_N} \setminus A^{\mathrm{c}}\}\Big] \ = \ \mu[\overline{A^{\mathrm{c}}} \setminus A^{\mathrm{c}}] \ \leq \ \mu[\partial A] = 0 & (2.57)\\ \lim_{N\to+\infty} \mu[A_{\epsilon_N} \setminus A] \ &= \ \mu\Big[\cap_{N\in\mathbb{N}} \{A_{\epsilon_N} \setminus A\}\Big] \ = \ \mu[\overline{A} \setminus A] \ \leq \ \mu[\partial A] = 0 \ . \qquad \blacksquare & (2.58)\end{aligned}$$

2.3 Characterization of compact sets in $\mathcal{P}(S)$

Definition 2.5 *A set $\Gamma \subset \mathcal{P}(S)$ is uniformly tight if for any $\epsilon > 0$ there exists $K \Subset S$ such that*

$$\forall\, \mu \ \in \Gamma \qquad \mu[K^{\mathrm{c}}] \ \leq \ \epsilon \ . \tag{2.59}$$

The matter is that every $\mu \in \mathcal{P}(S)$ is tight, *ie* for any $\epsilon > 0$ there exists a compact K such that $\mu[K] \ \geq \ 1 - \epsilon$. However, uniform tightness is a much stronger requirement in that it holds on a whole family of measures. In fact, the main result of this section, Prohorov's theorem, states that uniform tightness and relative compactness are equivalent notions. The former is however, in practice, much easier to verify.

In order to establish the above theorem, we need a few preparatory propositions.

Proposition 2.4 *Let $\mu_N \rightharpoonup \mu$ then $\Gamma \ = \ \{\mu_N \ : \ N \in \mathbb{N}\} \cup \{\mu\}$ is uniformly tight.*

Proof —

Since $\mu_N \rightharpoonup \mu$, one also has that $d_{LP}(\mu_N,\mu) \to 0$. Take $1 > \epsilon > 0$ and let $K \Subset S$ be a compact such that $\mu[S \setminus K] < \epsilon$. It then follows from the definition of the Levy-Prokhorov metric that

$$1 - \epsilon \ < \ \mu[K] \ \leq \ \mu_N\Big[K_{b_N}\Big] \ + \ b_N \qquad \text{with} \qquad b_N \ = \ d_{LP}(\mu_N,\mu) \ + \ \frac{1}{N} \ . \tag{2.60}$$

As a consequence,

$$a_N \ = \ \inf\Big\{\delta > 0 \ : \ \mu_N\Big[K_{b_N}\Big] > 1 - \epsilon\Big\} \quad \to \quad 0 \ . \tag{2.61}$$

The measure μ_N being tight in virtue of Ulam's theorem, one gets that there exists a compact

$$K_N \subset K_{a_N+1/N} \quad \text{such that} \quad \mu_N[K_N] > 1-\epsilon\ . \tag{2.62}$$

Then, set

$$L \;=\; K \cup \Big\{ \cup_{N\geq 1} K_N \Big\}\ . \tag{2.63}$$

By construction, for any $\nu \in \Gamma$, $\nu[S \setminus L] \;\leq\; \epsilon$. So it solely remains to show that L is relatively compact.

Let (x_n) be a sequence in L. There are two options. Either the sequence (x_n) contains a subsequence (x_{γ_n}) contained in K or one of the K_ℓ's. Then, by compactness of the latter, it does contain a convergent subsequence, and the job is done. Else, one can build a subsequence (x_{δ_n}) such that $x_{\delta_n} \in K_{\delta_n}$. However, due to $K_{\delta_n} \subset K_{a_{\delta_n}+1/\delta_n}$, there exists $y_n \in K$ such that

$$d(x_{\delta_n}, y_n) \;\leq\; a_{\delta_n} + 1/\delta_n \underset{n\to+\infty}{\longrightarrow} 0\ . \tag{2.64}$$

Furthermore, since (y_n) is a sequence in K, it admits a convergent subsequence y_{β_n} to some $y \in K$. Hence, it follows from the above bound on $d(x_{\delta_n}, y_n)$ that

$$d(x_{\delta_{\beta_n}}, y) \underset{n\to+\infty}{\longrightarrow} 0\ , \tag{2.65}$$

hence ensuring the compactness of L. ■

We shall admit the below structural result.

Lemma 2.3 *Let (X, d) be a separable metric space. Then, there exists a compact metric space $(Y, \mathfrak{d})$ and a homeomorphism T from X onto $T(X)$.*

This technical result allows one to get Prokhorov's theorem in the non-compact case as soon as its compact version is obtained. Hence, we now establish a selection theorem in the compact case.

Proposition 2.5 *Assume that (S, d) is a compact Polish space, then $(\mathcal{P}(S), d_{LP})$ is compact.*

Proof —

Since S is compact, $\mathscr{C}(S)$ is a Banach space once that it is equipped with

$$||f||_\infty \;=\; \sup_{s\in S} |f(s)|\ . \tag{2.66}$$

Let $\mathscr{C}'(S)$ denote its dual and set

$$\mathcal{G} \;=\; \Big\{ \varphi \in \mathscr{C}'(S) \;:\; ||\varphi|| \leq 1\ ,\ \varphi(1) \;=\; 1\ ,\ \varphi(f) \;\geq\; 0 \;\; \text{for any} \quad f \in \mathscr{C}(S)\ ,\ f \geq 0 \Big\} \tag{2.67}$$

The Riesz representation theorem then states that

$$T \;:\; \mu \to T_\mu \qquad T_\mu(f) \;=\; \int f(s) \cdot \mathrm{d}\mu(s)\ , \tag{2.68}$$

is a bijection from $\mathcal{P}(S)$ onto $\mathcal{G}$ that is, furthermore a sequential homeomorphism in respect to the weak-* topology on $\mathcal{G}$:

$$\mu_N \rightharpoonup \mu \Longrightarrow T_{\mu_N}(f) \underset{N\to+\infty}{\longrightarrow} T_\mu(f) \qquad \text{for any} \;\; f \in \mathscr{C}(S) \tag{2.69}$$

and

$$\varphi_N(f) \rightharpoonup \varphi(f) \quad \text{for any} f \ \in \mathscr{C}(S) \Longrightarrow \qquad T^{-1}(\varphi_N) \underset{N\to+\infty}{\longrightarrow} T^{-1}(\varphi)\,. \tag{2.70}$$

Recall that by Alaoglu's theorem, the set

$$\mathscr{B} \ = \ \left\{\varphi \in \mathscr{C}'(S) \ : \ \|\varphi\| \leq 1\right\} \tag{2.71}$$

is weak-* sequentially compact. Since $\mathcal{G}$ is weak-* sequentially closed in $\mathscr{B}$, $\mathcal{G}$ is also weak-* sequentially compact. Thus $\mathcal{P}(S)$ is compact. ■

We are finally in position so as to establish Prokhorov's theorem.

Theorem 2.5 *(***Prokhorov***)*
The following statements are equivalent

i) A subset $\Gamma \subset \mathcal{P}(S)$ *is uniformly tight ;*

ii) For any sequence (μ_N) *in* Γ *there exists a converging subsequence to a probability measure* $\mu \in \mathcal{P}(S)$*;*

iii) $\overline{\Gamma}$ *is compact in* $\mathcal{P}(S)$ *equipped with the weak convergence of probability measures ;*

iv) Γ *is totally bounded in respect to* d_{LP} *or, equivalently,* d_{BL}*.*

Proof —
i) $\Rightarrow$ *ii*) It follows that any sequence (μ_N) in Γ is uniformly tight.

We first show that $\overline{\Gamma}$ is uniformly tight. For any $1 > \epsilon > 0$ there exists a compact $K \Subset S$ such that

$$\text{for all} \quad \nu \in \Gamma \quad \nu[K] \ \geq \ 1 - \epsilon\,. \tag{2.72}$$

Let $\mu \in \overline{\Gamma}$. Then there exists a sequence (μ_N) in Γ converging to μ. Thence $\mu[K] \geq \limsup_{N\to+\infty} \mu_N[K] \geq 1 - \epsilon$, thus ensuring the uniform tightness of $\overline{\Gamma}$.

Let (μ_N) be a sequence in Γ. Let (Y, δ) be a compact metric space and $T : S \to Y$ a homeomorphism from S onto $T(S)$. Then, since T is continuous, one defines the measures ν_N on $(Y, \mathcal{B}(Y))$ as

$$\nu_N[B] \ = \ \mu_N[T^{-1}(B)]\,. \tag{2.73}$$

Then, (ν_N) is a sequence of probability measures on $\mathcal{P}(Y)$.Furthermore, $\mathcal{P}(Y)$ is compact since Y is compact. Hence, (ν_N) admits a converging subsequence ν_{N_k} converging to a probability measure $\nu \in \mathcal{P}(Y)$.

The whole point now is to translate the measure ν into a measure on S. The first step consists in showing that the mass of ν has not escaped too much out of $T(S)$. It is at this stage that uniform tightness plays a role.

We show that there exists $E \in \mathcal{B}(Y)$, $E \subset T(S)$ such that $\nu[E] = 1$. By uniform tightness of Γ, there exists a sequence of compacts K_ℓ such that

$$\rho[K_\ell] \ \geq 1 - \frac{1}{\ell} \qquad \forall \rho \in \Gamma\,. \tag{2.74}$$

The sets $T(K_\ell)$ are compact in Y, hence closed. Thus

$$\nu[T(K_\ell)] \ \geq \ \limsup_{\ell\to+\infty} \nu_{N_k}[T(K_\ell)] \ = \ \limsup_{\ell\to+\infty} \mu_{N_k}[T(K_\ell)] \ \geq \ 1 - \frac{1}{\ell}\,. \tag{2.75}$$

Thus $E = \sup_{\ell\geq 1} T(K_\ell)$ is a Borel set in Y such that

$$\nu[E] \;\geq\; \nu[K_\ell] \;\geq\; 1 - \frac{1}{\ell} \qquad \text{hence} \qquad \nu[E] \;\geq 1\;. \tag{2.76}$$

We now construct the limiting sequence $\mu \in \mathcal{P}(S)$ of the sequence μ_{N_k}. First, we restrict the measure ν to a measure $\widetilde{\nu}$ on $E \subset T(S)$ by

$$\widetilde{\nu}[A] \;=\; \nu[A \cap E] \qquad \text{for all} \;\; A \in \mathcal{B}(T(S))\;. \tag{2.77}$$

This is a well defined manipulation in that A being Borel in $\mathcal{B}(T(S))$, $A \cap E$ is Borel in E, and thus $A \cap E$ is Borel in Y since E is Borel in Y.

The restricted measure $\widetilde{\nu}$ is a finite Borel measure on $T(S)$ such that $\widetilde{\nu}[E] = \nu[E] = 1$. Set for any $A \in \mathcal{B}$

$$\mu[A] \;=\; \widetilde{\nu}[T(A)] \;=\; T^{-1}\#\widetilde{\nu}[A] \;=\; \widetilde{\nu}[(T^{-1})^{-1}(A)]\;. \tag{2.78}$$

Clearly, $\mu \in \mathcal{P}(S)$. It remains to show that $\mu_{N_k} \rightharpoonup \mu$. Let C be closed in S. Then $T(C)$ is closed in $T(S)$. Thus, there exists $Z \subset Y$ closed such that $T(C) \;=\; T(S) \cap Z$. Furthermore $T^{-1}(Z) \;=\; C$ since there are no points in $T(C)$ outside of $T(S)$. Furthermore, $Z \cap E \;=\; T(C) \cap E$. Thus,

$$\limsup_{k\to+\infty} \mu_{N_k}[C] \;=\; \limsup_{k\to+\infty} \nu_{N_k}[Z] \;\leq\; \nu[Z] \;=\; \nu[Z \cap E] \;+\; \nu[Z \cap E^{\mathrm{c}}] \;=\; \nu[T(C) \cap E] \;=\; \mu[C]\;. \tag{2.79}$$

Hence, by Portmanteau's theorem, $\mu_{N_k} \rightharpoonup \mu$.

ii) $\Rightarrow$ *iii*) Every sequence in $\overline{\Gamma}$ can be approximated by a sequence in A. By hypothesis, this new sequence has a converging subsequence to some element $\mu \in \overline{\Gamma}$.
iii) $\Rightarrow$ *iv*) Any compact set $\overline{\Gamma}$ is totally bounded, hence ensuring that Γ is totally bounded.
iv) $\Rightarrow$ *i*) Since $d_{LP} \;\leq\; 2\sqrt{d_{BL}}$, it is enough to deal with d_{LP}.

Since Γ is totally bounded, for any $\epsilon > 0$ there exists a finite subset B such that $\Gamma \subset B_\epsilon$. Furthermore, by Ulam's theorem, for any $\mu \in \Gamma$, there exists $K^{(\mu)} \Subset S$ such that

$$\mu[K^{(\mu)}] \;\geq\; 1 - \epsilon \tag{2.80}$$

Thence,

$$K_B \;=\; \cup_{\mu\in B} K^{(\mu)} \;\;\Subset\;\; S \qquad \text{and} \qquad \mu[K_B] \;\geq\; 1 - \epsilon\;. \tag{2.81}$$

Given any $\epsilon > 0$, take F a finite set such that $K_B \subset F_\epsilon$. Since $\Gamma \subset B_\epsilon$, for any $\nu \in \Gamma$ there exists $\mu \in B$ such that

$$d_{LP}(\mu, \nu) \;\leq\; \epsilon \qquad \Rightarrow \qquad 1 - \epsilon \;\leq\; \mu[K_B] \;\leq\; \mu[F_\epsilon] \;\leq\; \nu[F_{2\epsilon}] \;+\; \epsilon \quad \text{hence} \quad \nu[F_{2\epsilon}] \;\geq\; 1 - 2\epsilon\;. \tag{2.82}$$

Now, take $\delta > 0$ and take $\epsilon_\ell = \delta \cdot 2^{-\ell-1}$ above, hence giving rise to finite sets F_ℓ such that

$$1 - \frac{\delta}{2^\ell} \;\leq\; \nu\Big[(F_m)_{\frac{\delta}{2^\ell}}\Big] \qquad \Rightarrow \qquad \nu\Big[\cap_{\ell\geq 1} (F_\ell)_{\frac{\delta}{2^\ell}}\Big] \;\geq\; 1 - \sum_{\ell\geq 1} \nu\Big[(F_\ell)^{\mathrm{c}}_{\frac{\delta}{2^\ell}}\Big] \;\geq\; 1 - \sum_{\ell\geq 1} \frac{\delta}{2^\ell} \;=\; 1 - \delta\;. \tag{2.83}$$

Finally, given any finite set F introduce

$$U_\eta[F] \;=\; \cup_{x\in F} \overline{B}(x, \eta) \tag{2.84}$$

a finite union of balls of radius η. Then, in particular,

$$(F_\ell)_{\frac{\delta}{2^\ell}} \;\subset\; U_{\frac{\delta}{2^\ell}}[F_\ell]\;. \tag{2.85}$$

It follows that the set

$$L \;=\; \cap_{\ell\geq 1}(F_\ell)_{\frac{\delta}{2^\ell}} \tag{2.86}$$

is closed and totally bounded. Since S is complete, L is compact, hence ensuring the uniform tightness of Γ. ■

Prokhorov's theorem ensures, in particular, the completeness of $\mathcal{P}(S)$.

Corollary 2.1 *A Cauchy sequence in $(\mathcal{P}(S), d_{LP})$ is convergent.*

Proof —
Let (μ_N) be a Cauchy sequence in $\mathcal{P}(S)$. Then it is $\Gamma = \{\mu_N \;:\; N \in \mathbb{N}\}$ is totally bounded and thus, by Prokhrov's theorem, (μ_N) admits a convergent subsequence. Being a Cauchy sequence, it thus converges. ■

2.4 Separable character of $\mathcal{P}(S)$

Proposition 2.6 *The space $(\mathcal{P}(S), d_{LP})$ is separable.*

Proof —
Let $\mathcal{S} = \{x_k\}_{k\in\mathbb{N}}$ be a countable dense set in S. Then the set

$$\Gamma \;=\; \Big\{a_1\delta_{x_1} \;+\; \ldots \;+\; a_p\delta_{x_p} \;:\; a_k \in \mathbb{Q}\cap[\,0\,;1\,]\,,\; \sum_{k=1}^{p} a_k \;=\; 1\,,\; p\in\mathbb{N}\Big\} \tag{2.87}$$

is countable. It remains to show that it is dense.
Let $\mu \in \mathcal{P}(S)$. Given any $\ell \geq 1$, one has that

$$\bigcup_{k\geq 1} B(x_j, \frac{1}{\ell}) \;=\; S \qquad \text{thus there exists} \quad k_\ell \;:\; \mu\Big[\bigcup_{k\geq 1}^{k_\ell} B(x_j, \frac{1}{\ell})\Big] \;\geq\; 1 - \frac{1}{\ell}\;. \tag{2.88}$$

Decompose

$$\bigcup_{k\geq 1}^{k_\ell} B(a_j, \frac{1}{\ell}) \;=\; \bigcup_{k\geq 1}^{k_\ell} A_{k;\ell} \tag{2.89}$$

into a union of disjoint sets

$$A_{1;m} \;=\; B(x_1, \frac{1}{\ell}) \quad \ldots \quad A_{j;m} \;=\; B(x_j, \frac{1}{\ell}) \setminus \cup_{p=1}^{j-1} B(x_j, \frac{1}{\ell})\;. \tag{2.90}$$

Then, by construction,

$$\mu\Big[\cup_{k\geq 1}^{k_\ell} A_{k;\ell}\Big] \;=\; \sum_{k=1}^{k_\ell} \mu[A_{k;\ell}] \;\geq\; 1 \;-\; \frac{1}{\ell}\;. \tag{2.91}$$

It thus appears that a good approximate to μ would be the measure

$$\mu[A_{1;\ell}]\cdot\delta_{x_1} \;+\; \ldots \;+\; \mu[A_{k_\ell;\ell}]\cdot\delta_{x_{k_\ell}} \tag{2.92}$$

Yet, in order to deal with a measure belonging to Γ, one still has to slightly modify the coordinates so as to deal with rational ones. Thus, pick

$$a_{j;\ell}\in\mathbb{Q}\cap[\,0\,;1\,]\;:\;\sum_{j=1}^{k_\ell}a_{j;\ell} \;=\; 1 \quad \sum_{j=1}^{k_\ell}\big|\mu[A_{j;\ell}]\,-\,a_{j;\ell}\big| \;\leq\; \frac{2}{\ell} \tag{2.93}$$

The construction of such a sequence is left in exercise. Then, set

$$\mu_\ell \;=\; \sum_{j=1}^{k_\ell}a_{j;\ell}\cdot\delta_{x_j}\;. \tag{2.94}$$

Let g be bounded Lipschitz on S. Then,

$$\begin{aligned}\Big|\int g(s)\cdot\mathrm{d}\mu_\ell(s)\;-\;\int g(s)\cdot\mathrm{d}\mu(s)\Big| \;&\leq\; \sum_{j=1}^{k_\ell}\Big|a_{j;\ell}g(x_j)\;-\;\int_{A_{j;\ell}}g(s)\cdot\mathrm{d}\mu(s)\Big| \;+\; \int_{S\setminus\cup_{j=1}^{k_\ell}A_{j;\ell}}|g(s)|\cdot\mathrm{d}\mu(s)\\
\leq\; \|g\|_\infty\sum_{j=1}^{k_\ell}\big|\mu[A_{j;\ell}]\,-\,a_{j;\ell}\big| \;+\; &+\;\sum_{j=1}^{k_\ell}\Big|\int_{A_{j;\ell}}[g(x_j)\,-\,g(s)]\cdot\mathrm{d}\mu(s)\Big|\|g\|_\infty\mu\big[S\setminus\cup_{j=1}^{k_\ell}A_{j;\ell}\big]\\
&\leq\; \frac{2}{\ell}\|g\|_{BL}\;+\;\|g\|_{BL}\cdot\frac{1}{\ell}\;+\;\|g\|_{BL}\cdot\frac{1}{\ell}\;.\end{aligned} \tag{2.95}$$

As a consequence, $d_{BL}(\mu_\ell,\mu)\to 0$ in the $\ell\to+\infty$ limit. ■

We are thus finally in position to establish the following "lifting" of Polish space structure theorem

Theorem 2.6 *Let (S,d) be a Polish space. Then $(\mathcal{P}(S),d_{BL})$ is a Polish space.*

In particular, it follows from the above theorem that $(\mathcal{P}(\mathbb{R}),d_{BL})$ is a Polish space. This last result will be, in fact, used in full extend in the next section.

2.5 The large deviation principle

2.5.1 First definitions and basic properties

Definition 2.6 *A function $f:S\to\mathbb{R}$ is said to be lower semi-continuous (lsc) iff its level set $f^{-1}(]-\infty\,;c\,])$ are closed for all $c\in\mathbb{R}$ or, equivalently, $\liminf_{x_n\to x}f(x_n)\geq f(x)$.*

Lemma 2.4 *Let $\{f_I\}_{I\in\mathcal{I}}$ be a collection of lower semi-continuous functions. Then $f(x)=\sup_{I\in\mathcal{I}}[\,f_I(x)]$ is also lsc. Furthermore, if f is lsc then f attains its minimum on any $K\Subset S$, $K\neq\emptyset$.*

Proof —

Let $c>0$. Then, by definition, the sets

$$F_{I;c} \;=\; \{x\in S \;:\; f_I(x)\;\leq\;c\} \tag{2.96}$$

are closed. Let $F_c = f^{-1}(]-\infty\,; c\,])$. If $x \in F$ then, for all $I \in \mathcal{I}$ $f_I(x) \leq c$, *ie* $x \in F_{I;c}$ for all $I \in \mathcal{I}$. Reciprocally, if $f_I(x) \leq c$ then $\sup_{I \in \mathcal{I}} [\, f_I(x)] \leq c$, *ie* $x \in F_c$. Thus,

$$F_c = \cap_{I \in \mathcal{I}} F_{I;c} \tag{2.97}$$

and so is closed.

Let K be an non-empty compact of S. Then, let $I = \inf_{x \in K} f(x)$. For each $\lambda > I$, the sets

$$F_\lambda = \{x \in K \;:\; f(x) \;\leq\; \lambda\} \tag{2.98}$$

are closed by lower semi-continuity of f. Since F_λ is a decreasing non-empty sequence of closed sets of the compact set K,

$$F = \cap_\lambda F_\lambda \tag{2.99}$$

is not empty. However, if $y \in F$, then $f(y) \leq I$. Since, by definition, $f(y) \geq I$, we get $f(y) = \inf_{x \in K} f(x)$. ∎

Definition 2.7 *A sequence of probability measures $\mu_N \in \mathcal{P}(S)$ is said to satisfy a large deviation principle with speed a_N, $a_N \to +\infty$ and rate function J iff*

$$J : S \to [\,0\,; +\infty\,] \text{ is lsc} \tag{2.100}$$

$$\text{for any } F \subset S \text{ that is closed} \quad \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[F] \;\leq\; -\inf_F J \tag{2.101}$$

$$\text{for any } O \subset S \text{ that is open} \quad \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[O] \;\geq\; -\inf_O J \tag{2.102}$$

J is said to be a good rate function if J is a rate function and has compact level sets.

At this stage, it appears appropriate to make some general remarks and observations about the very formulation of a LDP.

- The first observation one can make is that the LDP is well ordered. Indeed, let B be a Borel-measurable set. Then one has

$$\liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[\mathring{B}] \;\leq\; \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B] \;\leq\; \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B] \;\leq\; \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[\overline{B}]\,, \tag{2.103}$$

  so that the way of ordering the limits in (2.101)-(2.102) does indeed make sense.

- In fact, the role played by the open and closed sets in the LDP is quite similar to the role played by open and closed sets in the weak convergence of measures:

$$\mu_N \rightharpoonup \mu \quad \Leftrightarrow \quad \forall F \subset S \;\; \text{closed} \quad \limsup_N \mu_N[F] \leq \mu[F] \quad \Leftrightarrow \quad \forall O \subset S \;\; \text{open} \quad \liminf_N \mu_N[O] \geq \mu[O]\,. \tag{2.104}$$

- The upper bound in the LDP (2.101) says that the mass of "large sets" (since for any Borel measurable set B, the set $\overline{B} \supset B$ is closed) is not too big whereas the lower-bound (2.102) says that the mass of "small sets" (since for any Borel measurable set B, the set $\mathring{B} \subset B$ is open) is not too small.

- The lsc requirement can always be met, in the sense that should a LDP be formulated with a rate function that is not lsc, then one can always construct a new rate function that will be lsc and drive a LDP that is equivalent to the initial one. Furthermore, the lsc requirement is also an optimal one in that it allows one to consider situations which cannot be reached by the sole use of continuous rate functions. Indeed, a continuous rate function would not allow to make a distinction between closed and open sets. Indeed, assume that B is Borel measurable and such that $\overline{\overset{\circ}{B}} = \overline{B}$. Then, by continuity

$$\inf_{x \in \overset{\circ}{B}} J(x) \;=\; \inf_{x \in \overline{B}} J(x) \;. \tag{2.105}$$

  In other words, continuous rate function would not allow one to control the subtle effect that could happen on a wide class of Borel sets. The need for such a control is typical in applications of LDP's to problem related with random matrix theory issued integrals.

  Nonetheless, it is quite possible that a give rate function does admit so-called J-continuous Borel set B, *ie* a set such that $\inf_{x \in \overset{\circ}{B}} J(x) \;=\; \inf_{x \in \overline{B}} J(x)$. In fact, for these sets the superior and inferior limits coincide, *viz* the limit itself exists.

- It could be tempting to alter the formulation of the LDP so as to only deal with limits and not superior and inferior limit, *ie* demand that for any Borel measurable set B

$$\lim_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B] \;=\; -\inf_{B} J \;. \tag{2.106}$$

  In many practical situations, such a formulation is simply useless as imposing too much important restriction. Indeed, suppose that the sequence μ_N has no-atoms for every N. Then taking $B = \{x\}$ would imply that the only possibility if to take $J(x) = +\infty$, this for any $x \in S$.

- Since S is closed and $\mu_N[S] = 1$, the bound (2.101) implies that $-\inf_S J \geq 0$, *ie* $\inf_S J \;=\; 0$. In particular, if J is a good rate function, then there exists an $x \in S$ such that $J(x) = 0$.

So as to summarize, the formulation of a LDP in terms of "weaker" limits provides one with a setting that is sufficiently "relaxed" so as to be able to hold in numerous interesting situations while still providing numerous informations of the sequence of probability measures being studied. In fact, the main purpose of a LDP is to answer the question of where the mass of the sequence of probability measures becomes concentrated in the large N limit. In other words, the LDP provides one with tools that allow to measure how events -represented by open and closed set of the space S on which the sequence of probability measures is defined- become "exponentially improbable" as soon as one moves away from the sets $J^{-1}(0)$ of "highest" probability. More precisely, assume that the rate function J admits its minimum at a unique point x. Let $\epsilon > 0$ and set $F = S \setminus B(x, \epsilon)$. Then, by the inequality for the superior limit, for any $1 > \eta > 0$ there exists N large enough such that

$$a_N^{-1} \cdot \ln \mu_N[F] \;\leq\; -\inf_F J \;+\; \eta \inf_F J \quad \Longrightarrow \quad \mu_N[F] \;\leq\; \mathrm{e}^{-a_N(1-\eta)\inf_F J} \;. \tag{2.107}$$

Thus, the mass of all points at uniformly in N finite distance to the minimum of J is exponentially small in the large N limit.

It is natural to wonder whether there could exist several rate functions for a given sequence of probability measures on S.

Lemma 2.5 *The rate function J associated with a LDP is unique.*

Proof —

Assume that a given sequence of probability measures admits two different rate functions J and H. Then there exists $x \in S$ such that, say, $J(x) > H(x)$. By lower semi-continuity, there exists an open neighbourhood O of x such that $\inf_{x\in\overline{O}}[J(x)] > H(x)$. Thus

$$-H(x) \leq -\inf_{y\in O}[H(y)] \leq \liminf_{N\to+\infty} \frac{1}{a_N} \ln \mu_N[O] \leq \limsup_{N\to+\infty} \frac{1}{a_N} \ln \mu_N[\overline{O}] \leq -\inf_{x\in\overline{O}}[J(x)] \leq -H(x) \,, \quad (2.108)$$

a contradiction. ■

In practice, in order to establish that a sequence $\mu_N \in \mathcal{P}(S)$ satisfies a LDP, it is often easier to establish first, a weaker version of the LDP and then some tightness property of the sequence of measures that is being studied. The two will then imply the full LDP.

Definition 2.8 *A sequence $\mu_N \in \mathcal{P}(S)$ is said to satisfy a weak LDP if it satisfies* (2.100), (2.101) *and* (2.102) *but with closed sets replaced by compact ones.*

Definition 2.9 *A sequence of probability measures $\mu_N \in \mathcal{P}(S)$ is said to be exponentially tight if there exists a sequence of compacts $K_L \Subset S$ such that*

$$\limsup_{L\to+\infty} \limsup_{N\to+\infty} \frac{1}{a_N} \ln \mu_N[K_L^{\mathrm{c}}] = -\infty \,. \quad (2.109)$$

These two properties do imply the LDP

Proposition 2.7 *Let $\mu_N \in \mathcal{P}(S)$ be an exponentially tight sequence that satisfies a weak LDP with rate function J and speed a_N. Then J is a good rate function and μ_N satisfies the LDP with rate function J and speed a_N.*

Proof —

By hypothesis, given any $\widetilde{L} > 0$ there exists $K_L \Subset S$ such that

$$\limsup_{N\to+\infty} \frac{1}{a_N} \ln \mu_N[K_L^{\mathrm{c}}] \leq -\widetilde{L} \quad (2.110)$$

Thus, given $F \subset S$ closed, one has

$$\mu_N[F] = \mu_N[F \cap K_L] + \mu_N[F \cap K_L^{\mathrm{c}}] \leq \mu_N[F \cap K_L] + \mu_N[K_L^{\mathrm{c}}] \,. \quad (2.111)$$

Hence,

$$\begin{aligned} \limsup_{N\to+\infty} \frac{1}{a_N} \ln \mu_N[F] &\leq \limsup_{N\to+\infty} \frac{1}{a_N} \ln \Big[2 \cdot \max\big\{\mu_N[F \cap K_L], \mu_N[K_L^{\mathrm{c}}]\big\}\Big] \\ &\leq \max\Big\{-\widetilde{L}, -\inf_{F\cap K_L} J\Big\} \leq \max\Big\{-\widetilde{L}, -\inf_{F} J\Big\} \end{aligned} \quad (2.112)$$

and the result follows by sending $L \to +\infty$.

It remains to establish that the rate function is good. By hypothesis, for any $\widetilde{L}$ there exists a compact $K_L > 0$ such that

$$\limsup_{N\to+\infty} \frac{1}{a_N} \ln \mu_N[K_L^{\mathrm{c}}] \leq -\widetilde{L} \quad (2.113)$$

Yet, since K_L^c is open, by the lower bound in the LDP,

$$- \inf_{x \in K_L^c} J(x) \;\le\; \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[K_L^c] \,, \tag{2.114}$$

we get that

$$\inf_{x \in K_L^c} J(x) \;\ge\; \widetilde{L} \quad \Rightarrow \quad \text{for any } c < \widetilde{L} \quad J^{-1}(]-\infty\,;c\,]) \subset K_L \,, \tag{2.115}$$

and thus the level sets corresponding to $c < \widetilde{L}$ are compact. Since $\widetilde{L}$ is arbitrary, the result follows. ■

Finally, in order to establish that a sequence $\mu_N \in \mathcal{P}(S)$ satisfies a weak LDP one usually establishes a technically easier to obtain result that is, however, equivalent to a weak LDP.

Proposition 2.8 *Assume that there exists a lsc function J such that for all $x \in S$*

$$-J(x) \;\ge\; \limsup_{\epsilon \to 0} \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B(x,\epsilon)] \qquad \text{and} \qquad -J(x) \;\le\; \liminf_{\epsilon \to 0} \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B(x,\epsilon)] \,. \tag{2.116}$$

Then μ_N satisfies a weak LDP with rate function J.

Proof —
Let O be open in S. Then, for any $\mu \in O$ there exists $\delta_\mu > 0$ such that $B(\mu,\eta) \subset O$ for all $\eta \in [\,0\,;\delta_\mu\,]$. Then, given any $\eta \in [\,0\,;\delta_\mu\,]$,

$$\mu_N[O] \;\ge\; \mu_N[B(\mu,\eta)] \Rightarrow \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[G] \;\ge\; \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B(\mu,\eta)] \tag{2.117}$$

so that, by taking $\liminf_{\eta \to 0}$ of both sides of the inequality, we get that

$$\liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[O] \;\ge\; -J(x) \,. \tag{2.118}$$

Then, optimizing over $x \in G$ we get (2.102). Further, let $K \Subset S$. Since,

$$-J(x) \;=\; \limsup_{\epsilon \to 0} \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B(x,\epsilon)] \tag{2.119}$$

we get that for any $\eta > 0$ there exists $\delta_x > 0$ such that

$$\limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B(x,\delta_x)] \;\le\; -J(x) + \eta \;\le -J_\eta(x) = -\min\{J(x) - \eta, \eta^{-1}\} \,. \tag{2.120}$$

$\cup_{x \in K} B(x,\delta_x)$ is an open cover of K and thus admits a finite subcover $\cup_{k=1}^m B(x_k,\delta_{x_k}) \supset K$. Thence,

$$\begin{aligned} \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[K] \;&\le\; \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[\,\cup_{k=1}^m B(x_k,\delta_{x_k})] \\ &\le\; \limsup_{N \to +\infty} \frac{1}{a_N} \ln \Big[m \cdot \max_{k \in [\![\,1\,;\,m\,]\!]} \mu_N[B(x_k,\delta_{x_k})] \Big] \;\le\; \max_{p \in [\![\,1\,;\,m\,]\!]} \{-I_\eta(x_p)\} \;\le\; -\inf_{x \in K} J_\eta(x) \end{aligned} \tag{2.121}$$

The LDP on compact sets then follows by sending $\eta \to 0^+$. ■

3 The large-N analysis of regular multiple integrals

3.0.2 A first example

We are now going to study an extremely simple example of a multiple integral. This will allow us to illustrate how the previously introduced formalism can be applied to the study of the large-N behaviour of multiple integrals and, also, how, in practice, one can establish that a sequence μ_N of probability measures on $\mathbb{R}$ does satisfy a LDP. Our model example will consist of the integral

$$\Big(\int_{\mathbb{R}} \mathrm{e}^{-NV(\lambda)} \cdot \mathrm{d}\lambda \Big)^N \tag{3.1}$$

Clearly, all information on its large N behaviour can be deduced from the Laplace principle for 1-dimensional integrals. We, however, shall take another route based on LDP. One can, in fact, interpret the integral (3.1) as the multiple integral

$$\int_{\mathbb{R}^N} \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)} \cdot \mathrm{d}^N\lambda \; = \; (I[V])^N \qquad \text{with} \qquad I[V] \; = \; \int_{\mathbb{R}} \mathrm{e}^{-NV(\lambda)} \cdot \mathrm{d}\lambda \; . \tag{3.2}$$

Then $(I[V])^N$ appears as the normalization constant for the probabiliy measure

$$\mathrm{d}\mathbb{P}_N(\boldsymbol{\lambda}_N) \; = \; \frac{1}{(I[V])^N} \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)} \cdot \mathrm{d}^N\lambda \qquad \text{on} \qquad \mathbb{R}^N \; . \tag{3.3}$$

In the following, we shall assume that the potential V is such that

- $V \in C^1(\mathbb{R})$;
- $V(x) \geq v_1|x| + v_2$ for some constants $(v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}$;
- $\exists \delta > 0$, $\sup_{x\in\mathbb{R}} \sup_{|y|\leq\delta} \Big| \dfrac{V'(x+y)}{V(x)} \Big| < +\infty$.

The law $\mathbb{P}_N$ allows one to define a random variable, the empirical measure, on $\mathcal{P}(\mathbb{R})$:

$$L_N^{(\boldsymbol{\lambda}_N)} \; = \; \frac{1}{N} \sum_{a=1}^{N} \delta_{\lambda_a} \; \in \; \mathcal{P}(\mathbb{R}) \qquad \text{where} \qquad \boldsymbol{\lambda}_N = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N \; . \tag{3.4}$$

In fact, $\mathbb{P}_N$ induces a probability measure μ_N on $\mathcal{P}(\mathbb{R})$ through $\mu_N \; = \; L_N^{(\boldsymbol{\lambda}_N)} \# \mathbb{P}_N$, *ie* for any Borel set $B \subset \mathcal{P}(\mathbb{R})$,

$$\mu_N[B] \; = \; P_N[\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \; : \; L_N^{(\boldsymbol{\lambda}_N)} \in B\}] \; . \tag{3.5}$$

Under the above assumptions, we are going to prove the

Proposition 3.1 *The sequence μ_N satisfies a LDP with speed N^2 and good rate function*

$$J[\mu] \; = \; \int_{\mathbb{R}} V(x) \cdot \mathrm{d}\mu(x) \; - \; \inf_{s\in\mathbb{R}} [V(s)] \; . \tag{3.6}$$

Furthermore,

$$\lim_{N\to+\infty} \frac{1}{N^2} \ln\big[(I[V])^N\big] \; = \; - \inf_{\mu\in\mathcal{P}(\mathbb{R})} \overline{J}[\mu] \; = \; - \inf_{s\in\mathbb{R}} [V(s)] \qquad \text{with} \qquad \overline{J}[\mu] \; = \; \int_{\mathbb{R}} V(x) \cdot \mathrm{d}\mu(x) \tag{3.7}$$

Proof —

• Exponential tightness

In order to establish the exponential tightness of the sequence μ_N the idea consists in tailoring a sufficiently nice sequence of compact sets $K_L \subset \mathcal{P}(\mathbb{R})$ such that obtaining the sought upper bound for $\mu_N[K_L]$ is relatively straightforward. Since the potential V naturally arises in the density of probability measure $\mathrm{d}\mathbb{P}_N(\boldsymbol{\lambda}_N)$, it seems natural to build the sequence K_L on the basis of functionals involving the potential V.

Let $v = \inf_{s\in\mathbb{R}} [V(s)]$ and define

$$K_L = \Big\{\mu \in \mathcal{P}(\mathbb{R}) \ : \ \int_{\mathbb{R}} [V(x) - v] \cdot \mathrm{d}\mu(x) \leq L\Big\} \,. \tag{3.8}$$

Since $V - v \geq 0$, by the monotone convergence theorem,

$$\int_{\mathbb{R}} [V(x) - v] \cdot \mathrm{d}\mu(x) = \sup_{M\in\mathbb{N}} \int_{\mathbb{R}} \min\Big\{V(x) - v, M\Big\} \cdot \mathrm{d}\mu(x) \tag{3.9}$$

we get that the *lhs* is lower semi-continuous as a supremum of a continous familiy of functions on $\mathcal{P}(\mathbb{R})$. Thus, K_L is closed as a level set of a lower semi-continuous function.

For any $\mu \in K_L$ one has

$$\begin{aligned} \mu\,[\mathbb{R} \setminus [\,-M\,;M\,]] \ &\leq\ \frac{1}{v_1 \cdot M + v_2 - v} \int_{\mathbb{R}\setminus[\,-M\,;M\,]} [V(x) - v] \cdot \mathrm{d}\mu(x) \\ &\leq\ \frac{1}{v_1 \cdot M + v_2 - v} \int_{\mathbb{R}} [V(x) - v] \cdot \mathrm{d}\mu(x) \ \leq\ \frac{L}{v_1 \cdot M + v_2 - v} \,. \end{aligned} \tag{3.10}$$

As a consequence,

$$K_L \subset \bigcap_{M\in\mathbb{N}} \Big\{\mu \in \mathcal{P}(\mathbb{R}) \ : \ \mu\,[\mathbb{R} \setminus [\,-M\,;M\,]] \leq \frac{L}{v_1 M + v_2 - v}\Big\} \ = \ \mathcal{K} \,. \tag{3.11}$$

Note that

$$\mathcal{F}_M : \mu \mapsto \int_{\mathbb{R}\setminus[\,-M\,;M\,]} \mathrm{d}\mu(s) \tag{3.12}$$

is lsc by the previous reasoning. Hence $\mathcal{K}$ is closed as an intersection of level sets of lower semi-continuous functions on $\mathcal{P}(\mathbb{R})$. Furthermore, $\mathcal{K}$ is uniformly tight. Indeed, given $\epsilon > 0$ there exists $M > 0$ such that

$$\frac{L}{v_1 M + v_2 - v} \ < \ \epsilon \,. \tag{3.13}$$

Thus for any $\mu \in \mathcal{K}$, $\mu[[\,-M\,;M\,]^{\mathrm{c}}] < \epsilon$.
Thence, by Prohorov's theorem, $\mathcal{K}$ is compact. As K_L is closed, it follows that it is compact.

Note that, this proof also ensures that J is a good rate function, *ie* the level sets of J are compact.

Having found a proper sequence of growing compacts, we now establish the exponential tightness of the sequence μ_N. For this, we estimate

$$\mu_N\left[K_L^{\mathrm{c}}\right] \equiv \mathbb{P}_N\Big[\Big\{\boldsymbol{\lambda}_N \equiv (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \ : \ L_N^{(\boldsymbol{\lambda}_N)} \in K_L^{\mathrm{c}}\Big\}\Big] \,. \tag{3.14}$$

In order to bound it from above, we need a lower bound for the normalization constant and, as well, further bound the integral part.

One has that

$$I[V] \; = \; \int\limits_{\mathbb{R}} \mathrm{e}^{-NV(\lambda)} \cdot \mathrm{d}\lambda \; \geq \; \mathrm{e}^{-(N-1)v} \; \cdot \; \int\limits_{\mathbb{R}} \mathrm{e}^{-V(\lambda)} \cdot \mathrm{d}\lambda \tag{3.15}$$

Thus,

$$\begin{aligned}
\mu_N[K_L^c] \; &\leq \; \frac{\mathrm{e}^{(N-1)Nv}}{\Big(\int\limits_{\mathbb{R}} \mathrm{e}^{-V(\lambda)} \mathrm{d}\lambda\Big)^N} \int\limits_{L_N^{(\lambda_N)} \in K_L^c} \mathrm{e}^{-\frac{N^2}{2} \int V(s) \mathrm{d}L_N^{(\lambda_N)}(s)} \cdot \prod_{a=1}^{N} \mathrm{e}^{-\frac{N}{2} V(\lambda_a)} \cdot \mathrm{d}^N \lambda \\
&\leq \; \frac{\mathrm{e}^{(N-1)Nv}}{\Big(\int\limits_{\mathbb{R}} \mathrm{e}^{-V(\lambda)} \mathrm{d}\lambda\Big)^N} \cdot \mathrm{e}^{-\frac{N^2}{2} v} \int\limits_{L_N^{(\lambda_N)} \in K_L^c} \underbrace{\mathrm{e}^{-\frac{N^2}{2} \int \big(V(s) - v\big) \cdot \mathrm{d}L_N^{(\lambda_N)}(s)}}_{\geq \, \exp\left[-N^2 L/2\right]} \cdot \prod_{a=1}^{N} \mathrm{e}^{-\frac{N}{2}\big(v_1 |\lambda_a| + v_2\big)} \cdot \mathrm{d}^N \lambda \\
&\leq \; \frac{\mathrm{e}^{(\frac{N}{2}-1)Nv}}{\Big(\int\limits_{\mathbb{R}} \mathrm{e}^{-V(\lambda)} \mathrm{d}\lambda\Big)^N} \cdot \mathrm{e}^{-\frac{N^2}{2} L} \cdot 2^N \frac{\mathrm{e}^{-\frac{N^2 v_2}{2}}}{v_1^N N^N} \; .
\end{aligned} \tag{3.16}$$

As a result,

$$\limsup_{N \to +\infty} \frac{1}{N^2} \ln \mu_N[K_L^c] \; \leq \; \frac{v - v_2}{2} \; - \; \frac{L}{2} \; , \tag{3.17}$$

so that taking $\limsup_{L \to +\infty}$ yields the exponential tightness of the sequence μ_N.

• A technical simplification

In the following we are going to prove estimates for shrinking balls. However, we shall not do it directly for the sequence μ_N but rather for $\overline{\mu}_N = \mu_N \cdot (I[V])^N$:

$$-\overline{J}[\mu] \; \geq \; \limsup_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{a_N} \ln \overline{\mu}_N[B(\mu, \delta)] \qquad \text{and} \qquad -\overline{J}[\mu] \; \leq \; \liminf_{\delta \to 0} \liminf_{N \to +\infty} \frac{1}{a_N} \ln \overline{\mu}_N[B(\mu, \delta)] \tag{3.18}$$

where

$$\overline{J}[\mu] \; = \; \int\limits_{\mathbb{R}} V(s) \cdot \mathrm{d}\mu(s) \; . \tag{3.19}$$

In such a way, by repeating the reasoning outlined in the proof of propositions 2.7 and 2.8, we are going to obtain similar bounds for all closed -upper bound- and open -lower bound- sets, namely for all open subsets O of $\mathcal{P}(\mathbb{R})$ and all closed subsets F of $\mathcal{P}(\mathbb{R})$

$$-\inf_{\mu \in F} \Big\{\overline{J}[\mu]\Big\} \; \geq \; \limsup_{N \to +\infty} \frac{1}{N^2} \ln \overline{\mu}_N[F] \qquad \text{and} \qquad \liminf_{N \to +\infty} \frac{1}{N^2} \ln \overline{\mu}_N[O] \; \geq \; -\inf_{\mu \in O} \Big\{\overline{J}[\mu]\Big\} \tag{3.20}$$

Then, by taking $O = F = \mathcal{P}(\mathbb{R})$, since $\mu_N[\mathcal{P}(\mathbb{R})] = 1$, one will get

$$- \inf_{\mu \in \mathcal{P}(\mathbb{R})} \Big\{ \overline{J}[\mu] \Big\} \ \leq \ \liminf_{N \to +\infty} \frac{1}{N^2} \ln \overline{\mu}_N[\mathcal{P}(\mathbb{R})] \ \leq \ \limsup_{N \to +\infty} \frac{1}{N^2} \ln \overline{\mu}_N[\mathcal{P}(\mathbb{R})] \ \leq \ - \inf_{\mu \in \mathcal{P}(\mathbb{R})} \Big\{ \overline{J}[\mu] \Big\} \tag{3.21}$$

thus since $\mu_N[\mathcal{P}(\mathbb{R})] = 1$ yielding

$$- \inf_{\mu \in \mathcal{P}(\mathbb{R})} \Big\{ \overline{J}[\mu] \Big\} \ = \ \lim_{N \to +\infty} \frac{1}{N} \ln I[V] \ . \tag{3.22}$$

In order to obtain (3.7), it remains to compute the minimum. Clearly, for any $\mu \in \mathcal{P}(\mathbb{R})$:

$$\overline{J}[\mu] \ \geq \ \int_{\mathbb{R}} \inf_{s \in \mathbb{R}} [V(s)] \cdot \mathrm{d}\mu(s) \ = \ \inf_{s \in \mathbb{R}} [V(s)] \ , \tag{3.23}$$

since μ is a probability measure. Furthermore, V attains its infimum at least at one point s_0. Then

$$\overline{J}[\delta_{s_0}] \ = \ V(s_0) \ = \ \inf_{s \in \mathbb{R}} [V(s)] \ , \tag{3.24}$$

so that indeed (3.7) follows.

This limit being established, one deduces from (3.20) estimates of μ_N on shrinking balls

$$-J[\mu] \ \geq \ \limsup_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B(\mu, \delta)] \qquad \text{and} \qquad -J[\mu] \ \leq \ \liminf_{\delta \to 0} \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mu_N[B(\mu, \delta)] \ , \tag{3.25}$$

thus leading, according to propositions 2.7 and 2.8 to the full LDP with speed N^2 and good rate function J for the sequence μ_N.

• Upper bound

In order to establish the upper bound on $\overline{\mu}_N[B(\mu, \delta)]$, one should integrate on the domain

$$\{\boldsymbol{\lambda}_N \ : \ L_N^{(\lambda_N)} \in B(\mu, \delta)\} \ . \tag{3.26}$$

The characterization of its geometric form is rather implict; however, it has a very good description in terms of the empirical measure. It is therefore convenient to recast the integrand in terms of integrals versus the empirical measure.

Set $\kappa_M = \| \min\{V, M\} \|_{BL}$. Then, for $M > 0$,

$$\begin{aligned} \overline{\mu}_N[B(\mu, \delta)] \ &= \ \int\limits_{\{\boldsymbol{\lambda}_N \, : \, L_N^{(\lambda_N)} \in B(\mu, \delta)\}} \exp\Big\{ - N(N-1) \int V(s) \cdot \mathrm{d}L_N^{(\lambda_N)}(s) \Big\} \prod_{a=1}^{N} \mathrm{e}^{-V(\lambda_a)} \cdot \mathrm{d}^N \lambda \\ &\leq \ \int\limits_{\{\boldsymbol{\lambda}_N \, : \, L_N^{(\lambda_N)} \in B(\mu, \delta)\}} \exp\Big\{ - N(N-1) \int \min\{V(s), M\} \cdot \mathrm{d}L_N^{(\lambda_N)}(s) \Big\} \prod_{a=1}^{N} \mathrm{e}^{-V(\lambda_a)} \cdot \mathrm{d}^N \lambda \ . \end{aligned} \tag{3.27}$$

It follows from the definition of the bounded-Lipschitz metric, since $s \mapsto [V(s) - v] \mathbf{1}_{|x| \leq M}(s)$ is a bounded Lipschitz function, that for $L_N^{(\lambda_N)} \in B(\mu, \delta)$

$$\int \min\{V(s), M\} \cdot \mathrm{d}L_N^{(\lambda_N)}(s) \ \geq \ \int \min\{V(s), M\} \cdot \mathrm{d}\mu(s) \ - \ \delta \cdot \| \min\{V, M\} \|_{BL} \tag{3.28}$$

Hence,

$$\overline{\mu}_N[B(\mu,\delta)] \;\leq\; \exp\Big\{ -N(N-1)\int \min\{V(s),M\}\cdot \mathrm{d}\mu(s)\Big\}\cdot \mathrm{e}^{\delta\cdot N(N-1)\kappa_M}\cdot\Big(\int \mathrm{e}^{-V(\lambda)}\cdot \mathrm{d}\lambda\Big)^N \;. \tag{3.29}$$

Thus, clearly,

$$\limsup_{N\to+\infty} \frac{1}{N^2}\ln\overline{\mu}_N[B(\mu,\delta)] \;\leq\; -\int \min\{V(s),M\}\cdot \mathrm{d}\mu(s) \;+\; \delta\cdot\kappa_M \;. \tag{3.30}$$

Further, sending $\delta\to 0$ leads to

$$\limsup_{\delta\to 0}\limsup_{N\to+\infty} \frac{1}{N^2}\ln\overline{\mu}_N[B(\mu,\delta)] \;\leq\; -\int \min\{V(s),M\}\cdot \mathrm{d}\mu(s) \;. \tag{3.31}$$

Finally, the sequence

$$f_M(s) \;=\; \min\{V(s),M\} \tag{3.32}$$

is increasing. Thus, by monotone convergence,

$$\lim_{M\to+\infty}\int \min\{V(s),M\}\cdot \mathrm{d}\mu(s) \;=\; \lim_{M\to+\infty}\int V(s)\cdot \mathrm{d}\mu(s) \;. \tag{3.33}$$

Hence, upon sending $M\to+\infty$ one gets

$$\limsup_{\delta\to 0}\limsup_{N\to+\infty} \frac{1}{N^2}\ln\overline{\mu}_N[B(\mu,\delta)] \;\leq\; -\overline{J}[\mu] \;. \qquad\blacksquare \tag{3.34}$$

It is important to note at this stage the very subtle effects that took place in taking the *ordered* limits. Taking the $N\to+\infty$ demanded the introduction of some regularizations. The latter would, *in fine* produce divergent factors should the $\delta\to 0^+$ not be taken first. Only after the $\delta\to 0^+$ limit, could the regularizing parameter M be sent to $+\infty$.

• Lower bound

Obtaining a lower bound is the most subtle procedure. Since one can only use properties of the bounded Lipschitz metric when integrating versus bounded-Lipschitz functions, one should start by approximating the measure $\mu\in\mathcal{P}(\mathbb{R})$ by a compactly supported one. Further, one will need to have a quite precise control on the shape of the domain of intergration. Although for numerous theoretical considerations, the below description

$$\Big\{\boldsymbol{\lambda}_N\in\mathbb{R}^N \;:\; d_{BL}(\mu,L_N^{(\lambda_N)})<\delta\Big\} \tag{3.35}$$

of the domain of integration seems sufficient, it appears extremely hard to say anything about its shape or more precise geometric properties. Therefore, for minoration purposes, it is convenient to build a domain contained in the latter that, however, has an explicit geometry.

Let $\mu \in \mathcal{P}(\mathbb{R})$. If $\int V(s)\cdot \mathrm{d}\mu(s) \;=\; +\infty$, then there is noting to prove. Hence, we may well assume that $\int V(s)\cdot \mathrm{d}\mu(s) \;<\; +\infty$. Furthermore, we pick $0<\eta'<\delta/2$ and $\widetilde{\mu}$ be an atomless measure such that

$$d_{\mathrm{BL}}(\widetilde{\mu},\mu) \;\leq\; \eta' \;. \tag{3.36}$$

For any $\delta>0$, there exists M_δ such that for any $M \geq M_\delta$ the measure

$$\mu^{(M)} \;=\; \frac{\mathbf{1}_{|x|\leq M}\cdot \widetilde{\mu}}{\widetilde{\mu}([\,-M\,;M\,])} \tag{3.37}$$

satisfies $d_{BL}(\mu,\mu^{(M)})<\delta$, where d_{BL} is the bounded Lipschitz metric. It then follows that

$$\Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \;:\; d_{BL}(\mu, L_N^{(\lambda_N)})<\delta\Big\} \subset \Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \;:\; d_{BL}(\mu^{(M)}, L_N^{(\lambda_N)})<2\delta\Big\} \;. \tag{3.38}$$

The measure $\mu^{(M)}$ allows one to introduce

$$x_1^N = \inf\Big\{x \;:\; \mu^{(M)}(]-\infty\,;x\,]) \geq \frac{1}{N+1}\Big\} \quad \text{and} \quad x_{a+1}^N = \inf\Big\{x \geq x_a^N \;:\; \mu^{(M)}(]\,x_a^N\,;x\,]) \geq \frac{1}{N+1}\Big\} \quad a=1,\dots,N-1 \tag{3.39}$$

Then, it can be shown that for any $\eta>0$, there exists N_η such that for any $N>N_\eta$

$$d_{BL}\Big(\mu^{(M)}, \underbrace{\frac{1}{N}\sum_{p=1}^{N}\delta_{x_p^N}}_{L_N^{(x_N)}}\Big)<\eta \;. \tag{3.40}$$

In the following we assume that $N>N_\eta$ where η is such that $0<\eta<\delta/2$. Then

$$\Omega_\delta \equiv \Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \;:\; \big|\lambda_a - x_a^N\big| < \frac{\delta}{2} \;\; a=1,\dots,N\Big\} \;\subset\; \Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \;:\; d(\mu^{(M)}, L_N^{(\lambda)})<\delta\Big\} \;. \tag{3.41}$$

It follows from the mean value theorem that, provided $|\lambda_a - x_a^N| \;\leq\; \delta/2$,

$$\Big|V(\lambda_a) \;-\; V(x_a^N)\Big| \;\leq\; \frac{\delta}{2}\cdot \sup_{|y|\leq \delta/2}\bigg|\frac{V'(x_a^N+y)}{V(x_a^N)}\bigg|\cdot |V(x_a^N)| \;\leq\; |V(x_a^N)|\cdot \sup_{x\in\mathbb{R}}\sup_{|y|\leq\delta/2}\bigg|\frac{V'(x+y)}{V(x)}\bigg| \tag{3.42}$$

As a consequence, given any $\boldsymbol{\lambda}_N \in \Omega_\delta$

$$-\sum_{a=1}^{N} V(\lambda_a) \;\geq\; -\sum_{a=1}^{N} V(x_a^N) \;-\; \delta\cdot \underbrace{\sup_{x\in\mathbb{R}}\sup_{|y|\leq\delta}\bigg|\frac{V'(x+y)}{V(x)}\bigg|}_{\varpi_\delta}\cdot \sum_{a=1}^{N}|V(x_a^N)| \;. \tag{3.43}$$

Therefore,

$$\begin{aligned}
\overline{\mu}_N[B(\mu,2\delta)] \;\geq\; \overline{\mu}_N[B(\mu^{(M)},\delta)] \;=\; & \int\limits_{\{\boldsymbol{\lambda}_N\in\mathbb{R}^N \,:\, d(\mu^{(M)},L_N^{(\lambda_N)})<\delta\}} \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)}\cdot \mathrm{d}^N\lambda \\
\leq\; \prod_{a=1}^{N}\mathrm{e}^{-NV(x_a^N)}\cdot \prod_{a=1}^{N}\mathrm{e}^{-N\delta\varpi_\delta|V(x_a^N)|}\cdot \int\limits_{\Omega_\delta}\mathrm{d}^N\lambda & \\
\leq\; \exp\Big\{-N^2\int V(s)\cdot \mathrm{d}L_N^{(\boldsymbol{x}_N)}(s) \;-\; N^2\delta\cdot\varpi_\delta\cdot\int \big|V(s)\big|\cdot \mathrm{d}L_N^{(\boldsymbol{x}_N)}(s)\Big\}\cdot\delta^N &
\end{aligned} \tag{3.44}$$

Then, using that $d_{BL}(L_N^{(\boldsymbol{x}_N)}, \mu^{(M)}) < \eta$, we get

$$\mu_N[B(\mu, 2\delta)] \;\geq\; \exp\Big\{ - N^2 \int \min\{V(s), V_M\} \cdot \mathrm{d}\mu^{(M)}(s) \;-\; N^2\delta \cdot \varpi_\delta \cdot \int \min\{|V(s)|, |V|_M\} \cdot \mathrm{d}\mu^{(M)}(s)$$

$$- \;N^2\eta(1 + \delta \cdot \varpi_\delta)\Big(\| \min\{V, V_M\}\|_{BL} \;+\; \| \min\{|V|, |V|_M\}\|_{BL}\Big)\Big\} \quad (3.45)$$

where we have set $V_M \;=\; \max\{V(s) \;:\; s \in [\,-M\,; M\,]\}$ and $|V|_M \;=\; \max\{|V(s)| \;:\; s \in [\,-M\,; M\,]\}$. Thus,

$$\liminf_{N\to+\infty} \frac{1}{N^2} \ln \mu_N[B(\mu, 2\delta)] \;\geq$$

$$- \int \min\{V(s), V_M\} \cdot \mathrm{d}\mu^{(M)}(s) \;-\; \delta \cdot \varpi_\delta \cdot \int \min\{|V(s)|, |V|_M\} \cdot \mathrm{d}\mu^{(M)}(s)$$

$$- \;\eta(1 + \delta \cdot \varpi_\delta)\Big(\| \min\{V, V_M\}\|_{BL} \;+\; \| \min\{|V|, |V|_M\}\|_{BL}\Big) \quad (3.46)$$

Sending η to 0^+ at this point, we get that

$$\liminf_{N\to+\infty} \frac{1}{N^2} \ln \mu_N[B(\mu, 2\delta)] \;\geq\; -\int \frac{V(s) \cdot \mathbf{1}_{|x|\leq M}(s)\mathrm{d}\widetilde{\mu}(s)}{\widetilde{\mu}([\,-M\,; M\,])} \;-\; \delta \cdot \varpi_\delta \cdot \int \frac{\big|V(s)\big| \cdot \mathbf{1}_{|x|\leq M}(s) \cdot \mathrm{d}\widetilde{\mu}(s)}{\widetilde{\mu}([\,-M\,; M\,])} \qquad (3.47)$$

Further, just as previously, we can send η' to 0, hence replacing the measure $\widetilde{\mu}$ by the measure μ. Then, since $V > 0$ for x large enough

$$\liminf_{N\to+\infty} \frac{1}{N^2} \ln \mu_N[B(\mu, 2\delta)] \;\geq\; -\int \frac{V(s) \cdot \mathbf{1}_{|x|\leq M}(s)\mathrm{d}\mu(s)}{\mu([\,-M\,; M\,])} \;-\; \delta \cdot \varpi_\delta \cdot \int \frac{\big|V(s)\big| \cdot \mathbf{1}_{|x|\leq M}(s) \cdot \mathrm{d}\mu(s)}{\mu([\,-M\,; M\,])} \,. \qquad (3.48)$$

So that, sending $M \to +\infty$ and then $\delta \to 0$, we get

$$\liminf_{\delta\to 0} \liminf_{N\to+\infty} \frac{1}{N^2} \ln \mu_N[B(\mu, 2\delta)] \;\geq\; -\int V(s) \cdot \mathrm{d}\mu(s) \;=\; -\overline{J}[\mu] \,. \qquad (3.49)$$

3.1 Varadhan's lemma and a first non-trivial application

In this subsection, we shall establish Varadhan's lemma which can be thought of as an infinite dimensional analogue of Laplace's method for extracting the leading large N behaviour of one-dimensional integrals of the type

$$\int_{\mathbb{R}} \mathrm{e}^{-NV(\lambda)} \cdot \mathrm{d}\lambda \,. \qquad (3.50)$$

We shall then apply the lemma so as to draw informations on the leading asymptotics of one of the multiple integrals that were introduced in the first part of the lectures.

Theorem 3.1 Varadhan's lemma

Let $\mu_N \in \mathcal{P}(S)$ satisfy a LDP with rate function J and speed a_N. Then, for any $f : S \mapsto \mathbb{R}$ that is continuous and bounded from above

$$\lim_{N\to+\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f}] \;=\; \sup_S [f \,-\, J] \qquad \text{with} \quad \mathbb{E}_N[g] \;=\; \int g(s) \cdot \mathrm{d}\mu_N(s) \qquad (3.51)$$

Proof —

We first estimate the lim sup. Let $M > 0$ be an integer. As f is continuous, there exists finitely many closed sets $F_1, \ldots, F_\ell$ such that

$$f \leq -M \text{ on } \Big(\cup_{p=1}^{\ell} F_p \Big)^{\mathrm{c}} \quad \text{and} \quad |f(x) - f(y)| \; < \; 1/M \quad \text{for any } x, y \in F_p \;, \; p = 1, \ldots, \ell \;. \tag{3.52}$$

Then,

$$\begin{aligned} \limsup_{N \to +\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f}] \; &\leq \; \max\Big\{ -M \;, \; \max_{p \in [\![1 \,;\, \ell]\!]} \Big(\limsup_{N \to +\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f} \mathbf{1}_{F_p}] \Big) \Big\} \\ &\leq \; \max\Big\{ -M \;, \; \max_{p \in [\![1 \,;\, \ell]\!]} \big[\sup_{x \in F_p} f(x) \; - \; \inf_{x \in F_p} J(x) \big] \Big\} \\ &\leq \; \max\Big\{ -M \;, \; \max_{p \in [\![1 \,;\, \ell]\!]} \big[\sup_{x \in F_p} \big(f(x) \; - \; J(x) + \frac{1}{M} \big) \big] \Big\} \\ &\leq \; \max\Big\{ -M \;, \; \sup_{x \in S} \big(f(x) \; - \; J(x) \big) \Big\} \; + \; \frac{1}{M} \end{aligned} \tag{3.53}$$

Letting $M \to +\infty$ yields

$$\limsup_{N \to +\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f}] \; \leq \; \sup_{x \in S} \Big[f(x) \; - \; J(x) \Big] \;. \tag{3.54}$$

In order to bound the lim inf, let $x \in S$. Then

$$\begin{aligned} \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f}] \; &\geq \; \liminf_{N \to +\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f} \mathbf{1}_{B(x,\delta)}] \\ &\geq \; \inf_{y \in B(x,\delta)} [f(y)] \; - \; \inf_{y \in B(x,\delta)} [J(y)] \; \geq \; \inf_{y \in B(x,\delta)} [f(y)] \; - \; J(x) \end{aligned} \tag{3.55}$$

By continuity of f, $\lim_{\delta \to 0} \inf_{y \in B(x,\delta)} [f(y)] = f(x)$, thus passing on to the $\delta \to 0^+$ limit yields

$$\liminf_{N \to +\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f}] \; \geq \; f(x) \; - \; J(x) \;. \tag{3.56}$$

It then remains to optimise in respect to x so as to get

$$\liminf_{N \to +\infty} \frac{1}{a_N} \ln \mathbb{E}_N[\mathrm{e}^{a_N f}] \; \geq \; \sup_{x \in S} \Big[f(x) \; - \; J(x) \Big] \;. \qquad \blacksquare \tag{3.57}$$

We are now in position to establish the

Theorem 3.2 *Let $W(\lambda, \mu)$ be such that $||W||_{BL}$ and $V \in C^1(\mathbb{R})$ be such that $V(x) \geq v_1|x| + v_2$ then*

$$\lim_{N \to +\infty} \frac{1}{N^2} \ln \Big[\int_{\mathbb{R}^N} \prod_{a,b=1}^{N} \mathrm{e}^{W(\lambda_a, \lambda_b)} \cdot \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)} \cdot \mathrm{d}^N \lambda \Big] \; = \; \sup_{\mu \in \mathcal{P}(\mathbb{R})} \Big[\int_{\mathbb{R}^2} W(s,t) \mathrm{d}\mu(s) \otimes \mathrm{d}\mu(t) \; - \; \int_{\mathbb{R}} V(s) \cdot \mathrm{d}\mu(s) \Big] \;. \tag{3.58}$$

Proof —

We have already established that the sequence of probability measures on $\mathbb{R}^N$ given by

$$\mathrm{d}\mathbb{P}_N(\boldsymbol{\lambda}_N) \;=\; \frac{1}{(I[V])^N}\prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)}\cdot \mathrm{d}^N\lambda\;, \qquad I[V] \;=\; \int_{\mathbb{R}} \mathrm{e}^{-NV(\lambda)}\mathrm{d}\lambda \tag{3.59}$$

induces a probability measure μ_N on $\mathcal{P}(\mathbb{R})$ through $\mu_N \;=\; L_N\#\mathbb{P}_N$ and that the sequence μ_N satisfies a LDP with speed N^2 and good rate function

$$J[\mu] \;=\; \int_{\mathbb{R}} V(x)\cdot \mathrm{d}\mu(x) \;-\; \inf_{s\in\mathbb{R}}\big[V(s)\big]\;. \tag{3.60}$$

Hence, one has

$$\begin{aligned}\int_{\mathbb{R}^N}\prod_{a,b=1}^{N}\mathrm{e}^{W(\lambda_a,\lambda_b)}\cdot\prod_{a=1}^{N}\mathrm{e}^{-NV(\lambda_a)}\cdot\mathrm{d}^N\lambda \;&=\; (I[V])^N\cdot\int_{\mathbb{R}^N}\exp\Big\{N^2\int W(s,t)\mathrm{d}L_N^{(\boldsymbol{\lambda}_N)}(s)\otimes\mathrm{d}L_N^{(\boldsymbol{\lambda}_N)}(t)\Big\}\cdot\mathrm{d}P_N(\boldsymbol{\lambda}_N)\\ &=\; (I[V])^N\cdot\mathbb{E}_N\Big[\mathrm{e}^{N^2\mathcal{W}}\Big]\end{aligned} \tag{3.61}$$

where the last equality is a mere restatement of the definition of the image measure μ_N, $\mathbb{E}_N$ refers to the expectation in respect to the measure μ_N. In order to be able to apply Varadhan's lemma, we still need to establish that

$$\mu\mapsto\mathcal{W}[\mu] \;=\; \int W(s,t)\cdot\mathrm{d}\mu(s)\otimes\mathrm{d}\mu(t) \tag{3.62}$$

is continuous and bounded from above. The latter follows from

$$\bigg|\int W(s,t)\cdot\mathrm{d}\mu(s)\otimes\mathrm{d}\mu(t)\bigg| \;\leq\; ||W||_{L^\infty(\mathbb{R}^2)} \tag{3.63}$$

since $\mu\in\mathcal{P}(\mathbb{R})$. Since, W is bounded Lipschitz in two variables, it is a quite direct consequence of the bounded Lipschitz metric that $\mathcal{W}$ is continuous on $\mathcal{P}(\mathbb{R})$. We leave the details to the reader as a character building exercise in real analysis. ■

4 Leading large-N asymptotic behaviour of β-ensembles

Let

$$\mathrm{d}\mathbb{P}_N^{(\beta)}(\boldsymbol{\lambda}_N) \;=\; \frac{1}{Z_N^{(\beta)}}\prod_{a<b}^{N}|\lambda_a-\lambda_b|^{2\beta}\cdot\prod_{a=1}^{N}\mathrm{e}^{-NV(\lambda_a)}\cdot\mathrm{d}^N\lambda \tag{4.1}$$

where $V:\mathbb{R}\to\mathbb{R}$ is a $C^1(\mathbb{R})$ function such that, for some $\beta'\geq\beta$

$$\liminf_{x\to\pm\infty}\frac{V(x)}{\beta'\cdot\ln|x|} \;>\; 1\;, \tag{4.2}$$

and

$$\limsup_{\delta\to 0} \varpi_\delta \;<\; +\infty \qquad \text{with} \qquad \varpi_\delta \;=\; \sup_{x\in\mathbb{R}} \sup_{|y|\le\delta} \Big|\frac{V'(x+y)}{V(x)}\Big| \tag{4.3}$$

Finally, the normalization constant or partition function reads and

$$Z_N^{(\beta)} \;=\; \int_{\mathbb{R}^N} \prod_{a<b}^{N} |\lambda_a-\lambda_b|^{2\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)} \cdot \mathrm{d}^N\lambda \;. \tag{4.4}$$

In the following we denote by μ_N the measure on $\mathcal{P}(\mathbb{R})$ induced by $\mathbb{P}_N^{(\beta)}$, *viz* $\mu_N = L_N^{(\lambda_N)} \# \mathbb{P}_N^{(\beta)}$.

4.1 The LDP for the associated sequence μ_N

Throughout this section, we are going to prove the

Theorem 4.1 *The sequence $\mu_N \in \mathcal{P}(\mathbb{R})$ satisfies a LDP with speed N^2 and good rate function*

$$J_\beta[\mu] \;=\; \int_{\mathbb{R}^2} f(s,t)\cdot \mathrm{d}\mu(s)\otimes\mathrm{d}\mu(t) \;-\; c_V \qquad \text{where} \qquad \left\{ \begin{array}{ccl} f(s,t) & = & \dfrac{V(s)}{2} \;+\; \dfrac{V(t)}{2} \;-\; \beta \ln|x-y| \\ c_V & = & \inf_{\mu\in\mathcal{P}(\mathbb{R})} \int_{\mathbb{R}^2} f(s,t)\cdot\mathrm{d}\mu(s)\otimes\mathrm{d}\mu(t) \end{array} \right. \tag{4.5}$$

Furthermore, the functional $I_V[\mu] \;=\; \int_{\mathbb{R}^2} f(s,t)\cdot\mathrm{d}\mu(s)\otimes\mathrm{d}\mu(t)$ attains its minimum at a unique compactly supported probability measure μ_V. Furthermore, if the potential V is analytic, then μ_V is continuous in respect to Lebesgue measure.

Note that this theorem is a typical example where one cannot apply Varadhan's lemma since the functional "perturbing" the decoupled measure studied in section 3.0.2

$$\mu \mapsto -\int_{\mathbb{R}^2} \ln|x-y|\cdot\mathrm{d}\mu(x)\otimes\mathrm{d}\mu(y) \tag{4.6}$$

is neither continuous nor bounded from above. Thus, the whole difficulty of the proof lies in circumventing these singularities. In fact, the steps of the proof decompose exactly as in the case of the "simple" example studied previously. Below, we shall establish the LDP and the limit of the partition function. In the next sub-section, based on several auxilliary lemmas we establish an explicit characterization of the minimizer of I_V, the so-called equilibrium measure.

Proof —
Note that the density $\mathrm{d}\mathbb{P}_N^{(\beta)}(\boldsymbol{\lambda}_N)$ can be represented as

$$\mathrm{d}\mathbb{P}_N^{(\beta)}(\boldsymbol{\lambda}_N) \;=\; \frac{1}{Z_V^{(\beta)}} \exp\Big\{ -N^2 \int_{x\neq y} f(x,y)\cdot\mathrm{d}L_N^{(\lambda_N)}(x)\otimes\mathrm{d}L_N^{(\lambda_N)}(y)\Big\} \cdot \prod_{s=1}^{N} \mathrm{e}^{-V(\lambda_s)}\cdot\mathrm{d}^N\lambda \tag{4.7}$$

• **Exponential tightness**

It follows from Jensen's inequality applied to the probability measure

$$\otimes^N \nu(\boldsymbol{\lambda}_N) \;=\; \prod_{s=1}^{N} \Bigg\{ \frac{\mathrm{e}^{-V(\lambda_s)}}{\int \mathrm{e}^{-V(\lambda)}\mathrm{d}\lambda} \Bigg\} \cdot \mathrm{d}^N \lambda \tag{4.8}$$

that

$$\begin{aligned} \ln Z_V^{(\beta)} \;&\geq\; N \ln\Big[\int \mathrm{e}^{-V(\lambda)}\mathrm{d}\lambda \Big] \;-\; N^2 \int_{\mathbb{R}^N} \Bigg\{ \int_{\substack{\mathbb{R}^2 \\ x\neq y}} f(x,y) \cdot \mathrm{d}L_N^{(\lambda_N)}(x) \otimes \mathrm{d}L_N^{(\lambda_N)}(y) \Bigg\} \cdot \otimes^N \nu(\boldsymbol{\lambda}_N) \\ &=\; N \ln\Big[\int \mathrm{e}^{-V(\lambda)}\mathrm{d}\lambda \Big] - N(N-1)\Big(\int \mathrm{e}^{-V(\lambda)}\mathrm{d}\lambda \Big)^{-2} \cdot \int_{\mathbb{R}^2} \Big(\frac{V(x)+V(y)}{2} - \beta \ln|x-y| \Big) \cdot \mathrm{e}^{-V(x)-V(y)} \mathrm{d}x\mathrm{d}y \;\geq\; -CN^2 \end{aligned} \tag{4.9}$$

for some $C > 0$.

Observe that

$$|x-y| \;\leq\; \sqrt{(x^2+1)(y^2+1)} \qquad \text{so that} \quad f(x,y) \;\geq\; \frac{1}{2}\Big(\psi_V(x) \;+\; \psi_V(y) \Big) \tag{4.10}$$

where

$$\psi_V(x) = V(x) - \beta \ln(x^2+1) \;. \tag{4.11}$$

Finally, it follows readily from (4.2) that there exists constants $v > 0$ and $c \in \mathbb{R}$ such that

$$f(x,y) \;\geq\; v(V(x) \;+\; V(y)) \;+\; c \;. \tag{4.12}$$

Let the compact K_L be defined as in (3.8)

$$K_L \;=\; \Big\{ \mu \in \mathcal{P}(\mathbb{R}) \;:\; \int V(x) \cdot \mathrm{d}\mu(x) \;\leq\; L \Big\} \,. \tag{4.13}$$

Then, one gets

$$\mu_N[K_L^{\mathrm{c}}] \;\leq\; \mathrm{e}^{N^2(C-c)} \mathrm{e}^{-2vN^2 L} \cdot \Big(\int_{\mathbb{R}} \mathrm{e}^{-V(\lambda)}\mathrm{d}\lambda \Big) \,. \tag{4.14}$$

Hence, μ_N is exponentially tight.

We proceed exactly as in the proof of the LDP in proposition 3.1. Hence, we define

$$\overline{\mu}_N \;=\; Z_N^{(\beta)} \cdot \mu_N \tag{4.15}$$

and introduce

$$I_V[\mu] \;=\; \int_{\mathbb{R}^2} f(s,t) \cdot \mathrm{d}\mu(s) \otimes \mathrm{d}\mu(t) \;. \tag{4.16}$$

• **Upper bound**

Given any $\mu \in \mathcal{P}(\mathbb{R})$, we shall establish the bound

$$\limsup_{\delta\to 0} \cdot \limsup_{N\to+\infty} \frac{1}{N^2} \ln \overline{\mu}_N[B(\mu,\delta)] \ \leq \ -I_V[\mu] \tag{4.17}$$

For any $M > 0$ set

$$f_M(x,y) \ = \ \min\Big[f(x,y), M \Big]. \tag{4.18}$$

Then

$$\overline{\mu}_N\Big[B(\mu,\delta)\Big] \ \leq \ \int\limits_{\substack{\big\{\lambda_N \,:\, d_{BL}(L_N^{(\lambda_N)},\mu)<\delta\big\} \\ \lambda_a\neq\lambda_b \ ,\, a\neq b}} \exp\Big\{ -N^2 \int\limits_{x\neq y} f_M(x,y)\cdot \mathrm{d}L_N^{(\lambda_N)}(x)\otimes \mathrm{d}L_N^{(\lambda_N)}(y)\Big\} \cdot \prod_{s=1}^{N} \mathrm{e}^{-V(\lambda_s)} \cdot \mathrm{d}^N\lambda \tag{4.19}$$

where we have used that under any measure absolutely continuous in respect to Lebesgue's one the integration variables are almost surely distinct. As a consequence, $L_N^{(\lambda_N)} \otimes L_N^{(\lambda_N)}(\{x=y\}) \ = \ N^{-1}$ and thus

$$\int\limits_{x\neq y} f_M(x,y)\cdot \mathrm{d}L_N^{(\lambda_N)}(x)\otimes \mathrm{d}L_N^{(\lambda_N)}(y) \ = \ \int f_M(x,y)\cdot \mathrm{d}L_N^{(\lambda_N)}(x)\otimes \mathrm{d}L_N^{(\lambda_N)}(y) \ + \ \frac{M}{N} \ . \tag{4.20}$$

Since f_M is bounded, the functional

$$I_V^{(M)} \ : \ \mu \mapsto \int f_M(x,y)\cdot \mathrm{d}\mu(x)\otimes \mathrm{d}\mu(y) \tag{4.21}$$

is continuous and there exists an f_M dependent constant C such that

$$\Big| I_V^{(M)}[\mu] - I_V^{(M)}[L_N^{(\lambda_N)}] \Big| \ \leq \ C\delta \tag{4.22}$$

provided $d_{BL}(L_N^{(\lambda_N)},\mu) < \delta$. As a consequence,

$$\limsup_{\delta\to 0} \cdot \limsup_{N\to+\infty} \frac{1}{N^2} \ln \mu_N[B(\mu,\delta)] \ \leq \ -I_V^{(M)}[\mu] \ . \tag{4.23}$$

Then, letting $M \nearrow +\infty$, one concludes by the monotone convergence theorem.

• **Lower bound**

For $\mu \in \mathcal{P}(\mathbb{R})$, we now establish the bound

$$\liminf_{\delta\to 0} \cdot \liminf_{N\to+\infty} \frac{1}{N^2} \ln \overline{\mu}_N[B(\mu,\delta)] \ \geq \ -I_V[\mu] \ . \tag{4.24}$$

If μ has atoms then $I_V[\mu] = +\infty$ and the bound is trivial. Hence, we may just as well assume that μ has no atoms. We then proceed as previously.

For any $\delta > 0$, there exists M_δ such that for any $M \geq M_\delta$ the measure

$$\mu^{(M)} \ = \ \frac{\mathbf{1}_{|x|\leq M}}{\mu([\,-M\,;M\,])} \tag{4.25}$$

satisfies $d_{BL}(\mu,\mu^{(M)}) < \delta$, where d_{BL} is the bounded Lipschitz metric. It then follows that

$$\Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \ : \ d(\mu, L_N^{(\lambda)}) < \delta)\Big\} \subset \Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \ : \ d(\tilde{\mu}, L_N^{(\lambda)}) < 2\delta\Big\} \, . \tag{4.26}$$

We use this measure μ to introduce

$$x_1^N = \inf\Big\{x \ : \ \mu(]-\infty\,;x\,]) \geq \frac{1}{N+1}\Big\} \quad \text{and} \quad x_{a+1}^N = \inf\Big\{x \geq x_a^N \ : \ \mu(]\,x_a^N\,;x\,]) \geq \frac{1}{N+1}\Big\} \quad a = 1,\dots,N-1 \tag{4.27}$$

Further, for any $\eta > 0$ there exists N_η such that for any $N > N_\eta$

$$d_{BL}\Big(\mu, \frac{1}{N}\sum_{p=1}^{N} \delta_{x_p^N}\Big) < \eta \, . \tag{4.28}$$

In the following, we assume that $N > N_\eta$ where η is such that $0 < \eta < \delta/2$. Then

$$\Omega_\delta \equiv \Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \ : \ \big|\lambda_a - x_a^N\big| < \frac{\delta}{2} \ \ a = 1,\dots,N\Big\} \ \subset \ \Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \ : \ d(\mu, L_N^{(\lambda)}) < \delta\Big\} \, . \tag{4.29}$$

Assume that $\boldsymbol{\lambda}_N \in \Omega_\delta$, then

$$-\sum_{a=1}^{N} V(\lambda_a) \ \geq \ -\sum_{a=1}^{N} V(x_a^N) \ - \ \delta\cdot\varpi_\delta\cdot\sum_{a=1}^{N}|V(x_a^N)| \, . \tag{4.30}$$

As a consequence,

$$\mu_N[B(\mu,2\delta)] \ \geq \ \overline{\mu}_N[B(\mu^{(M)},\delta)] \ \geq \ \exp\Big\{-N^2\int V(s)\cdot \mathrm{d}L_N^{(\boldsymbol{x}_N)}(s) \ - \ N^2\delta\cdot\varpi_\delta\cdot\int \big|V(s)\big|\cdot \mathrm{d}L_N^{(\boldsymbol{x}_N)}(s)\Big\}\cdot R_\delta \tag{4.31}$$

where

$$\begin{aligned} R_\delta \ = \ \int\limits_{|\lambda_a - x_a^N| < \frac{\delta}{2}} \prod_{b>a}^{N}\big|\lambda_a - \lambda_b\big|^{2\beta} \ &\geq \ \int\limits_{\substack{|\lambda_a| < \frac{\delta}{2} \\ \lambda_1 < \cdots < \lambda_N}} \prod_{b>a}^{N}\big|\lambda_b - \lambda_a + x_b^N - x_a^N\big|^{2\beta} \\ &\geq \ \prod_{b>a+1}^{N}\big|x_b^N - x_a^N\big|^{2\beta}\cdot\prod_{a=1}^{N-1}\big|x_{a+1}^N - x_a^N\big|^{\beta}\times \int\limits_{\substack{-\delta/2 \\ \lambda_1 < \cdots < \lambda_N}}^{\delta/2} \prod_{a=1}^{N-1}\big|\lambda_{a+1} - \lambda_a\big|^{\beta}\mathrm{d}^N\lambda \, . \end{aligned} \tag{4.32}$$

The last integral can be estimated through the change of variables

$$u_1 \ = \ \lambda_1 \quad u_j = \lambda_j - \lambda_{j-1} \quad j = 2,\dots,N \tag{4.33}$$

and the inclusion

$$\Big\{\boldsymbol{\lambda}_N \ : \ |\lambda_a| < \delta/2 \ \lambda_1 < \cdots < \lambda_N\Big\} \ \supset \ \Big\{\boldsymbol{\lambda}_N \ : \ |\lambda_1| < \delta/2N \ 0 < \lambda_{j+1} - \lambda_j < \frac{\delta}{2N}\Big\} \, . \tag{4.34}$$

Indeed, then,

$$\int\limits_{-\delta/2\lambda_1<\cdots<\lambda_N}^{\delta/2} \prod_{a=1}^{N-1} \big|\lambda_{a+1}-\lambda_a\big|^{2\beta} \mathrm{d}^N\lambda \;\geq\; \int\limits_{-\delta/2N}^{\delta/2N} \prod_{a=2}^{N} \big|u_a\big|^{\beta} \mathrm{d}^N u \;=\; \frac{\delta}{2N}\cdot\Big(\frac{2}{\beta+1}\Big(\frac{\delta}{2N}\Big)^{\beta+1}\Big)^{N-1} \tag{4.35}$$

Furthermore, since $x \mapsto \ln x$ is increasing

$$\begin{aligned}\int\limits_{\substack{x_1^N\\ x<y}}^{x_N^N} \ln|x-y|\cdot\mathrm{d}\mu^{(M)}(x)\otimes\mathrm{d}\mu^{(M)}(y) \;&=\; \sum_{a,b=1}^{N-1}\int\limits_{x_a^N}^{x_{a+1}^N}\int\limits_{x_b^N}^{x_{b+1}^N} \mathbf{1}_{x<y}(x,y)\ln|x-y|\cdot\mathrm{d}\mu_{\mathrm{eq}}(x)\otimes\mathrm{d}\mu_{\mathrm{eq}}(y)\\ &\leq\; \frac{1}{N^2}\sum_{a=1}^{N-1}\sum_{b=a+1}^{N-1}\ln(x_{b+1}^N-x_a^N) \;+\; \sum_{a=1}^{N-1}\ln(x_{a+1}^N-x_a^N)\cdot\frac{1}{2N^2}\;.\end{aligned} \tag{4.36}$$

Thus

$$\prod_{b>a+1}^{N}\big|x_b^N-x_a^N\big|^{2\beta}\cdot\prod_{a=1}^{N-1}\big|x_{a+1}^N-x_a^N\big|^{\beta} \;\geq\; \exp\Big\{2\beta\int\limits_{x_1^N}^{x_N^N}\mathbf{1}_{x<y}\cdot\ln|x-y|\cdot\mathrm{d}\mu^{(M)}(x)\otimes\mathrm{d}\mu^{(M)}(y)\Big\} \tag{4.37}$$

Putting all the pieces together and repeating the handlings outlined in the course of the setting of the first LDP established in proposition 3.1 and using that by the dominated convergence theorem

$$\int\limits_{x_1^N}^{x_N^N}\mathbf{1}_{x<y}\cdot\ln|x-y|\cdot\mathrm{d}\mu^{(M)}(x)\otimes\mathrm{d}\mu^{(M)}(y) \underset{N\rightarrow+\infty}{\longrightarrow} \frac{1}{2}\int\limits_{\mathbb{R}^2}\cdot\ln|x-y|\cdot\mathrm{d}\mu^{(M)}(x)\otimes\mathrm{d}\mu^{(M)}(y) \tag{4.38}$$

we get

$$\begin{aligned}\liminf_{N\rightarrow+\infty}\frac{1}{N^2}\ln\mu_N[B(\mu,2\delta)] \;\geq\; &-\int V(s)\cdot\mathrm{d}\mu^{(M)}(s) \;-\; \delta\cdot\varpi_\delta\cdot\int\big|V(s)\big|\cdot\mathrm{d}\mu^{(M)}(s)\\ &-\;\eta(1+\delta\cdot\varpi_\delta)\Big(||V\mathbf{1}_{|x|\leq M}||_{BL} \;+\; ||\,|V|\mathbf{1}_{|x|\leq M}||_{BL}\Big) \;+\; \beta\int\limits_{\mathbb{R}^2}\ln|x-y|\cdot\mathrm{d}\mu^{(M)}(x)\otimes\mathrm{d}\mu^{(M)}(y)\;.\end{aligned} \tag{4.39}$$

Sending first $\eta \rightarrow 0^+$, then relaxing $M \nearrow +\infty$ and finally taking $\liminf_{\delta\rightarrow 0}$ leads to

$$\liminf_{\delta\rightarrow 0}\;\liminf_{N\rightarrow+\infty}\frac{1}{N^2}\ln\mu_N[B(\mu,2\delta)] \;\geq\; I_V[\mu]\;. \qquad\blacksquare \tag{4.40}$$

4.2 Some applications of the LDP

In the present section we are going to apply the LDP for the eigenvalue distribution so as to establish several corollaries that allow one to answer positively to certain questions that have been raised in the introduction to this series of lectures.

We shall start with the convergence of the density of eigenvalues

Corollary 4.1 *Let f be bounded Lipschitz on $\mathbb{R}$ and $p_{1;N}^{(\beta)}$ the one-eigenvalue distribution function:*

$$p_{1;N}^{(\beta)}(\lambda) \;=\; \int\limits_{\mathbb{R}^{N-1}} P_N^{(\beta)}(\lambda,\lambda_2,\ldots,\lambda_N)\cdot\prod_{a=2}^{N}\mathrm{d}\lambda_a \;. \tag{4.41}$$

Then,

$$\lim_{N\to+\infty}\int\limits_{\mathbb{R}} f(\lambda)\cdot p_N^{(\beta)}(\lambda)\cdot\mathrm{d}\lambda \;=\; \int f(s)\cdot\mathrm{d}\mu_{\mathrm{eq}}(s) \tag{4.42}$$

Proof —

It is readily seen that

$$\int\limits_{\mathbb{R}} f(\lambda)\cdot p_N^{(\beta)}(\lambda)\cdot\mathrm{d}\lambda \;=\; \int\limits_{\mathbb{R}^N} f(\lambda_1)\cdot\mathrm{d}\mathbb{P}_N^{(\beta)}(\boldsymbol{\lambda}_N)$$
$$=\; \mathbb{P}_N^{(\beta)}\Big[\int f(s)\cdot\mathrm{d}L_N^{(\lambda_N)}(s)\Big] \;=\; \mu_N\Big[\int f(s)\cdot\mathrm{d}\mu(s)\Big]\;. \tag{4.43}$$

Recall that J_β attains a unique minimum at a compactly supported probability measure μ_{eq} on $\mathbb{R}$. This guarantees that, given $\epsilon>0$,

$$u_\epsilon \;=\; \inf_{\mathcal{P}(\mathbb{R})\setminus B(\mu_{\mathrm{eq}},\epsilon)} J_\beta[\mu] \;>\; 0. \tag{4.44}$$

Then, by the LDP for the sequence μ_N, there exists N_0 large enough such that

$$\mu_N\big[\mathcal{P}(\mathbb{R})\setminus B(\mu_{\mathrm{eq}},\epsilon)\big] \;\le\; \mathrm{e}^{-\frac{N^2}{2}u_\epsilon} \tag{4.45}$$

for any $N \;\ge\; N_0$. Furthermore,

$$\mu_N\Big[\int f(s)\cdot\mathrm{d}\mu(s) \;-\; \int f(s)\cdot\mathrm{d}\mu_{\mathrm{eq}}(s)\Big] \;=\; \underbrace{\mu_N\Big[\mathbf{1}_{B^c(\mu_{\mathrm{eq}},\epsilon)}\Big\{\int f(s)\cdot\mathrm{d}\mu(s) \;-\; \int f(s)\cdot\mathrm{d}\mu_{\mathrm{eq}}(s)\Big\}\Big]}_{|\cdot|\,\le\,2\|f\|_\infty\cdot\exp\left\{-N^2\frac{u_\epsilon}{2}\right\}}$$
$$+\;\underbrace{\mu_N\Big[\mathbf{1}_{B(\mu_{\mathrm{eq}},\epsilon)}\Big\{\int f(s)\cdot\mathrm{d}\mu(s) \;-\; \int f(s)\cdot\mathrm{d}\mu_{\mathrm{eq}}(s)\Big\}\cdot\mathrm{d}\mu(s)\Big]}_{\epsilon\cdot\|f\|_{BL}}\;. \tag{4.46}$$

In other words, given any $\epsilon>0$, one has that

$$\limsup_{N\to+\infty}\Big\{\int\limits_{\mathbb{R}} f(\lambda)p_N^{(\beta)}(\lambda)\cdot f(\lambda)\cdot\mathrm{d}\lambda \;-\; \int f(s)\cdot\mathrm{d}\mu_{\mathrm{eq}}(s)\Big\} \;\le\; \epsilon\;, \tag{4.47}$$

so that the claim follows. ■

Corollary 4.2 *Assume that the unique minimizer μ_{eq} of J_β is absolutely continuous in respect to Lebesgue's measure:*

$$\mathrm{d}\mu_{\mathrm{eq}}(s) \;=\; \rho(s)\cdot \mathrm{d}s \tag{4.48}$$

with connected support $[\,\alpha^{(-)}\,;\alpha^{(+)}\,]$. Let γ_a^*, $a = 1,\dots,N$ denote the "classical" positions of the integration variables which are understood to be ordered $\lambda_1 < \cdots < \lambda_N$, viz

$$\frac{a}{N} \;=\; \int\limits_{\alpha^{(-)}}^{\gamma_a} \mathrm{d}\mu_{\mathrm{eq}}(s)\;. \tag{4.49}$$

Given any $\epsilon > 0$,

$$\mathbb{P}_N^{(\beta)}\Big[\{\boldsymbol{\lambda}_N \in \mathbb{R}^N \;:\; \exists a \;\; |\lambda_a \,-\, \gamma_a^*| \;>\; \epsilon\}\Big] \;=\; \mathrm{O}(N^{-\infty})\;. \tag{4.50}$$

Proof —

Given $\epsilon > 0$, set

$$\Upsilon_\epsilon \;=\; \{\boldsymbol{\lambda}_N \in \mathbb{R}^N \;:\; \exists a \;\; |\lambda_a \,-\, \gamma_a^*| \;>\; \epsilon\}\;. \tag{4.51}$$

Let $\boldsymbol{\lambda}_N \in \mathbb{R}^N_{\uparrow}$. Pick $a \in [\![\, 1\,;\, N\,]\!]$ minimal such that $|\lambda_a - \gamma_a| > \epsilon$. Let $b \in [\![\, 1\,;\, N\,]\!]$ se such that $|\lambda_a - \gamma_b|$ is minimal. The density ρ of the equilibrium measure is in $L^1([\,\alpha^{(-)}\,;\alpha^{(+)}\,],\mathrm{d}x)$ and $\rho(x) > 0$ *a.e.* on $[\,\alpha^{(-)}\,;\alpha^{(+)}\,]$. Hence, there exists $\eta_\epsilon > 0$ such that for any

$$I \subset [\,\alpha^{(-)}\,;\alpha^{(+)}\,] \quad \text{with} \quad |I| > \epsilon/2 \qquad \Rightarrow \qquad \int\limits_I \mathrm{d}\mu_{\mathrm{eq}}(s) \;>\; \eta_\epsilon \tag{4.52}$$

As a consequence, provided that $N^{-1} < \eta$, one has that uniformly in

$$c \in [\![\, 1\,;\, N\,]\!] \qquad |\gamma_{c+1} - \gamma_c| \le \frac{\epsilon}{2}\;. \tag{4.53}$$

The latter bound implies that $|\lambda_a - \gamma_b| \;\le\; \epsilon/2$ what, in its turn, ensures that

$$|\gamma_b - \gamma_a| \;\ge\; \Big||\gamma_b - \lambda_a| \;-\; |\gamma_a - \lambda_a|\Big| \;\ge\; \frac{\epsilon}{2}\;. \tag{4.54}$$

Furthermore, one gets that

$$\begin{aligned} \bigg|\int \mathbf{1}_{]-\infty\,;\gamma_b\,]}(s)\cdot \mathrm{d}L_N^{(\gamma_N)}(s) \;-\; \int \mathbf{1}_{]-\infty\,;\gamma_b\,]}(s)\cdot \mathrm{d}L_N^{(\lambda_N)}(s)\bigg| \;&\ge\; \frac{b-a}{N} \\ &=\; \int\limits_{\gamma_a}^{\gamma_b} \mathrm{d}\mu_{\mathrm{eq}}(s) \;\ge\; \int\limits_{\gamma_a}^{\gamma_a+\frac{\epsilon}{2}} \mathrm{d}\mu_{\mathrm{eq}}(s) \;\ge\; \eta_\epsilon > 0 \end{aligned} \tag{4.55}$$

It thus follows that, for any $\boldsymbol{\lambda}_N \in \Upsilon_\epsilon$, $d_{BL}(L_N^{(\lambda_N)}, \mu_{\mathrm{eq}}) \;>\; \eta_\epsilon$, *ie*

$$\Upsilon_\epsilon \;\subset\; \Big\{\boldsymbol{\lambda}_N \in \mathbb{R}^N_{\uparrow} \;:\; L_N^{(\lambda_N)} \;\in\; B^{\mathrm{c}}(\mu_{\mathrm{eq}}, \eta_\epsilon)\Big\} \tag{4.56}$$

Thus, for N large enough,

$$\mathbb{P}_N^{(\beta)}\Big[\Upsilon_\epsilon\Big] \;\le\; \mu_N\Big[B^{\mathrm{c}}(\mu_{\mathrm{eq}}, \eta_\epsilon)\Big] \;=\; \mathrm{e}^{-CN^2}\;, \tag{4.57}$$

for some $C > 0$, as ensured by the uniqueness of the minimum of J_β and the LDP for the sequence μ_N. ■

4.3 Existence and uniqueness of the equilibrium measure

Proposition 4.1 *The functional*

$$I_V[\mu] = \int_{\mathbb{R}^2} f(x,y)\cdot \mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \qquad \text{with} \qquad f(x,y) = \frac{1}{2}\Big(V(x)+V(y)\Big) - \beta \ln|x-y| \;, \tag{4.58}$$

admits a minimum E_V on the space of probability measures on $\mathbb{R}$

$$E_V = \inf_{\mu\in\mathcal{P}(\mathbb{R})} I_V[\mu] \;. \tag{4.59}$$

This minimum is attained at a unique measure μ_V called the equilibrium measure. The support $\mathrm{supp}[\mu_V]$ *of the equilibrium measure is compact.*

Proof —

It follows from

$$|x-y| \;\le\; \sqrt{(x^2+1)(y^2+1)} \tag{4.60}$$

that for any probability measure on $\mathbb{R}$,

$$I_V[\mu] \;\ge\; -\frac{\beta}{2}\int_{\mathbb{R}} [\,\ln(x^2+1) \;+\; \ln(y^2+1)]\cdot \mathrm{d}\mu(x)\otimes \mathrm{d}\mu(y) \;+\; \int_{\mathbb{R}} V(x)\mathrm{d}\mu(x) \;\ge\; \kappa \;. \tag{4.61}$$

Above, we have used that

$$\psi_V(x) = V(x) - \beta\ln(x^2+1) \;, \tag{4.62}$$

is continuous and that $\psi_V \underset{|x|\to+\infty}{\longrightarrow} +\infty$, *ie* there exists a constant $\kappa\in\mathbb{R}$ such that $\psi_V \ge \kappa$.

Also, I_V is not identically $+\infty$, as follows by taking the probability measure

$$\mathrm{d}\mu(x) = \mathrm{e}^{-V(x)}\mathrm{d}x\cdot\Big\{\int_{\mathbb{R}} \mathrm{e}^{-V(x)}\mathrm{d}x\Big\}^{-1} \;. \tag{4.63}$$

This means that $E_V \in \mathbb{R}$. We now show that the minimum is attained. The functional I_V is lower semi-continuous as the supremum of lower semi continuous functionals. As a consequence, given any weakly convergent sequence μ_n to $\mu\in\mathcal{P}(\mathbb{R})$, $\mu_n \rightharpoonup \mu$ it follows that

$$\liminf_n I_V[\mu_n] \ge I_V[\mu] \;. \tag{4.64}$$

Next we show that any sequence of measures μ_n such that

$$E_V \;+\; \frac{1}{n} \ge I_V[\mu_n] \tag{4.65}$$

is tight. Assume that μ_n is not tight but satisfies (4.65). Then, there exists $\epsilon > 0$ such that for any compact K in $\mathbb{R}$ one has

$$\mu_n[\mathbb{R}\setminus K] > \epsilon \;. \tag{4.66}$$

As $\psi_V(t) \underset{|t|\to+\infty}{\longrightarrow} +\infty$, there exists an $M > 0$ such that $\psi_V \geq [E_V - \kappa + 1]/\epsilon$ on $\mathbb{R} \setminus [\,-M\,;M\,]$. Then, it follows from (4.61) that

$$E_V + \frac{1}{n} \;\geq\; \kappa \;+\; [E_V - \kappa + 1]/\epsilon \int\limits_{|x|\geq M} \mathrm{d}\mu_n(x) \;=\; E_V + 1\;, \tag{4.67}$$

a contradiction.

Yet, every sequence of tight probability measures admits a subsequence that is vaguely convergent to a probability measure. Hence, given a sequence μ_n satisfying (4.65), one has $\mu_{n_k} \rightharpoonup \widetilde{\mu} \in \mathcal{P}(\mathbb{R})$. It follows from (4.64) that $\widetilde{\mu}$ satisfies $E_V \geq I_V[\widetilde{\mu}]$. Thence, $\widetilde{\mu}$ is an equilibrium measure. The rest of the claim is a consequence of the below series of lemmas.

Lemma 4.1 *Every probability measure μ realizing the minimum of I_V is compactly supported.*

Proof —

Let μ be a probability measure on $\mathbb{R}$ such that $E_V = I_V[\mu]$ and $\mathscr{D} \subset \mathbb{R}$ such that $\mu[\mathscr{D}] > 0$. We then define, for $\epsilon \in \,]\,-1\,;1\,[$, the probability measure

$$\mu_\epsilon = \frac{1}{1+\epsilon\mu[\mathscr{D}]}\left(\mu + \epsilon\mu_{|\mathscr{D}}\right)\;. \tag{4.68}$$

$\epsilon \mapsto I_V[\mu_\epsilon]$ is smooth on $]\,-1\,;1\,[$ and attains a miniumu at $\epsilon = 0$. Hence

$$0 = \frac{\mathrm{d}}{\mathrm{d}\epsilon} I_V[\mu_\epsilon]\,|_{\epsilon=0} = -2\mu\,[\mathscr{D}]\,I_V[\mu] + 2\int\limits_{\mathbb{R}} f(x,y)\mathrm{d}\mu_{\mathscr{D}}(x)\mathrm{d}\mu(y) \geq \int\limits_{\mathbb{R}} [\psi_V(x) + \int\limits_{\mathbb{R}} \psi_V(y)\mathrm{d}\mu(y) - 2I_V[\mu]]\mathrm{d}\mu_{\mathscr{D}}(x) \tag{4.69}$$

One has by hypothesis that

$$\int \psi_V(y)\mathrm{d}\mu(y) \;<\; +\infty\;. \tag{4.70}$$

Since also $\psi_V(x) \to +\infty$ when $|x| \to +\infty$, it follows that there exists an $M > 0$ such that

$$\psi_V(x) \;+\; \int\limits_{\mathbb{R}} \psi_V(y)\mathrm{d}\mu(y) \;-\; 2I_V[\mu] \;\geq\; 1 \qquad \forall\; |x| \geq M\;. \tag{4.71}$$

Thus, if there exists a $\mathscr{D} \subset \mathbb{R} \setminus [\,-M\,;M\,]$, (4.69)- (4.71) would lead to a contradiction. In other words, the support of μ is compact. ■

Lemma 4.2 *Let $\mu = \mu_+ - \mu_-$ be a compactly supported signed measure on $\mathbb{R}$ of zero mean. Then one has the inequality*

$$-\int\limits_{\mathbb{R}} \ln|x-y|\,[\mathrm{d}\mu_+(x)\mathrm{d}\mu_+(y) + \mathrm{d}\mu_-(x)\mathrm{d}\mu_-(y)] \geq -\int\limits_{\mathbb{R}} \ln|x-y|\,[\mathrm{d}\mu_+(x)\mathrm{d}\mu_-(y) + \mathrm{d}\mu_-(x)\mathrm{d}\mu_+(y)] \tag{4.72}$$

Moreover, should the lhs be finite, one has in fact the equality

$$-\int_{\mathbb{R}} \ln|w(x)-w(y)|\,[\mathrm{d}\mu_+(x)\mathrm{d}\mu_+(y)+\mathrm{d}\mu_-(x)\mathrm{d}\mu_-(y)]$$

$$= -\int_{\mathbb{R}} \ln|w(x)-w(y)|\,[\mathrm{d}\mu_+(x)\mathrm{d}\mu_-(y)+\mathrm{d}\mu_-(x)\mathrm{d}\mu_+(y)] \;+\; 2\int_0^{+\infty} \left|\widehat{\mu}(u)\right|^2 \cdot \frac{\mathrm{d}u}{u} \quad (4.73)$$

where

$$\widehat{\mu}(u) = \int_0^{+\infty} \mathrm{e}^{ixs} \cdot \mathrm{d}s \tag{4.74}$$

Proof —

Given any $\epsilon > 0$, one has the identity

$$\ln(s^2+\epsilon^2) = \ln\epsilon^2 + 2\Im\Big(\int_0^{+\infty} \mathrm{d}u\mathrm{e}^{-\varepsilon u}\frac{\mathrm{e}^{isu}-1}{iu}\Big)\,. \tag{4.75}$$

Moreover, the function $\ln[(x-y)^2+\epsilon^2]$ is continuous on the compact support of μ. Applying Fubini's theorem and using the representation (4.75) along with the fact that the measure has a zero mean, one readily gets the equality

$$-\int_{\mathbb{R}} \ln[(x-y)^2+\epsilon^2]\cdot[\mathrm{d}\mu_+(x)\mathrm{d}\mu_+(y)+\mathrm{d}\mu_-(x)\mathrm{d}\mu_-(y)]$$

$$= -\int_{\mathbb{R}} \ln[(x-y)^2+\epsilon^2]\cdot[\mathrm{d}\mu_+(x)\mathrm{d}\mu_-(y)+\mathrm{d}\mu_-(x)\mathrm{d}\mu_+(y)] \;+\; 2\int_0^{+\infty}\frac{\mathrm{d}u}{u}\mathrm{e}^{-\epsilon u}\left|\widehat{\mu}(u)\right|^2\,. \quad (4.76)$$

The sequences of functions $-\ln[(x-y)^2+\epsilon^2]$ and $\mathrm{e}^{-\epsilon u}\left|\widehat{\mu}(u)\right|^2/u$ are increasing and the first one is bounded since the support of u is compact. One can thus apply the monotone convergence theorem leading to the claim. ■

Lemma 4.3 *There exists a unique probability measure μ_V on $\mathbb{R}$ such that $E_V = I_V[\mu_V]$.*

Proof —

Assume that μ_1 and μ_2 both satisfy $E_V = I_V[\mu_1] = I_V[\mu_2]$. Then, in virtue of lemma 4.1, μ_1 and μ_2 both have compact supports. Therefore, V being continuous, it is integrable in respect to μ_k, $k = 1,2$. As a consequence, $E_V < +\infty$ ensures that

$$-\ln|x-y| \tag{4.77}$$

is integrable in respect to $\mu_k\otimes\mu_k$, $k = 1,2$. By (4.72), it is thus integrable in respect to $\mu_1\otimes\mu_2+\mu_2\otimes\mu_1$. Therefore, it is integrable in respect to the measure $\tau_t = \mu_1 + t(\mu_2-\mu_1)$. Yet, as follows from (4.76), the function

$$\rho(t) = \int_{\mathbb{R}} f(x,y)\mathrm{d}\tau_t(x)\mathrm{d}\tau_t(y) \tag{4.78}$$

is convex as $\rho''(t) \geq 0$. Yet, from

$$E_V \;\leq\; I_V[\tau_t] \;\leq\; (1-t)I_V[\mu_1] + tI_V[\mu_2]\;, \tag{4.79}$$

it follows that ρ is constant on $[\,0\,;1\,]$. Thus, $\rho''(t) = 0$ leading to

$$\widehat{(\mu_1 - \mu_2)}(u) = 0 \tag{4.80}$$

Hence, $\mu_1 = \mu_2$, what proves the uniqueness.

Lemma 4.4 *In the case where this measure is continuous in respect to the Lebesgue measure,* $\mathrm{d}\mu_V(x) = \psi(x)\,\mathrm{d}x$, *with* $\psi(x)$ *continuous, one has that there exists a constant* ℓ *such that:*

$$\begin{aligned} V(x) \;-\; 2\beta \int_{\mathbb{R}} \ln|x-y|\psi(y)\,\mathrm{d}y \;\;&>\;\; c_V \qquad \text{for} \quad x \in \mathbb{R} \setminus \operatorname{supp}[\mu_V] \\ V(x) \;-\; 2\beta \int_{\mathbb{R}} \ln|x-y|\psi(y)\,\mathrm{d}y \;\;&=\;\; c_V \qquad \text{for} \quad x \in \operatorname{supp}[\mu_V]\;\;. \end{aligned} \tag{4.81}$$

Proof —
Let μ_V be the equilibrum measure and $\widetilde{\mu}$ another compactly supported probability measure of $\mathbb{R}$ such that $I_V[\widetilde{\mu}]$. Since $I_V[\mu_V] < +\infty$ and $I_V[\widetilde{\mu}] < +\infty$ we get that $\ln|x-y|$ is integrable in respect to both $\mathrm{d}\widetilde{\mu}(x) \otimes \mathrm{d}\widetilde{\mu}(y)$ and $\mathrm{d}\mu_V(x) \otimes \mathrm{d}\mu_V(y)$. It is thus integrable in respect to the signed measure $\mathrm{d}(\widetilde{\mu} - \mu_V)(x) \otimes \mathrm{d}(\widetilde{\mu} - \mu_V)(y)$ in virtue of (4.72). As a consequence $I_V[\tau_t]$ with $\tau_t = \mu_V + t(\widetilde{\mu} - \mu_V)$ is well defined and

$$I_V[\tau_t] \;=\; I_V[\mu_V] + t\int_{\mathbb{R}} \Big\{V(x) - 2\beta \int_{\mathbb{R}} \ln|x-y|\mathrm{d}\mu_V(y)\Big\}\mathrm{d}(\widetilde{\mu} - \mu_V)(x) - \beta t^2 \int_{\mathbb{R}} \ln|x-y| \cdot \mathrm{d}(\widetilde{\mu} - \mu_V)(x) \otimes \mathrm{d}(\widetilde{\mu} - \mu_V)(y)\;. \tag{4.82}$$

Further $t = 0$ is a minimum for $I_V[\tau_t]$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_V[\tau_t]_{|t=0} \geq 0 \quad \text{what leads to} \qquad \int_{\mathbb{R}} \Big\{V(x) - c_V - 2\beta \int_{\mathbb{R}} \ln|x-y|\mathrm{d}\mu_V(y)\Big\}\mathrm{d}\widetilde{\mu}(x) \;\geq\; 0\;. \tag{4.83}$$

Here we have set

$$c_V \;=\; \int_{\mathbb{R}} V(x)\mathrm{d}\mu_V(x) \;-\; 2\beta \int_{\mathbb{R}} \ln|x-y|\;\mathrm{d}\mu_V(x) \otimes \mathrm{d}\mu_V(y)\;. \tag{4.84}$$

Let

$$B = \Big\{x \in \mathbb{R} \;:\; V(x) \;-\; 2\beta \int_{\mathbb{R}} \ln|x-y| \cdot \mathrm{d}\mu_V(y) \;<\; c_V\Big\}\;. \tag{4.85}$$

Assume that $\widetilde{\mu}$ is such that $\widetilde{\mu}[B] > 0$. Then the probability measure $\widetilde{\mu}_B = \widetilde{\mu} \cdot \mathbf{1}_B / \widetilde{\mu}[B]$ has compact support and satisfies

$$\int_{\mathbb{R}} \Big\{V(x) - c_V - 2\beta \int_{\mathbb{R}} \ln|x-y|\mathrm{d}\mu_V(y)\Big\} \cdot \mathrm{d}\widetilde{\mu}_B(x) \;<\; 0 \tag{4.86}$$

a contracdiction. Thus $\widetilde{\mu}[B] = 0$ for any $\widetilde{\mu} \in \mathcal{P}(\mathbb{R})$ with compact support and such that $I_V[\widetilde{\mu}] < +\infty$. This holds true for $\widetilde{\mu} = \mu_V$ meaning that, by definition of c_V,

$$0 = \int_{\mathbb{R}} \Big\{ V(x) - c_V - 2\beta \int_{\mathbb{R}} \ln|x-y| \mathrm{d}\mu_V(y) \Big\} \mathrm{d}\mu_V(x) \;=\; \int_{\mathbb{R}\setminus B} \Big\{ U(x) - c_V - 2\beta \int_{\mathbb{R}} \ln|x-y| \mathrm{d}\mu_V(y) \Big\} \mathrm{d}\mu_V(x) \quad (4.87)$$

since $\mu_U[B] = 0$. Hence,

$$U(x) - c_V - 2\beta \int_{\mathbb{R}} \ln|x-y| \mathrm{d}\mu_V(y) = 0 \qquad \mu_V \; a.e. \; . \quad (4.88)$$

Note that, in the case when the measure μ_V is continuous in respect to the Lebesgue measure, with some continuous density ψ, one has that the function above is continuous on $\mathbb{R}$. As a consequence, (4.81) holds. ■

4.4 Explicit representation in the continuous case

In the remainder of this section, we assume that the equilibrium measure is continuous in respect to the Lebesgue measure. As a consequence, it is described by a density ψ whose support Σ consists of a union of disjoint intervals

$$\Sigma = \cup_{k=1}^{n} [\, \alpha_k \, ; \beta_k \,] \qquad \text{with} \quad \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_n \; . \quad (4.89)$$

Thus, the unknowns in the problem of characterizing the equlibrium measure are the density ψ, the number n of connected components of the support and the endpoints α_i, β_i of each connected component. In the following, we show that the density ψ admits the representation

$$\psi(x) = \frac{q_+(x)\, h(x)}{2i\pi} \mathbf{1}_{\Sigma}(x) \qquad \text{with} \quad q(z) = \prod_{k=1}^{n} (z-\alpha_k)^{\frac{1}{2}} (z-\beta_k)^{\frac{1}{2}} \; . \quad (4.90)$$

Furthermore, if V is analytic in some open neighborhood $\overset{\circ}{\mathscr{V}}(\Sigma)$ of Σ, then the function h entering in this decomposition is holomorphic on the same open neighborhood.

If the density of equilibrium measure exists, then, by taking the weak derivative of (4.81), one obtains a linear integral equation for the density of equilibrium measure:

$$V'(x) - 2\beta \fint_{\Sigma} \frac{\psi(y)}{x-y} \mathrm{d}y \;=\; 0 \; . \quad (4.91)$$

In order to solve this singular integral equation on Σ we introduce the function

$$F(z) = -\frac{1}{2i\pi q(z)} \int_{\Sigma} \frac{\psi(y)}{z-y} \mathrm{d}y \qquad \text{with} \quad q(z) = \prod_{k=1}^{n} (z-\alpha_k)^{\frac{1}{2}} (z-\beta_k)^{\frac{1}{2}} \; . \quad (4.92)$$

It is readily checked that F is the unique solution to below Riemann–Hilbert problem

- $F_+(x) - F_-(x) = -\dfrac{V'(x)}{2i\pi q_+(x)\beta} \quad$ for $x \in \overset{\circ}{\Sigma}$;
- $F \in \mathcal{O}(\mathbb{C} \setminus \Sigma)$;

- $F(z) \underset{z\to\infty}{\longrightarrow} 0$;

Thus, the solution to the RHP for F takes the form

$$F(z) = \int_{\Sigma} \mathrm{d}s \frac{1}{s-z} \frac{-V'(s)}{(2i\pi)^2\, q_+(s)\beta} \; . \tag{4.93}$$

In its turn, this leads to the representation for the density of equilibum measure

$$\psi(x) = q_+(x)(F_+(x) + F_-(x)) \; = \; q_+(x) \fint_{\Sigma} \frac{\mathrm{d}s}{\beta(2i\pi)^2} \cdot \frac{-V'(s)}{(s-x)q_+(s)} \; , \tag{4.94}$$

which can be recast into one more regular by using that V' in holomorphic on an open neighborhood $\mathring{\mathscr{V}}(\Sigma)$ of Σ. Thus, if $\Gamma(\Sigma)$ is a loop around Σ lying inside of $\mathring{\mathscr{V}}(\Sigma)$,

$$\fint_{\Sigma} \frac{\mathrm{d}s}{(2i\pi)^2\beta} \frac{-V'(s)}{(s-x)q_+(s)} \; = \; \oint_{\Gamma(\Sigma)} \frac{V'(s)}{(s-x)q(s)} \frac{\mathrm{d}s}{(2i\pi)^2\beta} \; . \tag{4.95}$$

It remains to write down the conditions fixing the number n and the endpoints α_i, β_i.
As $q(z) \sim z^n$ at $z \to \infty$ and $q(z)F(z) \sim -1/(2i\pi z)$ we get

$$-\int_{\Sigma} y^p \frac{V'(y)}{q_+(y)} \frac{\mathrm{d}y}{2i\pi} = \delta_{p,n} \qquad p = 0, \ldots, n \; . \tag{4.96}$$

The $n-1$ remaining conditions follow from the fact that the constant c_V is the same independently of the intervals $[\,\alpha_k\,; \beta_k\,]$:

$$\int_{\alpha_k}^{\beta_k} \left\{ V'(x) - 2\fint_{\Sigma} \frac{\psi(y)}{x-y} \mathrm{d}y \right\} \mathrm{d}x = 0 \; . \tag{4.97}$$

Note that should all of the above conditions be met, then the associated $\psi(y)$ is *the* density of the equilibrum measure $\mathrm{d}\mu_{\mathrm{eq}}$ in virtue of lemma 4.3.

One can also show that $\mathrm{d}\mu_{\mathrm{eq}}$ is continuous in respect to Lebesgue's measure so that one of these conditions is surely met. However, apart from exceptional cases, one cannot determine the associated parameters n, α_k, β_k explicitly.