The Thursday Colloquium
"The Algebra & Geometry of Modern Physics"

Lecture Notes

QUANTUM (FIELD) THEORY AS A FUNCTOR

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Cahier 4 *November 7th, 14th & 28th, 2013*

1° Motivation

1.1° Mathematical

→ compact orientable 3-manifolds through surgery on unknotted links in $\mathbb{S}^3$ (Alexander, Dehn, Lickorish, Wicky, Markov et al.);

→ Developed theory of topological invariants (in low dim.), with its known deficiencies (cf. Jones polynomial)

1.2° Physical

→ quantisation (à la Feynman-Dirac) as a structured assignment of Hilbert spaces & linear operators acting between histories of (charged) phys. entities (in interaction);

→ Feynman amplitudes (often) encode global topological properties of the cov. configuration bundle (e.g., when defined in terms

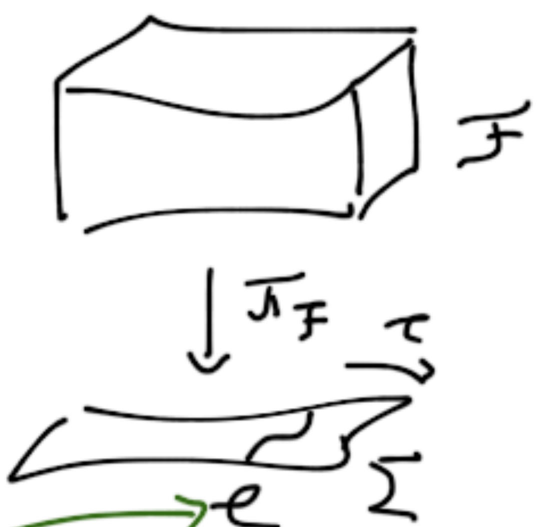
of differential characters
determined by geometric
objects realising classes
in the cohomology group
capturing the Dirac-quantised
topological charge
of the phys. entities.

2° Myth : Topological Quantum Field Theory as a scheme of generating topological invariants that generalise the celebrated Jones polynomial, etc

– since ca. 2000 (effectively) – a new scheme of quantisation of CFT.

3° Abstraction of the fundamental properties of the "quantum map":

classical field theory:



Cauchy hypersurface

$$S : \Gamma(\mathcal{F}) \to \mathbb{R}$$
$$\phi \mapsto S[\phi]$$

$\xrightarrow{\text{can. desc.}}$ SYMPL. STATE SPACE $(\mathcal{P}, \Omega)$

Cauchy data $(\phi|_e, \partial_\tau \phi|_e) \in \mathcal{P}$

NO LINEAR STRUCTURE!

Models phys. entity (particle, string ...)

quantum field theory :

$$e \longmapsto (\mathcal{H}_e, (\cdot,\cdot))$$

Hilbert space of states

For N entities take $\otimes_{\mathbb{C}}$

$$\Psi[\Phi] = \int_{\phi|_e = \Phi} \mathcal{D}\phi \; e^{iS[\phi]_{\tau \leq \tau_e}}$$

propagated by $\int \mathcal{D}\phi \, e^{iS[\phi]_{\tau_{e_1} \leq \tau \leq \tau_{e_2}}}$
(integral kernels)

Observation: Feynman amplitudes sensitive to global structure of $F \to \Sigma$.

Abstraction:

$$\mathcal{C} \longmapsto \mathcal{H}_{\mathcal{C}} \in \mathrm{Ob}\,\mathrm{Vect}_{\mathbb{K}}$$

$$\mathcal{C}_1 \xRightarrow{\Sigma_{1,2}} \mathcal{C}_2 \longmapsto X_{1,2} \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}_{\mathcal{C}_1}, \mathcal{H}_{\mathcal{C}_2})$$

We assume
$\mathcal{C}$ closed ($\partial\mathcal{C} = \phi$, compact)

More specifically,
we shall focus on Topological
FTs, i.e. those that depend
solely on top. class of Σ
(& that of $\mathcal{C}$)

Thus, we should have a realization of homeo's of $\mathcal{C}$ on $\mathcal{H}_C$ (through suitable cobordisms $\mathcal{C} \Longrightarrow f(\mathcal{C})$)

Among these,

$$\mathcal{C} \times I \;, \quad I := [0,1]$$

represents free propagation, with trivial effect on $\mathcal{H}_\mathcal{C}$.

Furthermore, we want to capture many-body propagation & interactions, & hence

$$\coprod_i \mathcal{C}_i \longmapsto \bigotimes_{i}{}_{\mathbb{C}} \mathcal{H}_{\mathcal{C}_i}$$

Finally, we note the factorisation property of the path integral,

$$\int_{\phi|_{e_i}=\Phi_i} \mathcal{D}\phi \, e^{iS[\phi]_{\Sigma_{1,2}}}$$

intermediate states

$$= \int_{\Phi_{3/2}:\, e_{3/2} \to F} \mathcal{D}\Phi_{3/2} \int_{\substack{\phi|_{e_1}=\Phi_1 \\ \phi|_{e_{3/2}}=\Phi_{3/2}}} \mathcal{D}\phi_1 \, e^{iS[\phi]_{\Sigma_{1,3/2}}} \int \mathcal{D}\phi_2 \, e^{iS[\phi]_{\Sigma_{3/2,2}}}$$

4° Axiomatisation

[Segal, Witten, Atiyah, Turaev, Reshetikhin et al.]

4.1° Geometric prerequisites:

First of all, we need a model of bounding surfaces, i.e. topological spaces which may carry additional structure (e.g., orientation, smooth str., markings etc.).

Def^{n} 1. Space structure

(in dimension $n \in \mathbb{N}$)

is a covariant functor

$$\mathcal{A} : n\mathrm{Top}_{\cong} \longrightarrow \mathrm{Set}_{\cong}$$

(with homeos) (with bijections),

$\ni$

$\emptyset$

i.e., X - top. space, $X \xrightarrow{f} Y$

$$\begin{array}{ccc} X & & X \xrightarrow{f} Y \\ \downarrow & & \downarrow \\ \mathcal{A}(X) \text{ - set} & & \mathcal{A}(X) \xrightarrow{\mathcal{A}(f)} \mathcal{A}(Y) \end{array}$$

$\mathcal{A}$-structures on X $\rightarrow$ $\mathcal{A}(X)$

$(X, \alpha \in \mathcal{A}(X))$ is called **$\mathcal{A}$-space**.

$\mathcal{A}(\emptyset) = \{*\}$ (singleton)

$\mathcal{A}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{A}(X)}$

E.g. Ori_n is the orientation functor

Diff_n is the differentiable-structure functor

Def^n 2. Let (X,α), (Y,β) be two $\mathcal{A}$-spaces.

An $\mathcal{A}$-homeo between (X,α) & (Y,β) is a homeo $f: X \to Y$

with the property

$$\mathcal{A}(f)(\alpha) = \beta .$$

E.g. - ~~orientation~~ -preserving homeos
- diffeos

In this manner, we get

category $\mathrm{nTop}^{\mathcal{A}}$

of $\mathcal{A}$-spaces with $\mathcal{A}$-homeos.

(Clearly, id_X is $\mathcal{A}$-homeo,
& composition of $\mathcal{A}$-homeos is $\mathcal{A}$-homeo.)

As we want to model multi-entity histories, we need

Defn 3.

Let $\mathcal{A}$ be a presheaf in Dim. $n \in \mathbb{N}$. It is termed

$\sqcup$-compatible iff

$\forall X, Y \in \mathrm{Ob}(n\mathrm{Top})$ $\exists$ canonical mapping $\mathcal{A}(X) \times \mathcal{A}(Y) \xrightarrow{\iota_{X,Y}} \mathcal{A}(X \sqcup Y)$

such that

(i)
$$\begin{tikzcd}
\mathcal{A}(X) \times \mathcal{A}(Y) \arrow[r, "\iota_{X,Y}"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{A}(X \sqcup Y) \arrow[d, "\mathcal{A}(\sigma^{\sqcup})", "\wr"'] \\
\mathcal{A}(Y) \times \mathcal{A}(X) \arrow[r, "\iota_{Y,X}"'] & \mathcal{A}(Y \sqcup X)
\end{tikzcd}$$

(ii) $\forall\, f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$:
$g \in \mathrm{Hom}_{\mathcal{A}}(Y, Y')$

$$\begin{tikzcd}
\mathcal{A}(X) \times \mathcal{A}(Y) \arrow[r, "\iota_{X,Y}"] \arrow[d, "\mathcal{A}(f)\times\mathcal{A}(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{A}(X \sqcup Y) \arrow[d, "\mathcal{A}(f \sqcup g)"] \\
\mathcal{A}(X') \times \mathcal{A}(Y') \arrow[r, "\iota_{X',Y'}"'] & \mathcal{A}(X' \sqcup Y')
\end{tikzcd}$$

(thus, $\iota : \mathcal{A} \circ \sqcup \Rightarrow \mathcal{A} \times \mathcal{A}$
is a natural transformation)

(iii) $\forall\, X, Y, Z \in \mathrm{Ob}(n\mathrm{Top})$:

$$\begin{tikzcd}
\mathcal{A}(X) \times \mathcal{A}(Y) \times \mathcal{A}(Z) \arrow[r, "\iota_{X,Y} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \iota_{Y,Z}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{A}(X \sqcup Y) \times \mathcal{A}(Z) \arrow[d, "\iota_{X \sqcup Y, Z}"] \\
\mathcal{A}(X) \times \mathcal{A}(Y \sqcup Z) \arrow[r, "\iota_{X, Y \sqcup Z}"'] & \mathcal{A}(X \sqcup Y \sqcup Z)
\end{tikzcd}$$

(iv) $\iota_{\emptyset, X}$ is induced by $\{*\} \times \mathcal{A}(X) = \mathcal{A}(X)$.

Since, furthermore, we want to consider oriented entities, in- & out-states, we give

Def^{n} 4.

The open functor $\mathcal{A}$ is **involutive** iff $\forall X \in \mathrm{Ob}(n\mathrm{Top})$ $\exists$ involution $\omega : \mathcal{A}(X) \circlearrowleft : \alpha \longmapsto -\alpha$ s.th. $\forall X, Y \in \mathrm{Ob}(n\mathrm{Top})$ $\forall f \in \mathrm{Hom}(X, Y)$: $\iota_{X,Y}$ & $\mathcal{A}(f)$ are ω-equivariant.

Denote $-X \equiv (X, -\alpha)$

Next, we want to model propagation of phys. entities (possibly involving the splitting-joining interactions), whence

Def^n 5. Let $\mathscr{A}$ be an involutive $\sqcup$-compatible space structure in dimension $n \in \mathbb{N}$, & let $(X_-, \alpha_-), (X_+, \alpha_+) \in \mathrm{Ob}(n\mathrm{Top}^{\mathscr{A}})$. Let $\mathscr{B}$ be a $\sqcup$-compatible space structure in dimension $n+1$, with $\partial_{\mathscr{A}}$-property:

$$\forall (M, \beta) \in \mathrm{Ob}((n+1)\mathrm{Top}^{\mathscr{B}}) : M \supset (\partial M, \alpha^{\partial}) \in \mathrm{Ob}(n\mathrm{Top}^{\mathscr{A}})$$

We call the latter the **$\mathcal{A}$-boundary** of M, assuming further that $\partial \circ \sqcup = \sqcup \circ \partial$.

We call $(M, \beta) \in \mathrm{Ob}((n+1)\mathrm{Top}^{\mathcal{B}})$ an **$(\mathcal{A}, \mathcal{B})$-cobordism from** (X_-, α_-) **to** (X_+, α_+) iff

$$(\partial M, \alpha^\partial) = (\partial_- M, \alpha^\partial_-) \sqcup (\partial_+ M, \alpha^\partial_+)$$

s.t. there exist $\mathcal{A}$-homeos

$$f^\pm : (X_\pm, \alpha_\pm) \longrightarrow (\partial_\pm M, \pm \alpha^\partial_\pm).$$

This we denote as bases

$$(X_-, \alpha_-) \xRightarrow{(M, \beta)} (X_+, \alpha_+),$$

or - if no confusion can arise -

as $X_- \xRightarrow{M} X_+$.

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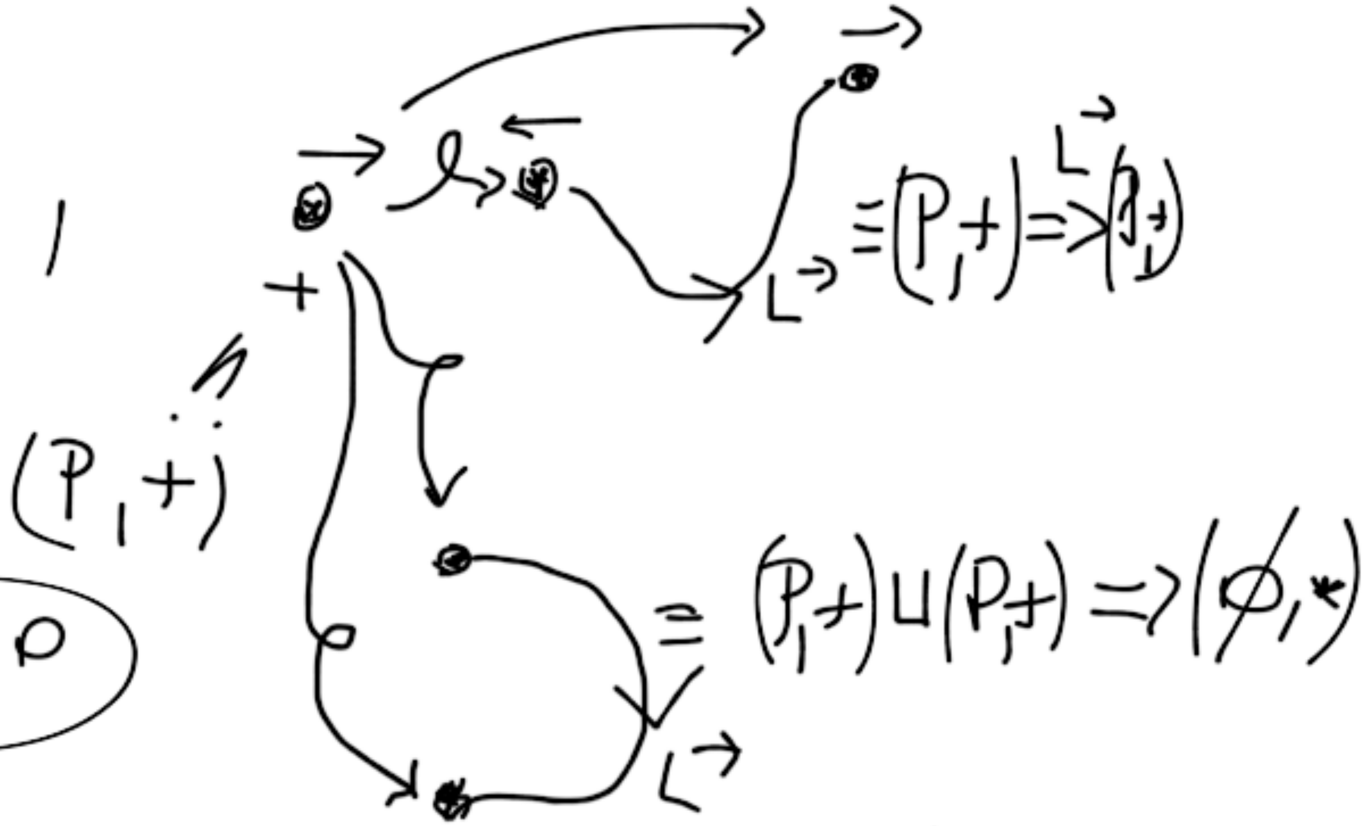
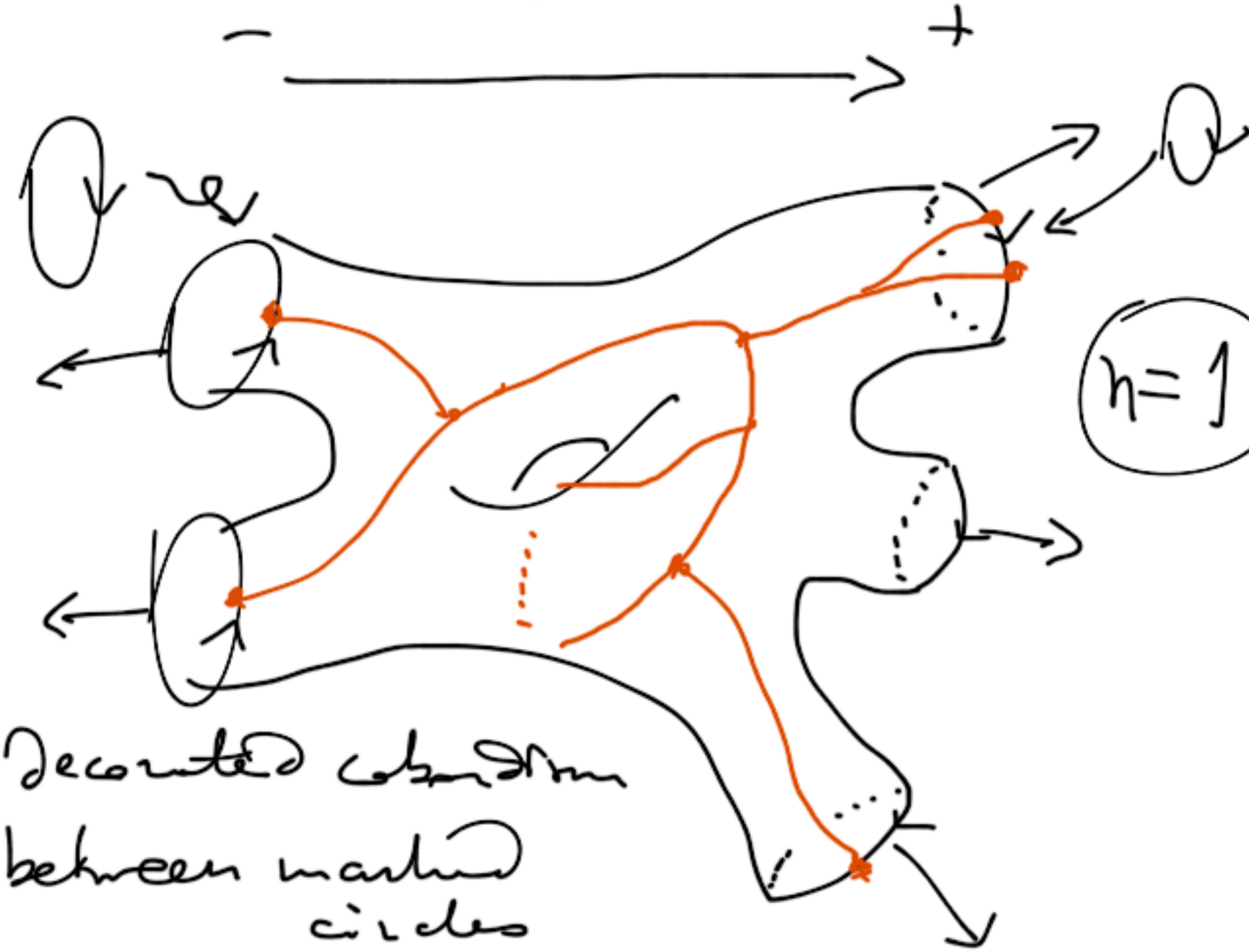
E.g. 1



$n = 0$



$n = 1$

decorated cobordism between marked circles

Def^n 6.

Given two (A, B)-cobordisms

$$(X^i_-, \alpha^i_-) \xrightarrow{(M_i, \beta_i)} (X^i_+, \alpha^i_+), \quad i \in \{1, 2\},$$

a ∂_+-mapping between them is a triple (g, f_-, f_+) composed of a B-homeo $g : (M_1, \beta_1) \to (M_2, \beta_2)$ & a pair of A-homeos $f_\pm : (X^1_\pm, \alpha^1_\pm) \to (X^2_\pm, \alpha^2_\pm)$ that render commutative the diagram

$$\begin{tikzcd}
(X^1_-, \alpha^1_-) \arrow[r, "f^1_-"] \arrow[d, "f_-"'] \arrow[dr, phantom, "\circlearrowleft"] & (M_1, \beta_1) \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & (X^1_+, \alpha^1_+) \arrow[l, "f^1_+"'] \arrow[d, "f_+"] \\
(X^2_-, \alpha^2_-) \arrow[r, "f^2_-"'] & (M_2, \beta_2) & (X^2_+, \alpha^2_+) \arrow[l, "f^2_+"]
\end{tikzcd}$$

This, altogether, leads us to

Def^n 7. Let $\mathcal{A}, \mathcal{B}$ be two $\sqcup$-compatible space structures, & assume further $\mathcal{A}$ to be involutive. The ***category of $(\mathcal{A},\mathcal{B})$-cobordisms*** (in dimension $n \in \mathbb{N}^\times$),

$$n\mathrm{Cob}^{(\mathcal{A},\mathcal{B})}, \quad \underline{\text{closed}}!$$

has $\mathrm{Ob}\left(n\mathrm{Cob}^{(\mathcal{A},\mathcal{B})}\right) = \left\{\begin{array}{l}\mathcal{A}\text{-spaces}\\ \text{in dimension } n-1\end{array}\right\}$

$$\forall\, (X_\pm, \alpha_\pm) : \mathrm{Hom}\left((X_-,\alpha_-),(X_+,\alpha_+)\right)$$

$$\|$$

$$\left\{(\mathcal{A},\mathcal{B})\text{-cobordisms } (X_-,\alpha_-) \xRightarrow{(M,\beta)} (X_+,\alpha_+)\right\},$$

where it is assumed that

(i) $\mathcal{B}$ admits <u>gluing</u>, i.e.,

for any two $(\mathcal{A},\mathcal{B})$-cobordisms

$$(X_-,\alpha_-) \xRightarrow{(M_1,\beta_1)} (X_+^1,\alpha_+^1) \sqcup (X_+^2,\alpha_+^2)$$

& $$(Y_-^1,\tilde{\alpha}_-^1) \sqcup (Y_-^2,\tilde{\alpha}_-^2) \xRightarrow{(M_2,\beta_2)} (Y_+,\tilde{\alpha}_+)$$

for which there exists an $\mathcal{A}$-homeo

$$g : (X_+^1,\alpha_+^1) \longrightarrow (Y_-^1,\tilde{\alpha}_-^1)$$

the pair (β_1,β_2) <u>canonically</u> induces a $\mathcal{B}$-structure

$$\beta_1 \sqcup_g \beta_2$$

on the pushout

$$M_1 \sqcup_g M_2 = \left(M_1 \sqcup M_2\right)/\sim_g$$

where $m_1 \sim_g m_2 \iff \exists x \in X^1_+ : \begin{cases} m_1 = f^1_+(x) \\ m_2 = \tilde{f}^1_- \circ g(x) \end{cases}$.

N.B.

$M_1 \sqcup_g M_2$ is the initial object for

$$\begin{tikzcd}
 & S & \\
 & M_1 \sqcup_g M_2 \arrow[u, dashed] & \\
M_1 \arrow[uur, "\tilde{i}_1"] \arrow[ur, "i_1"'] & & M_2 \arrow[uul, "\tilde{i}_2"'] \arrow[ul, "i_2"] \\
 & X^1_+ \arrow[ul, "f^1_+"] \arrow[ur, "\tilde{f}^1_- \circ g"'] &
\end{tikzcd}$$

Note :

$$\partial(M_1 \sqcup_g M_2) = \overbrace{(X_-, \alpha_-) \sqcup (Y^2_-, \tilde{\alpha}^2_-)}^{\partial_-}$$
$$\sqcup \underbrace{(X^2_+, \alpha^2_+) \sqcup (Y_+, \tilde{\alpha}_+)}_{\partial_+}$$

(ii) Gluings of (A,B)-cobs are natural wrt. morphisms of $n\mathrm{Cob}^{(A,B)}$, & commute with $\sqcup$.

(iii) Each $(X, \alpha) \in \mathrm{Ob}(n\mathrm{Cob}^{(A,B)})$ canonically induces a B-str. $\alpha \times I$ on the cylinder $X \times I$, $I = [0,1]$, with $\partial(X \times I, \alpha \times I) = (X, -\alpha) \times \{0\} \sqcup (X, \alpha) \times \{1\}$

& $I \circlearrowleft : t \mapsto 1-t$ transforming

$\alpha \times I \longmapsto -\alpha \times I$.

Hence, the induction

$$\alpha \longmapsto \alpha \times I$$

is natural wrt. homeos

& commutes with $\sqcup$.

(iv) Let $X^{(1)} = X = X^{(2)}$.

Then, $(X^{(1)} \times I) \sqcup_g (X^{(2)} \times I)$

for $g : X^{(1)} \times \{1\} \to X^{(2)} \times \{0\}$

$$(x, 1) \longmapsto (x, 0)$$

is isomorphic to $X \times I$ via k s.th. $k|_{X^{(1)} \times \{0\} \sqcup X^{(2)} \times \{1\}} = \mathrm{id}$.

Composition of morphisms
is determined by the gluing
(in an obvious manner);
the identity map
is determined by the cylinder
construction.

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Taking into account
the assumed $\sqcup$-compatibility
of the space structures involved,
we find

<u>Prop^y 1.</u> $\left(n\mathrm{Cob}^{(A,B)}, \sqcup, (\emptyset, \{*\}), \sigma^{\sqcup}\right)$
is a symmetric monoidal
category.

Remark:

The construction presented automatically enables us to model multi-body processes as it allows disconnected bases.

4.2° Algebraic prerequisites

We are now prepared to discuss the algebraic modelling of physical theories.

In the first step, we "quantise" the physical entities.

Def^n 8.

Let $\mathcal{A}$ be a $\sqcup$-compatible space structure. Fix a field $\mathbb{K}$.

An *$\mathcal{A}$-based modular functor with ground field $\mathbb{K}$ (in dimension n)*

is a STRONG cov. monoidal functor

$$(\mathcal{J}, \iota) : (n\mathcal{T}op^{\mathcal{A}}, \sqcup, \emptyset) \to (\mathrm{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$$

(with $\mathcal{A}$-homeos) (with $\mathbb{K}$-isos)

with coherence isomorphisms

$$\iota : \mathcal{J} \circ \sqcup \overset{\sim}{\Longrightarrow} \otimes_{\mathbb{K}} \circ (\mathcal{J} \times \mathcal{J}) .$$

The vector space

$$\mathcal{J}(X) \in \mathrm{Ob}(\mathrm{Vect}_{\mathbb{K}})$$

is termed the module of states over X & its elements are states on X.

For $\mathcal{A}$ involutive, the modular functor $\mathcal{J}$ is called self-dual iff for each $X \in \mathrm{Ob}(n\mathrm{Top}^{*})$ there exists a non-singular pairing

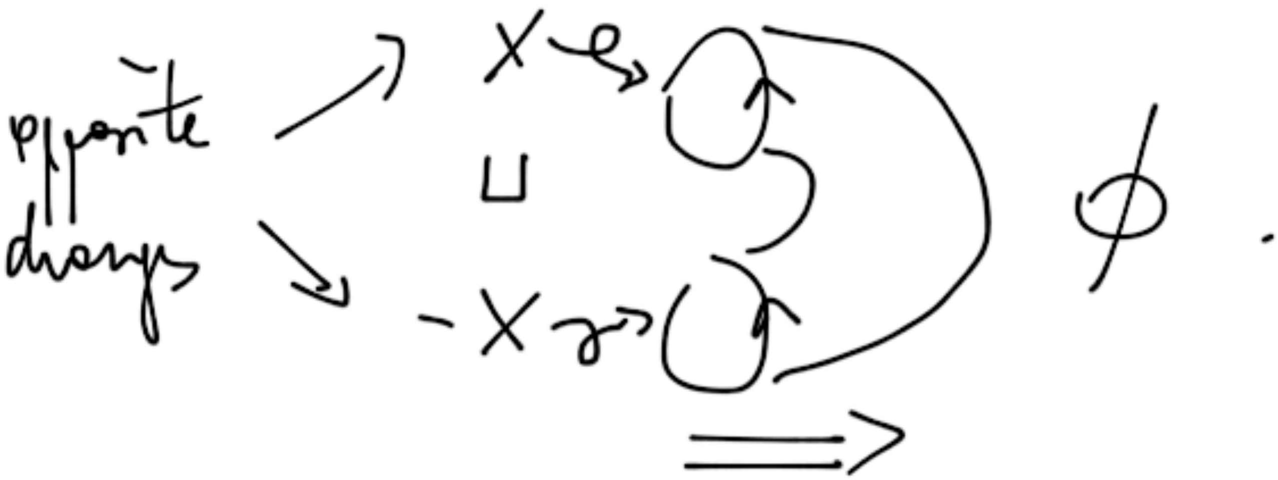
$$d_X : \mathcal{J}(X) \otimes_K \mathcal{J}(-X) \longrightarrow \mathbb{K}$$

& the $d_\cdot$ are natural wrt. $\mathcal{A}$-homeos, multiplicative wrt. $\sqcup$, & satisfy the identity

$$(S) \quad d_{-X} = d_X \circ \sigma_{\mathcal{J}(-X), \mathcal{J}(X)} .$$

Clearly, the last property is a mathematical transcription of the demand of the existence of a charge conjugation on phys. entities.

Indeed, we want to allow for the pair-annihilation processes captured by cobordisms of the form



In order to analyse them, we shall need ...

Lemma 1. Let $\mathbb{K}$ be a field, & consider $V, W \in \mathrm{Ob}(\mathrm{Vect}_{\mathbb{K}})$.
Let $b \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, V \otimes_{\mathbb{K}} W)$ & $d \in \mathrm{Hom}_{\mathbb{K}}(W \otimes V, \mathbb{K})$ satisfy the identities

$$(\mathrm{id}_V \otimes d) \circ (b \otimes \mathrm{id}_V) = k_1 \, \mathrm{id}_V$$

& $$(d \otimes \mathrm{id}_W) \circ (\mathrm{id}_W \otimes b) = k_2 \, \mathrm{id}_W$$

for some $k_1, k_2 \in \mathbb{K}^{\times}$.

Then, $k_1 = k_2$ & both b & d are non-singular in the sense that the induced homomorphisms

$$V^* \longrightarrow W \; : \; \varphi \longmapsto \langle b(1_{\mathbb{K}}), \varphi \otimes \mathrm{id}_W \rangle$$

$$W^* \longrightarrow V \; : \; \psi \longmapsto \langle b(1_{\mathbb{K}}), \mathrm{id}_V \otimes \psi \rangle$$

so

$$V \longrightarrow W^* : v \longmapsto \langle \cdot \otimes_{\mathbb{K}} v, d \rangle$$

$$W \longrightarrow V^* : w \longmapsto \langle w \otimes_{\mathbb{K}} \cdot, d \rangle$$

are isomorphisms, so

$$\infty > \dim_{\mathbb{K}} V = \dim_{\mathbb{K}} W$$
$$= k_1^{-1} \cdot (d \circ \sigma_{V,W} \circ b)(1_{\mathbb{K}}).$$

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4.3° Sekhet Hetepet

Defn 9. Fix a field $\mathbb{K}$.

Let $\mathcal{A}$ be an involutive U-compatible space structure in dimension $n-1$, & let $\mathcal{B}$ be a U-compatible space structure in dimension n, with the $\partial_{\mathcal{A}}$-property of Defn 5.

An **$(\mathcal{A},\mathcal{B})$-based Topological Quantum Field Theory** is a pair

$$(\mathcal{T}, \tau)$$

composed of an $\mathcal{A}$-based modular functor (with ground field $\mathbb{K}$)

$$\mathcal{J} : (n-1)\mathrm{Top}^{\mathcal{A}} \to \mathrm{Vect}_{\mathbb{K}}$$

& a map, termed operator invariant,

$$\tau : \mathrm{Mor}\left(n\,\mathrm{Cob}^{(\mathcal{A},\mathcal{B})}\right) \to \mathrm{Mor}\left(\mathrm{Vect}_{\mathbb{K}}\right)$$

$$\left(X_- \xrightarrow{M} X_+\right) \mapsto \left(\mathcal{J}(X_-) \xrightarrow{\tau(M,X_\pm)} \mathcal{J}(X_+)\right)$$

satisfying the following axioms:

(Naturality) For each $\partial_{\mathcal{A}}$-mapping

$(g, f_-, f_+) : (M_1, X_-^1, X_+^1) \to (M_2, X_-^2, X_+^2)$,

$$\begin{tikzcd}
\mathcal{J}(X_-^1) \arrow[r, "\tau(M_1, X_-^1, X_+^1)"] \arrow[d, "\mathcal{J}(f_-)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{J}(X_+^1) \arrow[d, "\mathcal{J}(f_+)"] \\
\mathcal{J}(X_-^2) \arrow[r, "\tau(M_2, X_-^2, X_+^2)"'] & \mathcal{J}(X_+^2)
\end{tikzcd}$$

(Multiplicativity)

$$\tau\left((M_1, X^1_-, X^1_+) \sqcup (M_2, X^2_-, X^2_+)\right) = \tau(M_1, X^1_-, X^1_+) \otimes \tau(M_2, X^2_-, X^2_+)$$

(Projective Functoriality)

For each pair of (A,B)-cobordisms (M_i, X^i_-, X^i_+), $i \in \{1,2\}$, composable along $g : X^1_+ \longrightarrow X^2_-$, there exists a scalar $k(M_1, M_2, g) \in \mathbb{K}^\times$ s.th.

$$\tau\left((M_2, X^2_-, X^2_+) \circ_g (M_1, X^1_-, X^1_+)\right)$$

$$= k(M_1, M_2, g) \cdot \tau(M_2, X^2_-, X^2_+) \circ J(g) \circ \tau(M_1, X^1_-, X^1_+).$$

We call it the gluing anomaly.

(Normalization)

$$\forall (X,\alpha) \in \mathrm{Ob}(n\mathrm{Top}^{\mathrm{st}}) : \tau(X \times I, X, X) = \mathrm{id}_{\mathcal{J}(X)}.$$

We call $(\mathcal{J}, \tau)$ <u>anomaly-free</u>

iff we can consistently set $k = 1_{\mathbb{K}}$.

It is natural to enquire about structural rel^{s} between TQFTs, in particular with view to their classification. This leads us to

Def^n 10. Adopt the notation of Def^n 9., & let $(\mathcal{T}_i, \tau_i)$, $i \in \{1,2\}$ be two $(\mathcal{A}, \mathcal{B})$-based TQFT with ground field $\mathbb{K}$ (in dimension n).

A morphism between $(\mathcal{T}_1, \tau_1)$ & $(\mathcal{T}_2, \tau_2)$ is a natural transformation

$$\eta : \mathcal{T}_1 \Rightarrow \mathcal{T}_2$$

that commuting with operator invariants. It is an isomorphism if the η_X are isomorphisms.

We thus obtain the category

$$n\mathrm{TQFT}_{\mathbb{K}}^{(\mathcal{A},\mathcal{B})}$$

of $(\mathcal{A}, \mathcal{B})$-based TQFTs with ground field $\mathbb{K}$ in dimension n

4.4° Properties of TQFT

Let us first examine the basic consequences of the axioms.

In what follows, $\mathbb{K}$ denotes a field, $\mathcal{G}$, $\mathcal{A}, \mathcal{B}$ are $\sqcup$-compatible space structures, of which the first is involutive, $\mathcal{G}$ and that there is a well-defined category $\mathrm{Cob}^{(\mathcal{A},\mathcal{B})}$.

We shall also use the notation:

$$\underline{\tau(M)} := \tau(M, \phi, \partial M)(1_{\mathbb{K}}) \in \mathcal{J}(\partial M).$$

— x —

We also have the auxiliary

Defn 11. Let $(\mathcal{J}, \tau)$ be an $(\mathcal{A},\mathcal{B})$-based TQFT. An anomaly 2-cocycle is a map

$$\alpha : \left\{\begin{matrix}\text{composable}\\ (\mathcal{A},\mathcal{B})\text{-cobordisms}\end{matrix}\right\} \longrightarrow \mathbb{K}^\times$$

determining the gluing anomaly of $(\mathcal{J},\tau)$, with the following properties:

(i) for every morphism between two pairs of $(\mathcal{A},\mathcal{B})$-cobordisms: (M_i, X_-^i, X_+^i), $i \in \{1,2\}$ & (N_j, Y_-^j, Y_+^j), $j\in\{1,2\}$, composable along $\mathcal{A}$-homeos g & h, respectively, i.e., a pair of morphisms between $(\mathcal{A},\mathcal{B})$-cobordisms:

$$\left(g_{(i)}, f_+^{(i)}, f_-^{(i)}\right) : (M_i, X_-^i, X_+^i) \longrightarrow (N_i, Y_-^i, Y_+^i),\ i\in\{1,2\},$$

that renders commutative the following diagram:

$$
\begin{tikzcd}
X^1_- \arrow[r, "f^1_-"] \arrow[d, "f^{(1)}_-"'] \arrow[dr, phantom, "\circlearrowleft"] & M_1 \arrow[d, "g^{(1)}"] \arrow[dr, phantom, "\circlearrowright"] & X^1_+ \arrow[l, "f^1_+"'] \arrow[r, "g"] \arrow[d, "f^{(1)}_+"] \arrow[dr, phantom, "\circlearrowleft"] & X^2_- \arrow[r, "f^2_-"] \arrow[d, "f^{(2)}_-"] \arrow[dr, phantom, "\circlearrowright"] & M_2 \arrow[d, "g^{(2)}"] \arrow[dr, phantom, "\circlearrowright"] & X^2_+ \arrow[l, "f^2_+"'] \arrow[d, "f^{(2)}_+"] \\
Y^1_- \arrow[r, "g^1_-"'] & N_1 & Y^1_+ \arrow[l, "g^1_+"] \arrow[r, "h"'] & Y^2_- \arrow[r, "g^2_-"'] & N_2 & Y^2_+ \arrow[l, "g^2_+"]
\end{tikzcd}
$$
,

the corresponding values of α satisfy

$$\alpha(M_1, M_2, g) = \alpha(N_1, N_2, h) \; ;$$

(ii) for any two pairs of (A,B)-cobordisms:

(M_i, X^i_-, X^i_+), $i \in \{1,2\}$ & (N_j, Y^j_-, Y^j_+), $j \in \{1,2\}$,

composable along A-homeos g & h, respectively, the following identity holds:

$$\alpha(M_1 \sqcup N_1, M_2 \sqcup N_2, g \sqcup h) = \alpha(M_1, M_2, g) \cdot \alpha(N_1, N_2, h);$$

(iii) for any (A,B)-cobordism (M, X_-, X_+),

$$\alpha(M, X_+ \times I, \mathrm{id}_{X_+}) = 1_{\mathbb{K}} \; ;$$

(iv) if (N, Y_-, Y_+) is an $(\mathcal{A},\mathcal{B})$-cobordism composable with the disjoint union $(M_1, X_-^1, X_+^1) \sqcup (M_2, X_-^2, X_+^2)$ of $(\mathcal{A},\mathcal{B})$-cobordisms along an $\mathcal{A}$-homeo

$$g_1 \sqcup g_2 : X_+^1 \sqcup X_+^2 \to Y_-^1 \sqcup Y_-^2 \cong Y_-,$$ then

$$\alpha(M_1 \sqcup M_2, N, g_1 \sqcup g_2) = \alpha(M_1, (N, Y_-^1, Y_+ \sqcup (-Y_-^2)), g_1)$$
$$\cdot \alpha(M_2, (N \sqcup_{g_1} M_1, Y_-^2, (-X_-^1) \sqcup Y_+), g_2);$$

(v) for any two $(\mathcal{A},\mathcal{B})$-cobordisms (M_i, X_-^i, X_+^i), $i \in \{1,2\}$ composable along an $\mathcal{A}$-homeo g, the following identity holds:

$$\alpha(M_1 \sqcup_{id} (X_+^1 \times I), M_2, X_+^1 \times \{1\} \xrightarrow{\cong} X_+^1 \xrightarrow{g} X_-^2)$$
$$= \alpha(M_1, M_2, g).$$

We are now ready to state

Th^{m} 1. (DUALITY)

The modular functor $\mathcal{T}$ of a TQFT $(\mathcal{T}, \tau)$ is self-dual.

———— × ————

Th^{m} 2. (SCALAR PRODUCT)

Let $M = M_1 \sqcup_g M_2$, $\partial M = \emptyset$

where $g : X_1^+ \to X_2^-$, $X_1^+ \simeq \partial_+ M_1 \equiv \partial M_1$, $-X_2^- \simeq \partial_- M_2 \equiv \partial M_2$

is the gluing A-homeo.

Denote $-g : -X_1^+ \to -X_2^-$

(the same map).

If $(\mathcal{T}, \tau)$ is anomaly-free, then

$$\tau(M) = d^{\tau}_{X_1^+}\left(\tau(M_1) \otimes_{\mathbb{K}} \mathcal{T}(-g)^{-1}(\tau(M_2))\right) = d^{\tau}_{-X_2^-}\left(\tau(M_2) \otimes_{\mathbb{K}} \mathcal{T}(g)(\tau(M_1))\right).$$

———— × ————

Thm 3. (DIMENSION OF THE MODULE OF STATES)

In any anomaly-free TQFT $(\mathcal{T}, \tau)$, we have

$\forall (X, \alpha) \in \mathrm{Ob}(n\mathcal{T}op^*)$: $\tau(X \times S^1) = \dim_{\mathbb{K}} \mathcal{T}(X)$.

Thm 4. (REALISATION OF SYMMETRIES) The module of states $\mathcal{T}(X)$ carries a projective linear representation of the mapping class group

$$1 \to \mathrm{Aut}_0^A(X) \to \mathrm{Aut}^A(X) \to MCG^*(X) \to 1$$

(isotopy classes of A-homeos)

If $(\mathcal{T}, \tau)$ admits an anomaly 2-cocycle.

We shall now give proofs of the above structural theorems.

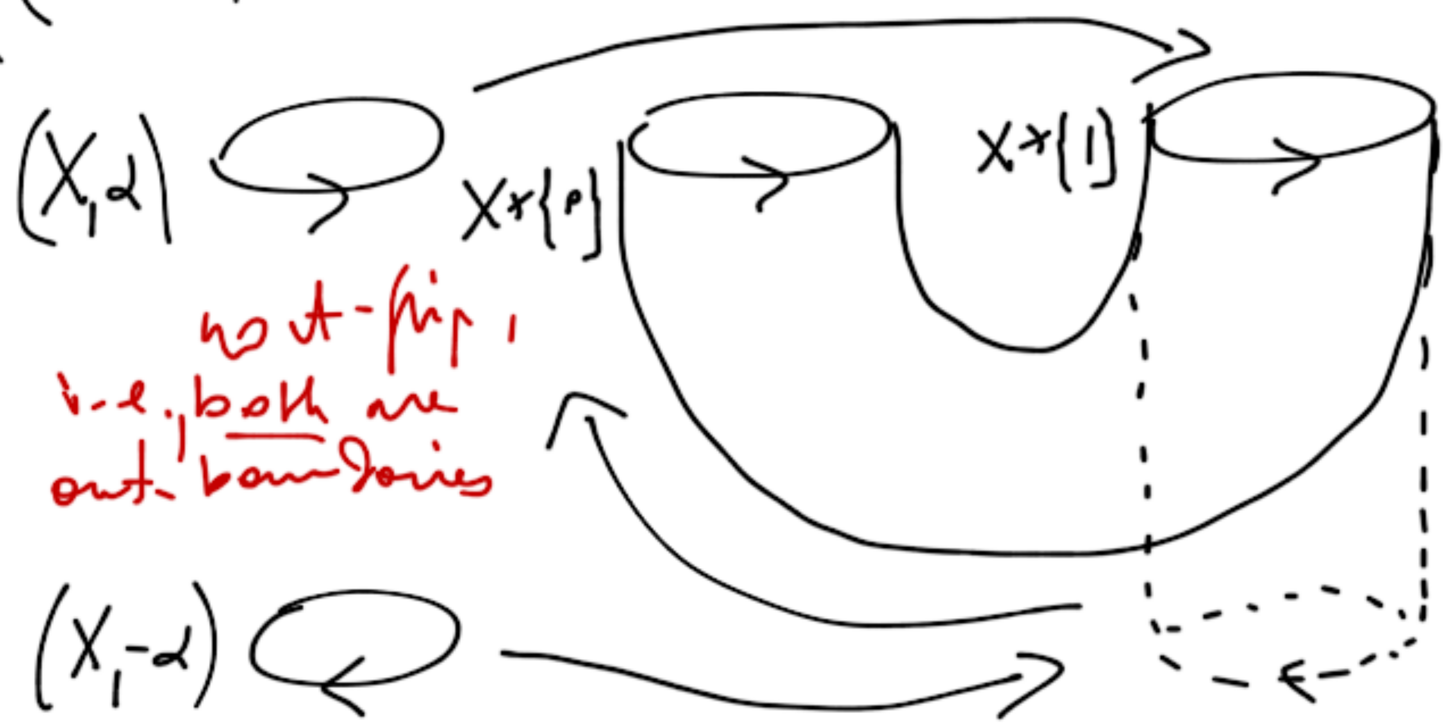
Proof of Th^m 1.

Consider $(X,\alpha) \in \mathrm{Ob}((n-1)\mathrm{Top}^{or})$

& write $V := \mathcal{T}(-X)$, $W := \mathcal{T}(X)$.

By axiom (7.iii), we get a pair of $(\mathcal{A},\mathcal{B})$-cobordisms:

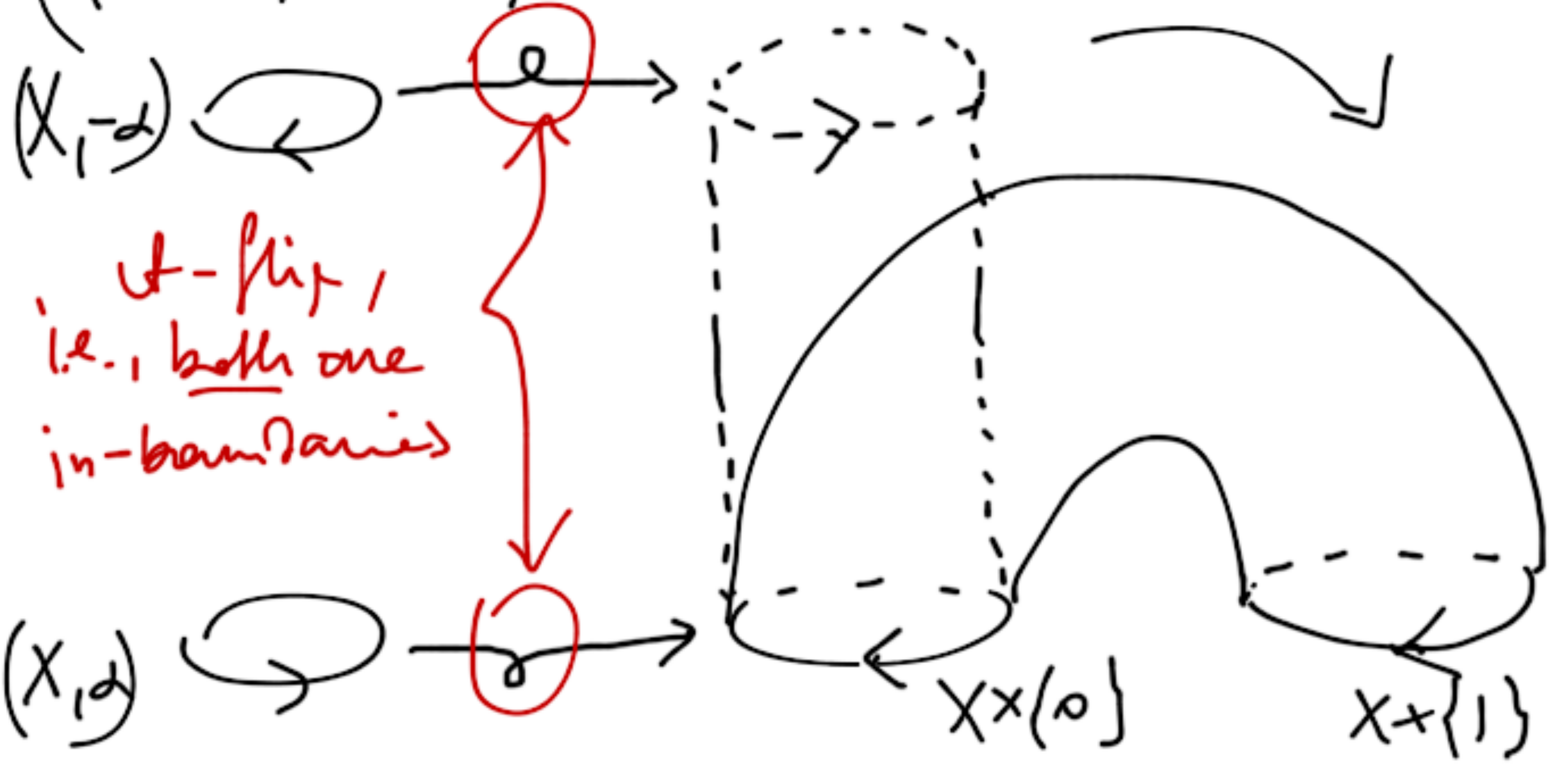
$$\left((X\times I, \alpha\times I), \emptyset, (X,-\alpha)\sqcup(X,\alpha)\right) =: B_X$$



with $b_X^{\tau} := \tau(B_X) : \mathbb{K} \longrightarrow \mathcal{T}(-X\sqcup X) \cong V \otimes_{\mathbb{K}} W$

Ex

$$\big((X\times I, \alpha\times I), (X,\alpha)\sqcup(X,-\alpha), \emptyset\big) =: D_X$$



with $d_X^\tau := \tau(D_X) : \mathcal{T}(X\sqcup(-X)) \longrightarrow \mathbb{K}$

$$\mathcal{T}(X\sqcup(-X)) \underset{(\text{up to} \sim)}{\cong} W\otimes_{\mathbb{K}} V$$

We want to demonstrate that the d_X^τ satisfy the defining condition of Def. 8. for the pairings.

Note, first of all, that they are
$\sqcup$-compatible & natural wrt.
$\mathcal{A}$-homeos by <u>construction</u>.
Consider, next, the homeo
$g : X \times I \circlearrowleft : (x,t) \mapsto (x, 1-t)$.
By axiom (7.iii), g is a $\mathcal{B}$-homeo
$X \times I \to (-X) \times I$, & it yields
a morphism of cobordisms:

$$D_X \to \left((X \times I, -\partial \times I), (X,-\partial) \sqcup (X,\partial), \phi \right)$$

& restricts to the $\mathcal{A}$-homeo:

$$W \otimes_{\mathbb{K}} V \longrightarrow V \otimes_{\mathbb{K}} W$$

which is just $\mathcal{T}(f_-) = \sigma_{W,V}$.

By the naturality of τ wrt.
$\mathcal{B}$-homeos, we have

$$d_{-X} \circ \sigma_{\mathcal{J}(X),\mathcal{J}(-X)} = \tau(D_{-X}) \circ \mathcal{J}(f_-)$$
$$= \mathcal{J}(f_+|_\phi) \circ \tau(D_X) = d_X . \quad (\delta d)$$

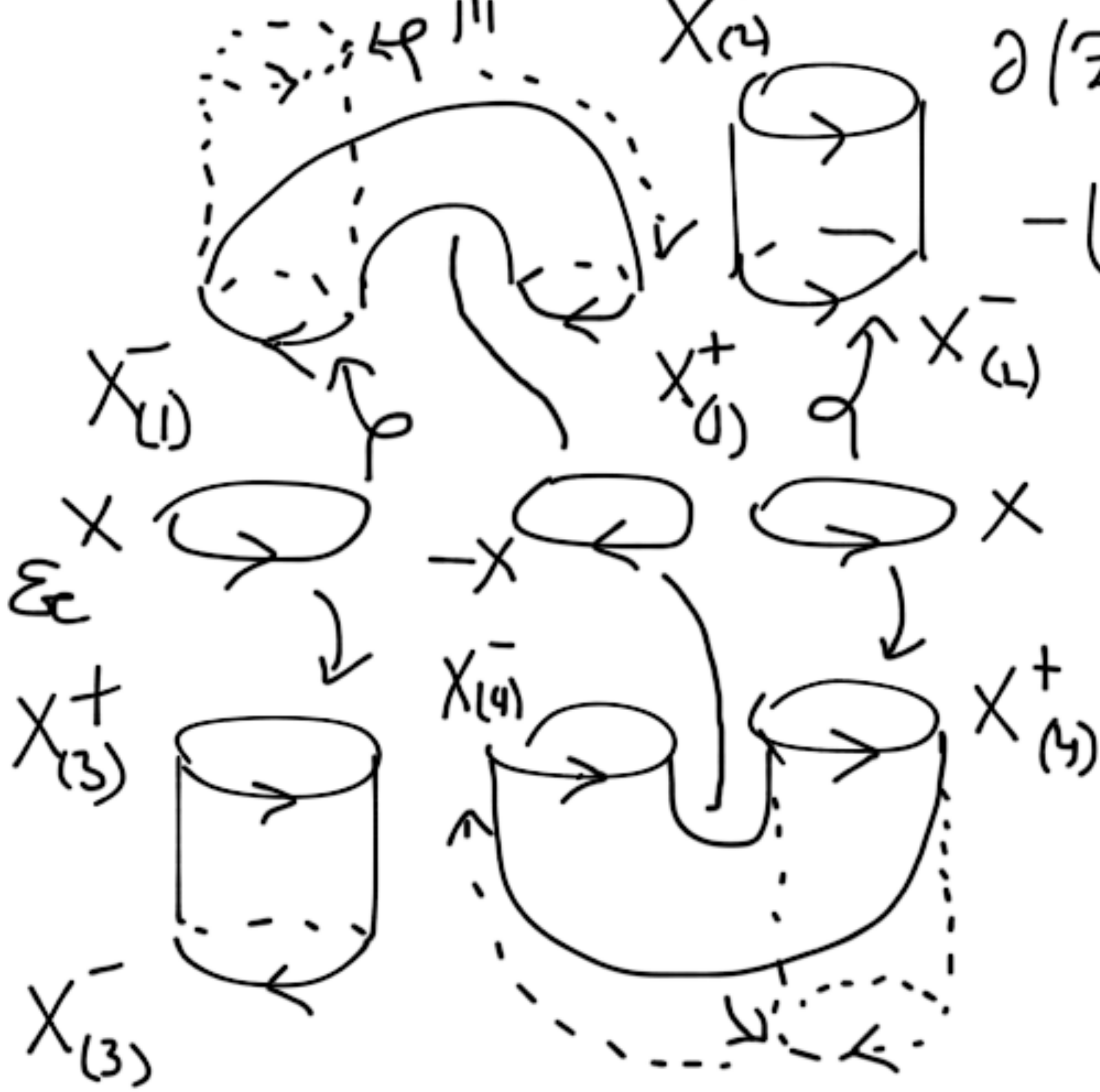
Similarly, we show $b_{-X} = \sigma_{\mathcal{J}(-X),\mathcal{J}(X)} \circ b_X$. (δb)

It remains to check
the non-singularity of the d_X.
For that, we prove that the b_X
are the corresponding copairings.

Take cylinders $Z_i := X \times I$
with $\partial Z_i = -X^-_{(i)} \sqcup X^+_{(i)}$, where $X^- \equiv X \times \{0\}$
& $X^+ \equiv X \times \{1\}$, & consider cobordisms:

• $\left(Z_1 \sqcup Z_2\,,\ -\partial Z_1 \sqcup X^-_{(2)}\,,\ X^+_{(2)}\right) =: C_1$

Recall:
$\partial(Z_1 \sqcup Z_2)$
$= -\left(-\partial Z_1 \sqcup X^-_{(2)}\right) \sqcup X^+_{(2)}$



• $\left(Z_3 \sqcup Z_4\,,\ X^-_{(3)}\,,\ X^+_{(3)} \sqcup \partial Z_4\right) =: C_2$

We have
$$\tau(C_1) = d^\tau_X \otimes \mathrm{id}_W\,,$$
$$\tau(C_2) = \mathrm{id}_W \otimes b^\tau_X\,.$$

Gluing C_2 to the bottom of C_1 along identity maps, we find — by axiom (PF) —

$$\tau(C_1 \circ C_2) = k_{C_1} \cdot (d_X^\tau \otimes id_W) \circ (id_W \otimes b_X^\tau)$$

for some $k_{C_1} \in \mathbb{k}^\times$.

On the other hand, by axiom (7.iv) $C_1 \circ C_2$ is $\mathcal{B}$-homeo to $X \times I$ along a map that restricts to id on the bases, so so $\tau(C_1 \circ C_2) = id_W$, whence altogether

$$(d_X^\tau \otimes id_W) \circ (id_W \otimes b_X^\tau) = k_{C_1}^{-1} \cdot id_W.$$

Upon exchanging $X \leftrightarrow -X$ in the above formulae & subsequently using identities (5d) & (5b) (1.47), in conjunction with the basic properties of the braiding $\sigma_{r,w}$, we find – for some $k_2 \in \mathbb{K}^\times$ –

$$k_2^{-1} \cdot \mathrm{id}_V = (d_{-X}^\tau \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes b_{-X}^\tau)$$
$$= (d_X^\tau \otimes \mathrm{id}_V) \circ (\sigma_{V,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \sigma_{V,W}) \circ (\mathrm{id}_V \otimes b_X^\tau)$$
$$= (d_X^\tau \otimes \mathrm{id}_V) \circ \sigma_{V \otimes_{\mathbb{K}} V, W} \circ (\mathrm{id}_V \otimes b_X^\tau)$$
$$= (\mathrm{id}_V \otimes d_X^\tau) \circ \sigma_{W \otimes_{\mathbb{K}} V, V} \circ \sigma_{V \otimes_{\mathbb{K}} V, W} \circ \sigma_{V \otimes_{\mathbb{K}} W, V} \circ (b_X^\tau \otimes \mathrm{id}_V)$$
$$= (\mathrm{id}_V \otimes d_X^\tau) \circ (b_X^\tau \otimes \mathrm{id}_V) .$$

The statement of the Thm now follows from Lemma L. $\square$

Proof of Thm 2.

Instrumental in the proof is the following

Lemma 2.

In an anomaly-free TQFT $(\mathcal{T},\tau)$, we have — for any $(M,\rho)\in \mathrm{Ob}(n\mathrm{Top}^{\mathcal{B}})$ —

$$\tau(M,X_-,\phi) = d^{\tau}_{-X_-}\left(\tau(M)\otimes \mathrm{id}_{\mathcal{T}(X_-)}\right)$$

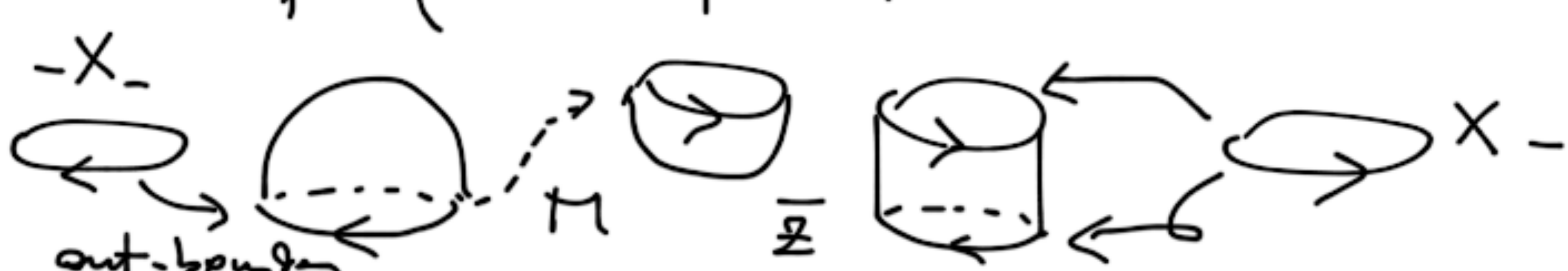
$-X_-\cong\partial M$ $\qquad : \mathcal{T}(-X_-)\to \mathbb{K}$.

Proof of Lemma 2.

Take cylinders $-X_-^{(1)}\times I =: Z$ & $X_-^{(2)}\times I =: \overline{Z}$ for $X_-^{(1)}\equiv X_-\equiv X_-^{(2)}$

& define cobordisms

$$C_1 := \left(M\sqcup \overline{Z}, X_-^{(2)}, (-X_-)\sqcup X_-^{(2)}\right)$$



& $C_2 := (Z, (-X_-^{(1)}) \sqcup X_-^{(1)}, \emptyset)$

with $\tau(C_1) = \tau(M) \otimes \mathrm{id}_{J(X_-)}$

& $\tau(C_2) = d^{\tau}_{-X_-}$.

Upon gluing along identity maps, we get

$$\tau(C_2 \circ C_1) = \underset{\text{NO ANOMALY}}{\underbrace{1_{\Bbbk}}} \cdot d^{\tau}_{-X_-} \circ \mathrm{id}_{J(-X_-) \otimes_{\Bbbk} J(X_-)} \circ (\tau(M) \otimes \mathrm{id}_{J(X_-)})$$

$$= d^{\tau}_{-X_-} \circ (\tau(M) \otimes \mathrm{id}_{J(X_-)})$$

By arguments similar to those adduced previously, we find

$C_2 \circ C_1 \quad \underset{\substack{\text{B-homeo} \\ \text{restricting to Id on base}}}{\cong}$ 

Invoking axiom (7.iii) & using the naturality of τ wrt. $\mathcal{B}$-homeos that extend id, we obtain

$$\tau(M, X_-, \phi) = \tau\Big($$

$$\Big) = id_{\mathbb{K}} \circ \tau(C_2 \circ C_1) \circ id_{\mathcal{J}(X_-)}$$

$$= d^{\tau}_{-X_-} \circ \left(\tau(M) \otimes id_{\mathcal{J}(X_-)}\right). \quad \square$$

Returning to the main proof, we write – by axiom (PF), Lemma 2, the naturality of $d_\cdot$ wrt $\mathcal{B}$-homeos & the symmetry (S) –

NO ANOMALY

$$\tau(M, \phi, \phi)(1_{\mathbb{K}}) = 1 \cdot \tau(M_2, X_2^-, \phi) \circ \mathcal{J}(g) \circ \tau(M_1)$$

$$= d^{\tau}_{-X_2^-} \circ \left(\tau(M_2) \otimes id_{\mathcal{J}(X_2^-)}\right) \circ \mathcal{J}(g)\left(\tau(M_1)\right)$$

$$= d^{\tau}_{-X_2^-} \circ \left(\tau(M_2) \otimes \mathcal{J}(g)(\tau(M_1))\right)$$

$$= d^{\tau}_{(-g)(-X_1^+)} \circ \Big(\tau(M_2) \otimes \mathcal{J}(g)\big(\tau(M_1)\big)\Big)$$

$$= d^{\tau}_{-X_1^+} \circ \Big(\mathcal{J}(-g)^{-1}\big(\tau(M_2)\big) \otimes \tau(M_1)\Big)$$

$$= d^{\tau}_{X_1^+} \circ \sigma_{\mathcal{J}(-X_1^+),\,\mathcal{J}(X_1^+)} \circ \Big(\mathcal{J}(-g)^{-1}\big(\tau(M_2)\big) \otimes \tau(M_1)\Big)$$

$$= d^{\tau}_{X_1^+} \circ \Big(\tau(M_1) \otimes \mathcal{J}(-g)^{-1}\big(\tau(M_2)\big)\Big). \quad \square$$

Proof of Thm 3.

Using the pairings d_X & copairing b_X from the proof of Thm 1., & involving Lemma 1., we calculate

$$\dim_{\mathbb{K}} \mathcal{J}(X) \overset{\text{NO ANOMALY}}{=} \big(d_X \circ \sigma_{\mathcal{J}(-X),\,\mathcal{J}(X)} \circ b_X\big)(1_{\mathbb{K}})$$

$$= d_{-X} \circ b_X(1_{\mathbb{K}})$$

formula from Lemma 1.

$= 1 \cdot \tau\Big($ 

$\Big) \circ \mathcal{J}(\mathrm{Id}) \circ \tau\Big($ 

$\Big)$

$= \tau\Big($ 

$\Big) \equiv \tau(X \times S^1).$

by axiom (7.iv)

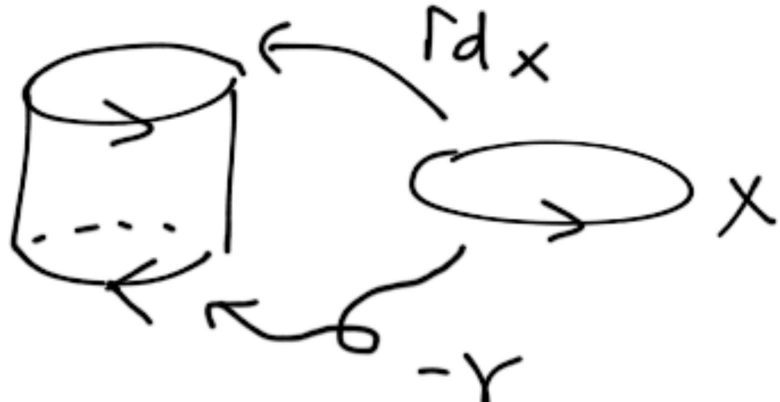
(by Lemma 1.) $\square$

Finally, we come to

<u>Proof of Thm 4.</u>

given an $\mathcal{A}$-homeo $\gamma : X \circlearrowleft$,
consider the corresponding
mapping cylinder

Z_γ



& denote

$$\varrho(\gamma) := \tau(Z_\gamma) \in \mathrm{End}_{\mathbb{k}} J(X).$$

For A-homeos γ & $\tilde{\gamma}$ related by isotopy

$$\Gamma : \Sigma \times I \longrightarrow \Sigma \text{ with } \begin{array}{l} \Gamma_0 := \gamma \\ \Gamma_1 := \tilde{\gamma} \end{array},$$

we find the homeo

$$\psi_{\gamma,\tilde{\gamma}} : \Sigma \times I \circlearrowleft \, : (x,t) \mapsto \left(\tilde{\gamma} \circ \Gamma_t^{-1}(x), t\right)$$

which manifestly renders commutative the diagram:

$$\begin{tikzcd}
 & Z_{\tilde{\gamma}} & \\
X \arrow[loop left, "\mathrm{id}_X"] \arrow[ur, "-\tilde{\gamma}"] \arrow[dr, "-\gamma"'] & & X \arrow[loop right, "\mathrm{id}_X"] \arrow[ul, "\mathrm{id}_X"'] \arrow[dl, "\mathrm{id}_X"] \\
 & Z_\gamma \arrow[uu, "\psi_{\gamma\tilde{\gamma}}"] &
\end{tikzcd},$$

whence

$$\tau_{\gamma,\tilde{\gamma}} \in \mathrm{Hom}\left(X \overset{M_\gamma}{\Rightarrow} X,\ X \overset{M_{\tilde{\gamma}}}{\Rightarrow} X\right).$$

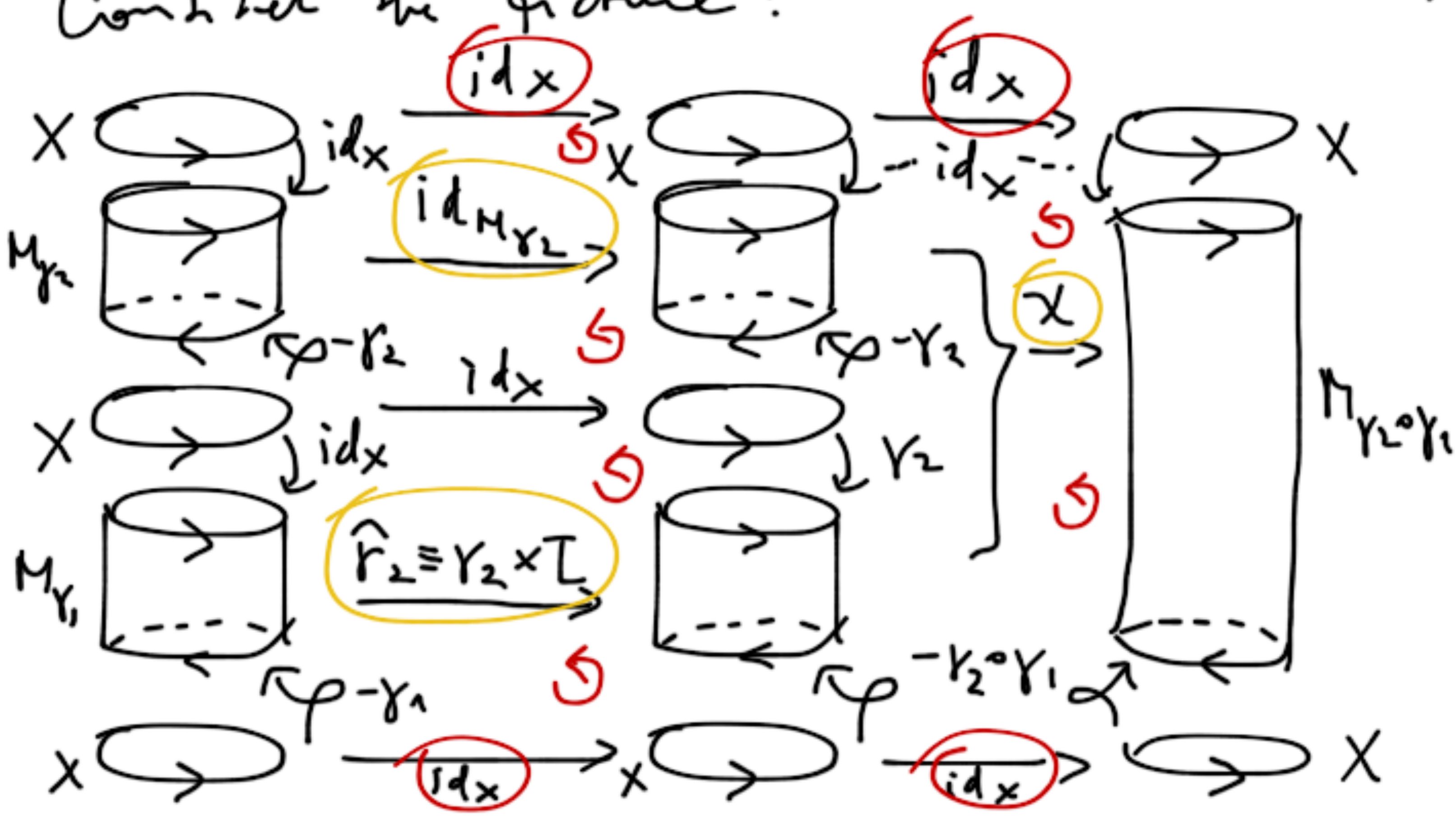
The invariance of τ wrt. and $\mathcal{B}$-homeos implies the equality

$$\varrho(\gamma) = \varrho(\tilde{\gamma}),$$

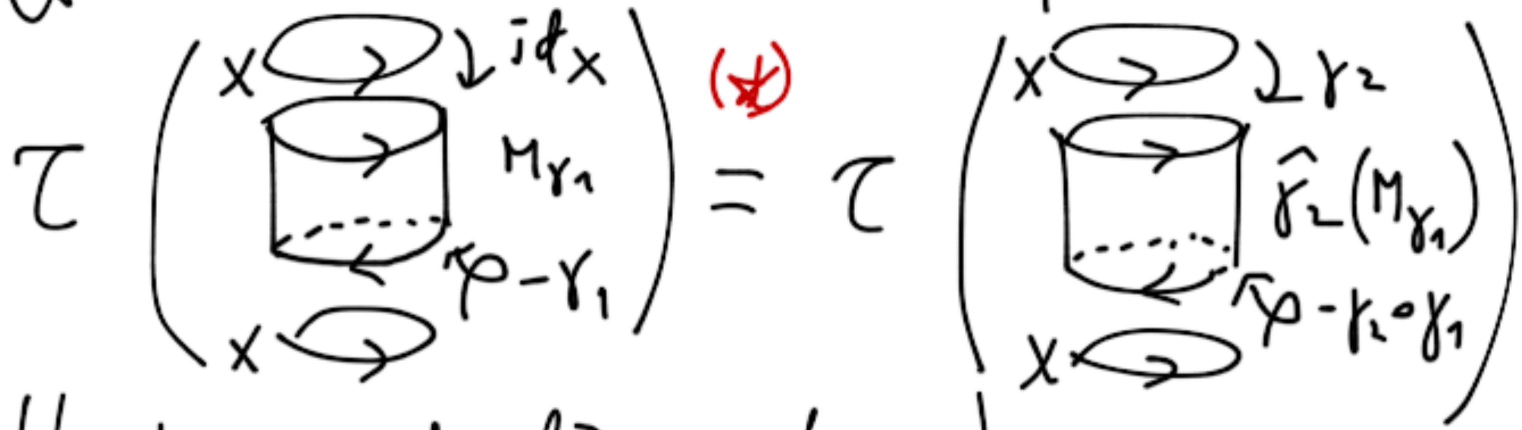
so ϱ restricts to $\hat{\varrho}$ on $\mathrm{Aut}^{\mathcal{A}}(X)/\mathrm{Aut}_0^{\mathcal{A}}(X)$.

By axiom (N), we have $\hat{\varrho}([id_X]) = id_{\mathcal{J}(X)}$.

Consider the picture:



Above, the $\mathcal{B}$-homeo $\gamma_2 \times I$ is judiciously put in so that it induces identity $\mathcal{A}$-homeos on the bases, whence



(by the naturality of τ),

So the existence of χ is ensured by axiom (7.iv). Altogether, then, we find

$$\hat{\jmath}([\gamma_2 \circ \gamma_1]) \equiv \tau(M_{\gamma_2 \circ \gamma_1}) = \tau(M_{\gamma_2} \circ \hat{\gamma}_2(M_{\gamma_1}))$$

$$= K(\hat{\gamma}_2(M_{\gamma_1}), M_{\gamma_2}, \mathrm{id}_X) \circ \tau(M_{\gamma_2}) \circ \mathrm{id}_{\mathcal{J}(X)} \circ \tau(\hat{\gamma}_2(M_{\gamma_1}))$$

$$\overset{(*)}{=} K^{!!}_{\gamma_1, \gamma_2} \circ \tau(M_{\gamma_2}) \circ \tau(M_{\gamma_1}) \equiv K_{\gamma_1, \gamma_2} \circ \hat{\jmath}(\gamma_2) \circ \hat{\jmath}(\gamma_1)$$

$\square$

2.5° Towards a classification of TQFT's.

Classification of TQFT's in all generality is a difficult problem (cf. recent work by Lurie). However, a smart choice of a subcategory within $n\mathrm{TQFT}^{(\mathrm{A},\mathrm{B})}$ may render classification tractable. In what follows, we restrict to one such subcategory, whereby we attain a classification of the relevant TQFT's in terms of quasi invariants for closed cobordisms.

Def$\underline{\text{n}}$ 10. Given $(X,\alpha) \in \mathrm{Ob}((n-1)\mathrm{Top}^{A})$, a B-space with X-parameterised boundary is a pair $((M,\beta), f)$ that consists of $(M,\beta) \in \mathrm{Ob}(n\mathrm{Top}^{B})$ & an A-homeo

$$f : (X,\alpha) \longrightarrow \partial(M,\beta),$$

termed the parameterising homeo. Denote the set of all B-spaces with X-parameterised boundary $\mathcal{B}_X$.

Given an (A,B)-based TQFT $(\mathcal{T},\tau)$, write - in the above notation -

$$\boxed{\tau^X((M,\beta),f)} := \tau(M) \in \mathcal{T}(X).$$

We call $(\mathcal{T},\tau)$ non-degenerate

iff $\forall (X,\alpha)\in Ob((n-1)Top^{\mathcal{A}})$:

$$\mathcal{T}(X) = \left\langle \tau^X((M,\beta),f) \mid ((M,\beta),f)\in \mathcal{B}_X \right\rangle_{\mathbb{K}}.$$

———— × ————

We then have

Thm 5.

Let $(\mathcal{T}_i,\tau_i)$, $i\in\{1,2\}$ be two $(\mathcal{A},\mathcal{B})$-based non-degenerate anomaly free TQFTs with ground field $\mathbb{K}$ in dimension $n\in\mathbb{N}^\times$.

Then,
$$(\mathcal{T}_1,\tau_1)\simeq(\mathcal{T}_2,\tau_2)$$

iff $\forall (M,\beta)\in Ob(nTop^{\mathcal{B}})$:

$$\partial M=\emptyset \Longrightarrow \tau_1(M)=\tau_2(M).$$

Proof of Thm 5.

$\boxed{\Rightarrow}$ Let $g : (\mathcal{T}_1, \tau_1) \xrightarrow{\cong} (\mathcal{T}_2, \tau_2)$ be an isomorphism. By Defn 10.1 the mapping

$$g_\phi : \mathcal{T}_1(\phi) \longrightarrow \mathcal{T}_1(\phi)$$

is an automorphism of $\mathbb{K}$, hence

$$g_\phi = k \cdot \mathrm{id}_{\mathbb{K}} \text{ for some } k \in \mathbb{K}^\times.$$

Taking into account the identity

$$k \cdot \mathrm{id}_{\mathbb{K}} = g_\phi = g_{\phi \sqcup \phi} = g_\phi \otimes g_\phi = k^2 \mathrm{id}_{\mathbb{K}} \otimes \mathrm{id}_{\mathbb{K}}$$
$$= k^2 \mathrm{id}_{\mathbb{K}},$$

we conclude that $k = 1$, so

$$g_\phi = \mathrm{id}_{\mathbb{K}}.$$

Given $(M,\beta) \in \mathrm{Ob}(\mathrm{nTop}^{\partial})$ with boundary $\partial M \cong X$ we then obtain – upon invoking the naturality of g. – the commutative diagram

$$\begin{tikzcd}
\mathcal{T}_1(\emptyset) \arrow[r, "g_\emptyset"] \arrow[d, "\tau_1(M)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{T}_2(\emptyset) \arrow[d, "\tau_2(M)"] \\
\mathcal{T}_1(X) \arrow[r, "g_X"'] & \mathcal{T}_2(X)
\end{tikzcd}$$

which yields the useful identity:

$$\tau_2(M) = g_X\left(\tau_1(M)\right).$$

For M closed, we infer from it the desired equality

$$\partial M = \emptyset \implies \tau_2(M) = \tau_1(M).$$

$\boxed{\Rightarrow}$ We commence by showing the existence of isomorphisms

$$\mathcal{T}_1(X) \cong \mathcal{T}_2(X).$$

For an arbitrary $((M,\beta),f) \in \mathcal{B}_X$, set

$$\eta_X\left(\tau_1^X((M,\beta),f)\right) := \tau_2^X\left((M,\beta),f\right)$$

Consider $k_i \in \mathbb{K}$ & $((M_i,\beta_i),f_i) \in \mathcal{B}_X$, $i \in \overline{1,N}$

s.th. $\sum_{i=1}^{N} k_i \triangleright \tau_1^X((M_i,\beta_i),f_i) = 0$

We then find $\sum_{i=1}^{N} k_i \triangleright \tau_2^X((M_i,\beta_i),f_i) = 0$.

Indeed, for any $((N,\gamma),g) \in \mathcal{B}_X$, we obtain

– by Thm 2. (with bases of M_i & N identified along id's)

& the assumed equality of τ_1 & τ_2

on closed $\mathcal{B}$-spaces –

$$d_X^{\tau_2}\left(\sum_{i=1}^{N} k_i \triangleright \tau_2^X\big((n_i,p_i),f_i\big) \otimes_{\mathbb{K}} \tau_2^X\big((N,\gamma),g\big)\right)$$

$$\equiv \sum_{i=1}^{N} k_i \cdot d_X^{\tau_2}\left(\tau_2(M_i) \otimes_{\mathbb{K}} \tau_2(N)\right)$$

$$\underset{\Pi^{-2}}{=} \sum_{i=1}^{N} k_i \cdot \tau_2\left(M_i \sqcup_{id_X} N\right) = \sum_{i=1}^{N} k_i \cdot \tau_1\left(M_i \sqcup_{id_X} N\right)$$

$$= d_X^{\tau_1}\left(\sum_{i=1}^{N} k_i \triangleright \tau_1^X\big((n_i,p_i),f_i\big) \otimes_{\mathbb{K}} \tau_1^X\big((N,\gamma),g\big)\right) = 0_{\mathbb{K}},$$

The claim now follows from the non-singularity of the pairing $d_X^{\tau_2}$.

We conclude that the maps η_X can be extended (by $\mathbb{K}$-linearity) to $\mathbb{K}$-linear maps

$$\overline{\eta}_X : \mathcal{T}_1(X) \longrightarrow \mathcal{T}_2(X).$$

By symmetry, we also obtain $\mathbb{K}$-linear maps

$$\widetilde{\theta}_X : \mathcal{J}_2(X) \longrightarrow \mathcal{J}_1(X)$$

that extend

$$\theta_X\left(\tau_2^X((M,\beta),f)\right) := \tau_1^X((M,\beta),f).$$

Maps $\widetilde{\eta}_X$ & $\widetilde{\theta}_X$ are mutually inverse (by construction), & so the $\widetilde{\eta}_X$ are isos.

It remains to verify that they compose a natural transformation

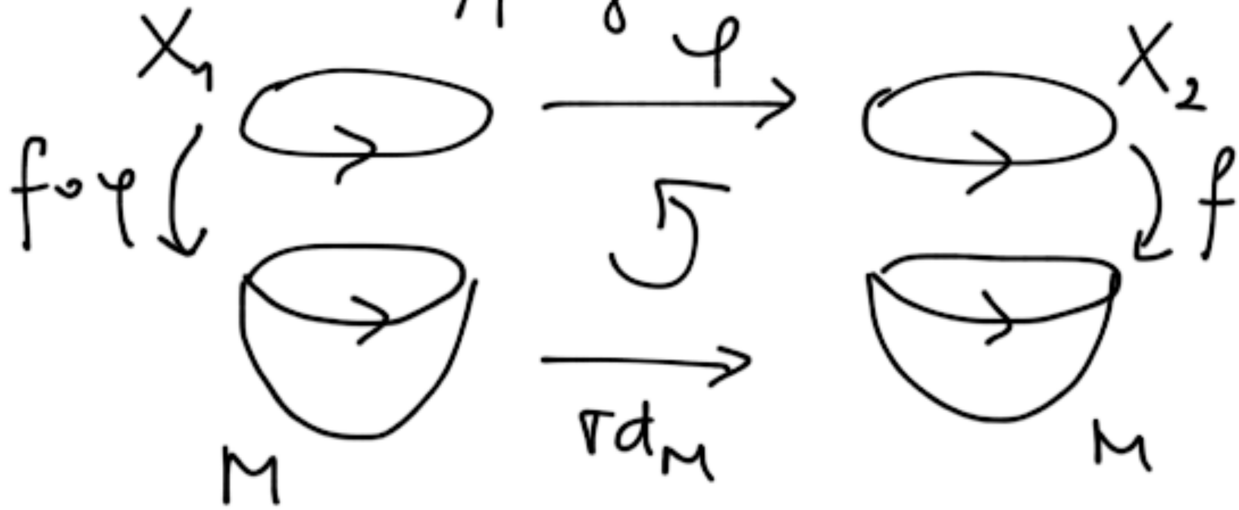
$$\widetilde{\eta} : (\mathcal{J}_1, \tau_1) \Longrightarrow (\mathcal{J}_2, \tau_2).$$

To this end, note – first of all – that the $\widetilde{\eta}_X$ are natural wrt. $\mathcal{A}$-homeos.

Indeed, given an $\mathcal{A}$-homeo

$$\varphi : (X_1, \alpha_1) \longrightarrow (X_2, \alpha_2)$$

& $((M,\beta), f) \in \mathcal{B}_{\varphi(X_1)}$, consider the $\partial_{\mathcal{A}}$-mapping



that yields – by axiom (Nat) – the identity

$$\tau_i^{X_2}((M,\beta),f) = T_i(\varphi)\left(\tau_i^{X_1}((M,\beta), f\circ\varphi)\right).$$

We may employ it in the calculation:

$$\tilde{\eta}_{\varphi(X_1)}\left(\tau_1^{\varphi(X_1)}((M,\beta),f)\right) \equiv \tau_2^{\varphi(X_1)}((M,\beta),f)$$

$$= T_2(\varphi)\left(\tau_2^{X_1}((M,\beta),f\circ\varphi)\right) = T_2(\varphi)\circ\tilde{\eta}_{X_1}\left(\tau_1^{X_1}((M,\beta),f\circ\varphi)\right)$$

$$= \mathcal{J}_2(\varphi) \circ \tilde{\eta}_{X_1} \circ \mathcal{J}_1(\varphi)^{-1}\left(\tau_1^{\varphi(X_1)}((M,\beta),f)\right)$$

that gives the desired naturality:

$$\mathcal{J}_2(\varphi) \circ \tilde{\eta}_{X_1} = \tilde{\eta}_{\varphi(X_1)} \circ \mathcal{J}_1(\varphi) .$$

In the next step, we note that the form of the mapping $\tilde{\eta}_\phi : \mathbb{K} \circlearrowleft$ is fixed by the value it takes on

$$\tau_1^\phi((\phi,*),\phi) = 1_{\mathbb{K}}$$

as $\tilde{\eta}_\phi(1_{\mathbb{K}}) = \tilde{\eta}_\phi\left(\tau_1^\phi((\phi,*),\phi)\right)$

$$= \tau_2^\phi((\phi,*),\phi) = 1_{\mathbb{K}} ,$$

whence $\tilde{\eta}_\phi \equiv \mathrm{id}_{\mathbb{K}}$.

Let us, now, write - for an arbitrary $((M,\beta),f) \in \mathcal{B}_{X_1 \sqcup X_2}$ or for $\mu_i \equiv ((M_i,\beta_i),f_i) \in \mathcal{B}_{X_i}$ -

$$\iota^A_{X_1,X_2}\left(\tau_A^{X_1\cup X_2}((M,\beta),f)\right) =: \sum_{\mu_i\in\mathcal{B}_{X_i}} k^A_{(\mu_i)}\triangleright\left(\tau_A^{X_1}(\mu_1)\otimes_{\mathbb{K}}\tau_A^{X_2}(\mu_2)\right).$$

We find, using the naturality of ι^*, the identity $\iota^A_{\phi,\phi}=\mathrm{id}_{\mathbb{K}}$, & axiom (M),

$$(\hat{\eta}_{X_1}\otimes\tilde{\eta}_{X_2})\circ\iota^1_{X_1,X_2}\left(\tau_1^{X_1\cup X_2}((M,\beta),f)\right)$$

$$=\sum_{\mu_i\in\mathcal{B}_{X_i}} k^1_{(\mu_i)}\triangleright\left(\tau_2^{X_1}(\mu_1)\otimes_{\mathbb{K}}\tau_2^{X_2}(\mu_2)\right)$$

$$=\sum_{\mu_i\in\mathcal{B}_{X_i}} k^1_{(\mu_i)}\triangleright\left(\tau_2(M_1)\otimes_{\mathbb{K}}\tau_2(M_2)\right)$$

$$=\sum_{\mu_i\in\mathcal{B}_{X_i}} k^1_{(\mu_i)}\triangleright\iota^2_{X_1,X_2}\left(\tau_2(M_1\cup M_2)\right)$$

$$=\iota^2_{X_1,X_2}\left(\sum_{\mu_i\in\mathcal{B}_{X_i}} k^1_{(\mu_i)}\triangleright\tau_2^{X_1\cup X_2}((M_1\cup M_2,\beta_1\cup\beta_2),f_1\cup f_2)\right)$$

$$= \overset{2}{\iota}_{X_1,X_2} \circ \tilde{\eta}_{X_1 \sqcup X_2} \left(\sum_{\mu_i \in \mathcal{B}_{X_i}} k^1_{(\mu_i)} \triangleright \tau_1^{X_1 \sqcup X_2}\big((M_1 \sqcup M_2, \beta_1 \sqcup \beta_2), f_1 \sqcup f_2\big) \right)$$

$$= \overset{2}{\iota}_{X_1,X_2} \circ \tilde{\eta}_{X_1 \sqcup X_2} \circ \left(\overset{1}{\iota}_{X_1,X_2}\right)^{-1} \left(\sum_{\mu_i \in \mathcal{B}_{X_i}} k^1_{(\mu_i)} \triangleright \left(\tau_1^{X_1}(\mu_1) \otimes_{\mathbb{k}} \tau_1^{X_2}(\mu_2)\right) \right)$$

$= \overset{2}{\iota}_{X_1,X_2} \circ \tilde{\eta}_{X_1 \sqcup X_2} \left(\tau_1^{X_1 \sqcup X_2}\big((M,\beta), f\big) \right)$, which shows

$$\left(\tilde{\eta}_{X_1} \otimes \tilde{\eta}_{X_2}\right) \circ \overset{1}{\iota}_{X_1,X_2} = \overset{2}{\iota}_{X_1,X_2} \circ \tilde{\eta}_{X_1 \sqcup X_2}.$$

At this stage, it remains to verify that the $\tilde{\eta}_\cdot$ intertwine operator invariants. That is we should prove the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{J}_1(X_-) \arrow[r, "\tilde{\eta}_{X_-}"] \arrow[d, "\tau_1(M, X_-, X_+)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{J}_2(X_-) \arrow[d, "\tau_2(M, X_-, X_+)"] \\
\mathcal{J}_1(X_+) \arrow[r, "\tilde{\eta}_{X_+}"'] & \mathcal{J}_2(X_+)
\end{tikzcd}$$

for any (A,B)-cobordism $(X_-,\alpha_-) \xRightarrow{(M,\beta)} (X_+,\alpha_+)$.
Taking into account the non-degeneracy of $(\mathcal{J}_1,\tau_1)$, we do that by verifying the ensuing identity

$$\tau_2(M,X_-,M_+)\circ\tilde{\eta}_{X_-} = \tilde{\eta}_{X_+}\circ\tau_1(M,X_-,X_+)$$

on the generators

$$\tau_1^{X_-}((N,\tilde{\beta}),\tilde{f}), \text{ with } ((N,\tilde{\beta}),\tilde{f})\in\mathcal{B}_{X_-}.$$

Define

$$\breve{N} := M\sqcup_{id_{X_-}} N, \text{ with } \partial\breve{N}\simeq X_+.$$

In the absence of the gluing anomaly, we have

$$\tau_i(\breve{N}) = \tau_i(M,X_-,X_+)\left(\tau_i^{X_-}((N,\tilde{\beta}),\tilde{f})\right)\in\mathcal{J}_i(X_+),$$

& so also (for $f_\pm : (X_\pm,\alpha_\pm)\to(\partial_\pm M,\pm\alpha^\partial)$)

$$\tau_2(M, X_-, X_+) \circ \tilde{\eta}_{X_-}\left(\tau_1^{X_-}((N,\tilde{\beta}),\tilde{f})\right)$$

$$= \tau_2(M, X_-, X_+)\left(\tau_2^{X_-}((N,\tilde{\beta}),\tilde{f})\right) = \tau_2(\breve{N})$$

$$\equiv \tau_2^{X_+}((\breve{N}, \beta \sqcup \mathrm{id}_{X_-}\tilde{\beta}), f_+)$$

$$= \hat{\eta}_{X_+}\left(\tau_1^{X_+}((\breve{N}, \beta \sqcup \mathrm{id}_{X_-}\tilde{\beta}), f_+)\right)$$

$$\equiv \tilde{\eta}_{X_+}\left(\tau_1(\breve{N})\right) = \tilde{\eta}_{X_+} \circ \tau_1(M, X_-, X_+)\left(\tau_1^{X_-}((N,\tilde{\beta}),\tilde{f})\right).$$

This completes the proof of the Th$\underline{m}$. $\square$

———— X ————

Remark:

The last Thm emphasises the clarifying rôle of the operator invariants for closed B-spaces. In fact, one can prove

Thm G. There is a bijection

$$\left\{\begin{array}{l}\text{Isoclasses of non-degenerate}\\ \text{anomaly-free } (\mathcal{A},\mathcal{B})\text{-based}\\ \text{TQFT's with ground ring } \mathbb{K}\\ \qquad\qquad\text{(in dimension } n\text{)}\end{array}\right\}$$

$$\Big\updownarrow \; 1:1$$

$$\left\{\begin{array}{l}\text{Quantum invariants, i.e., monoidal functors}\\ \tau_0 : \mathrm{Skel}(n\mathrm{Top}^{\mathcal{B}}, \sqcup, \emptyset) \longrightarrow (\mathbb{K}^{\times}, \cdot_{\mathbb{K}}, 1_{\mathbb{K}})\\ \qquad \text{with the splitting property [Th}\end{array}\right\}.$$

7° Models

In what follows, we shall demonstrate that

- the abstract definition of TQFT admits a nontrivial realisation;
- for natural choices of space structure, the algebraic structure behind the definition may sometimes be identified completely explicitly;
- the definition gives rise to a plethora of topological invariants that generalise, e.g., the celebrated LYMPHJOFLY polynomial.

7.1° 2TQFT for smooth oriented cobordisms vs Frobenius algebras

7.1.1° Categorical preliminaries

In the description of the geometric category $n\mathrm{Cob}^{(\mathrm{or},\partial)}$, it is often useful to reduce the apparent complexity of its structure by

- restricting to isoclasses of objects
- decomposing generic morphisms (cobordisms) into elementary LEGO bits.

Below, we formalise these ideas so as to ensure their applicability in the rigorous description of categories in list.

Def^{n} 11. Let $(\mathcal{C}, \otimes, 1)$ be a strict monoidal category, & let $\mathcal{F} \subset \mathrm{Mor}\,\mathcal{C}$.

__Words in $\mathcal{F}$ of__ __rank__ $n \in \mathbb{N}^{\times}$ are symbols of the form

(i) __$n = 1$__ : $[\mathrm{id}_V]$ for $V \in \mathrm{ob}\,\mathcal{C}$,

$[f]$ for $f \in \mathcal{F}$;

(ii) __$n > 1$__ : $w_1 \circ' w_2$, $w_1 \otimes' w_2$

for w_1, w_2 - arbitrary words of rank $< n$.

They compose a set $W(\mathcal{F}) = \bigcup_{\mathrm{rk}\, w \geq 1} \{w\}$.

To these, we assign morphisms as per

$$w \longmapsto \overline{w}$$

$$[\mathrm{id}_V] \longmapsto \mathrm{id}_V, \quad [f] \longmapsto f$$

$$w_1 \circ' w_2 \longmapsto \overline{w}_1 \circ \overline{w}_2, \quad w_1 \otimes' w_2 \longmapsto \overline{w}_1 \otimes \overline{w}_2.$$

Two words $w, w' \in W(\mathcal{F})$ are declared <u>equivalent</u>, which we write as $w \sim w'$, iff there exists a finite sequence:

$$\{ w \equiv w_0, w_1, w_2, \dots, w_n, w_{n+1} \equiv w' \} \subset W(\mathcal{F})$$

s.th. w_{i+1} is obtained from w_i through substitution of subwords according to the following rules:

$$([f] \circ' [g]) \circ' [h] \longleftrightarrow [f] \circ' ([g] \circ' [h]),$$

$$[\mathrm{Id}_{t(f)}] \circ' [f] \longleftrightarrow [f] \longleftrightarrow [f] \circ' [\mathrm{Id}_{s(f)}]$$

(t & s are the target & source of f, resp.),

$$[\mathrm{Id}_v] \circ' [\mathrm{Id}_v] \longleftrightarrow [\mathrm{Id}_v],$$

$$([f] \otimes' [g]) \otimes' [h] \longleftrightarrow [f] \otimes' ([g] \otimes' [h]),$$

$$[\mathrm{id}_1]\otimes'[f]\longleftrightarrow[f]\longleftrightarrow[f]\otimes'[\mathrm{id}_1],$$

$$[\mathrm{id}_V]\otimes'[\mathrm{id}_W]\longleftrightarrow[\mathrm{id}_{V\otimes W}]$$

& $$\big([f]\otimes'[g]\big)\circ'\big([f']\otimes'[g']\big)$$

$$\longleftrightarrow\big([f]\circ'[f']\big)\otimes'\big([g]\circ'[g']\big),$$

written for $f,g,h,f',g'\in\mathcal{F}$ & $V,W\in\mathrm{Ob}\,\mathcal{C}$.

Let, next, $\mathcal{R}(\mathcal{F})\subset W(\mathcal{F})\times W(\mathcal{F})$ be a relation such that

$$\forall (a,b)\in\mathcal{R}(\mathcal{F}):\ \bar{a}=\bar{b}.$$

Equivalence classes $[w]_\sim,[w']_\sim$ of words $w,w'\in\mathcal{F}$ are called <u>congruent with respect to</u> $\mathcal{R}(\mathcal{F})$, which we denote as

$$[w]_\sim \equiv [w']_\sim \mod \mathcal{R}(\mathcal{F}),$$

iff there exists a finite sequence:

$$\{w \equiv w_0, w_1, w_2, \dots, w_n, w_{n+1} \equiv w'\} \subset W(\mathcal{F})$$

s.t. w_{i+1} is obtained from w_i through substitution $a \leftrightarrow b$ for some $(a,b) \in \mathcal{R}(\mathcal{F})$.

This enables us to formulate the desired

Def$^{\text{n}}$ 12.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a strict monoidal category. It is <u>generated</u> by $\mathcal{F} \subset \mathrm{Mor}\, \mathcal{C}$ <u>through relations</u> $\mathcal{R}(\mathcal{F}) \subset W(\mathcal{F}) \times W(\mathcal{F})$ iff

(i) $\mathcal{F}$ generates $\mathcal{C}$, i.e., all $f \in \mathrm{Mor}\,\mathcal{C}$ can be obtained from elements of $\mathcal{F}$ & id_V, $V \in \mathrm{Ob}\,\mathcal{C}$ through a finite number of applications of $\circ$ & $\otimes$.

(ii) $\forall w, w' \in W(\mathcal{F})$:

$$[w]_\sim \equiv [w']_\sim \mod \mathcal{R}(\mathcal{F}) \iff \overline{w} = \overline{w}' .$$

We then call the pair $(\mathcal{F}, \mathcal{R}(\mathcal{F}))$ a presentation of $(\mathcal{C}, \otimes, \mathbb{1})$.

We shall also need

Def^{n} 13. Let $\mathcal{C}$ be a category. The skeleton $Sk(\mathcal{C})$ of $\mathcal{C}$ is the full subcategory whose object class contains a single representative of each iso class of $\mathrm{Ob}\,\mathcal{C}$.

Prop^{n} 1. $Sk(\mathcal{C}) \simeq \mathcal{C}$ (canonically)

7.1.2° The geometry of the TQFT

We may now introduce the object of interest:

Def^n 14. The category $2\mathrm{Cob}^{[\infty,\uparrow]}$ of smooth oriented cobordisms in dimension 2 has

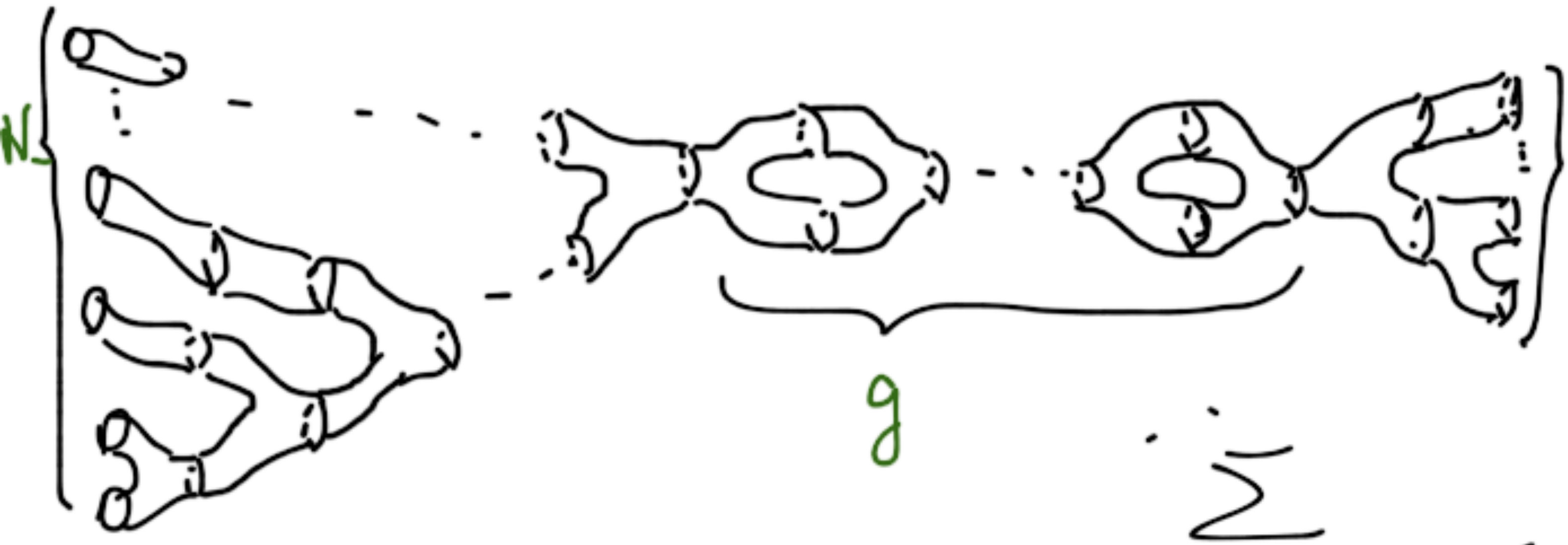
$\mathrm{Ob}\left(2\mathrm{Cob}^{[\infty,\uparrow]}\right) = \{$closed oriented 1-dim. C^∞-manifolds$\}$

$\mathrm{Mor}\left(2\mathrm{Cob}^{[\infty,\uparrow]}\right) = \{$diffeo classes of oriented 2-dim. C^∞-smooth cobordisms between objects$\}$.

Much of the subsequent discussion is based on the following classical

Thm 7.

Two connected compact oriented surfaces with oriented boundary are diffeomorphic iff they have the same genus (g) & the same number of in- (N_-) & out- (N_+) boundaries. Each such surface can be reduced (diffeomorphically) to a normal form



Using this, in conjunction with the basic observation

Prop. 2.

Every 1-dim. closed oriented manifold is diffeomorphic to S^1.

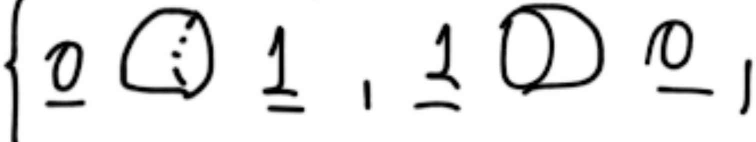
& the (sub)additivity of the Euler characteristic

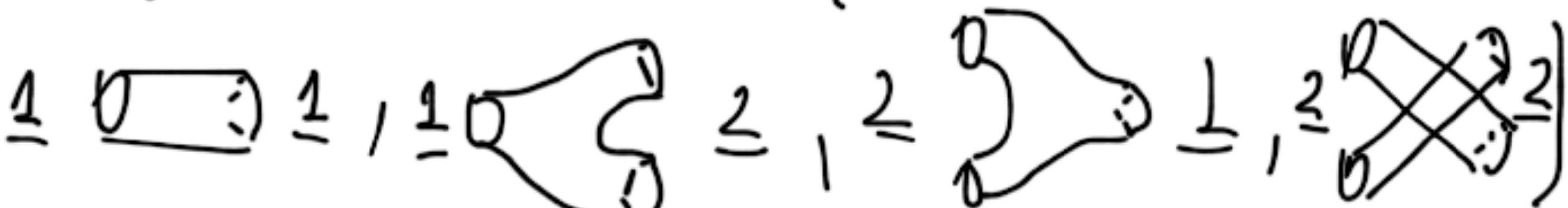
$$\chi(\Sigma) = 2 - 2g - (N_+ + N_-),$$

we readily prove (cf. the proof in [Kock])

Th^{m} 8. The twisted monoidal category $\mathrm{Skel}\left(2\mathrm{Cob}^{[\infty,\uparrow]}, \sqcup, \emptyset, \sigma^{\sqcup}_{\cdot,\cdot}\right)$ has $\mathrm{Ob}\left(\mathrm{Skel}\left(2\mathrm{Cob}^{[\infty,\uparrow]}\right)\right) = \mathbb{N}$

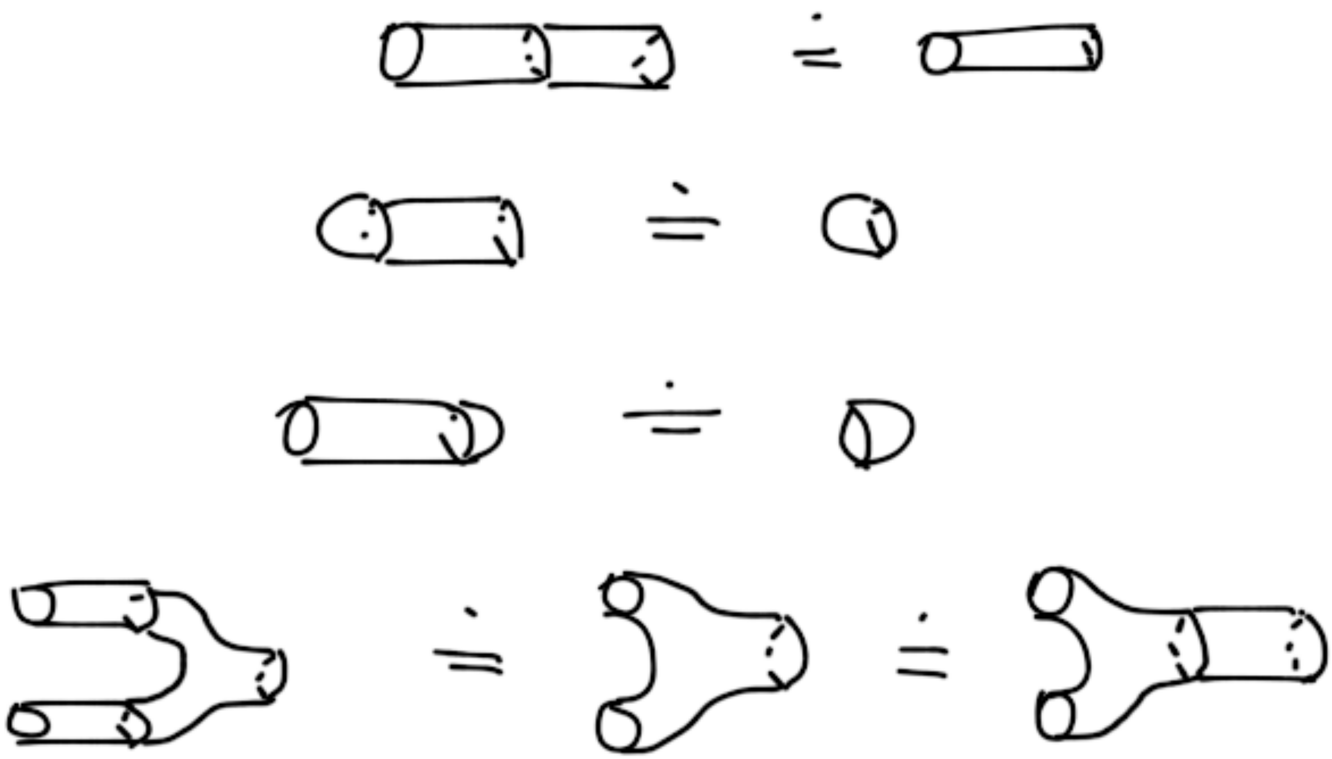
& generators: $\mathcal{F} = \{$

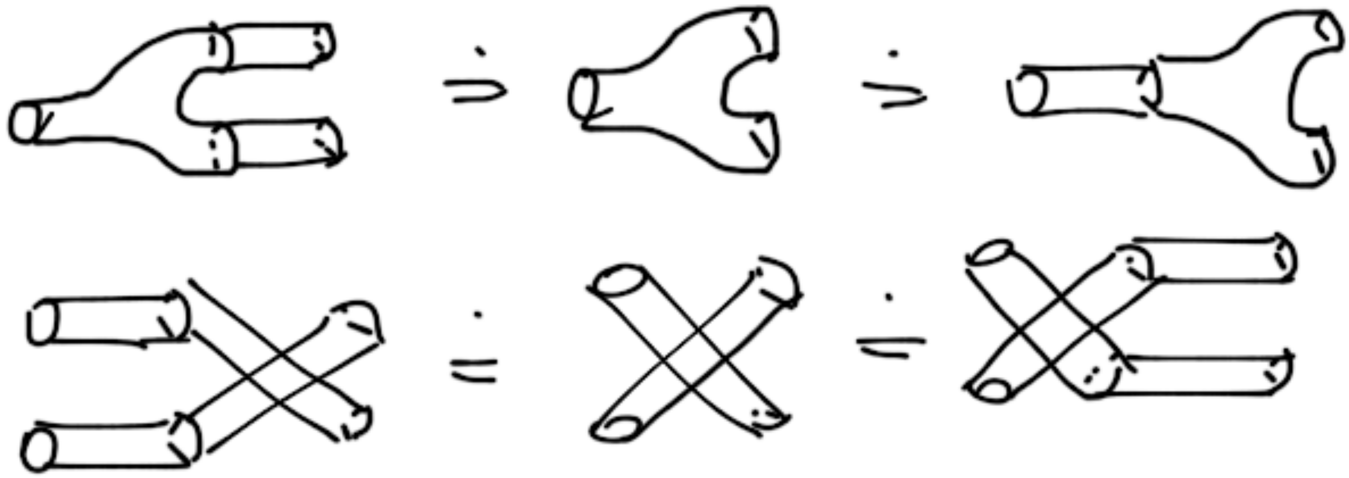




subject to the relations $\mathcal{R}(\mathcal{F})$:

(IDENTITY REL$^{\underline{n}}$s)





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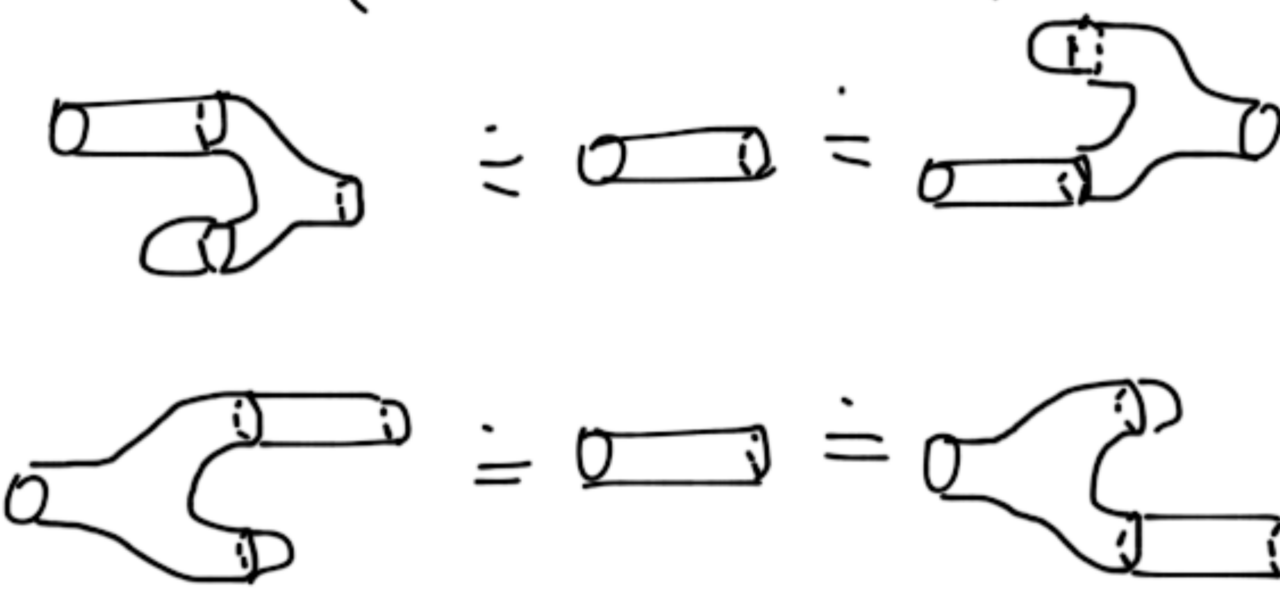
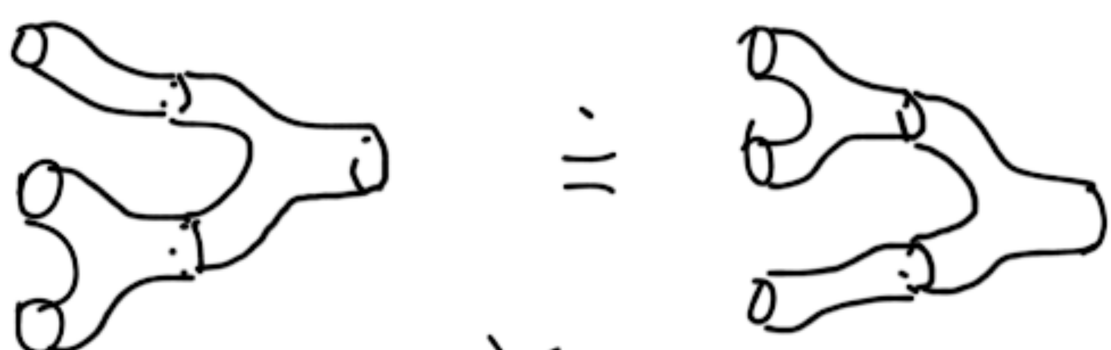
(SEWING IN DISCS)



(ASSOCIATIVITY)



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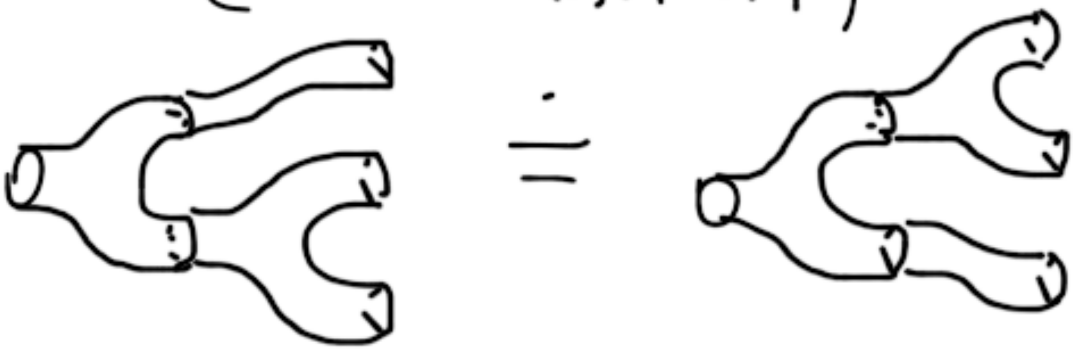
(COASSOCIATIVITY)



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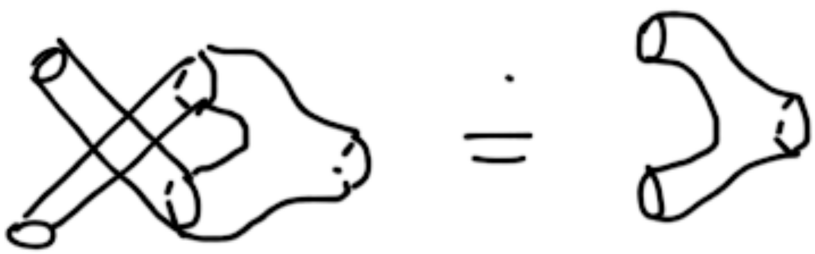
(COMMUTATIVITY)



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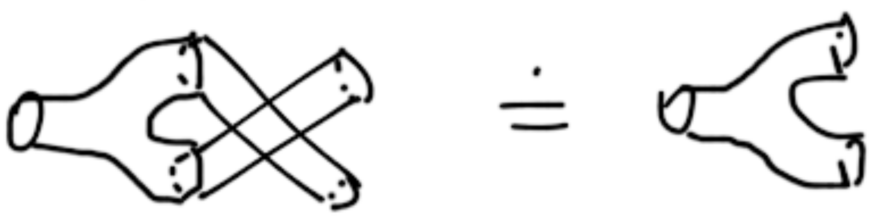
(COCOMMUTATIVITY)



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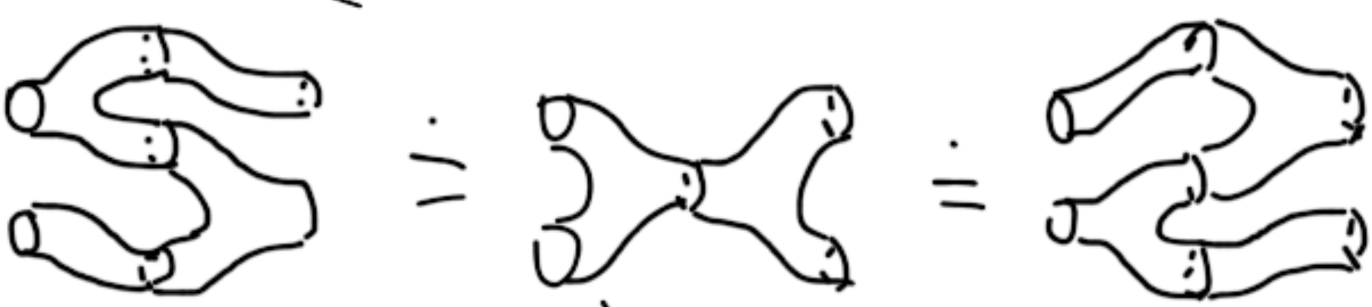
(FROBENIUS REL$^{\underline{N}}$)



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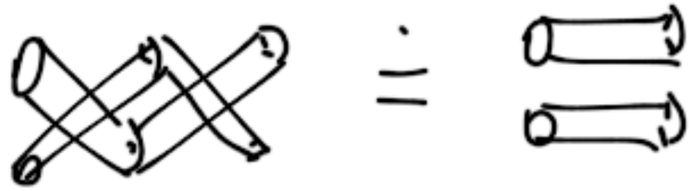
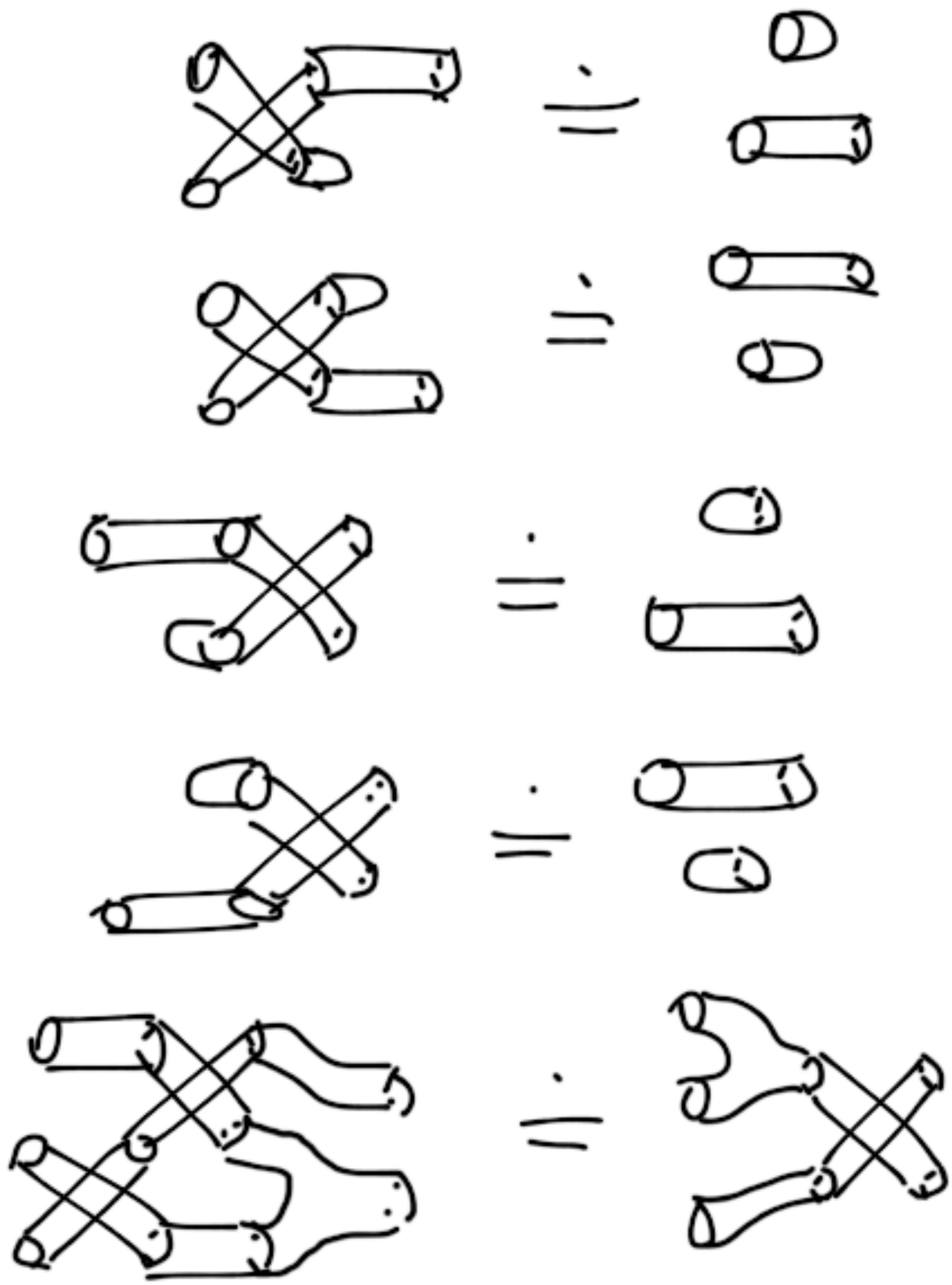
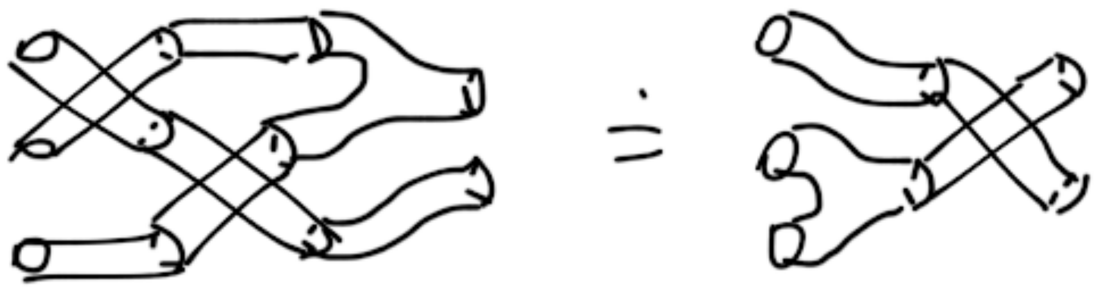
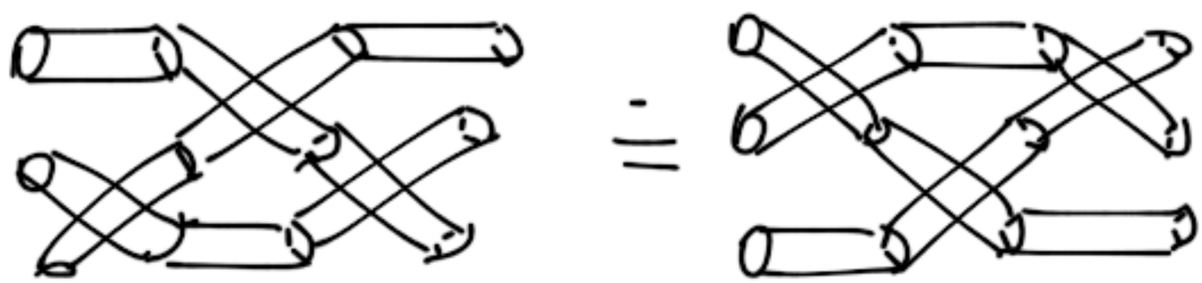
(SYMMETRY)



(NATURALITY OF BRAIDING)







<u>Proof</u>: Uses elementary combinatorics of surfaces, alongside the observation that all diffeos of S^1 are homotopically trivial ($\Leftarrow$ winding number necessarily $= 0$).

We shall next discuss the algebra of the TQFT realisation of the skeleton.

7.1.3° The algebra of the TQFT.

We set the stage for our considerations by recalling

Def$^{\underline{n}}$ 15.

An algebra over field $\mathbb{K}$ (of dimension D) is a septuple $(A, +, P, \cdot \mapsto 0_A, D, \mu, \eta)$ composed of a vector space $(A, +, P, \cdot \mapsto 0_A, D)$ over $\mathbb{K}$ (of $\dim_{\mathbb{K}} A = D$) & the maps

- multiplication $\mu : A \otimes_{\mathbb{K}} A \longrightarrow A$,

- unit $\eta : \mathbb{K} \longrightarrow A$.

subject to the axioms expressed by the following commutative diagrams:

(Associativity)

$$\begin{tikzcd}
 & A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A \arrow[dl, "\mu \otimes id_A"'] \arrow[dr, "id_A \otimes \mu"] & \\
A \otimes_{\mathbb{K}} A \arrow[dr, "\mu"'] & \circlearrowleft & A \otimes_{\mathbb{K}} A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}$$

(Unit)

$$\begin{tikzcd}
\mathbb{K} \otimes_{\mathbb{K}} A \arrow[r, "\eta \otimes id_A"] \arrow[dr, "\simeq"', "\text{(can.)}"'] & A \otimes_{\mathbb{K}} A \arrow[d, "\mu"] & A \otimes_{\mathbb{K}} \mathbb{K} \arrow[l, "id_A \otimes \eta"'] \arrow[dl, "\simeq", "\text{(can.)}"] \\
 & A &
\end{tikzcd}$$

——— X ———

Next, we introduce the main object of interest:

Defn 1.6.

A **Frobenius algebra over field** $\mathbb{K}$ is an octuple $(A, +, P, \cdot \mapsto 0_A, \Delta, \mu, \eta, \varepsilon)$ composed of an algebra $(A, +, P, \cdot \mapsto 0_A, \Delta, \mu, \eta)$ over $\mathbb{K}$, of dimension $\dim_{\mathbb{K}} A < \infty$, and a $\mathbb{K}$-linear functional $\varepsilon : A \to \mathbb{K}$, termed the **Frobenius form**, such that

$$\mathrm{Null}(\varepsilon) := \{ v \in A \mid \varepsilon(v) = 0_{\mathbb{K}} \}$$

contains no non-trivial (principal) left ideals.

———— X ————

It is useful to seek alternative definitions of a Frobenius algebra. To this end, we give

Prop.^{n} 3. Let $(A, +, \mathbb{K}\cdot \mapsto 0_A, \Delta, \mu, \eta)$ be an algebra over field $\mathbb{K}$, & let $\varepsilon \in A^*$. Denote the associative pairing on A induced by ε as

$$\langle \cdot , \cdot \rangle := \varepsilon \circ \mu : A \otimes A \to \mathbb{K}.$$

Then, the following are equivalent:

(i) $\langle \cdot , \cdot \rangle$ is non-singular;

(ii) $\mathrm{Null}(\varepsilon)$ contains no nontrivial left ideals;

(iii) ——— " ——— right ideals.

———X———

Proof:

$$\begin{array}{ccccc} \langle A, x \rangle = 0 & \Rightarrow & x = 0 & & (i) \\ ||| & & \Updownarrow & & \\ \varepsilon(A \cdot x) = 0 & \Rightarrow & x = 0 & & (ii) \\ & & \Updownarrow & & \\ & & (iii) & & \end{array}$$

by non-degeneracy in the other argument. $\square$

The above leads us to

Defn 16'.

A **Frobenius algebra over field** $\mathbb{K}$ is an algebra over $\mathbb{K}$ of finite dimension, equipped with an associative non-singular pairing $\beta : A \otimes_{\mathbb{K}} A \to \mathbb{K}$, termed the **Frobenius pairing**.

Of course, the existence of a non-singular pairing gives us yet another alternative

Defn 16''.

A **Frobenius algebra over field** $\mathbb{K}$ is an algebra over $\mathbb{K}$ of finite dimension, equipped with a left/right A-linear isomorphism $A \cong A^*$.

Finally, we come to

Def^n 17.

A Frobenius algebra over field $\mathbb{K}$ is called <u>symmetric</u> iff one of the following equivalent conditions holds:

(i) the Frobenius form $\varepsilon \in A^*$ is <u>central</u>, i.e.,

$$\forall a, b \in A : \varepsilon(a \cdot b) = \varepsilon(b \cdot a) \quad (\text{TRACE COND}^{\underline{n}});$$

(ii) the Frobenius pairing $\langle \cdot, \cdot \rangle \in (A \otimes_{\mathbb{K}} A)^*$ is <u>symmetric</u>, i.e.,

$$\forall a, b \in A : \langle a, b \rangle = \langle b, a \rangle;$$

(iii) the left (resp. right) A-linear iso $A \cong A^*$ is also right (resp. left) A-linear.

———×———

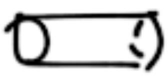
E.g. 1 (i) $(\mathbb{K}, \mathrm{id}_{\mathbb{K}})$ (trivial)

(ii) $(\mathbb{C}, \mathrm{Re})$

(iii) $(\mathrm{Mat}(N\times N;\mathbb{K}), \mathrm{Tr})$ (matrix algebra)

———— × ————

The axioms of the FA admit a suggestive graphical presentation based on the pictionary:

$\mathrm{id}_A \doteq$  ; $\mu \doteq$  ; η : 

$\varepsilon \doteq$  ; $\beta \doteq$  $\equiv$  $\doteq \varepsilon\circ\mu$

We may employ the ensuing pictography to extract further structure from the axioms.

Def$^{\underline{n}}$ 18. A coalgebra over field $\mathbb{K}$ (of dimension D) is a septuple $(A, +, P, \cdot \mapsto 0_A, D, \delta, \varepsilon)$ composed of a vector space $(A, +, P, \cdot \mapsto 0_A, D)$ over $\mathbb{K}$ (of $\dim_{\mathbb{K}} A = D$) & the two maps:

- comultiplication $\delta : A \to A \otimes_{\mathbb{K}} A$,
- counit $\varepsilon : A \to \mathbb{K}$,

subject to the axioms expressed by the following commutative diagrams:

(Coassociativity)

$$\begin{tikzcd}
 & A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A \arrow[dl, "\mu \otimes id_A"'] \arrow[dr, "id_A \otimes \mu"] & \\
A \otimes_{\mathbb{K}} A \arrow[dr, "\mu"'] \arrow[rr, phantom, "\circlearrowleft"] & & A \otimes_{\mathbb{K}} A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}$$

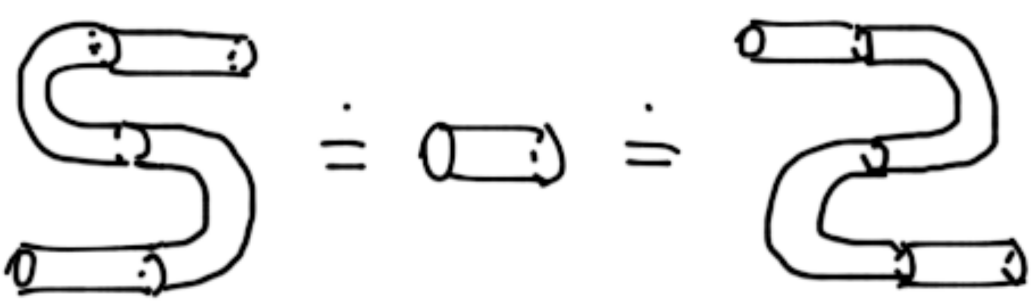
(Commit)

$$\begin{tikzcd}
\mathbb{K}\otimes_{\mathbb{K}} A \arrow[r, "\eta\otimes \mathrm{id}_A"] \arrow[dr, "\cong"', "(\mathrm{can.})"] & A\otimes_{\mathbb{K}} A \arrow[d, "\mu"'] & A\otimes_{\mathbb{K}}\mathbb{K} \arrow[l, "\mathrm{id}_A\otimes\eta"'] \arrow[dl, "\cong", "(\mathrm{can.})"'] \\
& A &
\end{tikzcd}$$

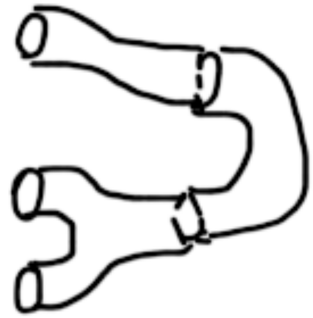
———— X ————

With view to constructing δ for a given FA, we first extract from its def^{n} that of a copairing s.th.

$$(\mathrm{id}_A\otimes\beta)\circ(\gamma\otimes\mathrm{id}_A)=\mathrm{id}_A=(\beta\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\gamma)$$



The construction uses the auxiliary object:

$\phi := (\mu \otimes \mathrm{id}_A) \circ \beta \doteq$  $\equiv$  ,

3-POINT FUNCTION

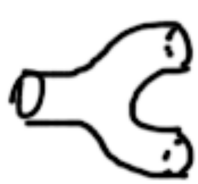
for which we readily prove

 $\doteq$  $\doteq$  .

This further shows that

 $\doteq$  ,

& so we may define

FROBENIUS COMULTIPLICATION  $:=$ 

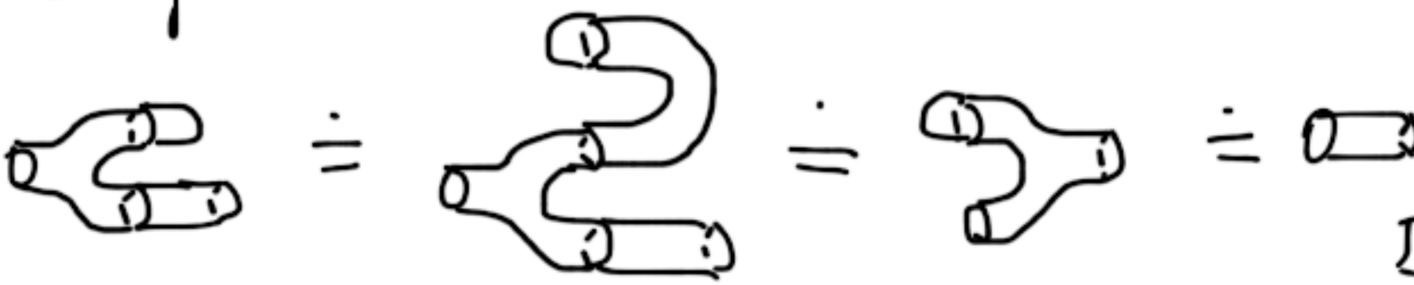
We readily check that

Prop^n 4.

The Frobenius form is a counit for the Frobenius comultiplication.

——— x ———

Proof:

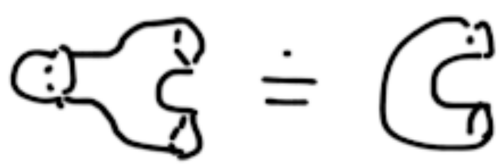


□

——— x ———

Prop^n 5.

The Frobenius comultiplication is coassociative, satisfies



& the Frobenius rel^n (cf p. 85).

——— x ———

The uniqueness of the copairing associated with a given pairing on A implies

Prop.^{n} 6.

For every FA, there exists a unique coassociative comultiplication whose counit is the Frobenius form, & such that the Frobenius rel^{n} is obeyed.

—————— x ——————

In fact, even more is true.

Prop.^{n} 7.

Let $(A, +, P, \cdot \mapsto 0_A, \triangleright)$ be a vector space over field $\mathbb{K}$, equipped with multiplication $\mu : A \otimes_{\mathbb{K}} A \to A$, unit $\eta : \mathbb{K} \to A$, comultiplication $\delta : A \to A \otimes A$ & counit $\varepsilon : A \to \mathbb{K}$ for which the Frobenius rel^{n} holds true. Then,

(i) $\dim_{\mathbb{K}} A < \infty$;

(ii) μ is associative ($\Rightarrow$ $\mathbb{K}$-algebra th.);

(iii) ε is a Frobenius form
($\Rightarrow$ Frobenius algebra th.).

Proof: Write $\beta := \varepsilon \circ \mu$ & prove its non-degeneracy ($\equiv$ snake rel^{n} with $\gamma := \delta \circ \eta$). $\square$

The last notion that we need is

Def^{n} 19.
A homomorphism of FA's is an algebra & coalgebra homomorphism. Thus, in particular, it preserves the Frobenius form.

Our hitherto discussion yields the category

$$(c)\ \mathsf{FA}_{\mathbb{K}}$$

of (commutative) Frobenius algebras over field $\mathbb{K}$.

We have the important

Prop. 9.

Let $(A_i, +_i, P_i, \hookrightarrow O_{A_i}, \Delta_i, \mu_i, \eta_i, \varepsilon_i)$, $i \in \{1, 2\}$ be two FA's, & let $\phi : A_1 \to A_2$ be an algebra homomorphism compatible with the Frobenius forms, i.e., such that the diagram

$$\begin{tikzcd} A_1 \arrow[rr, "\phi"] \arrow[dr, "\varepsilon_1"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & A_2 \arrow[dl, "\varepsilon_2"] \\ & \mathbb{K} & \end{tikzcd}$$

is commutative.

Then, ϕ is a monomorphism.

——— x ———

Proof: $\mathrm{Null}(\phi) \subset \mathrm{Null}(\mathcal{E}_1)$ & it is an ideal since $\phi(a \cdot_1 b) = \phi(a) \cdot_2 \phi(b)$.
However, $\mathrm{Null}(\mathcal{E}_1)$ contains no non-trivial ideals, whence $\mathrm{Null}(\phi) = \{0_{A_1}\}$. $\square$

In conclusion, we state

Prop.^n 10. (c)FA_K is a groupoid.

Proof: ϕ is comultiplicative & it respects counits as a coalgebra homo.
Therefore, the dual map $\phi^*: A_2^* \to A_1^*$ is multiplicative & respects units of the dual FA's. By Prop.^n 9.,
the latter is injective, & so the former is surjective. But it is also injective by Prop.^n 9., whence the claim. $\square$

The foregoing discussion prepares the ground for the fundamental

Thm 8.

There exists a canonical equivalence of categories

$$2\mathrm{TQFT}^{[\varnothing,1]}_{\mathbb{K}} \simeq \mathrm{cFA}_{\mathbb{K}}.$$

(with monoidal natural transformations)

Proof: Obvious from the dictionary of p. 94 (cf. Thm 8.).

7.2° The Turaev-Reshetikhin TQFT

7.2.1° Geometric preliminaries

Thm 9. [The Heegaard thm]

Every compact orientable (triangulable) 3-manifold admits a decomposition

$$M \simeq H_g \sqcup_{\chi} \widetilde{H}_g$$ HEEGARD SPLITTING

$H_g, \widetilde{H}_g$ - solid handlebodies of genus $g \in \mathbb{N}$

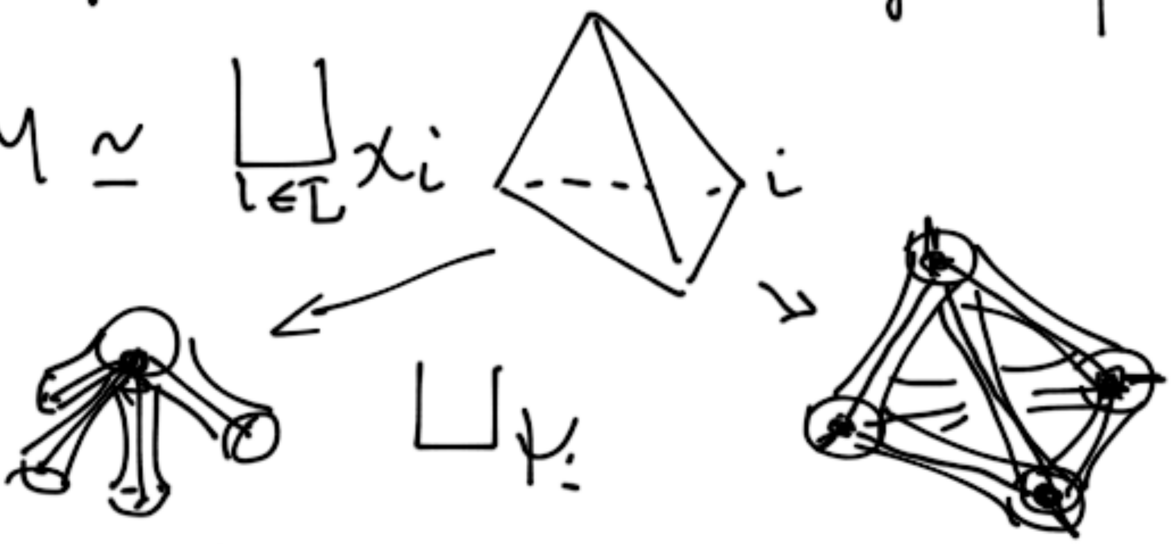
$$\chi : \Sigma_g = \partial H_g \xrightarrow{\simeq} -\partial \widetilde{H}_g = -\widetilde{\Sigma}_g$$

gluing homeo

———— x ————

Proof: 2^{nd} barycentric subdivision of a given triangulation of M yields

$$M \simeq \bigsqcup_{i \in I} X_i$$



$$\sqcup\, Y_i$$

E.g., $\mathbb{S}^3 \simeq \mathbb{T}^2 \sqcup_\chi \widetilde{\mathbb{T}}^2$, where χ glues the meridian μ (resp. the equator ε) of $\mathbb{T}^2$ to the equator $\tilde{\varepsilon}$ (resp. the meridian $\tilde{\mu}$) of $\widetilde{\mathbb{T}}^2$.

If, on the other hand, we glue by $\check{\chi} : (\mu, \varepsilon) \mapsto (\tilde{\mu}, \tilde{\varepsilon})$, we obtain

$$\mathbb{T}^2 \sqcup_{\hat{\chi}} \widetilde{\mathbb{T}}^2 \simeq \mathbb{S}^2 \times \mathbb{S}^1.$$

Important corollary:

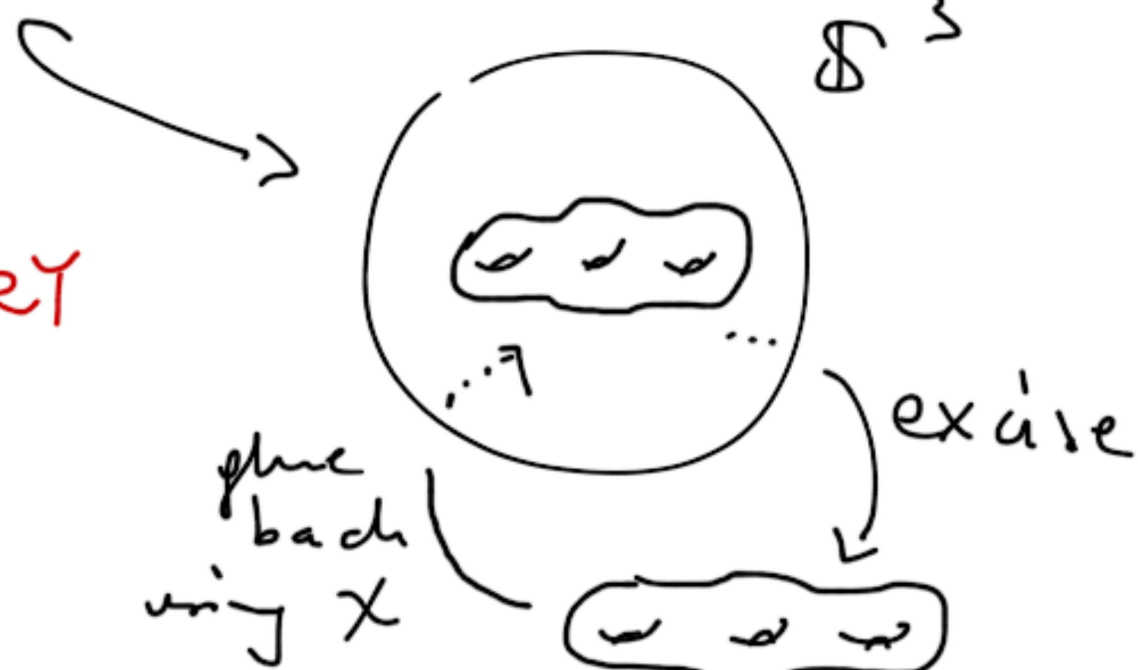
Topology of a 3-manifold encoded in the gluing homeo

$\chi : \Sigma_g \xrightarrow{\simeq} \tilde{\Sigma}_g$.

Here:

$M \simeq H_g \sqcup_\chi \tilde{H}_g$ $\qquad$ $H_g \sqcup_{\chi_0} \tilde{H}_g \cong S^3$

SURGERY

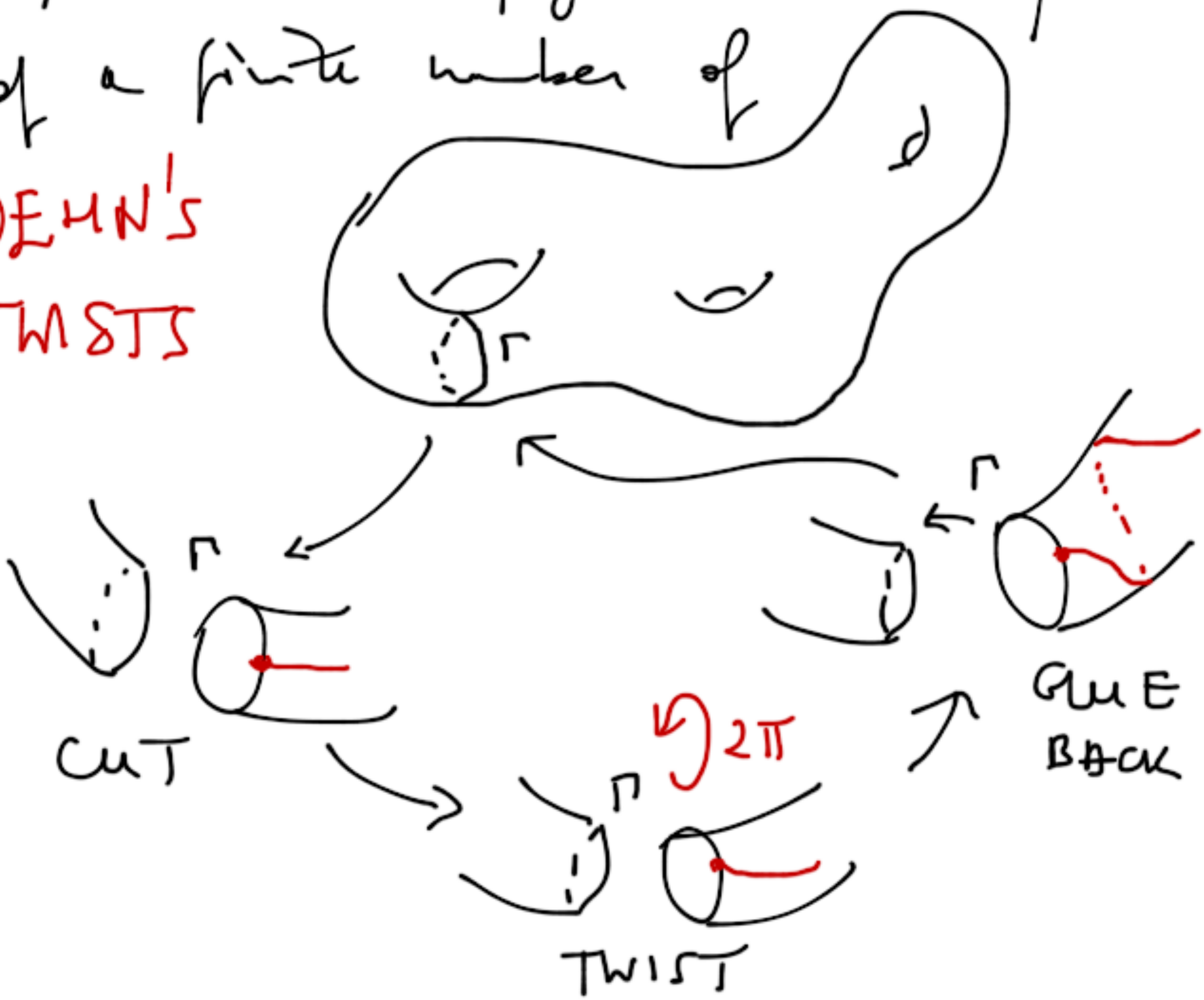


Question: How to classify 2-homeos?

$\text{Th}^{\underline{m}}$ 10. [The Dehn–Lickorish $\text{Th}^{\underline{m}}$]

Every orientation-preserving homeo of an oriented closed 2-manifold can be presented, up to isotopy, as the composition of a finite number of

DEHN'S TWISTS



Proof : uses
- classification of 2-manifolds (genus, # of connected boundary cycles)
- classification of homeos of a punctured disc (twists around cycles)

We thus arrive at the all-important

Thm 11. [The Surgery Thm]
Every compact orientable 3-manifold can be obtained through **Dehn's surgery** on S^3 along an unknotted framed link.

Idea of proof:

Apply

$$
\begin{tikzcd}
X_1 \arrow[r, "\xi_1"] \arrow[d, "\varphi"', "\partial"] \arrow[dd, bend left] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{X}_1 \arrow[d, "\partial"', "\tilde{\varphi}"] \arrow[dd, bend left] \\
X_2 \arrow[r, "\xi_2"'] \arrow[d, "\sqcup"'] \arrow[dr, phantom, "\Downarrow"] & \tilde{X}_2 \arrow[d, "\sqcup"] \\
M \arrow[r, "\simeq"] & \tilde{M}
\end{tikzcd}
$$

to

$$
\begin{array}{l}
M = H_g \sqcup_X \tilde{H}_g \\
\wr\!\!\mid \quad \xi_1 \downarrow \qquad\qquad \downarrow \xi_2 = id \\
\tilde{M} = H_g \sqcup_{\tilde{X}} \tilde{H}_g \subset S^3
\end{array}
$$

with ξ_1 fixed by $H_g = (H_g \setminus \mathbb{T}^2) \sqcup_{id} \mathbb{T}^2$

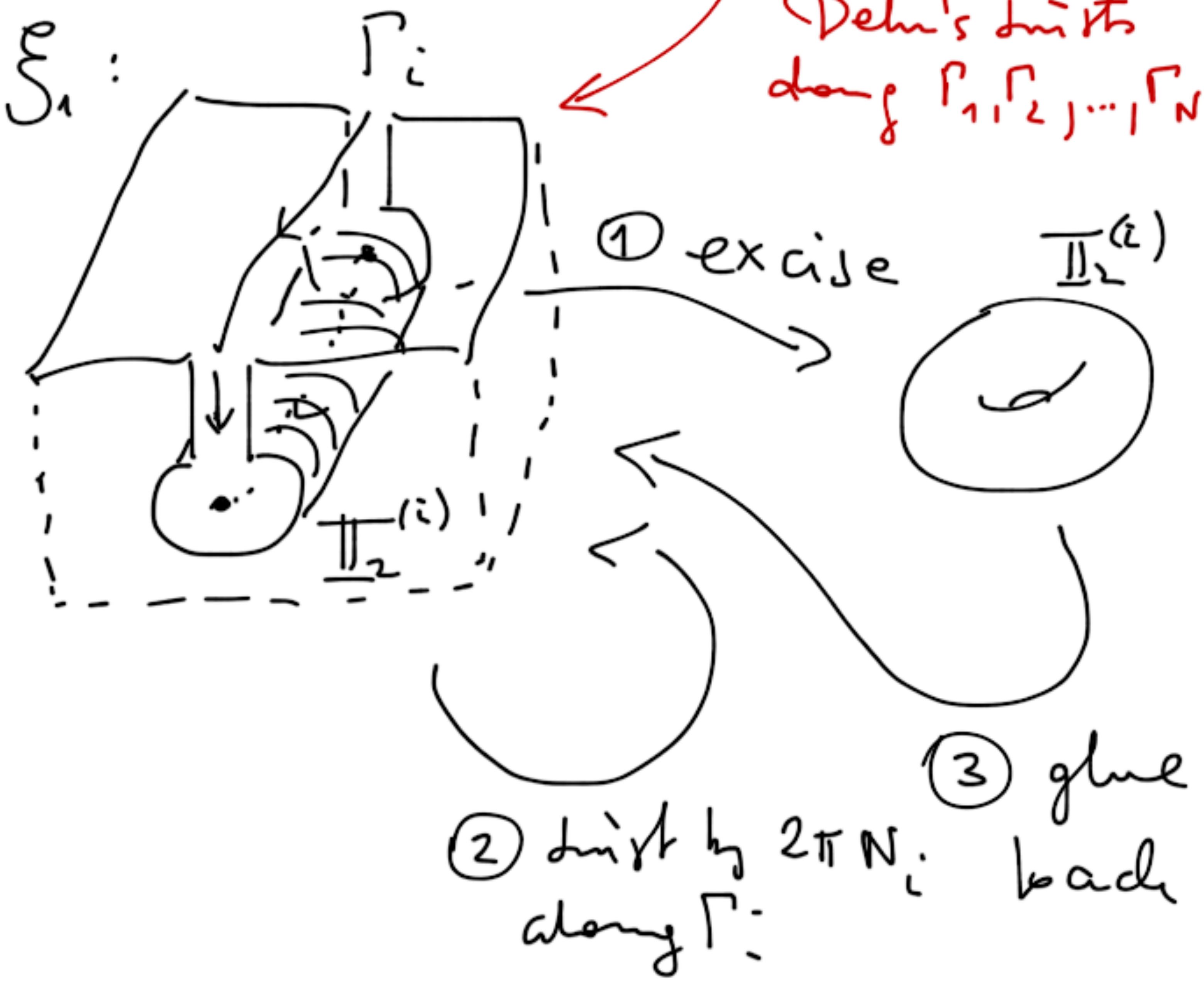
as follows:

Let the re-gluing homeo

$$\underline{\chi} : \partial H_g \longrightarrow \partial \tilde{H}_g \subset S^3$$

be given as $\underline{\chi} = D_1 \circ D_2 \circ \dots \circ D_N$

Dehn's twists along $\Gamma_1, \Gamma_2, \dots, \Gamma_N$



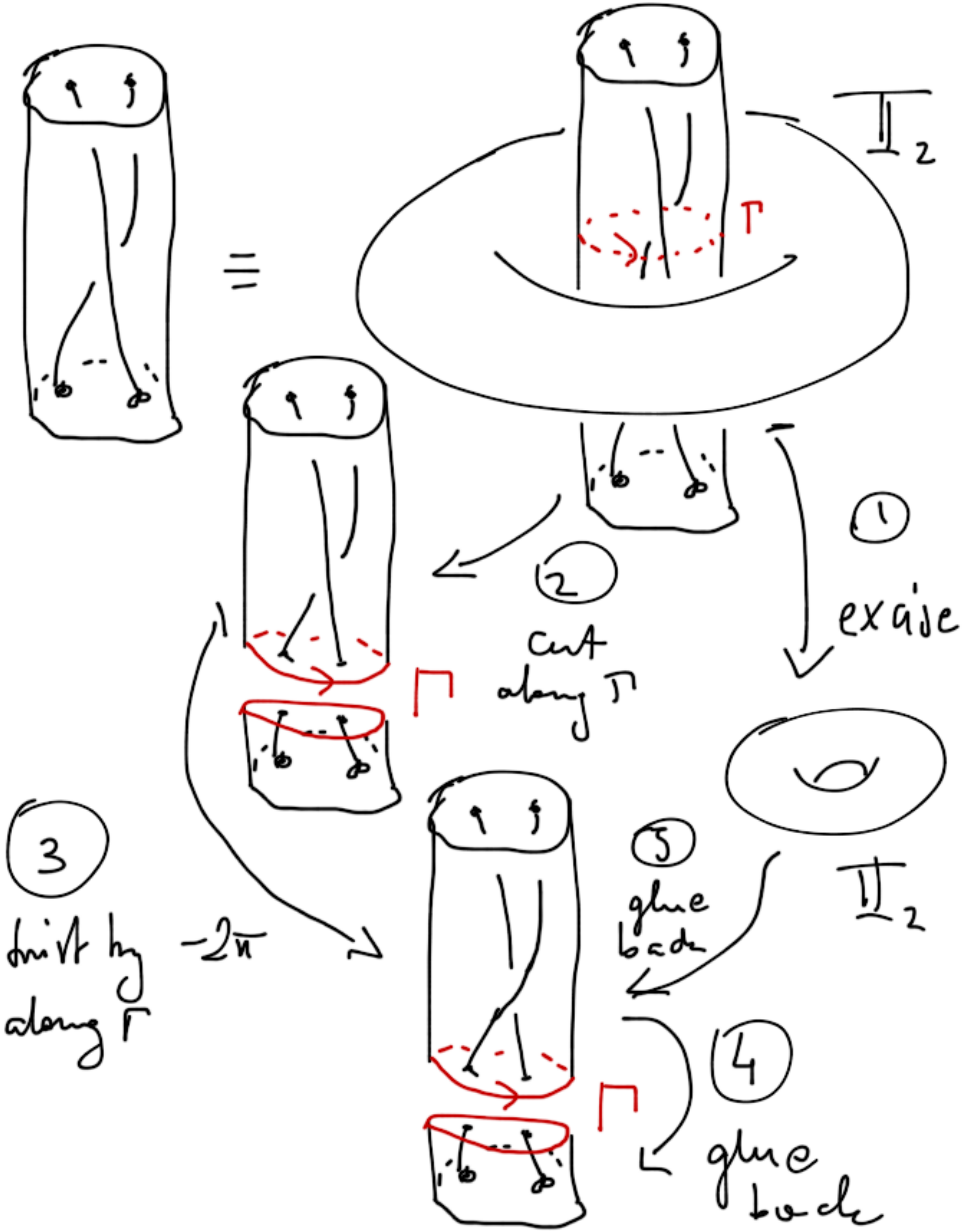
Net effect:

Topology of M encoded in the gluing of the solid tori $\mathbb{T}_2^{(i)}$ into $H_g \subset H_g \sqcup_{\chi_0} \overline{H}_g \simeq S^3$

as per ς: $\mu_i \longmapsto \varepsilon_i + N_i\mu_i$

$\varepsilon_i \longmapsto -\mu_i$

The Γ_i might be knotted, but a homeomorphic re-crossing operation unties all knots, at the expense of extra Dehn's surgeries.

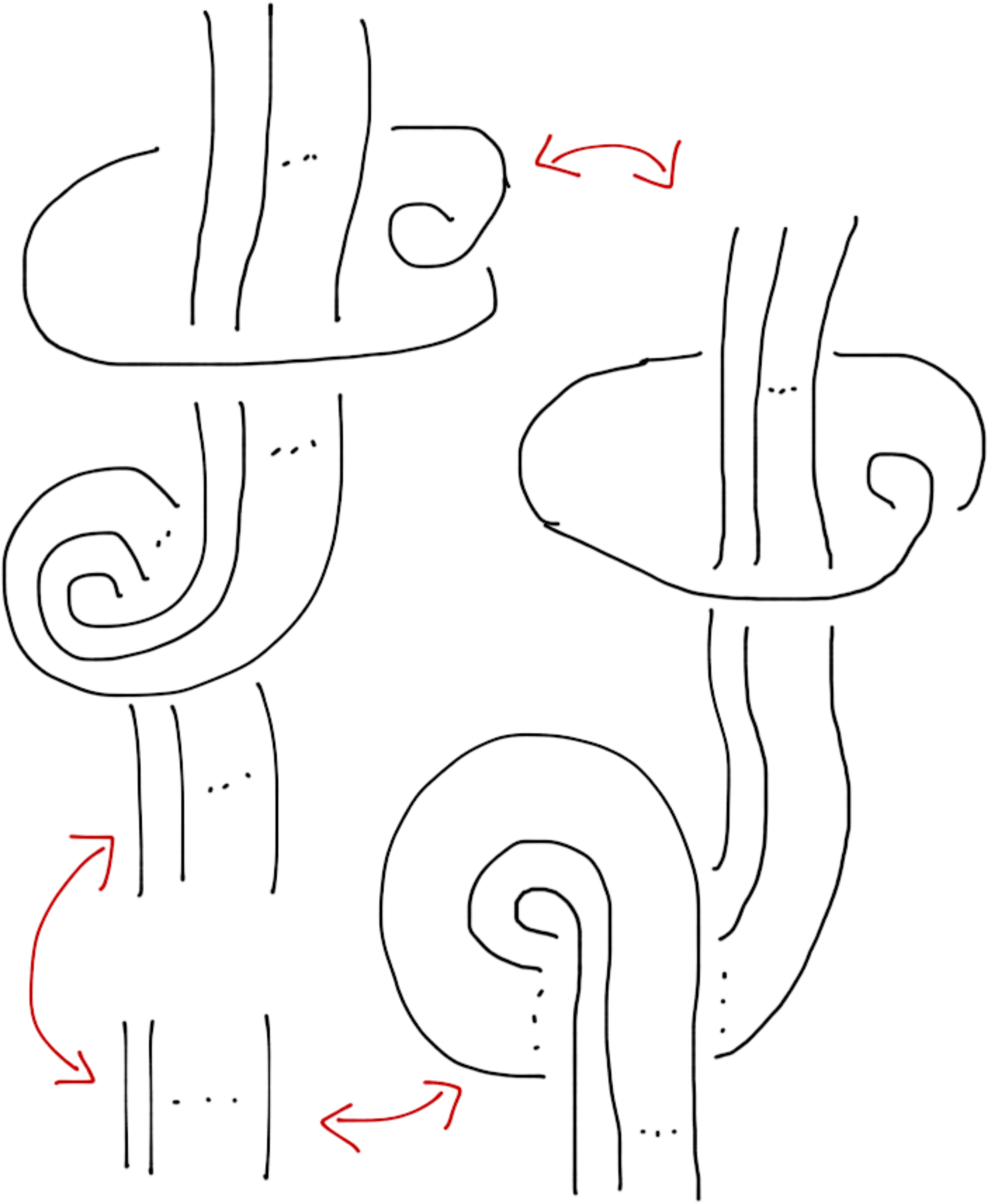


=
$\amalg_2$
Γ
1
excise
2
cut along Γ
Γ
3
twist by -2π along Γ
5
glue back
$\amalg_2$
4
Γ
glue back

Thus, the essential information on the mapping S can be encoded in a link embedded in S^3 & composed of UNKNOTS Γ_i with FRAMINGS N_i. $\square$

We have the crucial

Thm 12. [The Kirby-Fenn-Rourke Thm]
Two framed links in S^3 yield homeomorphic 3-manifolds via Dehn's surgery iff they are related by FR moves & isotopies,



Upshot: We should construct

FRAMED-LINK INVARIANTS

i.e. mappings

$$\left\{ \begin{array}{c} \text{Isotopy classes} \\ \text{of framed links} \\ \text{in } S^3 \end{array} \right\} \longrightarrow \mathbb{K} \text{ (field)}$$

invariant under the KFR calculus.

To this end, we shall first categorify the geometric notions introduced heretofore.

By way of illustration of the idea of encoding topological information in an algebraic structure, we briefly review the construction of knot polynomials. To this end, we introduce

Def^{n} 20.

A triple (L_+, L_-, L_0) of oriented links in $\mathbb{R}^3$ is termed a Conway triple iff the three links can be represented by link diagrams (D_+, D_-, D_0) in $\mathbb{R}^2$ isotopic to one another outside the respective discs $\mathbb{D}_+, \mathbb{D}_-$ & $\mathbb{D}_0$, & the tangles $D_x \cap \mathbb{D}_x$, $x \in \{+, -, 0\}$

are isotopic – respectively – to

$X_+ \doteq$  , $X_- \doteq$  & $X_0 \doteq$  .

———— x ————

In order to set the stage for the subsequent algebraic discussion, we give also

Def^n 21. Let R be a commutative ring, & let <u>Link</u> denote the set of isotopy classes of oriented links in $\mathbb{R}^3$.

Consider the free R-module $R[\underline{\text{Link}}]$ of (formal) R-linear combinations of elements of <u>Link</u>. A <u>Conway module</u> is a submodule $\text{Conway} \subset R[\underline{\text{Link}}]$ generated R-linearly by elements of the form

$$r_1[L_+] + r_2[L_-] + r_3[L_0] \quad \text{with } r_1, r_2, r_3 \in R^\times$$

& (L_+, L_-, L_0) arbitrary Conway triples.
In this context, the relation

$$r_1[L_+] + r_2[L_-] + r_3[L_0] = 0 \quad (sk)$$

is termed the skein relation.

A skein module is a quotient module $R[\underline{\text{Links}}]/\text{Conway}$.

×

<u>Remark</u>: Skein rel^{ns} of the "recrossing" type (sk) have played a central rôle in the construction of algebraic invariants of knots. They can be regarded as multi-variable generalisations of the basic skein rel^{n} for unoriented knots: $[L_+] = r[L_0] + r^{-1}[\quad]$ (of the unknotting type), due to Kauffman.

We are now ready to phrase

Thm 13. [The LYMPHTOFU Polynomial]

Let $\mathbb{K}$ be a field, & let $R := \mathbb{K}[x, x^{-1}, y, y^{-1}]$ be the ring of Laurent polynomials in the variables x, y, with coefficients from $\mathbb{K}$. There exists a unique map

$$\mathcal{P} : \underline{\mathrm{Link}} \longrightarrow R : [L] \longmapsto \mathcal{P}[L]$$

that satisfies the conditions

(Skein)

$$x\,\mathcal{P}[L_+] - x^{-1}\mathcal{P}[L_-] = y\,\mathcal{P}[L_0]$$

for every Conway triple (L_+, L_-, L_0).

(Normalisation)

$$\mathcal{P}[\bigcirc] = 1_{\mathbb{K}}.$$

the unknot

<u>Proof</u>: Central to the proof is the following

<u>Lemma 3.</u> Let $R := \mathbb{K}[x, x^{-1}, y, y^{-1}]$ be as above, & let Skein be the skein module given by the coset of $R[\underline{\text{Link}}]$ wrt. the Conway submodule generated by elements

$$x[L_+] - x^{-1}[L_-] - y[L_0] \quad (\underline{Sk})$$

labelled by Conway triples (L_+, L_-, L_0).
The R-linear map

$$Q : R \to \text{Skein}$$

that satisfies the condition

$$Q(1_R) = [O] + \text{Conway}$$

is an <u>isomorphism</u> of R-modules.

——— × ———

Partial proof of Lemma 3.

We begin by demonstrating the surjectivity of Q, or – equivalently – the identity

$$\mathrm{Skein} = \langle [O] \rangle_R .$$

Let us first show that

trivial links

$$\mathrm{Skein} = \langle [O^{\otimes n}] \rangle_R^{n \in \mathbb{N}^\times} .$$

To this end, we consider the filtration

$$\mathrm{Skein}_0 \subset \mathrm{Skein}_1 \subset \ldots \subset \mathrm{Skein}_m \subset \ldots \subset \bigcup_{m \in \mathbb{N}} \mathrm{Skein}_m \equiv \mathrm{Skein}$$

of Skein by R-submodules Skein_m generated by isotopy classes of links with at most m crossing points. We proceed by induction on m.

1° m=0 – trivial

2° Assume that the assertion holds true for all m < M. We pick up from $[L]$ an arbitrary representative with precisely M crossing points. Fix a crossing for which $L = L_+$ or L_- & form the corresponding Conway triple (L_+, L_-, L_0). Clearly, $[L_0] \in \mathrm{Skein}_{M-1}$. Using the skein rel^n (Sk), we thus establish $[L_+] = x^{-2}[L_-] \bmod \mathrm{Skein}_{M-1}$. But every link can be trivialised by a finite number of recrossings, e.g.

& so L can be reduced to a trivial link modulo Skein_{M-1}, the latter being generated by the trivial links by the induction hypothesis.

In order to complete the proof of the surjectivity of Q, we should still reduce $[O^{\otimes n}]$ to an R-translate of $[O]$. For that, take an arbitrary representative of $[O^{\otimes n}]$ & pick up one of the n unknots. The latter can be isotopically deformed to

, thus defining $L_+ \in [O^{\otimes n}]$. Form the corresponding Conway triple (L_+, L_-, L_0), with $L_0 \supset$

$\equiv O^{\otimes 2}$, whence also $L_0 \in [O^{\otimes (n+1)}]$. We find

$$x[O^{\otimes n}] - x^{-1}[O^{\otimes n}] \equiv x[L_+] - x^{-1}[L_-]$$

$$= y[L_0] \equiv y[O^{\otimes(n+1)}],$$

& so $[O^{\otimes n}] = \left(\frac{x - x^{-1}}{y}\right)^{n-1}[O]$.

A proof of the surjectivity of Q, essentially algebraic in its nature, calls for lots of extra structure (enhanced R-matrices, functorial realisations of the category of tangles in the algebraic category $\mathrm{Vect}_{\mathbb{K}}$ etc.), so it will not be presented here.

◪

The existence of the isomorphism $Q : R \to$ Skein immediately implies the unique definition of the sought-after map

$$\mathcal{P}[L] := Q^{-1}([L] + \text{Conway}),$$

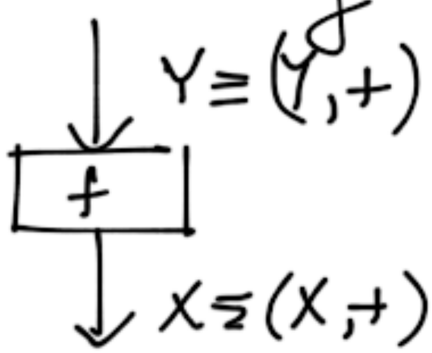
which concludes the proof of the Thm. □

7.2.2° Algebraic preliminaries

Our goal is to find a "minimal" algebraic structure that captures the geometry of framed links, or – more generally (anticipating the applications) – arbitrary ribbon graphs. In other words, we seek to define a category with the following property:

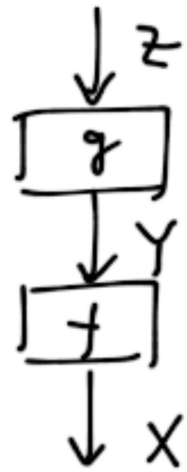
If we think of ribbons as objects of an abstract category, then the standard geometric operations performed on the (crossings, twists & bendings) should be reflected in the existence of *distinguished morphisms* in the abstract category, whereas isotopies of diagrams should be reflected in *coherence constraints* imposed upon those morphisms (& their consequences).

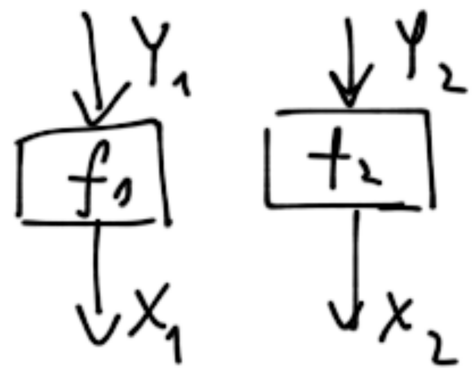
Thus, if we set up the following pictionary:

$$X \xrightarrow{f} Y \doteq$$ 

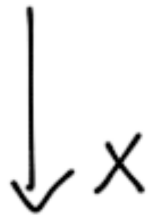
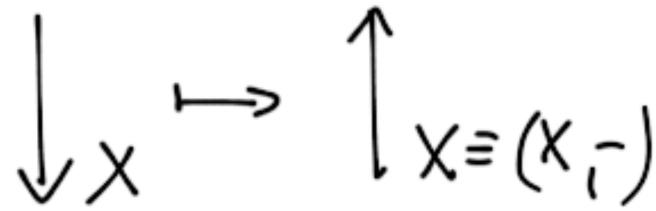
(isotopy classes)

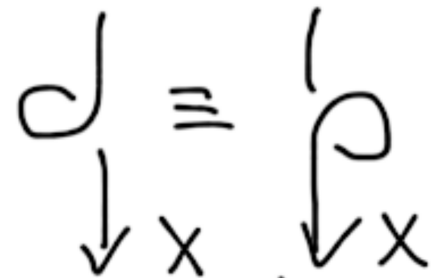
with the basic rules:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \doteq$$ 

$$X_1 \otimes X_2 \xrightarrow{f_1 \otimes f_2} Y_1 \otimes Y_2 \doteq$$ 

& naturality understood as "possibility to pull an arbitrary morphism étiquette through", we are led to request the existence of special morphisms:

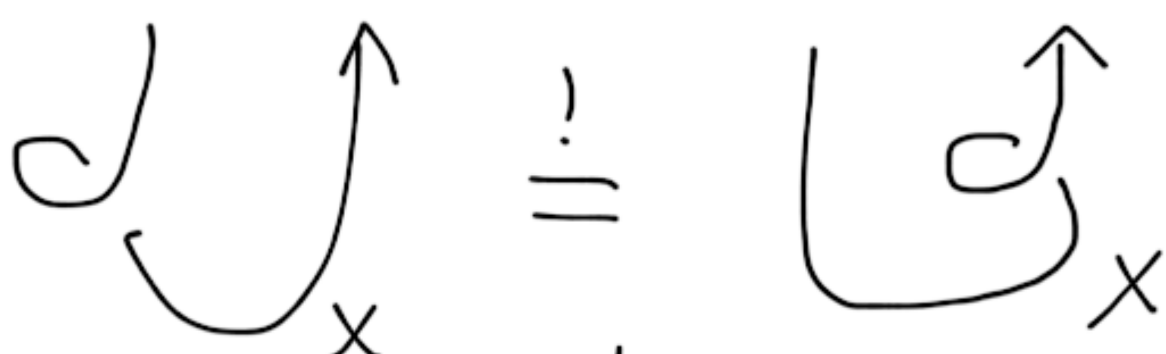
$\mathrm{id}_X \doteq$  $\ast :$ 

$c_{X,Y} \doteq$  $\theta_X \doteq$ 

(twist of the ribbon)

$b_X \doteq$  $d_X \doteq$  ,

subject to obvious "geometric" coherence constraints, e.g.,



i.e. $(\theta_X \otimes \mathrm{id}_{X^*}) \circ b_X \overset{!}{\doteq} (\mathrm{id}_X \otimes \theta_{X^*}) \circ b_X$.

We thus arrive at

Def$^{\underline{n}}$ 22. A strict **ribbon category** is an octuple

$$(\mathcal{C}, \otimes, \mathbb{1}, c, \theta, *, b, d)$$

composed of a monoidal category

$$(\mathcal{C}, \otimes, \mathbb{1}) \quad (\text{strict})$$

with **braiding** (transposition)

$$c : \otimes \xrightarrow[(\text{nat.})]{\cong} \otimes \circ \tau,$$

twist $\forall X \in \mathrm{Ob}\,\mathcal{C}: \theta_X : X \circlearrowleft^{\cong} (\text{nat.})$

& **duality** $X \mapsto (X^*, b_X, d_X)$,

where $b_X : \mathbb{1} \to X \otimes X^*$ & $d_X : X^* \otimes X \to \mathbb{1}$, subject to the following axioms:

(Hexagon)

$$
\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] \arrow[d, equal] & (Y \otimes Z) \otimes X \arrow[d, equal] \\
(X \otimes Y) \otimes Z \arrow[d, "c_{X,Y} \otimes \mathrm{id}_Z"'] \arrow[r, phantom, "\circlearrowleft"] & Y \otimes (Z \otimes X) \\
(Y \otimes X) \otimes Z \arrow[r, equal] & Y \otimes (X \otimes Z) \arrow[u, "\mathrm{id}_Y \otimes c_{X,Z}"']
\end{tikzcd}
$$

& a similar one for $(X \otimes Y) \otimes Z$;

(Rigidity)

$$
\begin{tikzcd}
& \mathbb{1}\otimes X \arrow[dl, equal] \arrow[dr, "b_X \otimes id_X"] & \\
X \arrow[dr, equal] & & X\otimes X^{*}\otimes X \arrow[dl, "id_X \otimes d_X"] \\
& X\otimes \mathbb{1} &
\end{tikzcd}
$$

& a similar one for X^{*};

(Balancing)

$$
\begin{tikzcd}
X\otimes Y \arrow[r, "\theta_X \otimes \theta_Y"] \arrow[d, "\theta_{X\otimes Y}"'] \arrow[dr, phantom, "\circlearrowleft"] & X\otimes Y \arrow[d, "c_{X,Y}"] \\
X\otimes Y & Y\otimes X \arrow[l, "c_{Y,X}"]
\end{tikzcd}
\quad ;
$$

(Compatibility)

$$\begin{tikzcd}
 & \mathbb{1} \arrow[dl, "b_X"'] \arrow[dr, "b_X"] & \\
X\otimes X^* \arrow[dr, "\theta_X\otimes \mathrm{id}_{X^*}"'] \arrow[rr, phantom, "\circlearrowleft"] & & X\otimes X^* \arrow[dl, "\mathrm{id}_X\otimes\theta_{X^*}"] \\
 & X\otimes X^* &
\end{tikzcd}$$

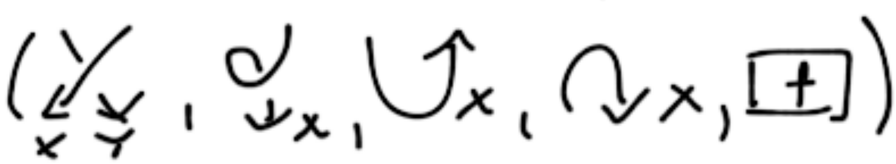
We have - for $\mathcal{R}ib_{\mathcal{C}}$: $\left| \begin{array}{l} \mathrm{Ob}\,\mathcal{R}ib_{\mathcal{C}} \ni (X,\pm) \\ \mathrm{Mor}\,\mathcal{R}ib_{\mathcal{C}} \ni \text{isotopy classes of } \mathcal{C}\text{-marked ribbon graphs} \end{array}\right.$

Thm 14. [Reshetikhin - Turaev]

$\exists!$ covariant tensor functor

OPERATOR INVARIANT $\quad F_{RT} : \mathcal{R}ib_{\mathcal{C}} \longrightarrow \mathcal{C}$

s.t. $F(X,+) = X$, $F(X,-) = X^*$ &

F: () $\mapsto (c_{X,Y}, \theta_X, d_X, b_X, f)$

Proof: Uses a presentation of $\mathcal{R}ib_{\mathcal{C}}$ & Reidemeister's theory of isotopies of (framed) links.

— X —

Elementary consequences of the axioms:

1° $\forall X \in \mathrm{Ob}\,\mathcal{C} \ \exists \ (X^*)^* \underset{(\text{nat.})}{\simeq} X$

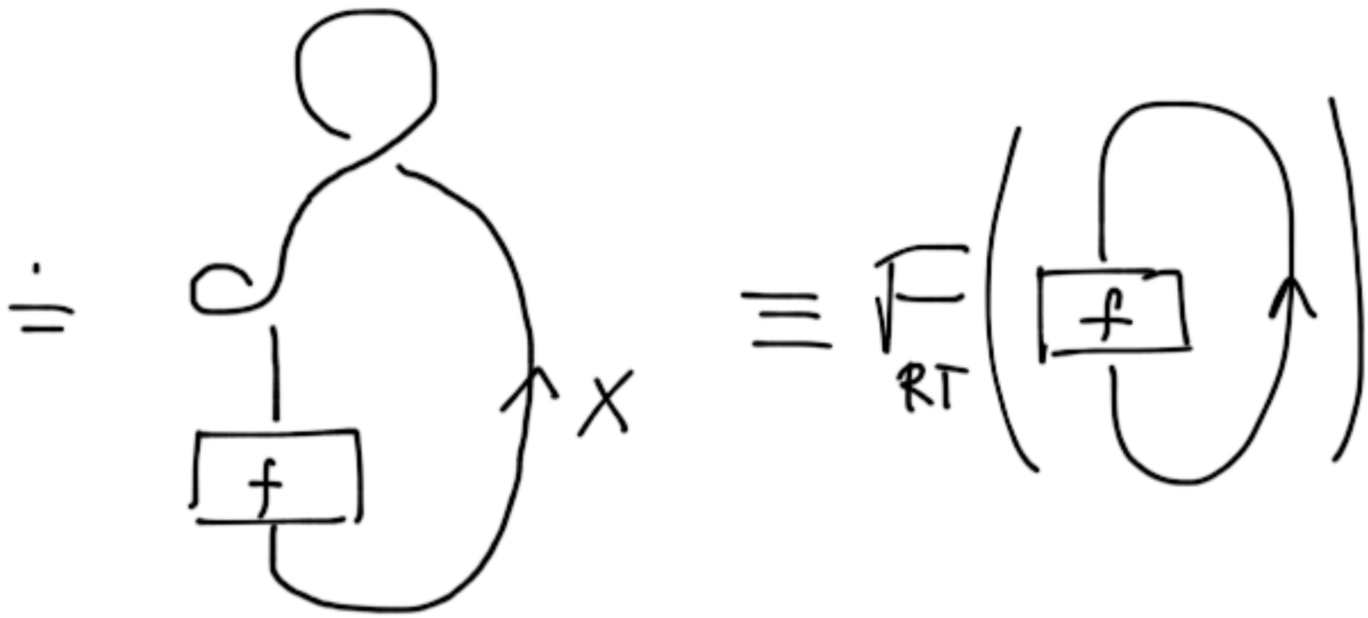
2° $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a commutative semigroup,

$$k \circ k' = (k \otimes id_{\mathbb{1}}) \circ (id_{\mathbb{1}} \otimes k') = k \otimes k'$$
$$= (id_{\mathbb{1}} \otimes k') \circ (k \otimes id_{\mathbb{1}}) = k' \circ k .$$

3° We have a meaningful

Def$^{\underline{n}}$ 23. Let $X \in \mathrm{Ob}\,\mathcal{C}$.
The <u>categorical trace</u> of $f \in \mathrm{End}_{\mathcal{C}} X$
is

$$\underline{\mathrm{tr}}\, f := d_X \circ c_{X,X^*} \circ (\theta_X \circ f \otimes \mathrm{id}_{X^*}) \circ b_X$$



& the <u>categorical dimension</u> of X
is

$$\underline{\dim}\, X := \underline{\mathrm{tr}}\, \mathrm{id}_X .$$

It turns out that we need further structure on $\mathcal{C}$ to be able to use it in a definition of a TQFT.

Def$\underline{\text{n}}$ 24. A *modular category* is a (strict) ribbon category

$$(\mathcal{C}, \otimes, \mathbb{1}, c, \theta, *, b, d)$$

which is *pre-abelian*, i.e., its Hom-sets are abelian groups & both $\circ$ & $\otimes$ are bilinear, & has a finite family

$$\{X_i\}_{i \in I}, \quad |I| < \infty$$

of *simple objects*, i.e. such that

$$R_{\mathcal{C}} := \mathrm{End}_{\mathcal{C}} \mathbb{1} \longrightarrow \mathrm{End}_{\mathcal{C}} X_i : k \mapsto k \otimes id_{V_i}$$

GROUND RING

is a bijection,

subject to the following axioms:

(Normalisation)

$$\exists\, 0 \in I : X_0 = \mathbb{1}$$

(Duality)

$$\forall i \in I\ \exists\, i^* \in I : X_{i^*} \simeq X_i^*$$

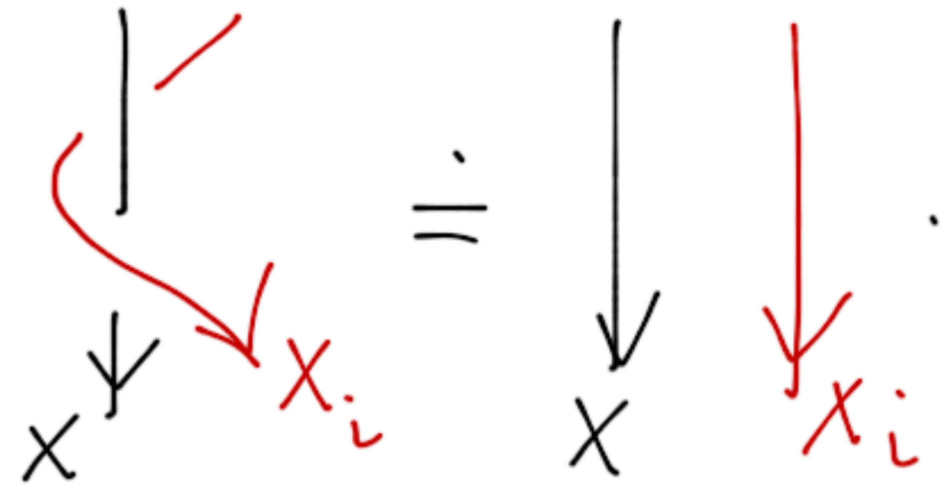
(Dominance)

$$\forall X \in \mathrm{Ob}\, \mathcal{C}\ \exists \begin{cases} \iota_i : X_i \to X \\ \pi_i : X \to X_i \end{cases} : \mathrm{id}_X = \sum_{i \in I} \iota_i \circ \pi_i$$

(Non-Degeneracy)

$$S := \Big(S_{ij} = \underline{\mathrm{tr}}\left(c_{X_j, X_i} \circ c_{X_i, X_j} \right) \Big)_{i,j \in I} \overset{!}{\in} GL(|I|, \mathbb{F}_{\mathcal{C}})$$

<u>Remark</u>: While there seems to be no <u>a priori</u> geometric justification of the first three axioms, the non-degeneracy is equivalent to the <u>non</u>-existence of transparent simple objects for which



<u>Elementary consequences of the axioms</u>:

1° $R_{\mathcal{C}} = \mathrm{End}_{\mathcal{C}} \mathbb{1} = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathbb{1})$ is a commutative ring.

2° $\mathrm{Hom}_{\mathcal{C}}(X_i,X_j) = \delta_{ij} R_{\mathcal{C}}$ (Schur Lemma)

3° The objects

$$\hat{s} := \mathcal{D}_e^{-1} \cdot S$$

$$\hat{t} := \left(\theta_i^{-1} \delta_{ij}\right)_{i,j \in I}$$

written in terms of

$$\begin{cases} \mathcal{D}_e := \sqrt[2]{\sum_{i \in I} \left(\underline{\dim} X_i\right)^2}, \\ \theta_{X_i} := \theta_i \otimes \mathrm{id}_{X_i} \end{cases}$$

RANK

satisfy identities

$$\hat{s}^4 = 1 \quad \& \quad \left(\hat{t}\,\hat{s}\right)^3 = \Delta_e \mathcal{D}_e^{-1} \hat{s}^2$$

with

$$\Delta_e := \sum_{i \in I} \theta_i^{-1} \cdot \left(\underline{\dim} X_i\right)^2 ,$$

& so they furnish a projective linear representation of $SL(2, \mathbb{Z})$.

E.g., Representations of any (extended) quasitriangular Hopf algebra form a ribbon category. In particular, the finite-dimensional irreps of highest (& lowest) weight of the Drinfeld-Jimbo quantum universal enveloping algebra $U_q(\mathfrak{g})$ over Lie algebra $\mathfrak{g}$, at q a root of unity of an even order compose such a category that actually extends to a full-fledged modular category [Turaev-Reshetikhin].

7.2.3° Sekhet Hetepet (Reloaded)

→ Fix a modular category $\mathcal{C}$.

→ Let M be a closed oriented 3-manifold with an embedded $\mathcal{C}$-coloured ribbon graph Ω.

→ Suppose M is the result of Dehn's surgery on $\mathbb{S}^3$ along an oriented framed link $L = \coprod_{k=1}^{N} L_k$.

→ Use an isotopy to have $\Omega \cap \overline{\mathrm{Tub}(L)} = \emptyset$.

→ Write $\mathrm{col}(L) := \{$colourings of the L_k by the $X_i\}$ & $L_\lambda \equiv L$ coloured as per $\lambda \in \mathrm{col}(L)$.

This prepared, we may state

$\underline{\text{Th}^{\text{m}}}$ 15. [The Reshetikhin–Turaev Topological Invariant]

$$\tau_{RT}(M,\Omega) := \mathcal{N}_{\mathcal{C}}(M) \cdot \sum_{\lambda \in col(L)} \prod_{k=1}^{m} \underline{\dim}\left(X_{\lambda(L_k)}\right) F_{RT}(L_\lambda \cup \Omega)$$

(with the scalar $\mathcal{N}_{\mathcal{C}}(M) \in R_{\mathcal{C}}$ expressed as the ratio of certain integer powers of $\mathcal{D}_{\mathcal{C}}$ & $\Delta_{\mathcal{C}}$)

is a topological invariant of (M,Ω).

$\underline{\text{Proof}}$: Uses all of it, & more.

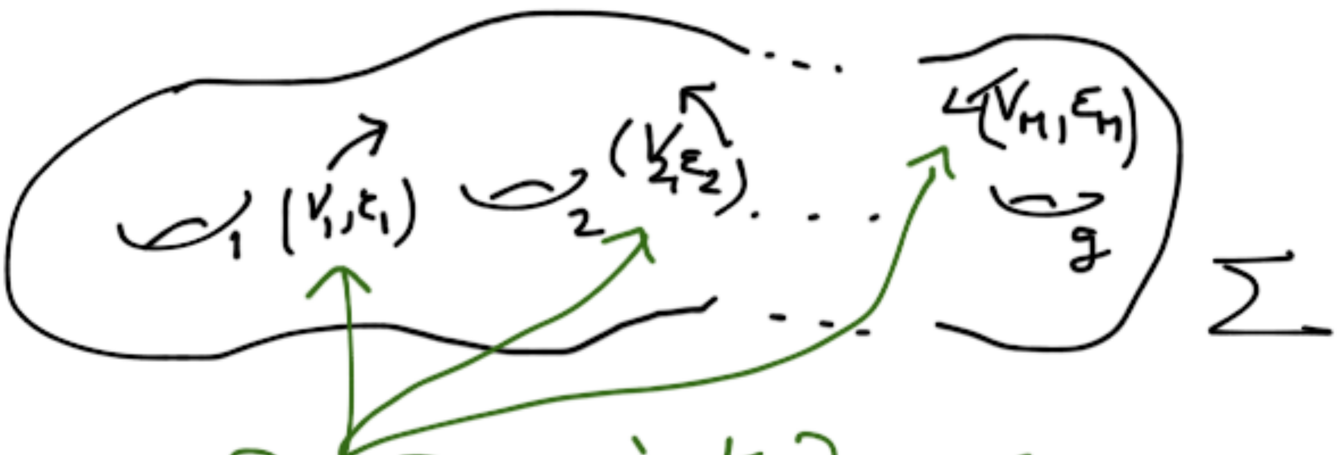
Def^{n} 25. Fix a modular category $\mathcal{C}$.

Involutive $A_{\mathcal{C}}$-structure : $\mathcal{C}$-decorated closed connected oriented 2-surfaces Σ

fixes homeo class of Σ

of TYPE $t := (g; (V_k, \varepsilon_k), k \in \overline{1,M})$

$V_k \in \mathrm{Ob}\,\mathcal{C}$, $\varepsilon_k \in \{\pm\}$



ordered oriented arcs

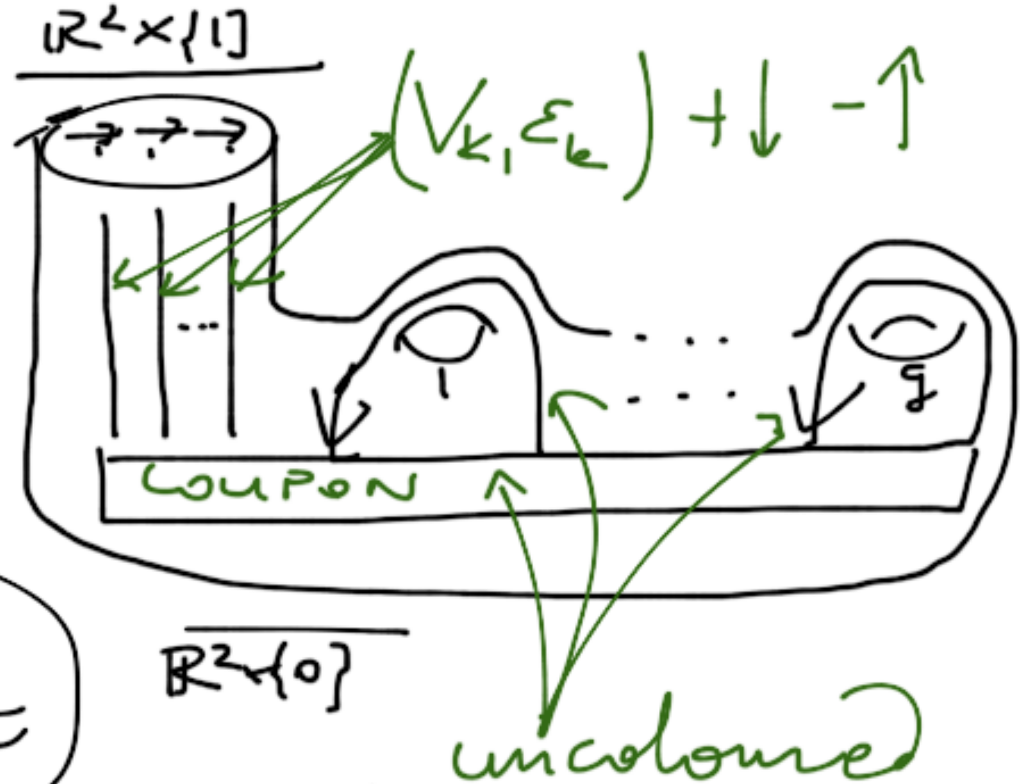
with involution $\equiv$ orientation flip

$\mathbb{Z}_2$

$$t \longmapsto -t := (g; (V_k, -\varepsilon_k), k \in \overline{1,M})$$

Standard ℓ-decorated surface:

$\mathbb{R}^2\times[0,1]\supset$



$$\partial H_t \equiv \Sigma_t$$

parametrise others

Modular functor: $\quad J(\Sigma \xleftarrow{x} \Sigma_t) := J(\Sigma_t)$

$$J(\Sigma_t) := \bigoplus_{i\in L^g} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \Phi(t;i))$$

module of states

where

$$\Phi(t;i) := V_1^{\varepsilon_1}\otimes V_2^{\varepsilon_2}\otimes\cdots\otimes V_n^{\varepsilon_n}\otimes\bigotimes_{r=1}^{g}\left(X_{i_r}\otimes X_{i_r}^*\right),$$

so $J(f)\equiv \mathrm{id}_{J(\Sigma)}$.

$\mathcal{B}_\mathcal{C}$-structure: $\mathcal{C}$-decorated 3-manifolds (with corners!)

Standard handlebodies: H_t & their mirrors $\tilde{H}_t$ wrt. $\mathbb{R}^2 \times \{\frac{1}{2}\}$ (with the cups reversed)

Given a $\mathcal{B}_\mathcal{C}$-space (M, Σ_-, Σ_+),

$$\tau(M) : \mathcal{T}(\Sigma_-) \longrightarrow \mathcal{T}(\Sigma_+)$$

is the adjoint of the $R_\mathcal{C}$-linear map $\mathcal{T}(\Sigma_-) \otimes_{R_\mathcal{C}} \mathcal{T}(\Sigma_+)^* \to R_\mathcal{C}$ induced as follows: Cap M with appropriate standard handlebodies & apply τ_{RT}, depending $R_\mathcal{C}$-linearly on colourings of coupons & cups/caps.

Th$^{\underline{m}}$ 16. [The Reshetikhin-Turaev TQFT]

$(\mathcal{T}, \tau)$ is a non-degenerate (anomalous) TQFT.

__Proof__: Well, it is a LONG way to Tipperary ...

———————— X ————————

The End